New analytical results for massive inflationary correlators at tree and loop orders

Zhong-Zhi Xianyu (鲜于中之)

Department of physics, Tsinghua University

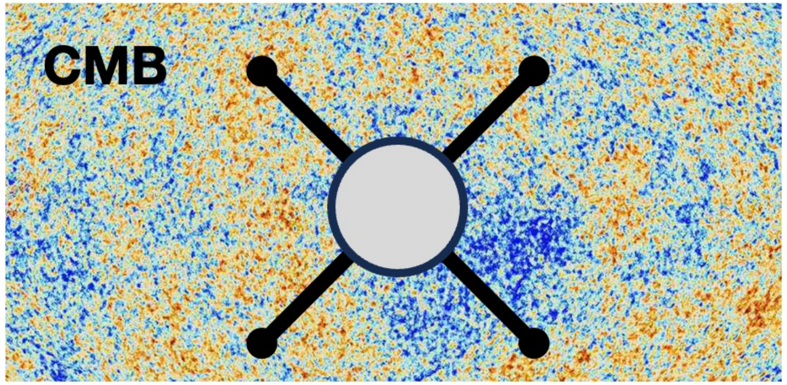
Looping in the primordial universe

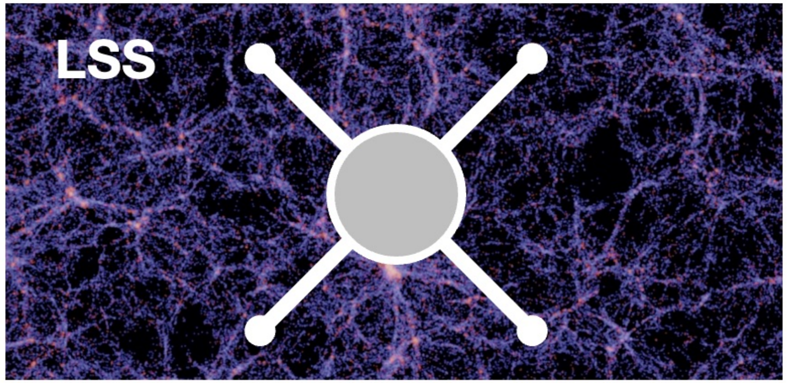
CERN | October 31, 2024

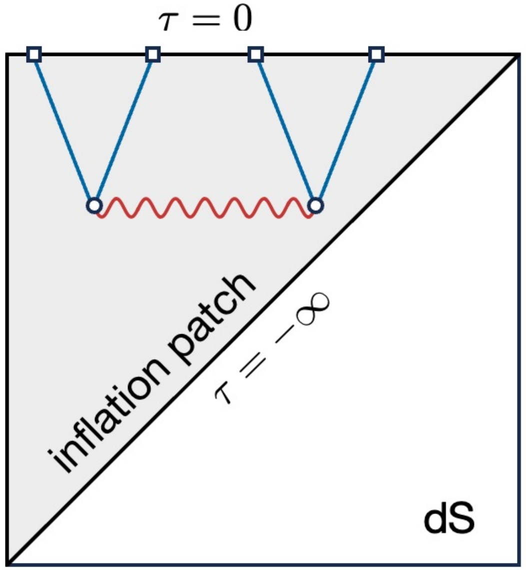
Based on: Haoyuan Liu, Zhehan Qin, ZX, 2407.12299; Haoyuan Liu, ZX, to appear

A Cosmological collider program

[Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more]



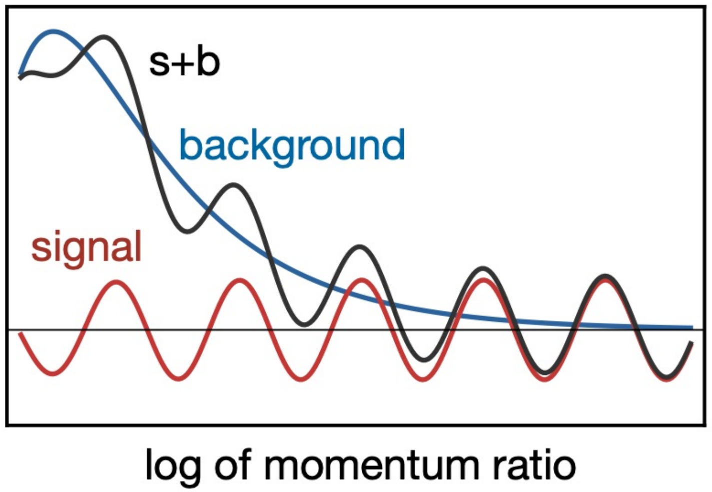




Inflation ~ dS

particle production

mass ~ 10^{14} GeV

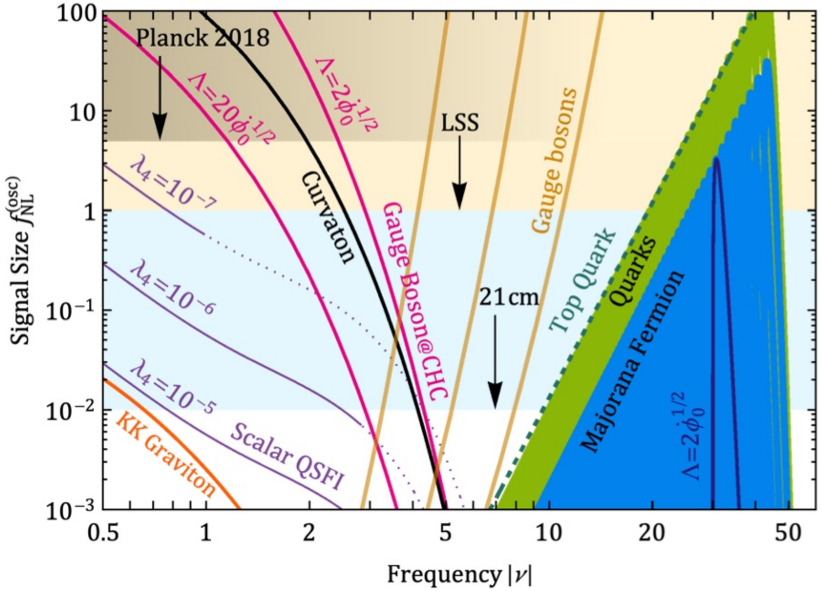


superhorizon resonance

mass, spin, coupling, etc

amplitude nonanalyticity

A Cosmological collider program



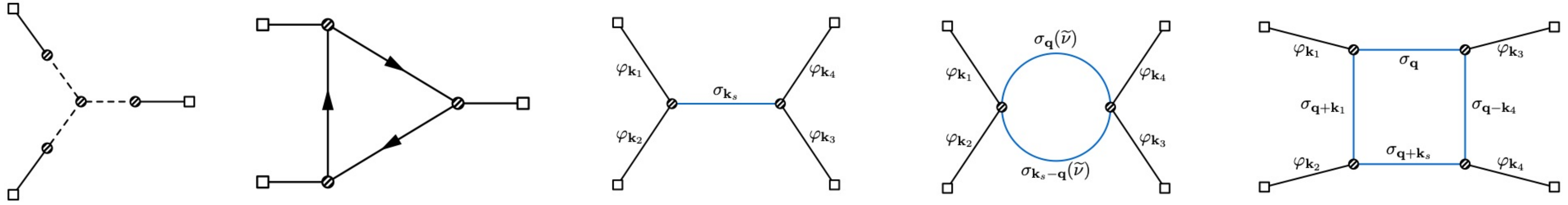
[Lian-Tao Wang, ZX, 1910.12876]

Over the years, many particle models identified in SM/BSM, with large signals

Many types of diagrams (tree + loop) involved

Understanding the amplitudes!

