Aspen 2012

The Hunt for New Particles, from the Alps to the Plains to the Rockies

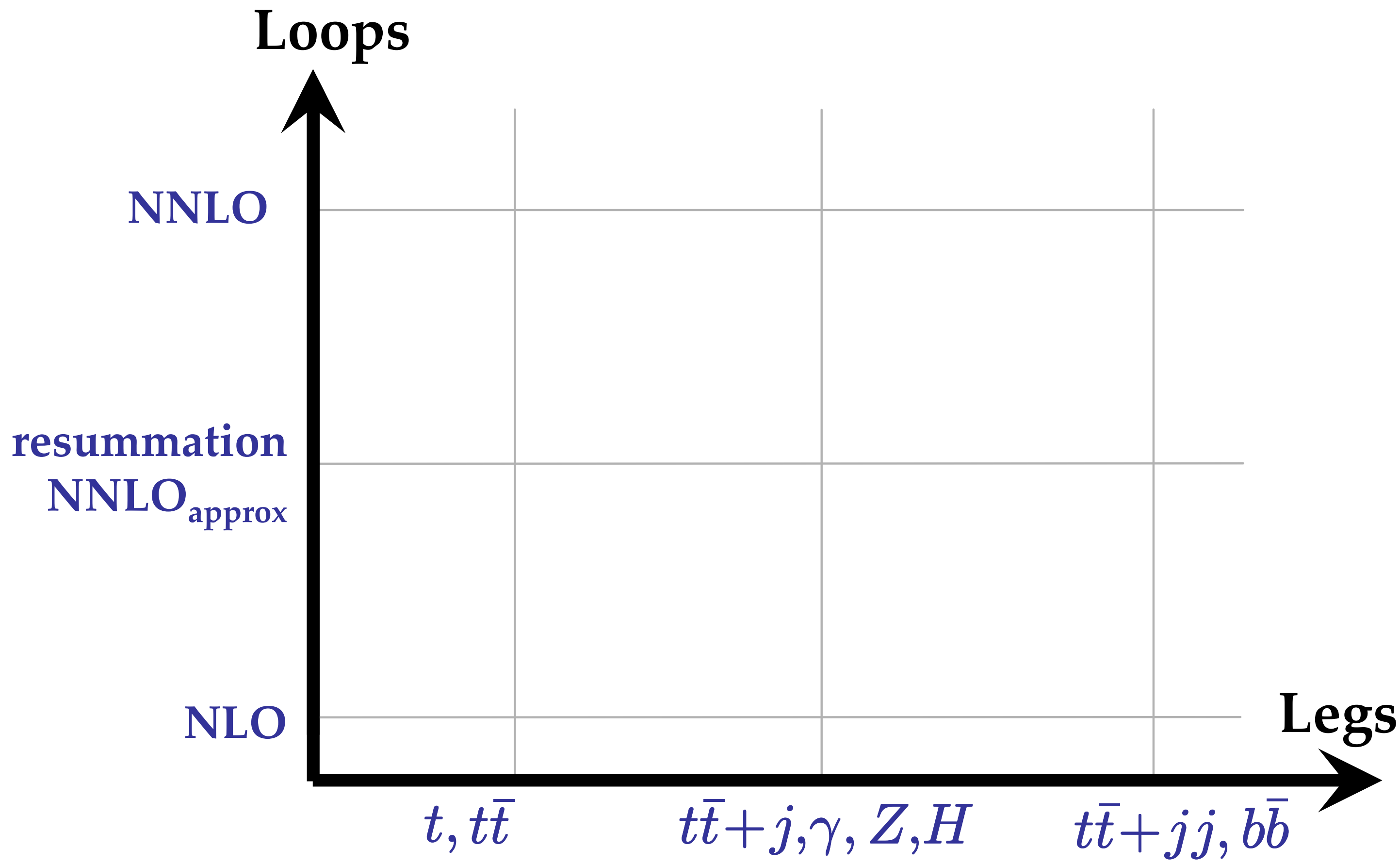
Standard Model
Top Quark Theory

Markus Schulze

Argonne NATIONAL LABORATORY

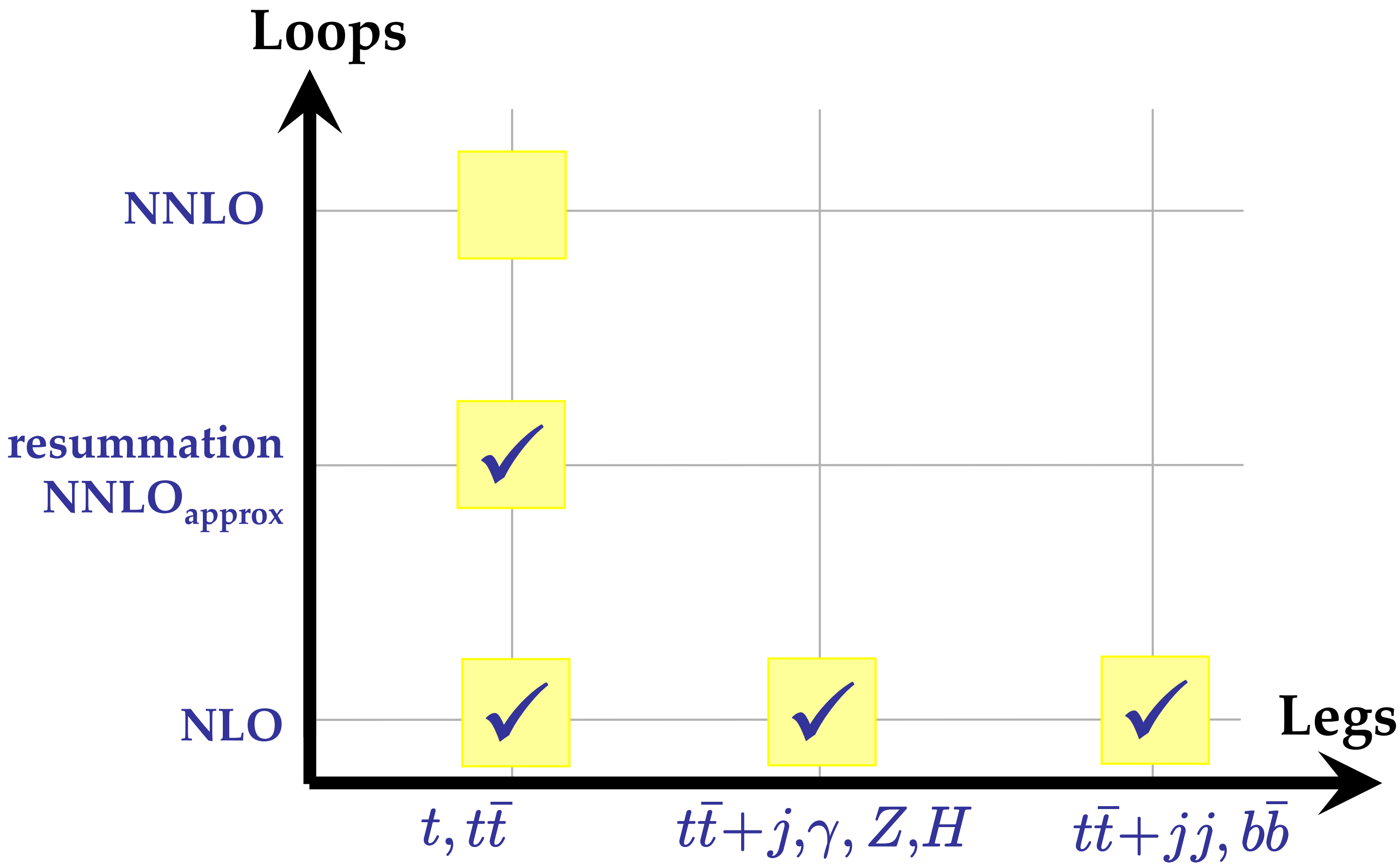
Top quark physics: a wide field

Directions



Loops
NNLO
resummation
$\mathrm{NNLO}_{\mathrm{approx}}$
NLO
Legs
$t, t\bar{t}$
$t\bar{t}+j, \gamma, Z, H$
$t\bar{t}+jj, b\bar{b}$

Directions



Loops
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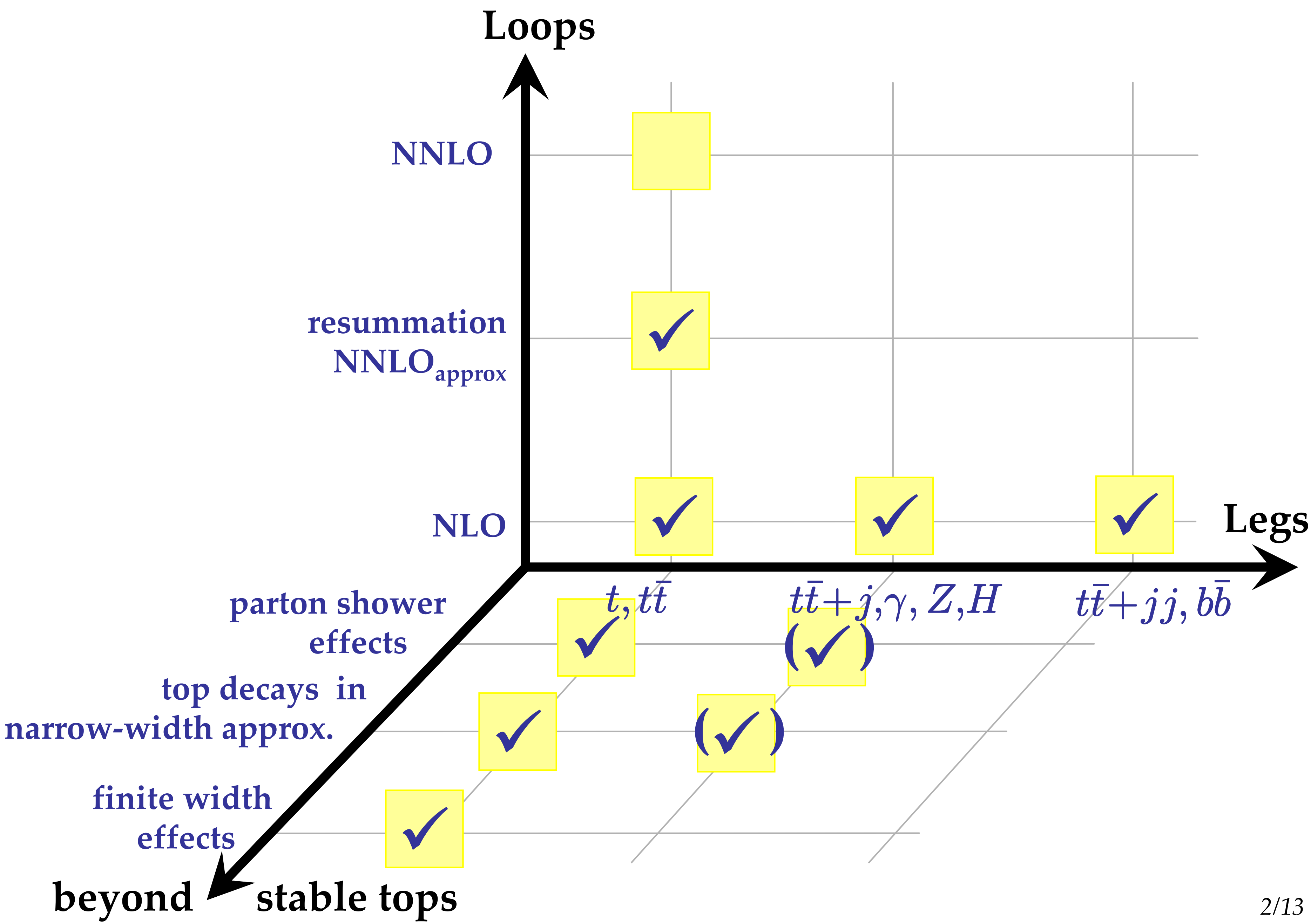