- efficient numerical implementation
- analytical structure



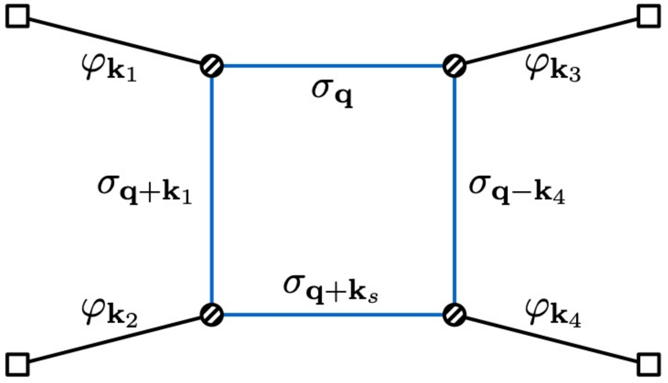
Massive inflation correlators

[See Chen, Wang, ZX, 1703.10166 for a review]

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{q} \times (-\tau)^p \times e^{\mathrm{i}E\tau} \times \mathrm{H}_{\mathrm{i}\tilde{\nu}}\Big[-K(\boldsymbol{q},\boldsymbol{k})\tau\Big] \times \theta(\tau_i - \tau_j)$$

vertex int　loop int　ext line　bulk line

- Massless / conformal external lines + (principal) massive internal lines
- Challenges:
  - Mode functions (Hankel, Whittacker, …)
  - Loop momentum integrals
  - Nested time integrals
- Complexity increases with # of loops and # of vertices



Massive inflation correlators: Tree level

- "Cosmological bootstrap"
  Diff eqs and solutions for single exchange [Arkani-Hamed, Baumann, Lee, Pimentel, 1811.00024]
  Single exchange in closed form for 3pt [Qin, ZX, 2301.07047]
  Diff eqs and solutions for double exchange [Aoki, Pinol, Sano, Yamaguchi, Zhu, 2404.09547]
- Cosmological polytopes and kinematic flow: "Conformal scalar amplitudes"
  Energy integrand [Arkani-Hamed, Benincasa, Postnikov, 1709.02813]
  Diff eqs for any number of exchanges [Arkani-Hamed et al., 2312.05303]
- Partial Mellin-Barnes + Family tree decomposition [Qin, ZX, 2205.01692, 2208.13790]
  Conformal amplitudes: full analytical results [Fan, ZX, 2403.07050]
  Massive: any exchanges reduced to a mechanical procedure [ZX, Zang, 2309.10849]
- Numerical package also available: CosmoFlow [Werth, Pinol, Renaux-Petel, 2302.00655]
- Massive exchanges essentially solved, but not optimally (too many layers of summation)
  Can we find better results? Can we directly get the result without doing any computation?

Partial Mellin-Barnes + family tree decomposition

[Qin, ZX, 2205.01692, 2208.13790]

Partial Mellin-Barnes rep: MB rep for all bulk lines; Special functions => powers

For example: Massive scalar propagator [Hankel function]

$$\mathrm{H}_\nu^{(1)}(-k\tau) = \frac{1}{\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\left(\frac{k}{2}\right)^{-2s}(-\tau)^{-2s}e^{(2s-\nu-1)\pi\mathrm{i}/2}\Gamma\left[s-\frac{\nu}{2}, s+\frac{\nu}{2}\right]$$

Time and momentum factorized

All time and momentum integrals factorized; We can deal with them separately:

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d\boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\right] \times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j)\right]$$

bulk lines → Left poles

loop int → Left poles, Right poles

nested time int → Right poles

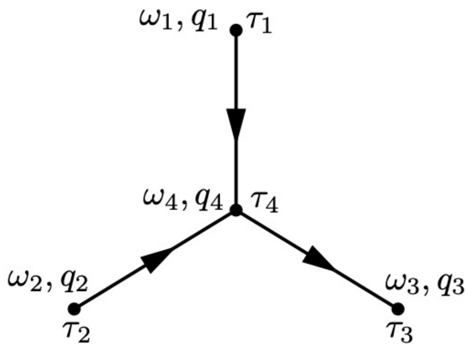
Partial Mellin-Barnes + family tree decomposition

$$\mathcal{T}\big(\{\boldsymbol{k}\}\big) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d\boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^{\alpha}\right] \times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^{\beta} \times \theta(\tau_i - \tau_j)\right]$$

bulk lines — loop int — nested time int

The most general time integral: $$(-\mathrm{i})^N \int_{-\infty}^{0} \prod_{\ell=1}^{N} \left[\mathrm{d}\tau_\ell \, (-\tau_\ell)^{q_\ell - 1} e^{\mathrm{i}\omega_\ell \tau_\ell}\right] \prod \theta(\tau_j - \tau_i)$$

It naturally acquires a graphic representation [NOT original Feynman diagrams]:



$$= (-\mathrm{i})^4 \int \prod_{\ell=1}^{4} \left[\mathrm{d}\tau_\ell \, (-\tau_\ell)^{q_\ell - 1} e^{\mathrm{i}\omega_\ell \tau_\ell}\right] \theta(\tau_4 - \tau_1)\theta(\tau_4 - \tau_2)\theta(\tau_3 - \tau_4)$$

Family tree decomposition

[ZX, Zang, 2309.10849]

Family tree decomposition:

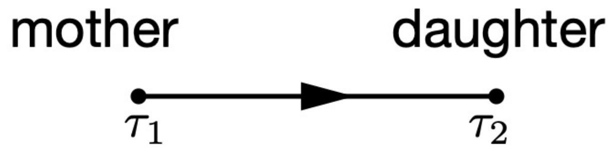
We flip the directions such that all nested graphs are partially ordered

$$\theta(\tau_1 - \tau_2) + \theta(\tau_2 - \tau_1) = 1$$

Partial order:

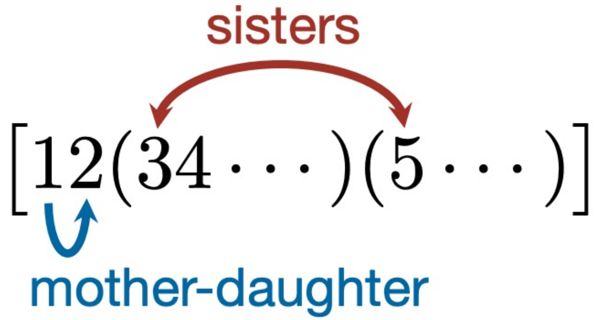
A mother can have any number of daughters

but a daughter must have only one mother



Every resulting nested graph can be interpreted as a maternal family tree

A useful notation for family trees:



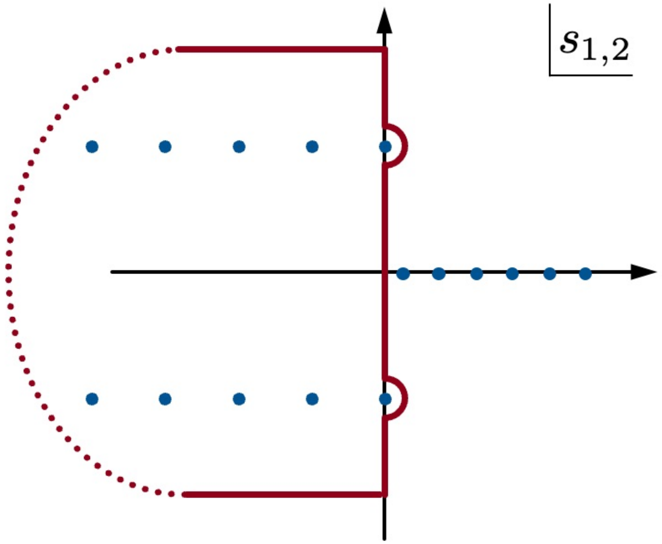
Computing the family tree

- The family tree has a one-line series expansion, in terms of $1/E_{max}$

$$[\mathscr{P}(\widehat{1}2\cdots N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}} \sum_{n_2,\cdots,n_N=0}^{\infty} \Gamma(q_{1\cdots N} + n_{2\cdots N}) \prod_{j=2}^{N} \frac{(-\omega_j/\omega_1)^{n_j}}{(\widetilde{q}_j + \widetilde{n}_j)n_j!}$$

earliest site   sum of all q's on Site j and her descendants $(q_{12\cdots} \equiv q_1 + q_2 + \cdots)$

- Mellin integrands typically meromorphic [only poles]
- Final results by residue theorem: pole collecting
- Pole structure encodes rich physics!
- A massive tree with *I* internal lines reduced to a series of 3*I*-fold summation [2*I* from PMB, *I* from family tree]



- Simpler family trees sum to named hypergeometric functions:

$$[12] = \frac{-1}{(\mathrm{i}\omega_1)^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right] \qquad [2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix} q_1, q_3 \\ q_1+1, q_3+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$

Gauss hypergeo — Appell

$$[123] = \frac{\mathrm{i}}{(\mathrm{i}\omega_1)^{q_{123}}}\ {}^{2+1}\mathcal{F}_{1+1}\left[\begin{matrix} q_{123}, q_{23} \\ q_{23}+1 \end{matrix}\middle|\begin{matrix} -, q_3 \\ -, q_3+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right]$$

Kampé de Fériet

$$[1(2)\cdots(N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}}\mathcal{F}_A\left[q_{1\cdots N}\middle|\begin{matrix} q_2, \cdots, q_N \\ q_2+1, \cdots, q_N+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, \cdots, -\frac{\omega_N}{\omega_1}\right]$$

Lauricella

- More complicated family trees not yet named, but the many different ways of expanding them provide a convenient tool for numerical evaluation and analytical continuation

$$[12] = [12] \qquad \frac{1}{\omega_1^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right] = \frac{\Gamma[q_2]}{\omega_{12}^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} 1, q_{12} \\ q_2+1 \end{matrix}\middle| \frac{\omega_2}{\omega_{12}}\right]$$

$$[12] + [21] = [1]\,[2] \qquad \frac{1}{\omega_1^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right] + \frac{1}{\omega_2^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_1, q_{12} \\ q_1+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}\right] = \frac{\Gamma[q_1, q_2]}{\omega_1^{q_1}\omega_2^{q_2}}$$

$$[123] + [2(1)(3)] = [1]\,[23] \qquad \frac{1}{\omega_1^{q_{123}}}\ {}^{2+1}\mathcal{F}_{1+1}\left[\begin{matrix} q_{123}, q_{23} \\ q_{23}+1 \end{matrix}\middle|\begin{matrix} -, q_3 \\ -, q_3+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right] + \frac{1}{\omega_2^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix} q_1, q_3 \\ q_1+1, q_3+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right] = \frac{\Gamma[q_1]}{\omega_1^{q_1}\omega_2^{q_{23}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_3, q_{32} \\ q_3+1 \end{matrix}\middle| -\frac{\omega_3}{\omega_2}\right]$$

A byproduct: conformal-scalar amplitudes in FRW

[Fan, ZX, 2403.07050]

A nice toy model: conformal scalar with non-conformal self-interactions

$$S[\phi_c] = -\int \mathrm{d}^{d+1}x\sqrt{-g}\left[\frac{1}{2}(\partial_\mu\phi_c)^2 + \frac{1}{2}\xi R\phi_c^2 + \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi_c^n\right] \qquad \xi \equiv (d-1)/(4d)$$

Rule for conformal amplitudes: 1. Fix a partial order; 2. Write the uncut tree; 3. Cut!

Example: 4-site star: Full analytical expressions in terms of family tress in two lines

$$\begin{aligned}\widetilde{\psi}_{\text{4-star}} = \sum_{\mathsf{a,b,c}=\pm} \mathsf{abc}\Big\{ & \left[4_{1^{\mathsf{a}}2^{\mathsf{b}}3^{\mathsf{c}}}(1_{\bar{1}^{\mathsf{a}}})(2_{\bar{2}^{\mathsf{b}}})(3_{\bar{3}^{\mathsf{c}}})\right] + \Big(\left[4_{\bar{1}^{\mathsf{a}}2^{\mathsf{b}}3^{\mathsf{c}}}(2_{\bar{2}^{\mathsf{b}}})(3_{\bar{3}^{\mathsf{c}}})\right]\left[1\right] + 2\text{ perms}\Big) \\ & + \Big(\left[4_{\bar{1}^{\mathsf{a}}\bar{2}^{\mathsf{b}}3^{\mathsf{c}}}3_{\bar{3}^{\mathsf{c}}}\right]\left[1_1\right]\left[2_2\right] + 2\text{ perms}\Big) + \left[4_{\bar{1}^{\mathsf{a}}\bar{2}^{\mathsf{b}}\bar{3}^{\mathsf{c}}}\right]\left[1_1\right]\left[2_2\right]\left[3_3\right]\Big\}\end{aligned}$$

Compared with kinematic flow: 64 coupled diff eqs!
Actually, much easier to derive diff eqs from family trees [He, Jiang, Liu, Yang, Zhang, 2407.17715]