Loops
NNLO
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$t, t\bar{t}$
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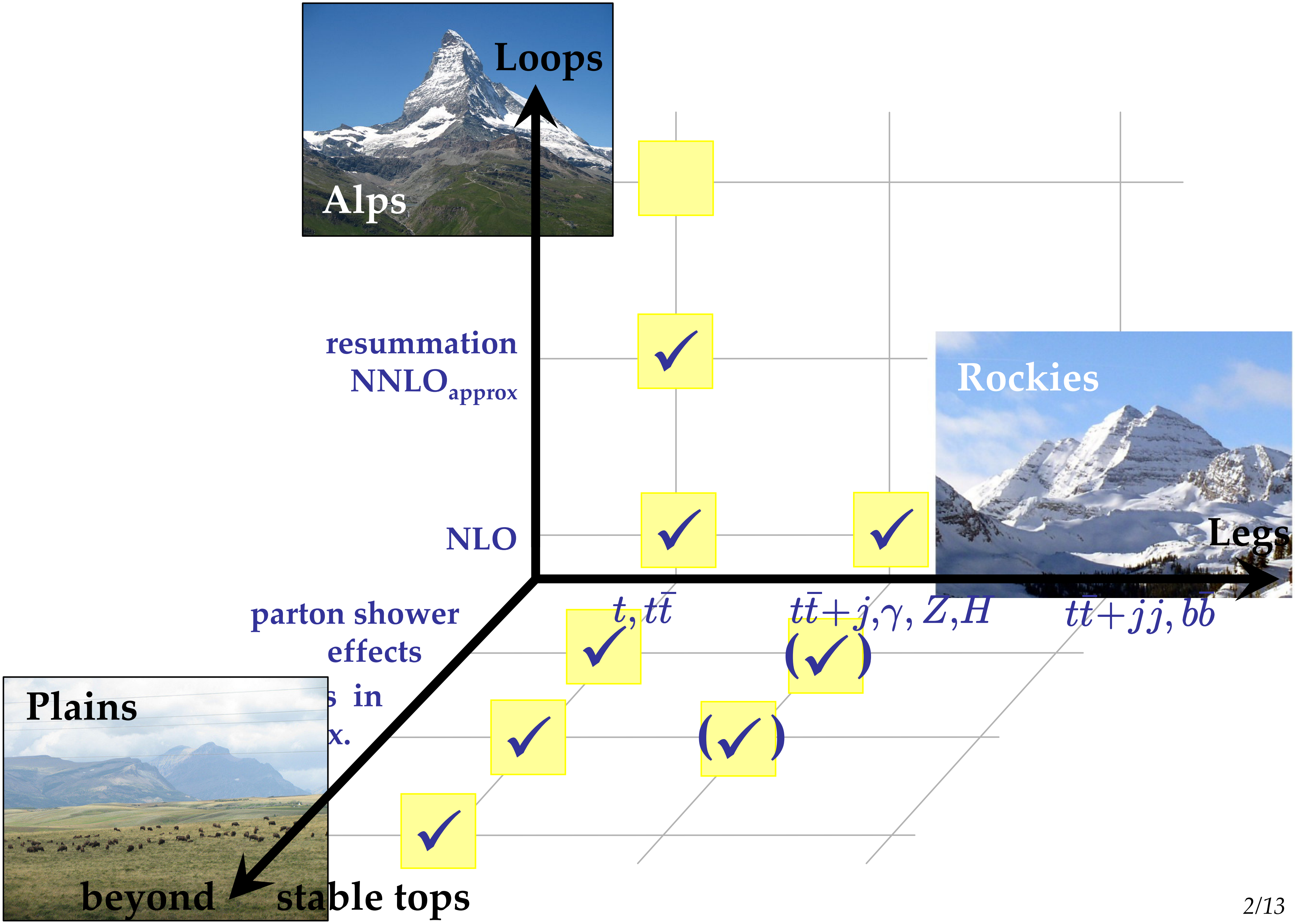
parton shower effects
top decays in narrow-width approx.
finite width effects
beyond stable tops

Directions


Loops
Alps
resummation
NNLO$_{\text{approx}}$
NLO
Rockies
Legs
$t, t\bar{t}$
$t\bar{t}+j, \gamma, Z, H$
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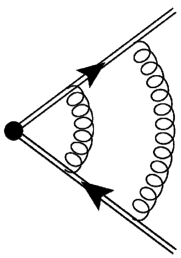
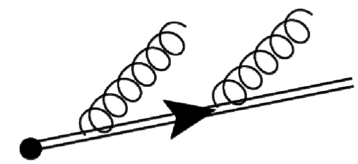
parton shower
effects
in
(✓)
(✓)
Plains
beyond
stable tops

Loops:
Towards $t\bar{t}$ production at NNLO QCD

Resummation and $\text{NNLO}_{\text{approx}}$

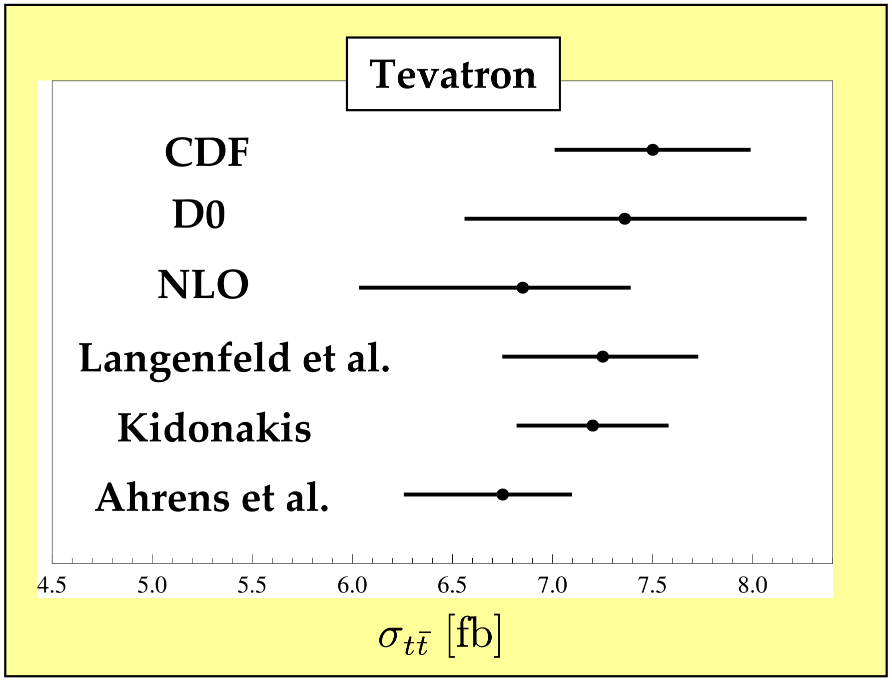


Close to threshold, the total cross section receives velocity (β) enhanced contributions due to Coulomb interaction and soft gluon emission



$$\sigma^{(2)} \sim \sigma_{\text{LO}} + \frac{\alpha_s}{4\pi}\left(\#\,\frac{1}{\beta} + \#\,\log^{1,2}(\beta) + \text{finite}\right)$$
$$+ \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\#\,\frac{1}{\beta^2} + \frac{1}{\beta}\{\# + \#\,\log^{1,2}(\beta)\} + \#\,\log^{1,2,3,4}(\beta) + \text{finite}\right)$$

Resummation and NNLO$_{approx}$



[Bonciani,Catani,Mangano,Nason]

[Langenfeld,Moch,Uwer]

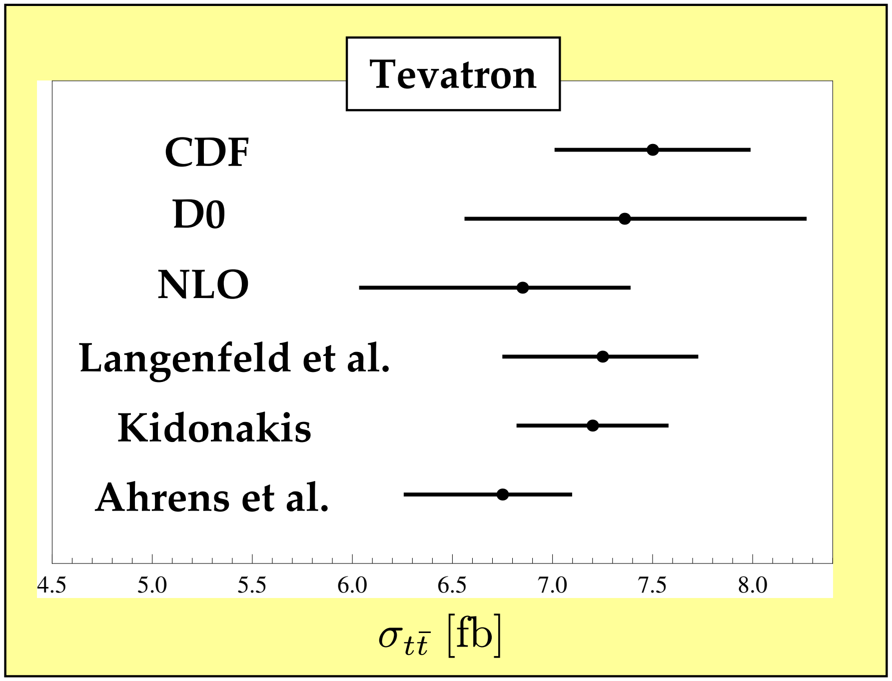
[Beneke,Czakon,Falgari,Klein,Mitov,Schwinn]

[Kidonakis]

[Ahrens,Ferroglia,Neubert,Pecjak,Yang]

Slightly different central values result from ambiguities in treating approximate higher order contributions. At the Tevatron, the spread of different results is about as large as NLO uncertainty.

Resummation and NNLO$_{approx}$



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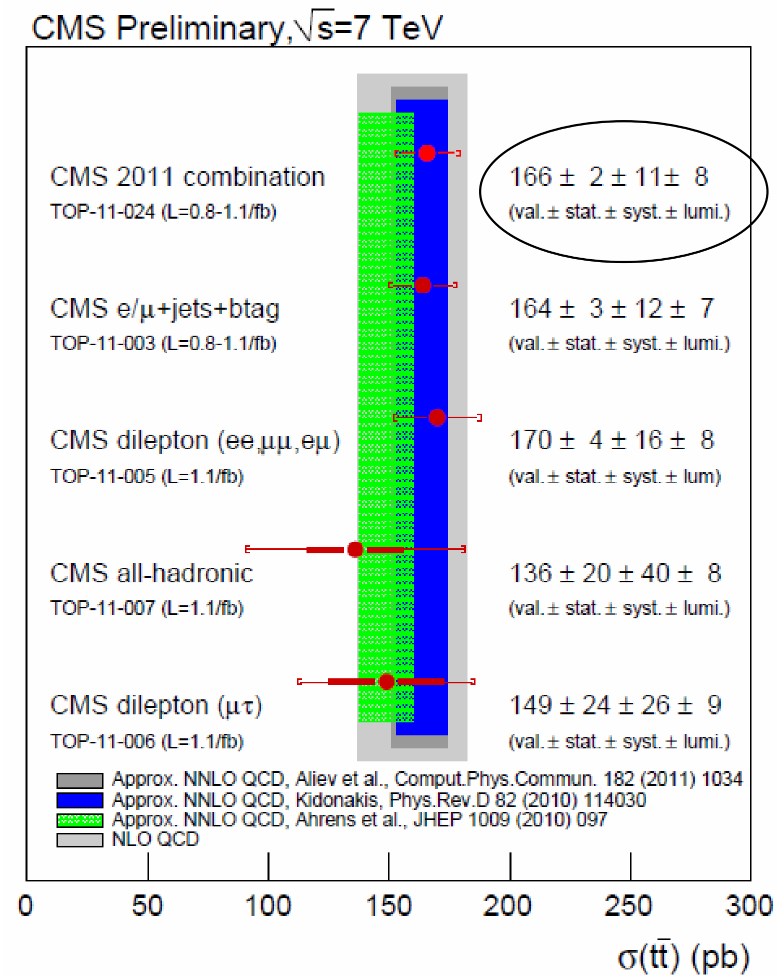
[Kidonakis]
[Ahrens,Ferroglia,Neubert,Pecjak,Yang]

	$A_{FB}^{p\bar{p}}$ [%]
NLO	$4.81^{+0.45+0.13}_{-0.39-0.13}$
NLO+NNLL	$4.88^{+0.20+0.17}_{-0.23-0.18}$
NNLO$_{\text{approx.}}$	$5.2^{+0.0}_{-0.6}$

Slightly different central values result from ambiguities in treating approximate higher order contributions. At the Tevatron, the spread of different results is about as large as NLO uncertainty.

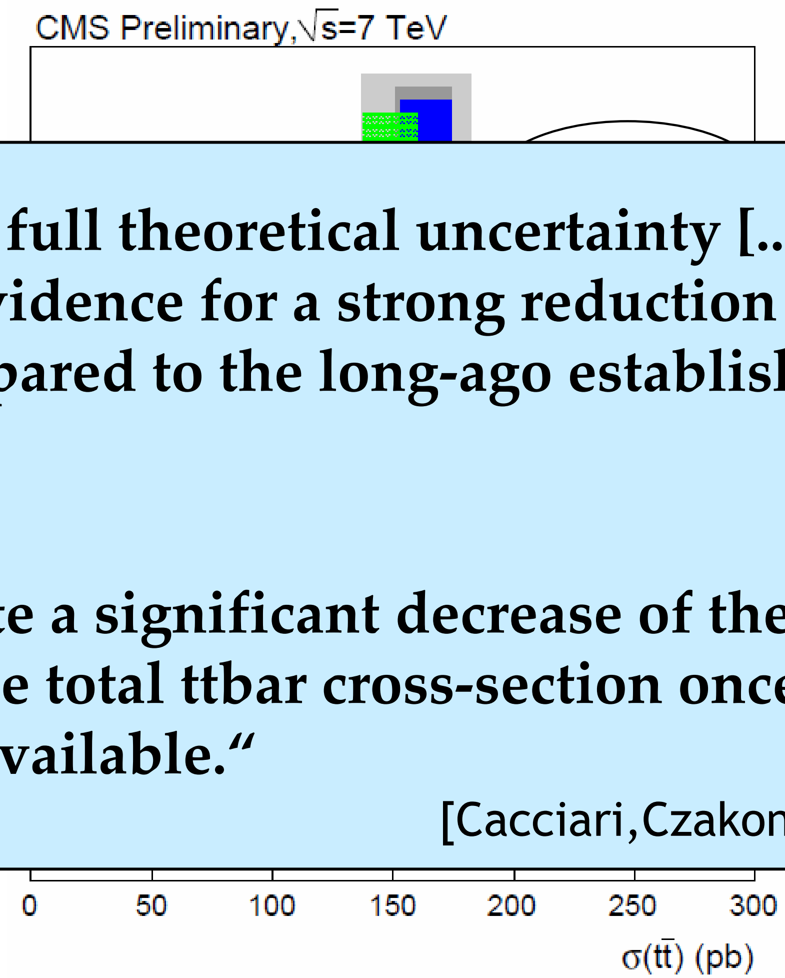
Resummation and NNLO$_{approx}$



$$\sigma_{\rm tot}^{\rm NLO+NNLL}(7\,{\rm TeV}) = 158.7^{+12.2(7.7\%)}_{-13.5(8.5\%)}\ [{\rm scale}]\ ^{+4.3(2.7\%)}_{-4.4(2.8\%)}\ [{\rm MSTW\ pdf}]\ \ {\rm pb}$$

[Cacciari, Czakon, Mangano, Mitov, Nason] (2011)

Resummation and NNLO$_{\text{approx}}$



"Addressing the full theoretical uncertainty [...], we conclude that we see no evidence for a strong reduction of the theoretical uncertainty compared to the long-ago established NLO+NLL analysis."

"We [...] speculate a significant decrease of the theoretical uncertainty in the total ttbar cross-section once the full NNLO result becomes available."

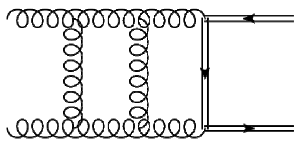
[Cacciari,Czakon,Mangano,Mitov,Nason]

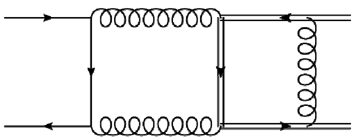
$$\sigma_{\text{tot}}^{\text{NLO+NNLL}}(7\,\text{TeV}) = 158.7^{+12.2(7.7\%)}_{-13.5(8.5\%)}\ [\text{scale}]\ ^{+4.3(2.7\%)}_{-4.4(2.8\%)}\ [\text{MSTW pdf}]\ \ \text{pb}$$

[Cacciari,Czakon,Mangano,Mitov,Nason] (2011)

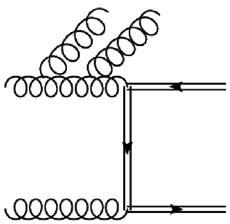
Full NNLO QCD

1. two-loop virtual correction

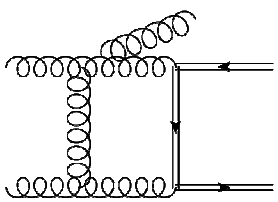




2. double-real emission



3. real-virtual (one-loop) corrections



Full NNLO QCD

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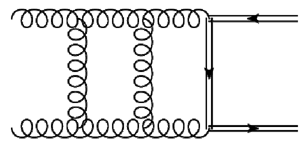

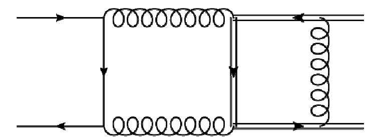

$q\bar{q}$ numerical [Czakon] (2008)

partial analytic [Bonciani,Ferroglia,Gehrmann,Maitre,Studerus] (2008-9)

gg work in progress [Czakon,Bärnreuther]

analytic leading-N_c [Bonciani,Ferroglia,Gehrmann,Manteuffel,Studerus] (2010)

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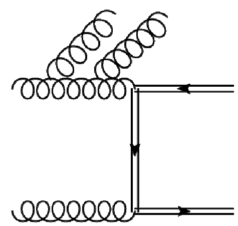

antenna subtraction: work in progress [Boughezal,Gehrmann-DeRidder,Ritzmann], [Glover,Pires]

sector decomposition [Czakon] (2011)

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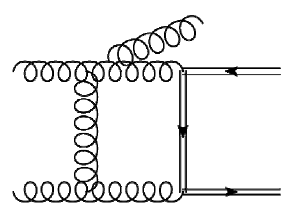
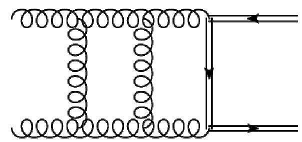


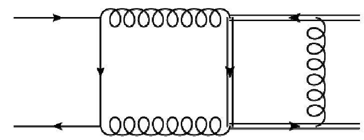
one-loop soft/collinear limits

[Bern,DelDuca,Kilgore,Schmidt] (1989), [Catani,Grazzini] (2000)

[Bierenbaum,Czakon,Mitov] (2011), [Gehrmann,Monni] (2011)

Full NNLO QCD





1. two-loop virtual correction

$q\bar{q}$ numerical [Czakon] (2008)

partial analytic [Bonciani,Ferroglia,Gehrmann,Maitre,Studerus] (2008-9)

gg …)

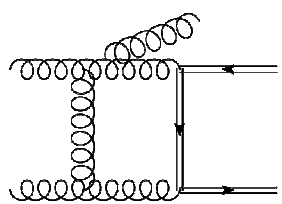
Since 2011, a complete set of tools/techniques is available to calculate the *full* NNLO corrections to hadronic ttbar production.

2. d…

antenna subtraction: work in progress [Boughezal,Gehrmann-DeRidder,Ritzmann], [Glover,Pires]

sector decomposition [Czakon] (2011)

3. real-virtual (one-loop) corrections



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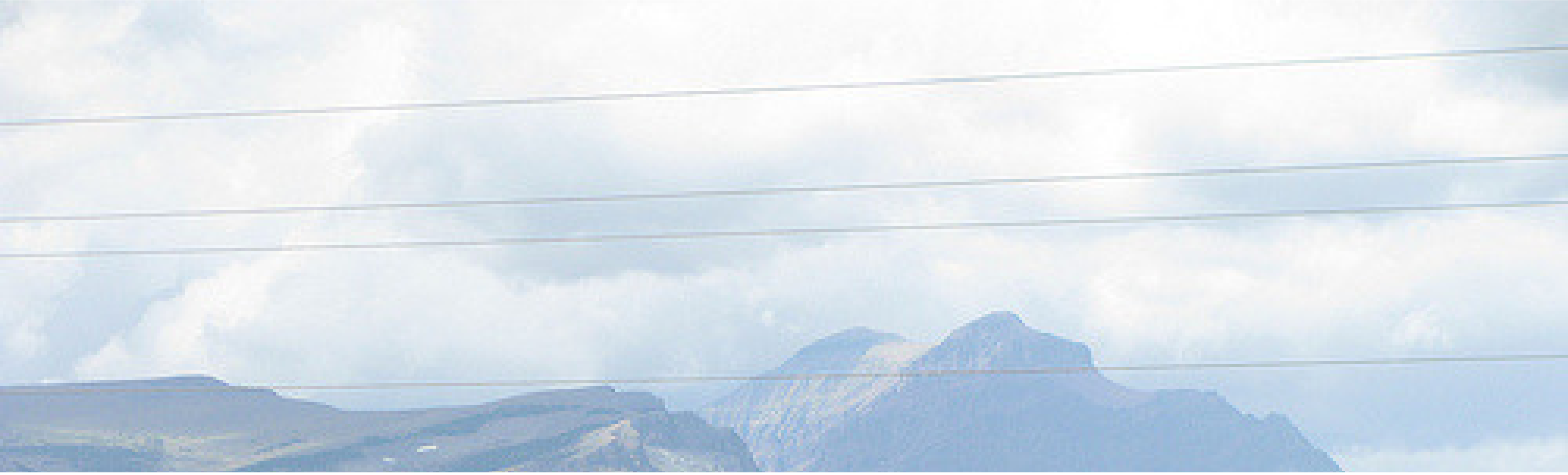
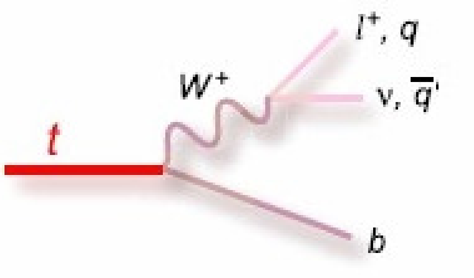


Beyond stable tops:
Top quark decays and finite-width effects

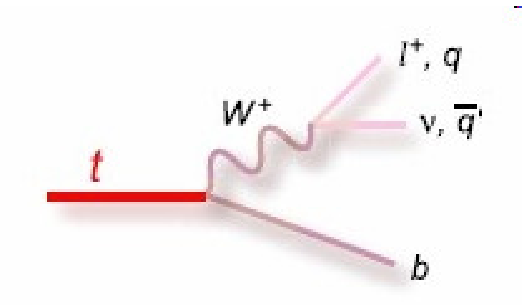
Top quark decays



Top quarks are not stable particles. We only observe their decay products (jets,leptons) in experiments.

- **The decay process is subject to higher order corrections as well (virtual corrections, additional jet radiation).**
- **Those corrections affect acceptance cuts and production dynamics through spin correlations.**

Top quark decays



Top quarks are not stable particles. We only observe their decay products (jets,leptons) in experiments.

- **The decay process is subject to higher order corrections as well (virtual corrections, additional jet radiation).**
- **Those corrections affect acceptance cuts and production dynamics through spin correlations.**

$\Rightarrow$ **There is no reason why corrections to the decay process are equal to / less important than the production process.**

Example: $$\sigma_{\rm tot} = \frac{N_{\rm obs}}{\mathcal{L}} \cdot \frac{1}{A}$$ with $A = \frac{\sigma_{\rm cuts}}{\sigma_{\rm tot}}$

To claim that the total cross section has been measured with NLO accuracy, one needs to calculate A at NLO QCD. Otherwise potential biases are introduced.

Narrow-width approximation

Thanks to their short lifetime and narrow width, top quarks can be treated in a parametric approximation which is valid up to $\mathcal{O}(\Gamma_t/m_t)$.

→ **Neglects largely off-shell and non-resonant diagrams, "intuitive" separation of production and decay**

Selection of recent NLO QCD predictions:

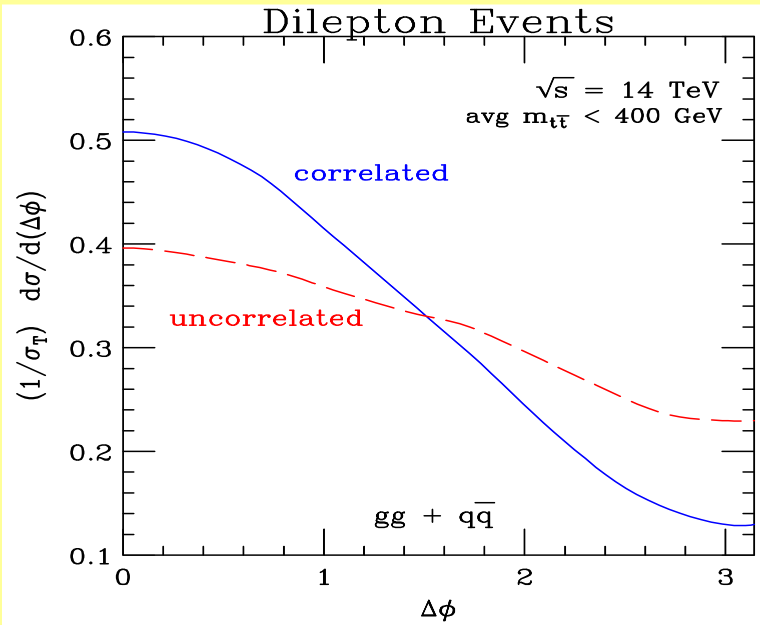
- **Top quark spin correlations** [Bernreuther,Brandenburg,Si,Uwer] [Mahlon,Parke], [Melnikov, M.S.]
- **Top mass determination through kinematic distributions** [Biswas,Melnikov,M.S.]
- **Lepton charge asymmetry at the Tevatron** [Bernreuther,Si]