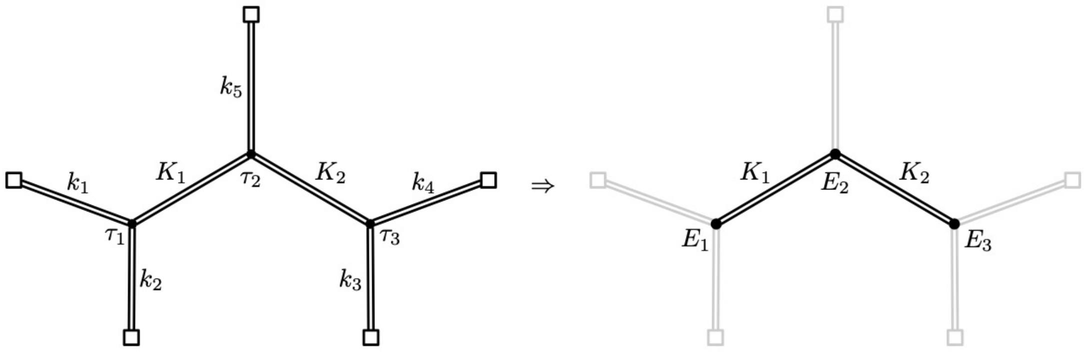
Arbitrary massive tree: differential equations

[Liu, ZX, to appear]

- A massive graph is fully specified by:
- a vertex energy *E* and a twist *p* for each vertex, a line energy *K* and a mass *ν* for each line

$$\mathcal{G}(\{E\},\{K\}) = \sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V=\pm}\int_{-\infty}^{0}\prod_{i=1}^{V}\left[\mathrm{d}\tau_i\,\mathrm{i}\mathsf{a}_i(-\tau_i)^{p_i}e^{\mathrm{i}\mathsf{a}_iE_i\tau_i}\right]\prod_{\alpha=1}^{I}D^{(\widetilde{\nu}_\alpha)}_{\mathsf{a}_\alpha\mathsf{a}'_\alpha}(K_\alpha;\tau_\alpha,\tau'_\alpha)$$

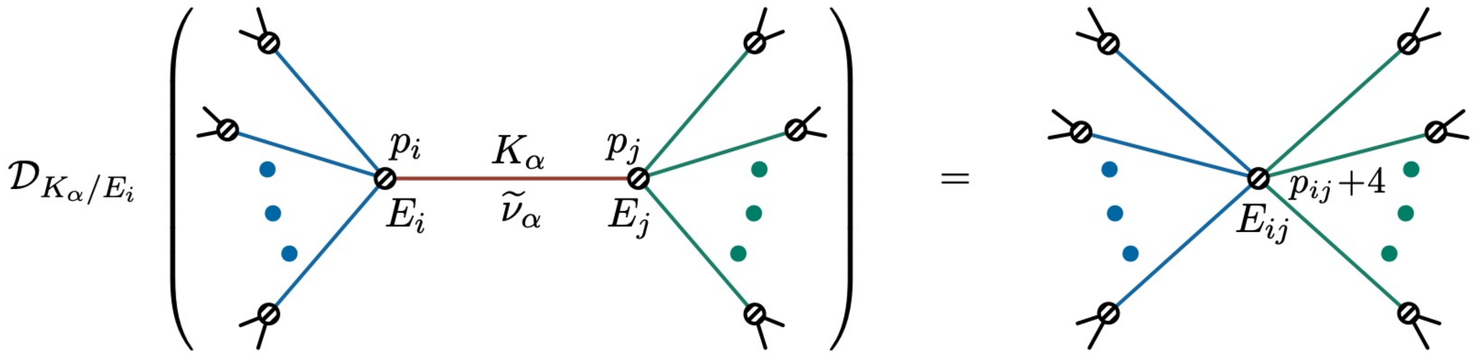
- *I* = # lines, *V* = # vertices. In tree diagrams, *V* = *I* + 1. 2*I* independent kinematic variables
- In particular, external leges are irrelevant: $\prod_{n=1}^{A}C_{\mathsf{a}_i}(k_n;\tau_i)(-\tau_i)^{P_i} = \left(\prod_{n=1}^{A}\frac{-\tau_f}{2k_n}\right)e^{+\mathrm{i}\mathsf{a}_iE_i\tau_i}(-\tau_i)^{P_i+A}$



Arbitrary massive tree: differential equations

- An internal line (bulk propagator) is collapsed to 0 or δ by a Klein-Gordon operator
- The KG operator can be pulled out of the integral with IBP at a given vertex
- We obtain a 2nd order diff eq for the graph by picking up a line + one of its two endpoint

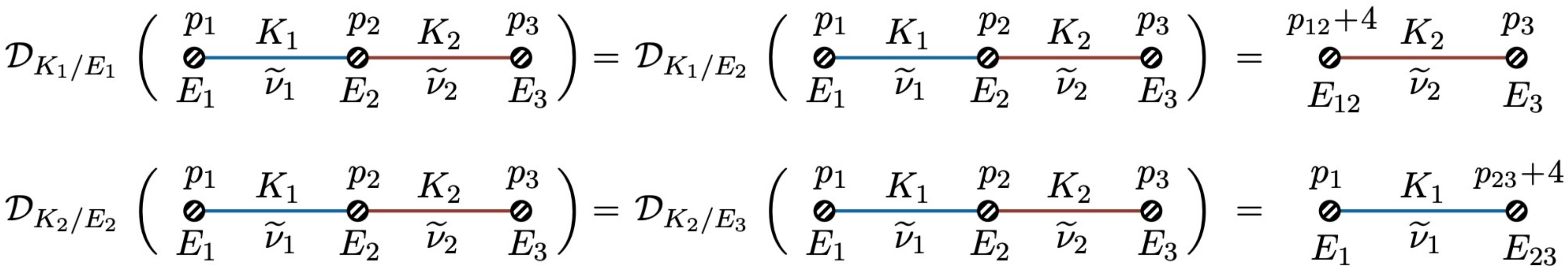
$$\mathcal{D}_{K_\alpha/E_i}\mathcal{G} = \mathsf{C}_\alpha[\mathcal{G}]$$



- There are a total of 2*I* choices => 2*I* diff eqs for 2*I* indep energy ratios. A complete set!

Arbitrary massive tree: differential equations

- An example of 3-vertex chain:

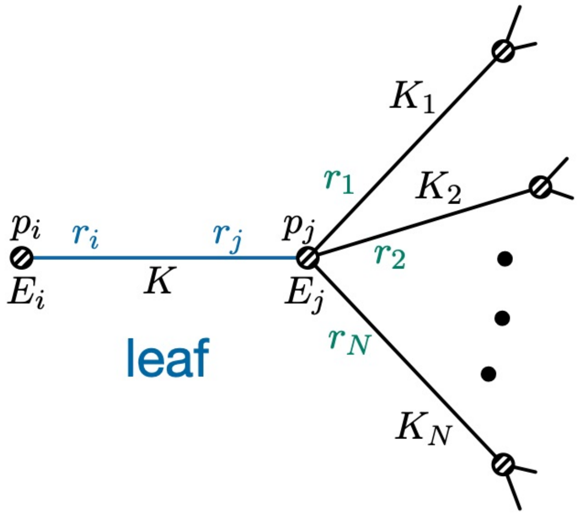


$$
\begin{aligned}
&\left[\left(\vartheta_{K_1}+\tfrac{3}{2}\right)^2+\widetilde{\nu}_1^2-\frac{K_1^2}{E_1^2}\left(p_1+5+\vartheta_{K_1}\right)\left(p_1+4+\vartheta_{K_1}\right)\right]\mathcal{G}^{p_1p_2p_3}_{\widetilde{\nu}_1\widetilde{\nu}_2}(E_1,E_2,E_3;K_1,K_2)\\
=&\left[\left(\vartheta_{K_1}+\tfrac{3}{2}\right)^2+\widetilde{\nu}_1^2-\frac{K_1^2}{E_2^2}\left(p_2+8+\vartheta_{K_1}+\vartheta_{K_2}\right)\left(p_2+7+\vartheta_{K_1}+\vartheta_{K_2}\right)\right]\mathcal{G}^{p_1p_2p_3}_{\widetilde{\nu}_1\widetilde{\nu}_2}(E_1,E_2,E_3;K_1,K_2)\\
=&\,\mathcal{G}^{(p_{12}+4)p_3}_{\widetilde{\nu}_2}(E_{12},E_3;K_2)
\end{aligned}
$$