Narrow-width approximation

Thanks to their short lifetime and narrow width,

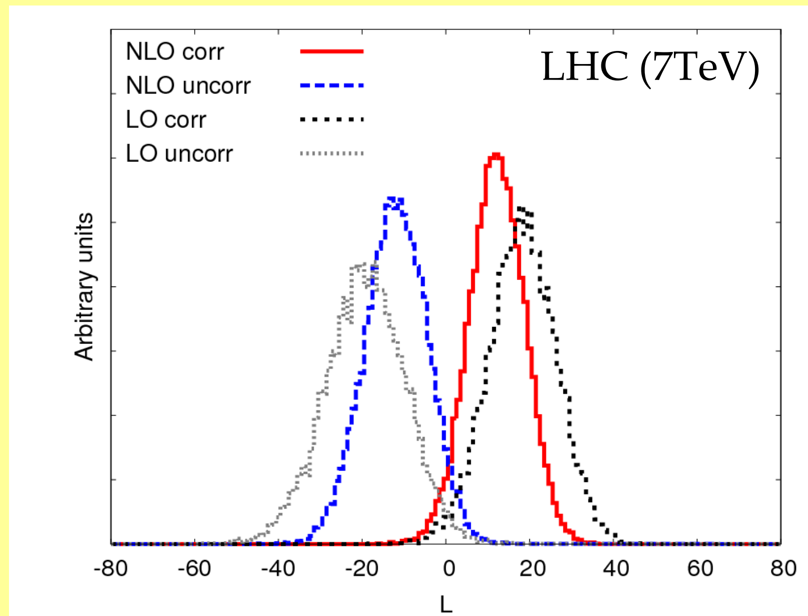
Top quark spin correlations

[Mahlon,Parke]



lepton opening angle
in laboratory frame

[Melnikov, M.S.]



statistical test which includes
all event information at NLO QCD
$\Rightarrow$ ~500 events sufficient for 4σ

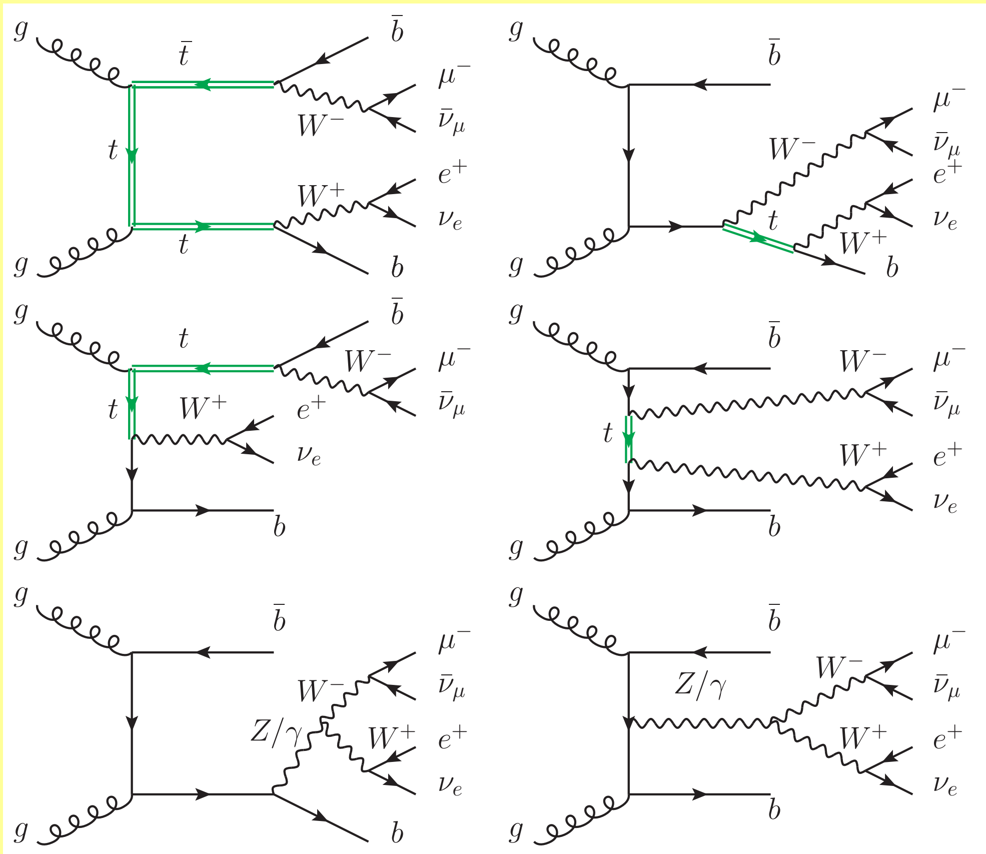
- **Lepton charge asymmetry at the Tevatron**

[Bernreuther,Si]

Finite-width effects in ttbar production

In late 2010, full NLO QCD calculations for $WWb\bar{b}$ became available.

[Bevilacqua,Czakon,vHameren,Papadopoulos,Worek], [Denner,Dittmaier,Kallweit,Pozzorini]



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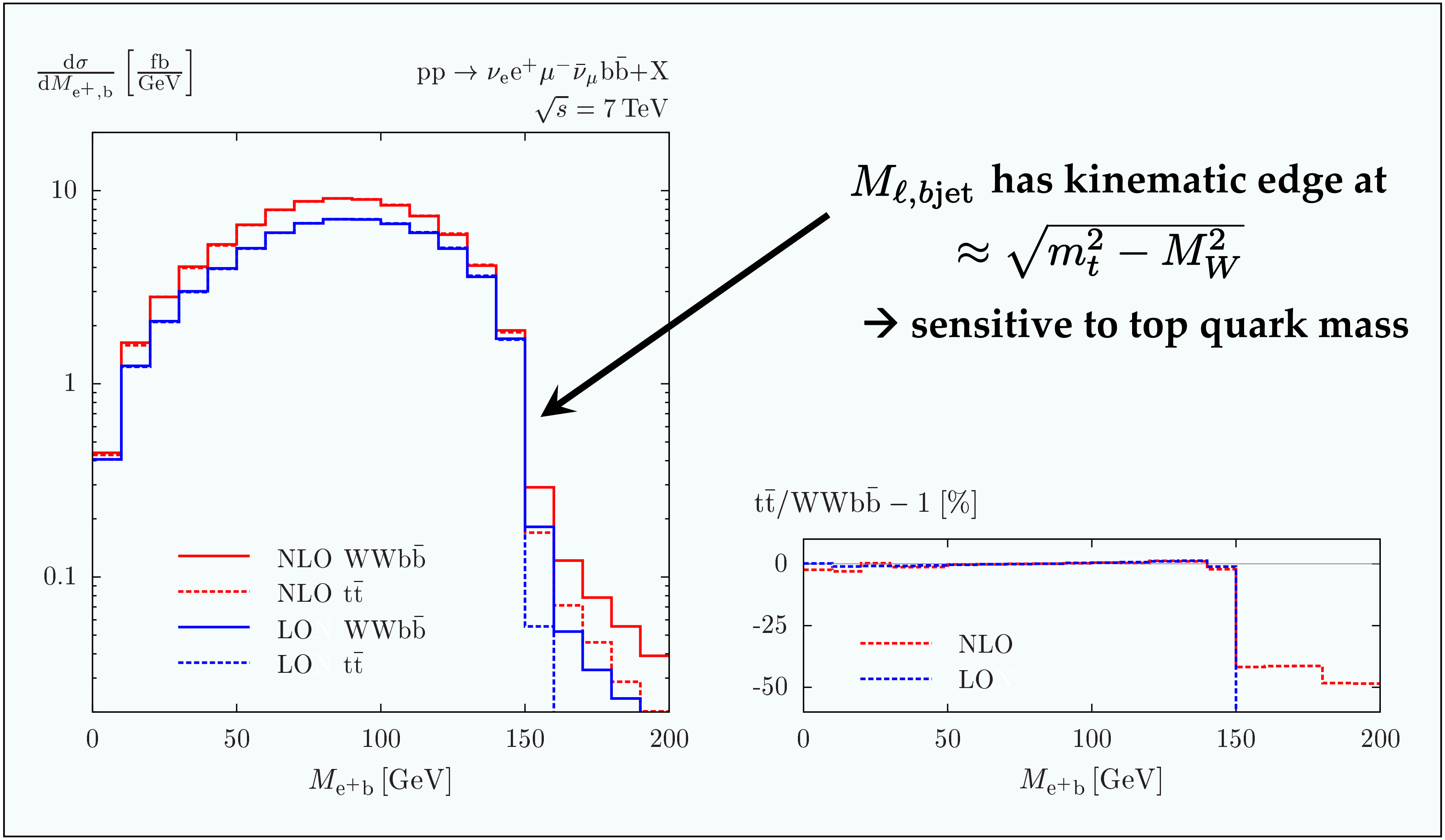
Tevatron: $A^t_{\mathrm{FB}} = 5.1(1)\%$

Tuned benchmark comparison to determine finite-width effects:

[Denner,Dittmaier,Kallweit,Pozzorini,M.S.] (2012)

Collider	$\sqrt{s}$ [TeV]	approx.	$\sigma_{t\bar{t}}$ [fb]	σ_{WWbb} [fb]	$\sigma_{t\bar{t}}/\sigma_{WWbb} - 1$
Tevatron	1.96	LO	$44.691(8)^{+19.81}_{-12.58}$	$44.310(3)^{+19.68}_{-12.49}$	$+0.861(19)\%$
		NLO	$42.16(3)^{+0.00}_{-2.91}$	$41.75(5)^{+0.00}_{-2.63}$	$+0.98(14)\%$
LHC	7	LO	$659.5(1)^{+261.8}_{-173.1}$	$662.35(4)^{+263.4}_{-174.1}$	$-0.431(16)\%$
		NLO	$837(2)^{+42}_{-87}$	$840(2)^{+41}_{-87}$	$-0.41(31)\%$

Finite-width effects in ttbar production



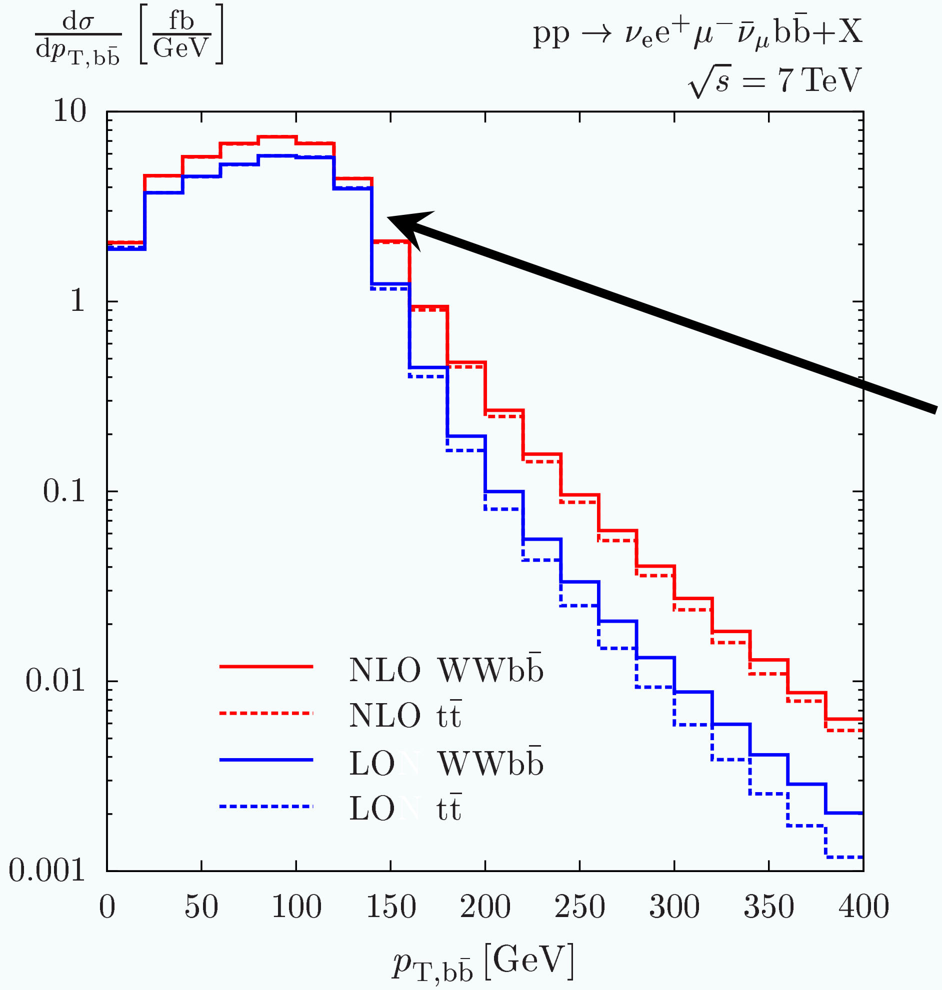
$M_{\ell,bjet}$ **has kinematic edge at**

$$\approx \sqrt{m_t^2 - M_W^2}$$

→ sensitive to top quark mass

[Denner,Dittmaier,Kallweit,Pozzorini,M.S.]