$$
\vartheta_{K_\alpha}\equiv K_\alpha\partial_{K_\alpha}
$$

Solving the massive-tree differential equations

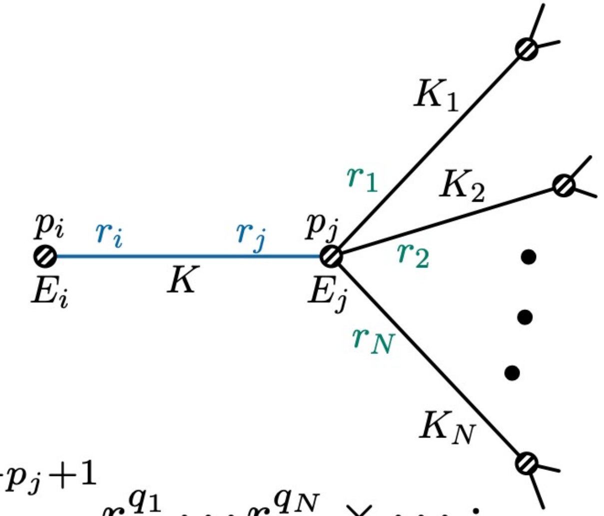
- Eventually, solutions in multivariate hypergeometric functions
  => The best we can hope: series solutions in regions of interest (i.e., the physical region)
- An arbitrary tree graph can be reduced to a single point by recursively removing its leaves
  => Conversely: recursively constructing a tree by adding leaves to a single point
- Assuming we know a V-site graph (series sol) => construct (V+1)-site graph (series sol)
- Solving the diff eq for a leaf: 2nd order ODE with a source; hom sol + inhom sol
- Homogeneous solutions ~ factorized time integral ("easy")
  inhomogeneous solutions ~ nested time integral ("hard")
- Strategy:
  tackling the hardest part first (completely inhom sol)
  The easy part obtained by taking cuts



Inhomogeneous recursion formulae

- Assuming we know the series solution for a V-vertex tree:

$$\widetilde{\mathcal{G}}_V = \sum_{\{n\}} c_{\{n\}} r_1^{q_1} \cdots r_N^{q_N} \times \cdots$$



- We can solve the diff eq of the "new leaf"
  to get the nested part of the (V+1)-vertex tree:

$$\left[\widetilde{\mathcal{G}}_{V+1}\right]_{\text{inh}} = \sum_{\ell,m=0}^{\infty} \sum_{\{n\}} d_{\ell m\{n\}} \left(\frac{r_i}{2}\right)^{2m+3} \left(\frac{r_i}{r_j}\right)^{\ell+q_{1\ldots N}+p_j+1} r_1^{q_1} \cdots r_N^{q_N} \times \cdots ;$$

$$d_{\ell m\{n\}} = \frac{2(-1)^\ell (q_{1\ldots N} + p_{ij} + 5)_{\ell+2m}}{\ell! \left(\frac{\ell+q_{1\ldots N}+p_j}{2} + \frac{5}{4} \pm \frac{i\widetilde{\nu}}{2}\right)_{m+1}} c_{\{n\}}$$

- The energy order is important; Define an energy flow (from large to small) ⇔ time flow
- The above solution is for an "ingoing line" ($E_i > E_j$). "Outgoing solution" also obtainable

Completely inhomogeneous solution: massive family trees

- Applying the inhomogeneous recursion formulae for any tree graph, we find a compact one-line (or two-line) formula for the completely inhomgeneous solution

$$\text{CIS}\big[\widetilde{\mathcal{G}}_V\big] = \sum_{\{\ell,m\}} 2^V \cos(\pi p_{1\cdots V}/2)\Gamma(q_1+p_1+1) \times \prod_{i=2}^{V} \frac{(-1)^{\ell_i}}{\ell_i!\big(\frac{\ell_i+q_i+p_i}{2}+\frac{5}{4}\pm\frac{\mathrm{i}\widetilde{\nu}_i}{2}\big)_{m_i+1}}\Big(\frac{K_i}{2E_1}\Big)^{2m_i+3}\Big(\frac{E_i}{E_1}\Big)^{\ell_i+p_i+1}$$

- The solution is expanded in the reciprocal of the largest vertex energy (E_1), and is indep of orders of other energies
- Picking up a largest energy automatically generates a partial order: **massive family tree**
  q: a "family parameter" encoding the tree structure: $q_i \equiv \widetilde{\ell_i} + 2\widetilde{m}_i + \widetilde{p}_i + 4N_i$
- Graph ⇔ solution; WYSIWYG: $\text{CIS}\big[\widetilde{\mathcal{G}}_V\big] = [\![\hat{1}2\cdots V]\!]$

Homogeneous solutions: cuts of massive family trees

- The homogeneous solutions are obtained by executing appropriate cuts:

$$\underset{K_\alpha}{\mathrm{Cut}}\,[\widetilde{\mathcal{G}}_V] = [\![\widehat{1}\cdots i^\sharp \cdots V_1]\!]\Big\{[\![(V_1+1)\cdots j^\sharp\cdots V]\!] + [\![(V_1+1)\cdots j^\flat\cdots V]\!]\Big\} + \text{ c.c.}$$

- The cut involves certains dressings of massive family trees: augmentation and flattening:

$$[\![\cdots i^\sharp \cdots]\!] \equiv \sum_{m=0}^{\infty}\mathcal{A}_m\Big(\frac{K_\alpha}{2E_i}\Big)^{2m+\mathrm{i}\widetilde{\nu}_\alpha+3/2}[\![\cdots i\cdots]\!]_{p_i\to p_i+2m+\mathrm{i}\widetilde{\nu}_\alpha+3/2}$$

$$[\![\cdots i^\flat \cdots]\!] \equiv \sum_{m=0}^{\infty}\mathcal{F}_m\Big(\frac{K_\alpha}{2E_i}\Big)^{2m-\mathrm{i}\widetilde{\nu}_\alpha+3/2}[\![\cdots i\cdots]\!]_{p_i\to p_i+2m-\mathrm{i}\widetilde{\nu}_\alpha+3/2}$$

- The cut is directional:
  Subgraph with the largest energy augmented only; the other both augmented and flattened
  Consistent with the cosmological collider signal cutting rule