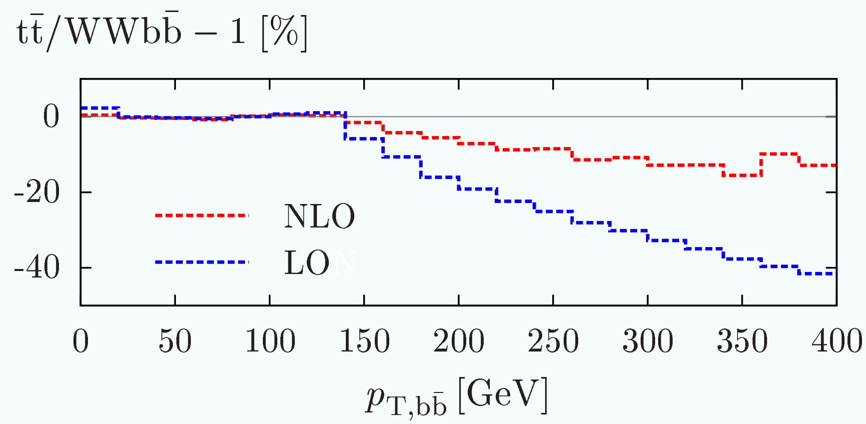
Finite-width effects in ttbar production



kinematic boundary for

$$p_T^{bjet} \approx \frac{m_t^2 - M_W^2}{2m_t} \approx 65\ \text{GeV}$$

beyond $p_T^{b\bar{b}} > 2 \times 65$ GeV
$b\bar{b}$**-pair needs to be boosted**

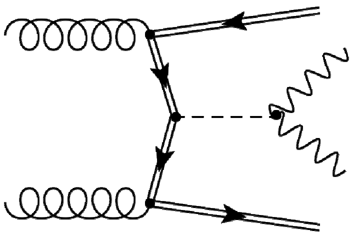


[Denner,Dittmaier,Kallweit,Pozzorini,M.S.]

Legs:
Associated top quark pair production

Associated top quark pair production

$t\bar{t} + H$:

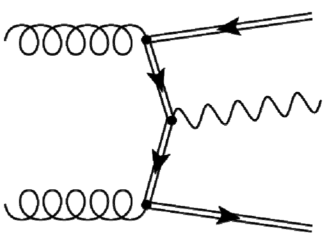


$$\sigma^{\rm NLO}_{t\bar{t}H}(14\,{\rm TeV};\, m_H = 125\,{\rm GeV}) = 611\ {\rm fb}\ \pm 9\%({\rm scale})\ \pm 9\%({\rm PDFs})$$

→ after branchings: $\sim \mathcal{O}(5\,{\rm fb})$

[Beenakker,Dittmaier,Krämer,Plümper,Spira,Zerwas], [Dawson,Jackson,Orr,Reina,Wackeroth]
[Frederix,Frixione,Hirschi,Maltoni,Pittau,Torielli], [Garzelli,Kardos,Papadopoulos,Trocsanyi]

$t\bar{t} + Z$:

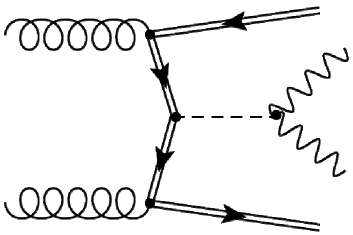


$$\sigma^{\rm NLO}_{t\bar{t}+Z}(14\,{\rm TeV}) = 1.1\ {\rm pb}$$ → after branchings: $\sim \mathcal{O}(1\,{\rm fb})$

[Lazopoulos,McElmurry,Melnikov,Petriello]
[Garzelli,Kardos,Papadopoulos,Trocsanyi]

Associated top quark pair production

$t\,\bar{t} + H$:



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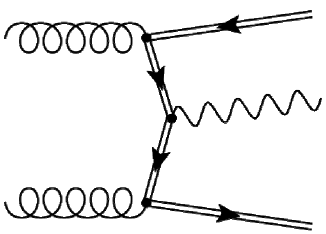

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[Lazopoulos,McElmurry,Melnikov,Petriello]
[Garzelli,Kardos,Papadopoulos,Trocsanyi]

We have to wait for the 13 TeV run to observe these signals.

Associated top quark pair production

The situation is different for emission of light particles from tops.

$t\,\bar{t} + \gamma$: [Duan,Ma,Zhang,Han,Guo,Wang]
[Melnikov,Scharf,M.S.]

$t\,\bar{t} + \mathrm{jet(s)}$: [Dittmaier,Uwer,Weinzierl], [Melnikov,Scharf,M.S], [Alioli,Moch,Uwer]
[Kardos,Papadopolous,Trocsanyi]
[Bredenstein,Denner,Dittmaier,Pozzorini], [Bevilacqua,Czakon,Papadopoulos,Worek]

$\Rightarrow$ **QCD corrections and radiation in top quark decays are crucial for describing** $t\bar{t}{+}\gamma$ and $t\bar{t}{+}\mathrm{jet}$ production.

[Melnikov,Scharf,M.S.] (2011)

E.g., in the process $t\,\bar{t} + \gamma$ more than 50% of all photons come from radiation in the top decays.

$$\sigma^{\rm NLO}_{t\bar{t}\gamma} = 138 \text{ fb} = 61(\text{prod}) + 77(\text{decay}) \text{ fb}$$

Associated top quark pair production

$t\bar{t} + \mathrm{jet}$

$$\mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}+1j} = \Gamma^{-2}_{t,\mathrm{tot}} \left[\mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}+1j} \mathrm{d}\Gamma^{\mathrm{LO}}_{t\bar{t}} + \mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}} \mathrm{d}\Gamma^{\mathrm{LO}}_{t\bar{t}+1j} \right]$$

Associated top quark pair production

$t\bar{t} + \text{jet}$

$$\mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}+1j} = \Gamma^{-2}_{t,\mathrm{tot}} \left[\mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}+1j} \mathrm{d}\Gamma^{\mathrm{LO}}_{t\bar{t}} + \mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}} \mathrm{d}\Gamma^{\mathrm{LO}}_{t\bar{t}+1j} \right]$$

$$\mathrm{d}\sigma^{\delta\mathrm{NLO}}_{t\bar{t}+1j} = \Gamma^{-2}_{t,\mathrm{tot}} \Bigg[\overbrace{\left(\mathrm{d}\sigma^{\mathrm{virt}}_{t\bar{t}+1j} + \mathrm{d}\sigma^{\mathrm{real}}_{t\bar{t}+2j} \right) \mathrm{d}\Gamma^{\mathrm{LO}}_{t\bar{t}}}^{\text{production}} + \overbrace{\mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}} \left(\mathrm{d}\Gamma^{\mathrm{virt}}_{t\bar{t}+1j} + \mathrm{d}\Gamma^{\mathrm{real}}_{t\bar{t}+2j} \right)}^{\text{decay}}$$
$$+ \underbrace{\mathrm{d}\sigma^{\mathrm{real}}_{t\bar{t}+1j} \mathrm{d}\Gamma^{\mathrm{real}}_{t\bar{t}+1j} + \mathrm{d}\sigma^{\mathrm{virt}}_{t\bar{t}} \mathrm{d}\Gamma^{\mathrm{LO}}_{t\bar{t}+1j} + \mathrm{d}\sigma^{\mathrm{LO}}_{t\bar{t}+1j} \mathrm{d}\Gamma^{\mathrm{virt}}_{t\bar{t}}}_{\text{mixed}} \Bigg].$$

[Melnikov,Scharf,M.S.] (2011)

$$\sigma^{\mathrm{LO}}_{t\bar{t}+\mathrm{jet}} = 350^{+215}_{-123}\ \mathrm{fb} = 317(\mathrm{prod}) + 33(\mathrm{decay})\ \mathrm{fb}$$

$$\sigma^{\mathrm{NLO}}_{t\bar{t}+\mathrm{jet}} = 288^{+46}_{-18}\ \mathrm{fb} = 323(\mathrm{prod}) + 40.5(\mathrm{decay}) - 75.5(\mathrm{mixed})\ \mathrm{fb}$$

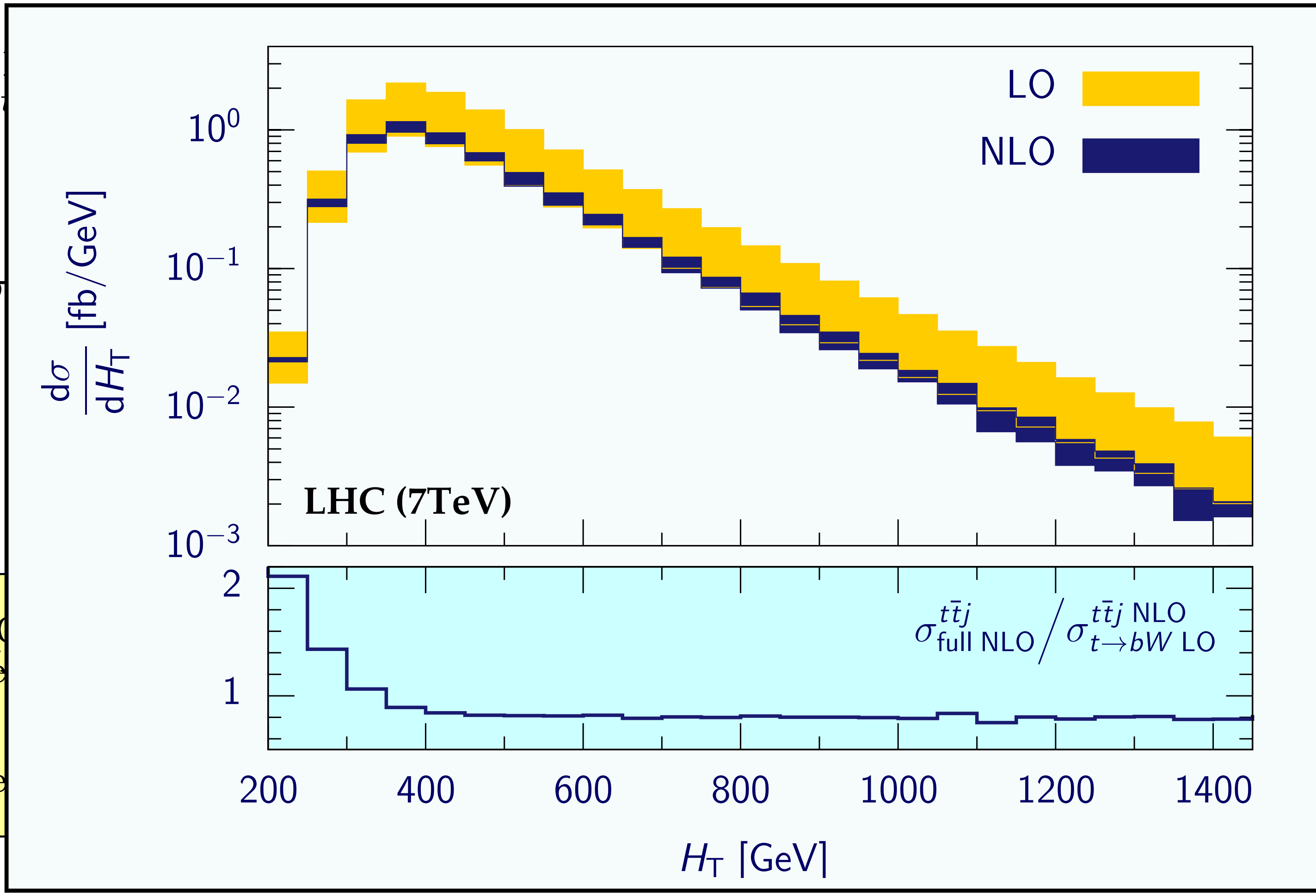
We also observe significant shape changes in kinematic distributions due to radiation in decay.

Associated top quark pair production

$t\bar{t} + \text{jet}$

$d\sigma$ … $d\sigma$ … $\text{real}_{t\bar{t}+2j}$ …]. …harf,M.S.] (2011)

$\sigma^{\text{LO}}_{t\bar{t}+\text{je}}$ … $\sigma^{\text{NLO}}_{t\bar{t}+\text{je}}$ …) fb



We also observe significant shape changes in kinematic distributions due to radiation in decay.