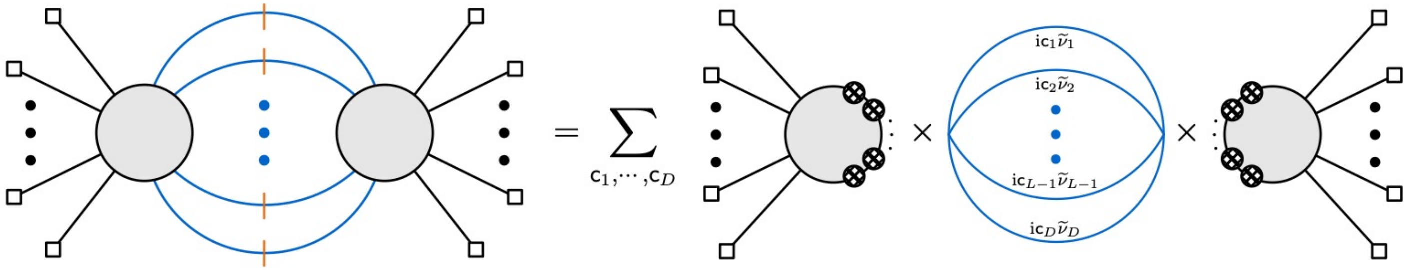
- Multiple cuts similarly defined; Summing over all cuts (including no cut) give the final answer

Massive tree: a summary

- A differential equation system and its solution obtained for arbitrary massive exchanges
- Diff Eqs: A coupled Lauricella system
- Solutions: A massive family tree plus all its cuts
- Nice correspondence with Cosmo Collider pheno:
  Massive family tree => Background
  Symmetric part of the cut (♯♯ + c.c.) => Nonlocal signal
  Directional part of the cut (♯♭ + c.c.) => Local signal
- The series for a *V*-vertex graph involves a 2(*V*-1)-fold summation:
  match the # of indep variables => Optimal in the sense of transcendentality (?)
  [In comparison: 3(*V*-1)-fold from PMB + family tree]
- Finally, we can write down the analytical answer for arbitrary massive tree graph in inflation without doing any computation, like amplitudes in flat space!

Massive cosmological correlators: Loop level

- Spectral decomposition: first complete result for 1-loop bubble diagram [ZX, Zhang, 2211.03810]
- Partial PM: Factorization theorems, cutting rule, & analytical result for leading nonlocal signals at arbitrary loops order [Qin, ZX, 2304.13295; 2308.14802]

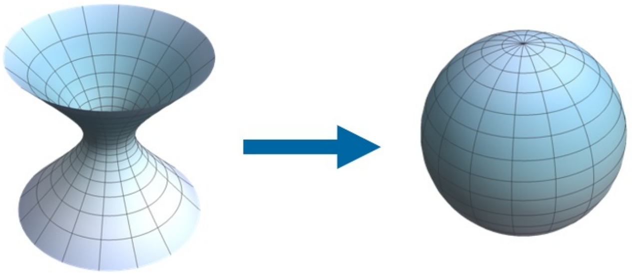


$$\mathfrak{M}_{c_1\cdots c_D}(P) \equiv \frac{P^{3(D-1)}}{(4\pi)^{(5D-3)/2}}\Gamma\begin{bmatrix} -\sum_{i=1}^{D} c_i\mathrm{i}\widetilde{\nu}_i - \frac{3}{2}(D-1) \\ \frac{3}{2}D + \sum_{i=1}^{D} c_i\mathrm{i}\widetilde{\nu}_i \end{bmatrix} \prod_{\ell=1}^{D}\left\{\Gamma\left[\tfrac{3}{2} + c_\ell\mathrm{i}\widetilde{\nu}_\ell, -c_\ell\mathrm{i}\widetilde{\nu}_\ell\right]\left(\frac{P}{2}\right)^{2\mathrm{i}c_\ell\widetilde{\nu}_\ell}\right\}$$

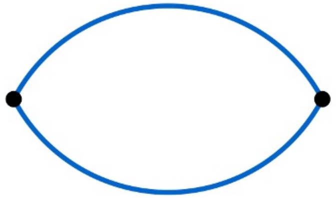
- Diff eqs for loop integrands; loop integrals remain challenging [He, Jiang, Liu, Yang, Zhang, 2407.17715, Baumann, Goodhew, Lee, 2410.17994]
- A new boostrap combining spectral decomposition and dispersion techniques, with result neatly organized as a sum over quasi-normal modes, and free of UV divergence

Spectral decomposition

- dS isometries => the bubble function as linear superposition of principal massive propagators



The spectral density obtainable by Wick-rotating dS to sphere or AdS



$$D^2_{\widetilde{\nu}}(x,y) \sim \sum_{\vec{L}} B_{\widetilde{\nu}}(L) \times$$

EdS bubble function

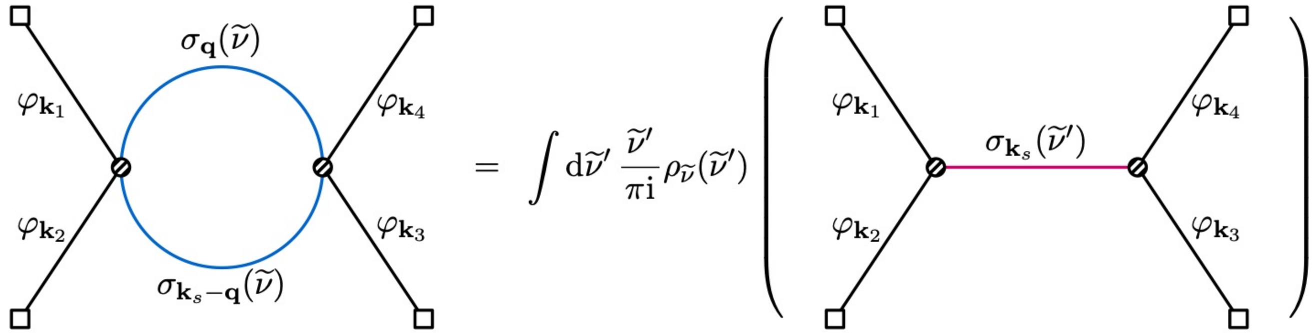
Explicitly computable by known results of Y's

- Wick rotating back to dS, a spetral function obtained in arbitrary dimensions, dim reg automatic [Marolf, Morrison, 1006.0035] [See also Loparco, Penedones, Vaziri, Sun, 2306.00090]

$$\rho^{\rm dS}_{\widetilde{\nu}}(\widetilde{\nu}') = \frac{1}{(4\pi)^{(d+1)/2}} \frac{\cos[\pi(\frac{d}{2}-{\rm i}\widetilde{\nu})]}{\sin(-\pi{\rm i}\widetilde{\nu})} \Gamma\left[\begin{matrix} \frac{3-d}{2}, \frac{d}{2}-{\rm i}\widetilde{\nu} \\ \frac{2-d}{2}-{\rm i}\widetilde{\nu} \end{matrix}\right]$$
$$\times\, {}_7\mathcal{F}_6\left[\begin{matrix} \frac{2-d}{2}+{\rm i}\widetilde{\nu}'-{\rm i}\widetilde{\nu}, \frac{3-d/2+{\rm i}\widetilde{\nu}'-{\rm i}\widetilde{\nu}}{2}, \frac{2-d}{2}, \frac{2-d}{2}-{\rm i}\widetilde{\nu}, \frac{2-d}{2}+{\rm i}\widetilde{\nu}', \frac{{\rm i}\widetilde{\nu}'-2{\rm i}\widetilde{\nu}+d/2}{2}, \frac{{\rm i}\widetilde{\nu}'+d/2}{2} \\ \frac{1-d/2+{\rm i}\widetilde{\nu}'-{\rm i}\widetilde{\nu}}{2}, 1+{\rm i}\widetilde{\nu}'-{\rm i}\widetilde{\nu}, 1+{\rm i}\widetilde{\nu}', 1-{\rm i}\widetilde{\nu}, \frac{4+{\rm i}\widetilde{\nu}'-3d/2}{2}, \frac{4+{\rm i}\widetilde{\nu}'-2{\rm i}\widetilde{\nu}-3d/2}{2} \end{matrix}\middle| 1\right]$$
$$+ (\widetilde{\nu} \to -\widetilde{\nu}).$$