Associated top quark pair production

$t\bar{t} + \text{jet}$

dσ

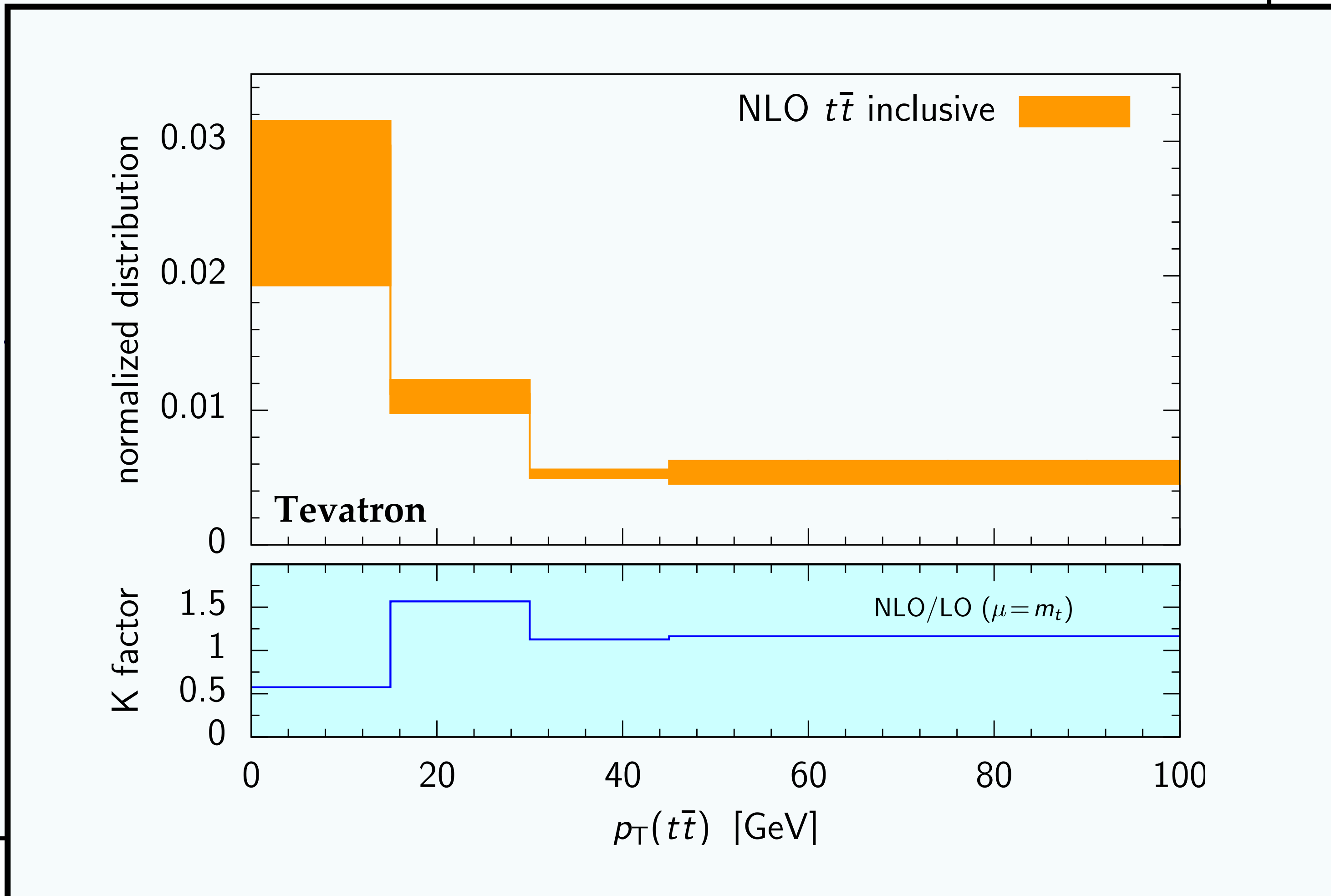
dσ

$_j$)

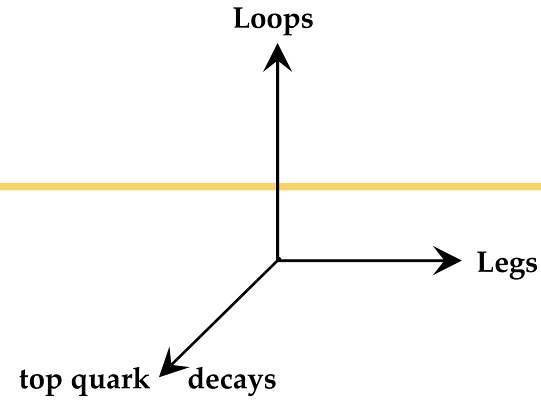
M.S.] (2011)

$\sigma^{\text{LO}}_{t\bar{t}+\text{je}}$

$\sigma^{\text{NLO}}_{t\bar{t}+\text{je}}$



kinematic distributions due to radiation in decay.



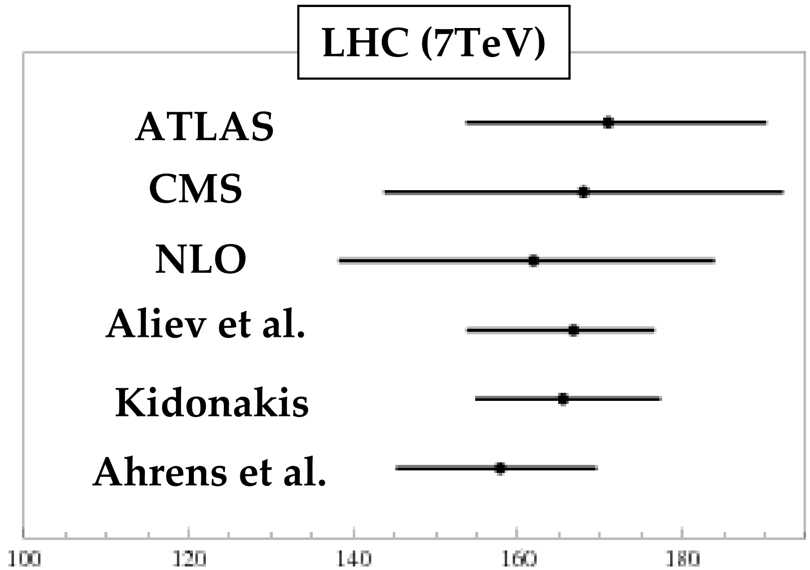
Conclusions

- **Top quark theory is well-positioned for high precision measurements**
  - size of experimental uncertainties ($1fb^{-1}$) of the total cross section are comparable to the best theory predictions
  - predictions of the FB asymmetry seem robust
  - only a full NNLO QCD calculation will bring more insight
- **A realistic description requires top quark decays beyond LO**
  - corrections in the decay are as important as for the production process
  - NWA works well; 5-30% off-shell effects in some kinematic tails
- **The study of associated top quark pair production will mark the post-Tevatron era in top quark physics**
  - electroweak top quark couplings, background to many NP searches
  - radiation in decay is sizable contribution that cannot be missed

Extras

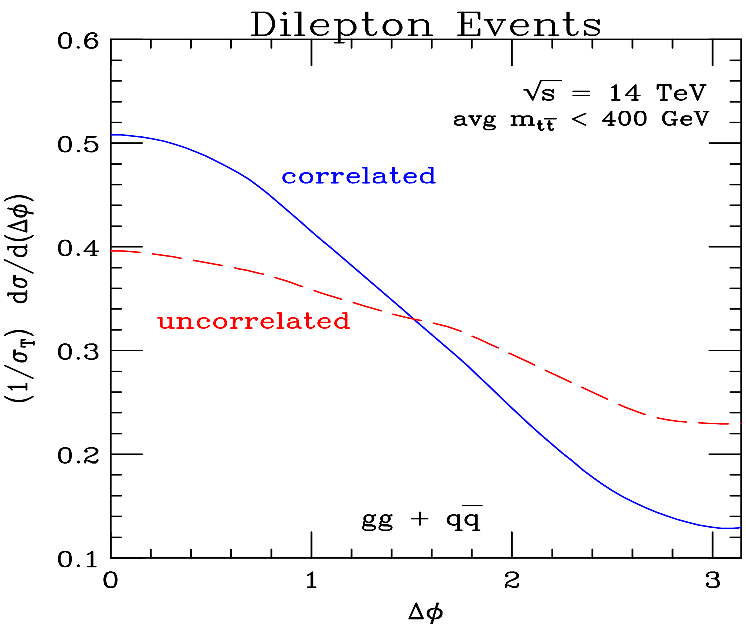
Extras

$t\,\bar{t}$:

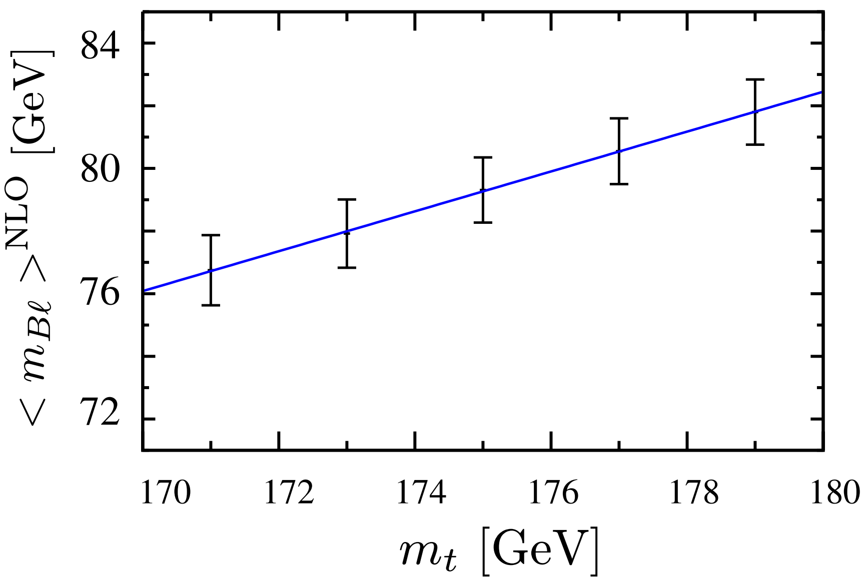


Extras

$t\,\bar{t}$:



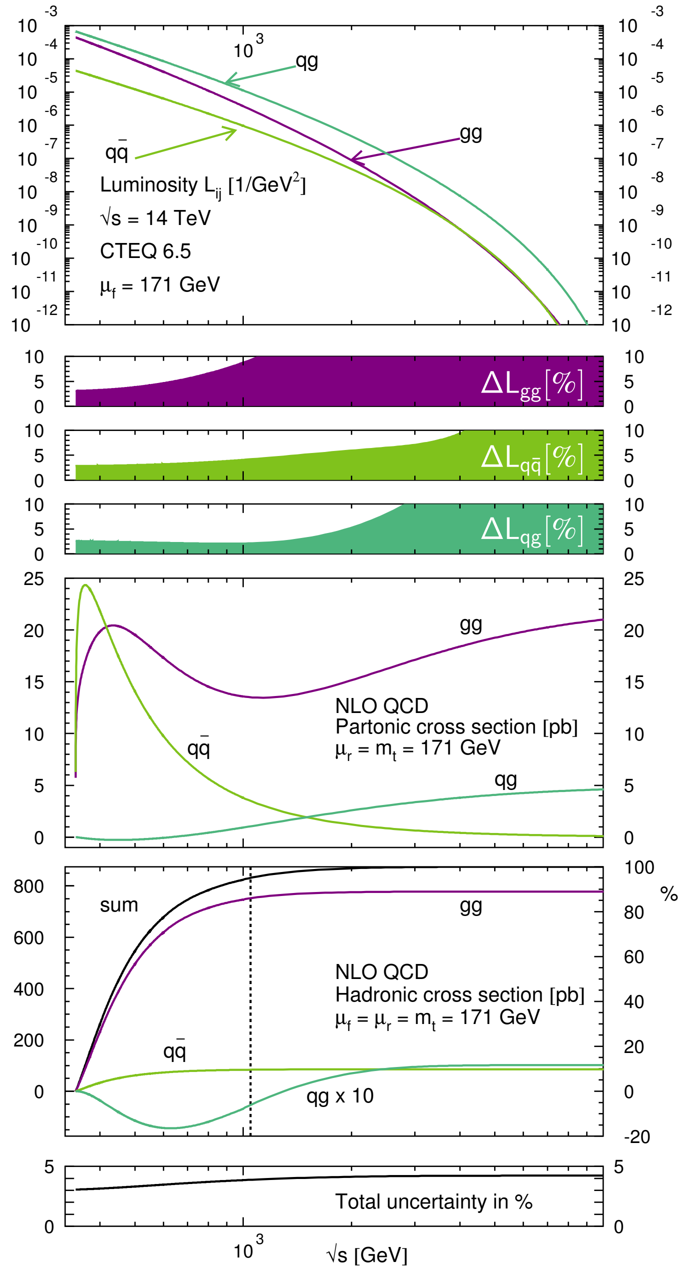
[Mahlon,Parke]



$$< m_{B\ell} >^{\rm prod}_{\rm NLO} = 0.64 m_{\rm top} - 32.12 {\rm GeV}$$

[Melnikov, M.S.]