1-loop trispectrum and bispectrum

- 1-Loop bubble = a spectral integral of tree: finish the integral by collecting correct residues



- The spectral function has simple residues (Gamma products) corresponding to the weights of quasi-normal mode. Especially transparent in the factorized part (the signal):

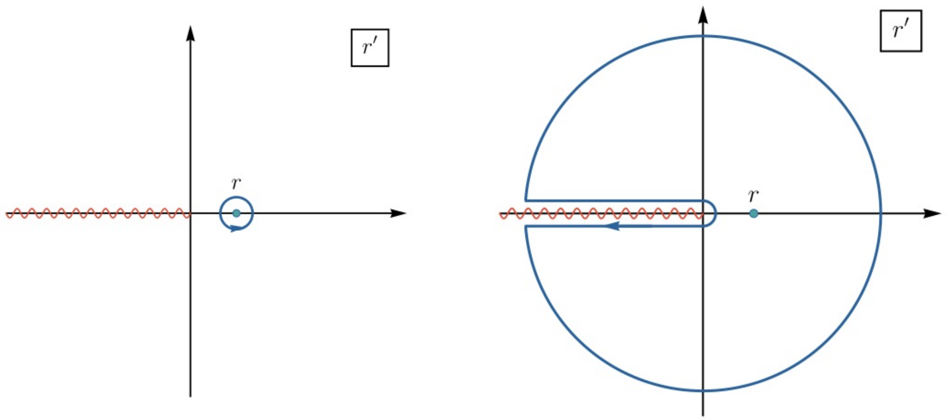
$$\widehat{\mathcal{J}}_{\mathrm{NS}} = \frac{2(r_1 r_2)^{3/2+2\mathrm{i}\widetilde{\nu}}}{\pi^2 \cos(2\pi \mathrm{i}\widetilde{\nu})} \sum_{n=0}^{\infty} \frac{(1+n)_{\frac{1}{2}} \left[(1+\mathrm{i}\widetilde{\nu}+n)_{\frac{1}{2}}\right]^2 (1+2\mathrm{i}\widetilde{\nu}+n)_{\frac{1}{2}}}{(1+2\mathrm{i}\widetilde{\nu}+2n)_2} \left(\tfrac{3}{2}+2\mathrm{i}\widetilde{\nu}+2n\right)$$
$$\times\, {}_2\mathcal{F}_1\left[\begin{matrix} 2+\mathrm{i}\widetilde{\nu}+n,\ \frac{5}{2}+\mathrm{i}\widetilde{\nu}+n \\ \frac{5}{2}+2\mathrm{i}\widetilde{\nu}+2n \end{matrix}\middle| r_1^2\right] {}_2\mathcal{F}_1\left[\begin{matrix} 2+\mathrm{i}\widetilde{\nu}+n,\ \frac{5}{2}+\mathrm{i}\widetilde{\nu}+n \\ \frac{5}{2}+2\mathrm{i}\widetilde{\nu}+2n \end{matrix}\middle| r_2^2\right] (r_1 r_2)^{2n} + \mathrm{c.c.}.$$

- The nested part can then be computed by a dispersion method

Dispersion relations for massive correlators

[Liu, Qin, ZX, 2407.12299]

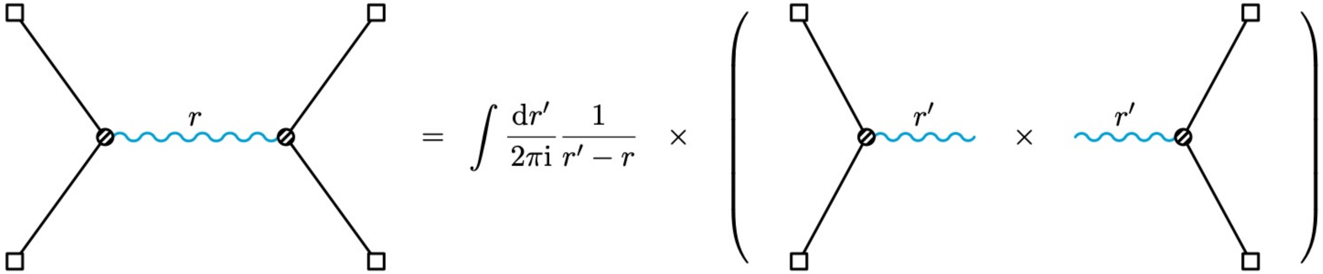
- Dispersion integral: relating the value of a function with the integral along its branch cut



$$f(r) = \int_{C'} \frac{\mathrm{d}r'}{2\pi\mathrm{i}} \frac{f(r')}{r'-r} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}r'}{2\pi\mathrm{i}} \frac{\mathrm{Disc}_{r'} f(r')}{r'-r}$$

The potential divergence of the large circle can be easily removed by a "subtraction"

- The disc. is often much easier to calculate than the full answer [factorization; cutting rule; optical theorem] [Melville, Pajer, 2103.09832; Goodhew, Jazayeri, Lee, Pajer, 2104.06587]

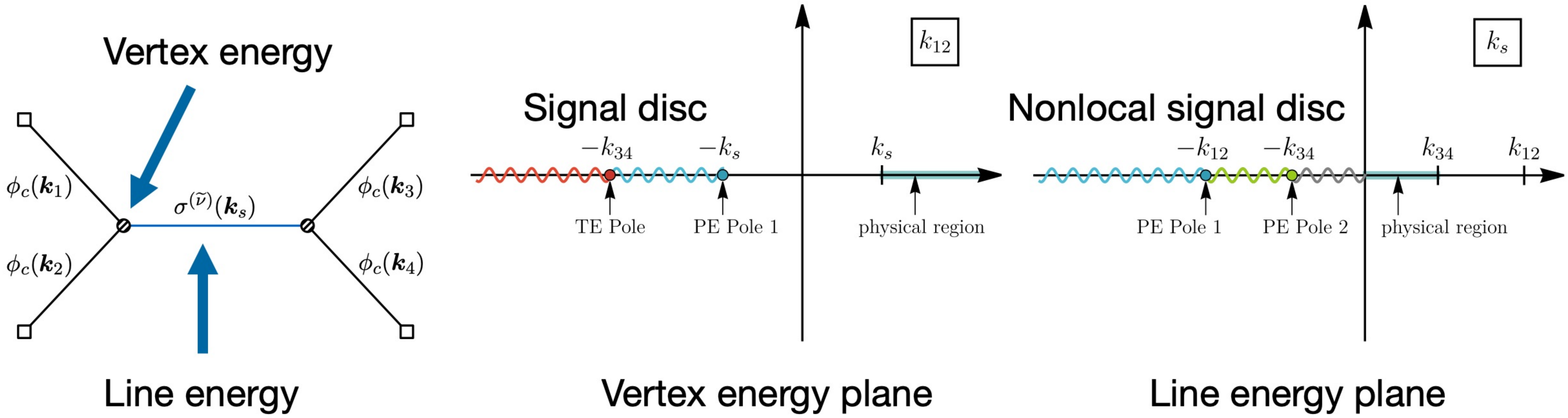


Dispersion relations for massive correlators

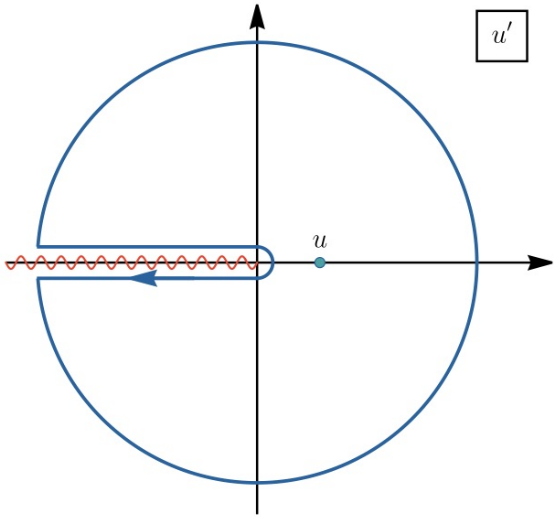
[Liu, Qin, ZX, 2407.12299]

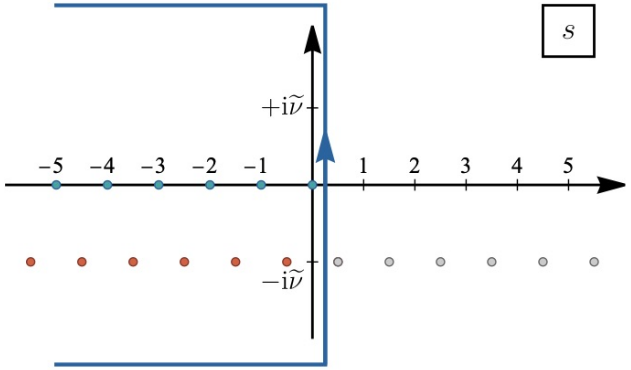
- Two types of dispersion relations: complex vertex and line energies
  [See also Werth, 2409.02072, for a dispersion relation of massive trees on the complex mass plane]

- In vertex dispersion, the Disc totally from the signal (nonlocal + local)
  In line dispersion, the Disc from the nonlocal signal alone

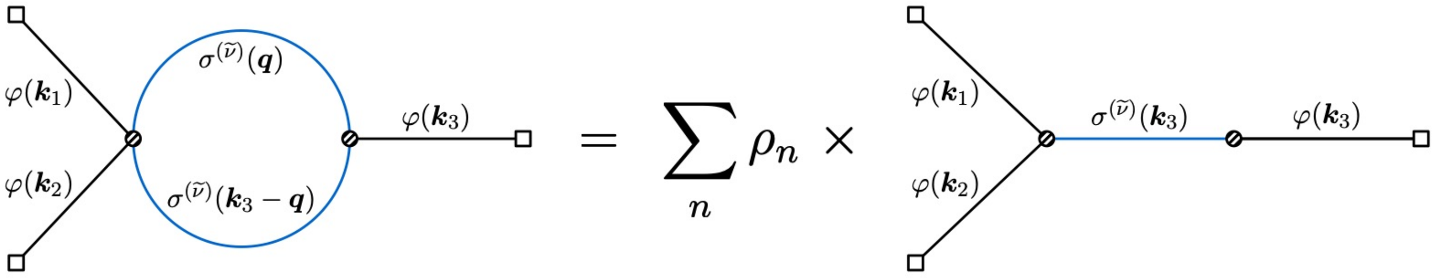


Finish the dispersion integral by going to the Mellin plane => Pole collecting





$$u = 2k_3/k_{123}$$



- The result expressed as a sum over quasi-normal mode contributions; signal + background

$$\mathcal{J}^{0,-2}(u) = Cu^3 - \frac{u^4}{128\pi\sin(2\pi i\tilde{\nu})}\sum_{n=0}^{\infty}\frac{(3+4i\tilde{\nu}+4n)(1+n)_{\frac{1}{2}}(1+2i\tilde{\nu}+n)_{\frac{1}{2}}}{(\frac{1}{2}+i\tilde{\nu}+n)_{\frac{1}{2}}(\frac{3}{2}+i\tilde{\nu}+n)_{\frac{1}{2}}}$$
$$\times\left\{{}_2\mathcal{F}_1\left[\begin{matrix}2+2i\tilde{\nu}+2n, 4+2i\tilde{\nu}+2n\\ 4+4i\tilde{\nu}+4n\end{matrix}\middle| u\right]u^{2n+2i\tilde{\nu}} - {}_3\mathcal{F}_2\left[\begin{matrix}1,2,4\\ 1-2n-2i\tilde{\nu}, 4+2n+2i\tilde{\nu}\end{matrix}\middle| u\right]\right\}$$
$$+(\tilde{\nu}\to-\tilde{\nu})$$

Massive loop: a summary

- No UV divergence ever shows up in the calculation
  [UV properties encoded in the divergences in the large circle; removed by the subtraction, at the expense of generating new local terms of unknown coefficients: renormalization!]
- Lesson: UV divergence and regularization is largely an artefact of the ordinary Feynman-diagram computation procedure, and can be totally avoided in a dispersive calculation
- On the other hand, renormalization is physical and cannot be determined by the calculation alone: The computation can determine a UV sensitive 1-loop graph only up to additive tree graphs; the unknown coefficients of these tree graphs should be determined by renormalization conditions
- The divergence-free calculation makes the dispersive method a potentially useful tool for numerical computation

Final thoughts

- Unlike flat-space amplitudes such as Parke-Taylor, massive inflation correlators are not the sort of "hundred pages of midsteps collapsed to a one-line formula" thing
- They belong to a GKZ hypergeometrical system, carrying their own transcendental weights Whatever method we take, we have to reach that transcendentality
- Then, what does analytical computation mean other than identifying these hypergeometrics?
- GKZ might be too general; desirable to develop new and more specialized techniques for understanding massive inflationary correlators
- **[Analyticity]** Series expansions around all possible singularities can be very powerful in understanding the analytical structure and phenomenology of these object
- **[Numerical & pheno]** A complementarity between analytical and numerical methods: Typically, squeezed limit easy / hard for analytical / numerical; equilateral limit otherwise

Thank you!