Extras

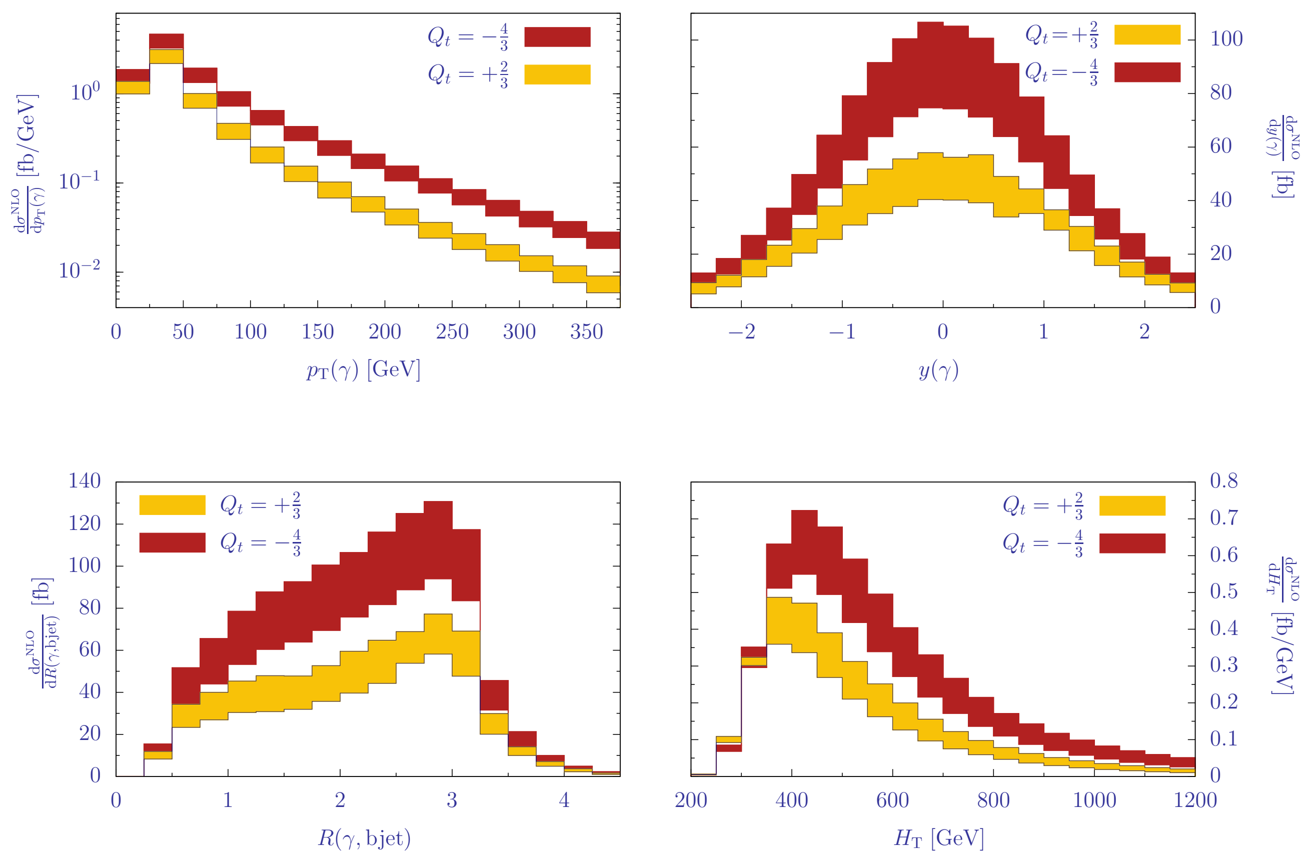


$t\,\bar{t}$:

[Moch, Uwer]

Extras

$$t\,\bar{t} + \gamma:$$

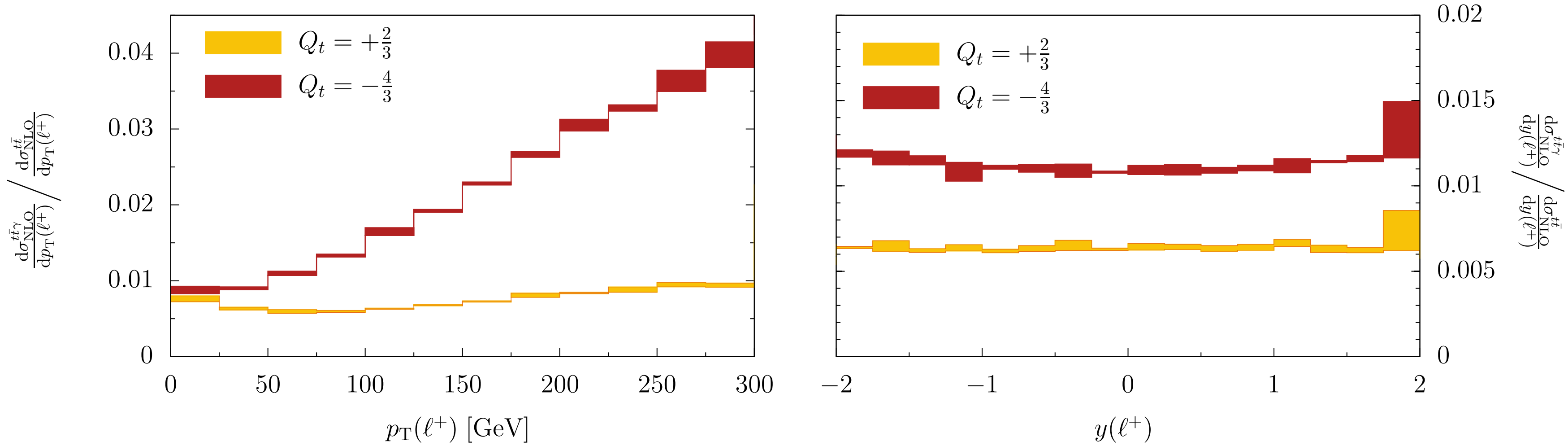


$$\sigma^{\mathrm{NLO}}_{t\bar{t}\gamma} = 138 \text{ fb} \xrightarrow{Q_t = \frac{2}{3} \to -\frac{4}{3}} \sigma^{\mathrm{NLO}}_{t\bar{t}\gamma} = 243 \text{ fb}$$

Naive expectation of Q_t^2 scaling fails !

$t\bar{t} + \gamma$:

Ratio of cross sections

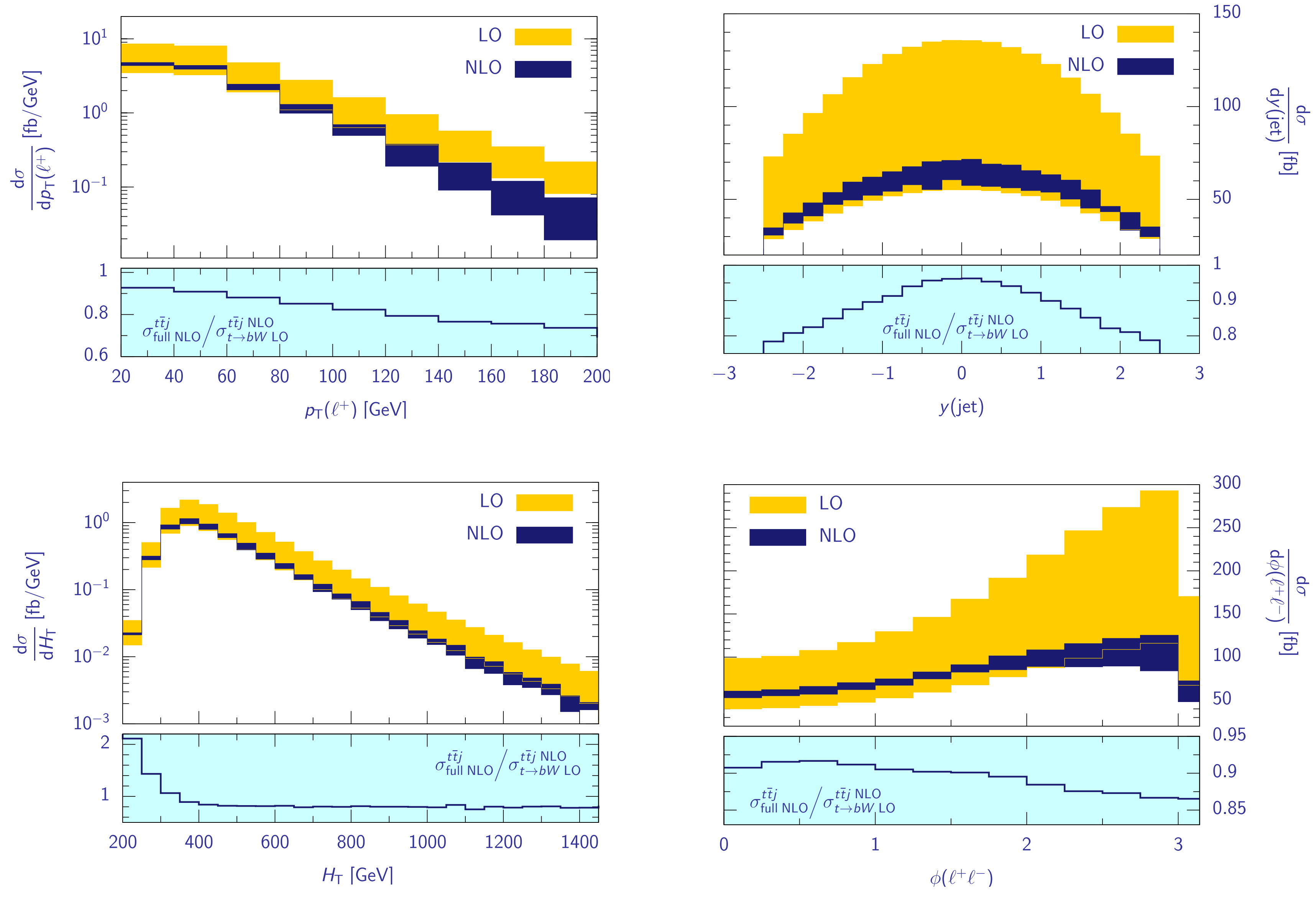


$$\frac{\sigma^{Q_t=2/3}_{t\bar{t}\gamma}}{\sigma_{t\bar{t}}} = \begin{cases} 5.66^{+0.03}_{-0.02} \times 10^{-3}, & \text{LO;} \\ 6.33^{+0.26}_{-0.14} \times 10^{-3}, & \text{NLO,} \end{cases} \qquad \frac{\sigma^{Q_t=-4/3}_{t\bar{t}\gamma}}{\sigma_{t\bar{t}}} = \begin{cases} 10.4^{+0.2}_{-0.2} \times 10^{-3}, & \text{LO;} \\ 11.2^{+0.3}_{-0.2} \times 10^{-3}, & \text{NLO.} \end{cases}$$

- Ratios are significantly more stable against NLO corrections
- Small scale uncertainties
- Some experimental uncertainties cancel

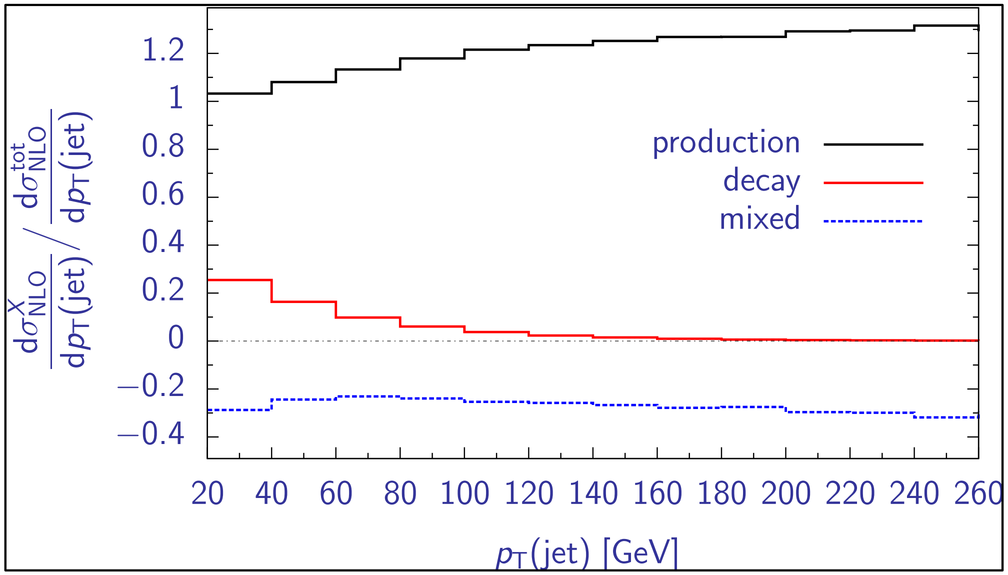
Extras

$t\bar{t} + \text{jet}$



Extras

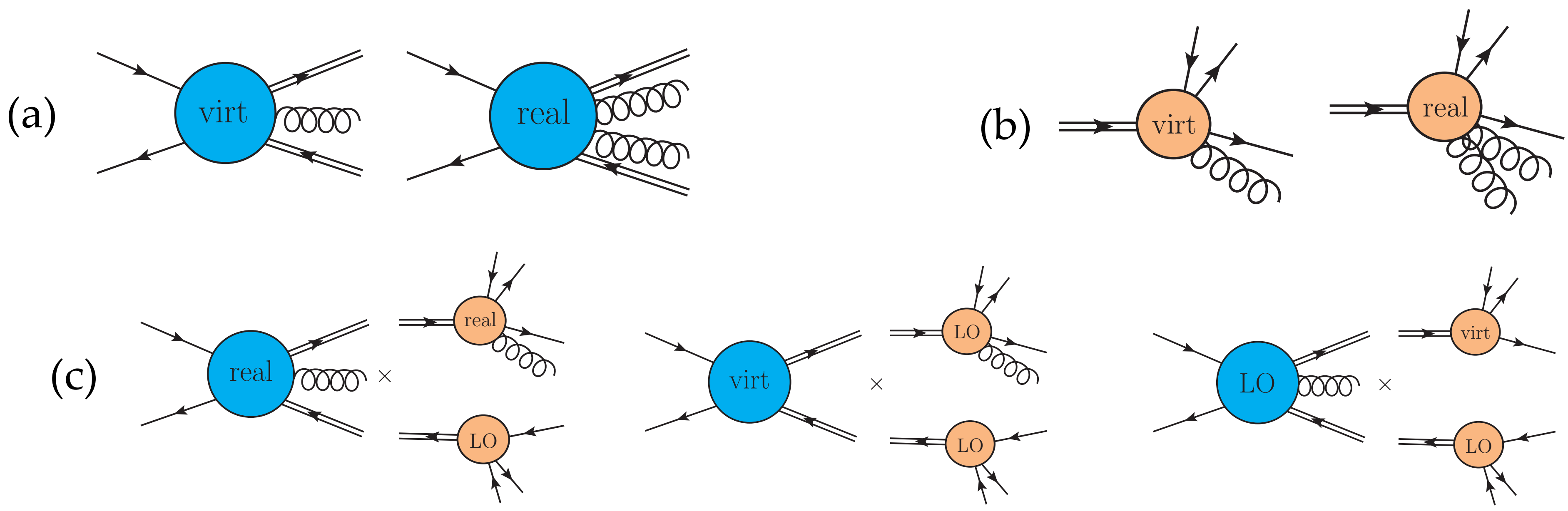
$t\bar{t} + \mathrm{jet}$



Extras

$t\bar{t} + \text{jet}$

$$\mathrm{d}\sigma^{\text{NLO}}_{t\bar{t}+1j} = \Gamma^{-2}_{t,\text{tot}}\Big(\mathrm{d}\sigma^{\text{LO}}_{t\bar{t}+1j}\mathrm{d}\Gamma^{\text{LO}}_{t\bar{t}} + \mathrm{d}\sigma^{\text{LO}}_{t\bar{t}}\mathrm{d}\Gamma^{\text{LO}}_{t\bar{t}+1j} + \overbrace{\left(\mathrm{d}\sigma^{\text{virt}}_{t\bar{t}+1j} + \mathrm{d}\sigma^{\text{real}}_{t\bar{t}+2j}\right)\mathrm{d}\Gamma^{\text{LO}}_{t\bar{t}}}^{(a)\ \textbf{production}}$$
$$+ \underbrace{\mathrm{d}\sigma^{\text{LO}}_{t\bar{t}}\left(\mathrm{d}\Gamma^{\text{virt}}_{t\bar{t}+1j} + \mathrm{d}\Gamma^{\text{real}}_{t\bar{t}+2j}\right)}_{(b)\ \textbf{decay}} + \underbrace{\mathrm{d}\sigma^{\text{real}}_{t\bar{t}+1j}\mathrm{d}\Gamma^{\text{real}}_{t\bar{t}+1j} + \mathrm{d}\sigma^{\text{virt}}_{t\bar{t}}\mathrm{d}\Gamma^{\text{LO}}_{t\bar{t}+1j} + \mathrm{d}\sigma^{\text{LO}}_{t\bar{t}+1j}\mathrm{d}\Gamma^{\text{virt}}_{t\bar{t}}}_{(c)\ \textbf{mixed}}\Big).$$



Extras

$t\bar{t} + \text{jet}$

Comparison with parton shower (Tevatron)

Powheg+HELAS+Pythia

[Kardos,Papadopoulos,Trocsanyi]
Phys.Lett. B705 (2011) 76

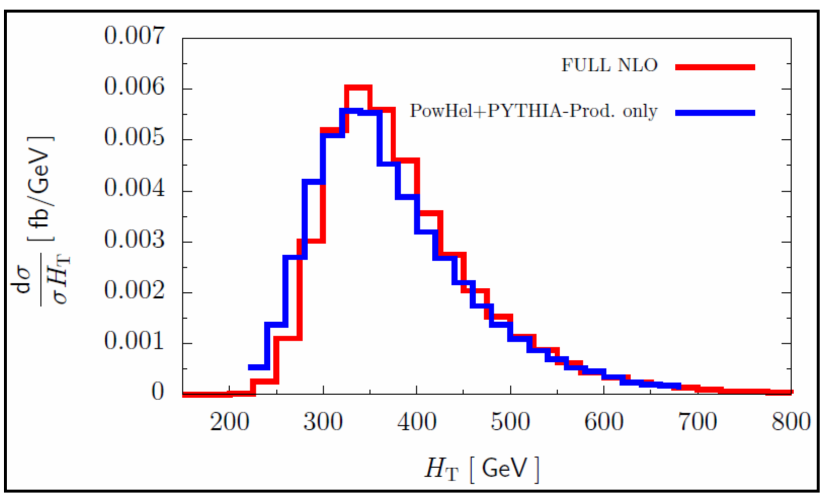
NLO QCD in production and decay

[Melnikov,Scharf,M.S.]

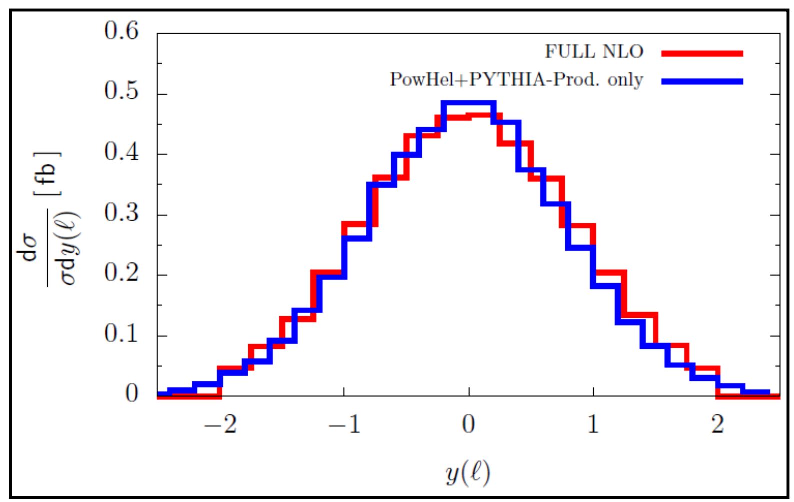
$\sigma \approx 48$ fb

Normalization

$$\sigma = 78.9 = 47.7(\text{prod}) + 36.7(\text{decay}) - 5.5(\text{mix}) \text{ fb}$$

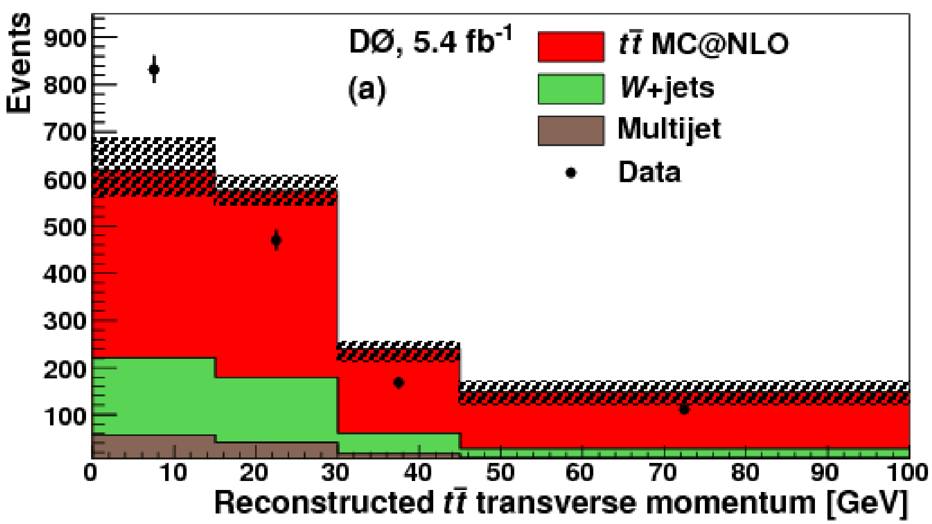


Shapes



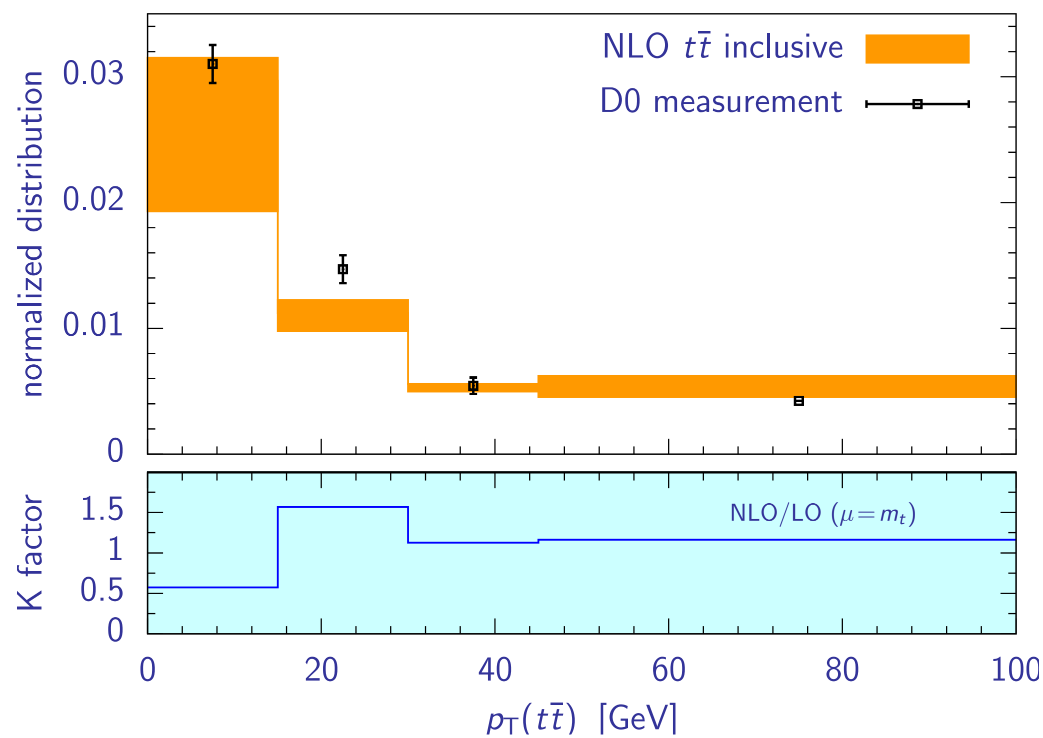
Extras

$$t\bar{t} + \text{jet}$$



[DZero] Phys.Rev.D84 (2011) 112005

"resolution is half the bin width"



[Melnikov, Scharf, M.S.]

PRELIMINARY ?

Resummation and NNLO$_{\text{approx}}$

Close to threshold, the total cross section receives contributions proportional to *1/β* and log(*β*).

$$\sigma_{ij}^{(2)}(\mu,\beta) = \sigma_{ij}^{(0)}(\mu,\beta)\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2\left[\sigma_{ij}^{(2,0)} + \sigma_{ij}^{(2,1)}\log\left(\frac{\mu^2}{m_t^2}\right) + \sigma_{ij}^{(2,2)}\log^2\left(\frac{\mu^2}{m_t^2}\right)\right]$$

$$\sigma_{ij}^{(2,n)} = \sum_{a=0,1,2}\ \sum_{b=0,4-2a}^{a+b>0} k_{ij}\,\frac{1}{\beta^a}\log^b\beta + C_{ij}^{(2,n)} + \mathcal{O}(\beta)$$

$C_{ij}^{(2,0)}$ can only be be determined from a full next-to-next-to-leading order (NNLO) calculation