Extraction of Parton Structure including Lattice QCD

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Parton Structure

For various flavors and spin combinations

Wigner Distribution/
Generalized Transverse Momentum Distribution (GTMD)

$F(x, b_T, k_T)$

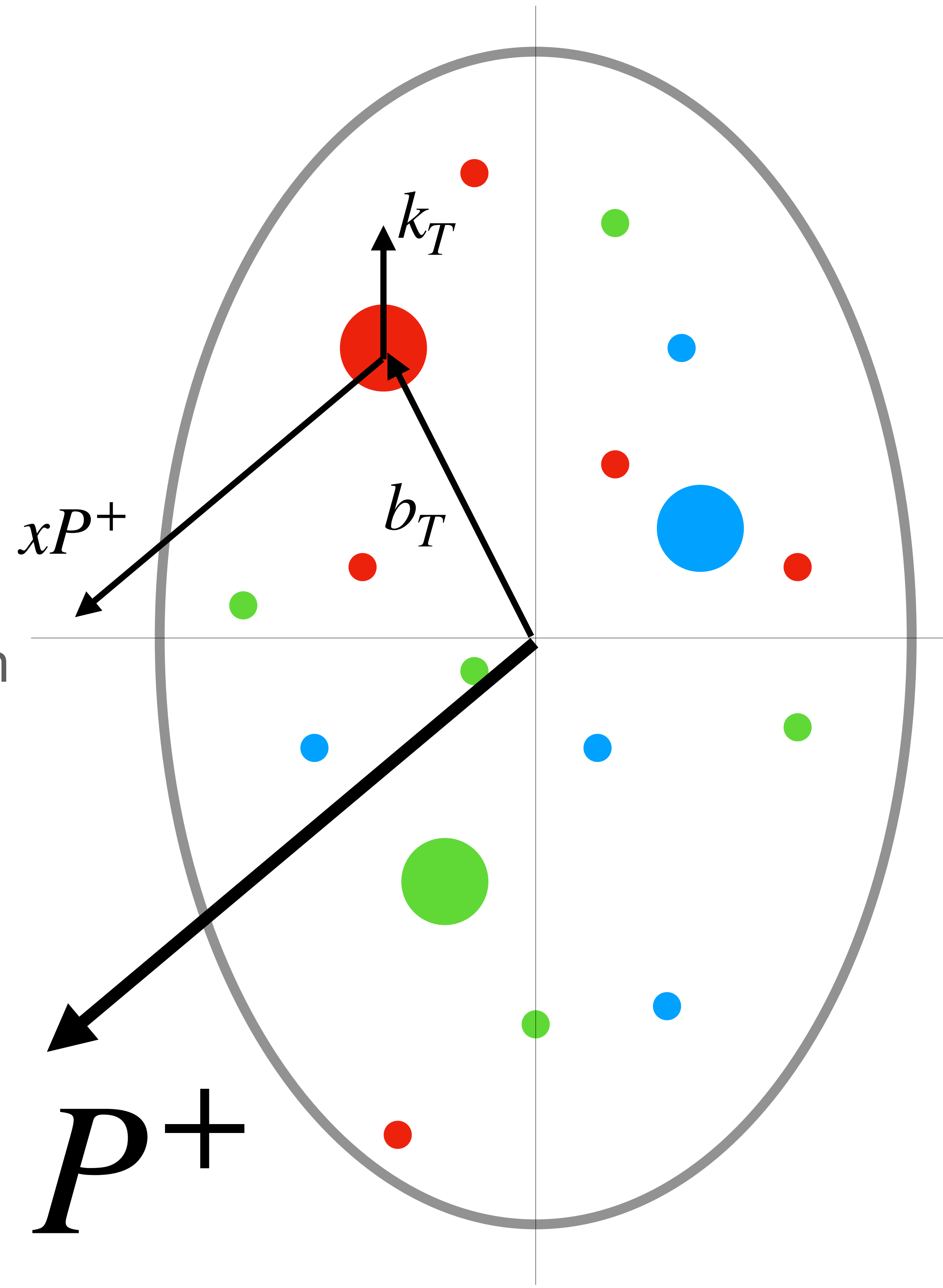
$\int d^2 b_t$ $\int d^2 k_t$

Transverse Momentum Distribution (TMD)
$f(x, k_T)$

Generalized Parton Distribution (GPD)
$f(x, b_T)$

$\int d^2 k_t$ $\int d^2 b_t$

Parton Distribution Function (PDF)
$f(x)$



What is this workshop about?

The focus of this workshop is on Uncertainty Quantification

A **list of Key Questions** is attached to the Indico page of the workshop

1. Accessing PDFs: global analyses and lattice computations

   $\longrightarrow$ How does PDF determination work in global analyses and lattice QCD?

2. Global QCD analyses: inverse problem and objective functions

   $\longrightarrow$ How is the inverse problem entailed by PDF determination addresed?

3. Lattice QCD: considerations on the validity of the perturbative matching

   $\longrightarrow$ How is the equivalence between zP_z and $\xi^- P^+$ defined?

4. Setting up a common language: definitions and benchmarks

   $\longrightarrow$ How to benchmark lattice moments and quasi-/pseudo-PDFs with global analyses?

5. Combining lattice and experimental data to determine PDFs

   $\longrightarrow$ What are the efforts/limitations to incorporate lattice data in PDF determinations?

6. Uncertainty quantification and bias/variance trade-off

   $\longrightarrow$ How are aleatoric and epistemic uncertainties combined? How is a model chosen?

Parton and Ioffe Time distributions

- Unpolarized Ioffe time distributions

Ioffe time: $\nu = p \cdot z$

"Ioffe time distributions instead of parton momentum distributions in description of DIS"
V. Braun, P. Gornicki, L. Mankiewicz
Phys Rev D 51 (1995) 6036-6051

- $$I_q(\nu, \mu^2) = \frac{1}{2p^+} \langle p \,|\, \bar{\psi}_q(z^-)\gamma^+ W(z^-;0)\psi_q(0) \,|\, p\rangle_{\mu^2}$$

$$z^2 = 0$$

$$I_g(\nu, \mu^2) = \frac{1}{(2p^+)^2} \langle p \,|\, F_{+i}(z^-) W(z^-;0) F^i_+(0) \,|\, p\rangle_{\mu^2}$$

$$i = x, y$$

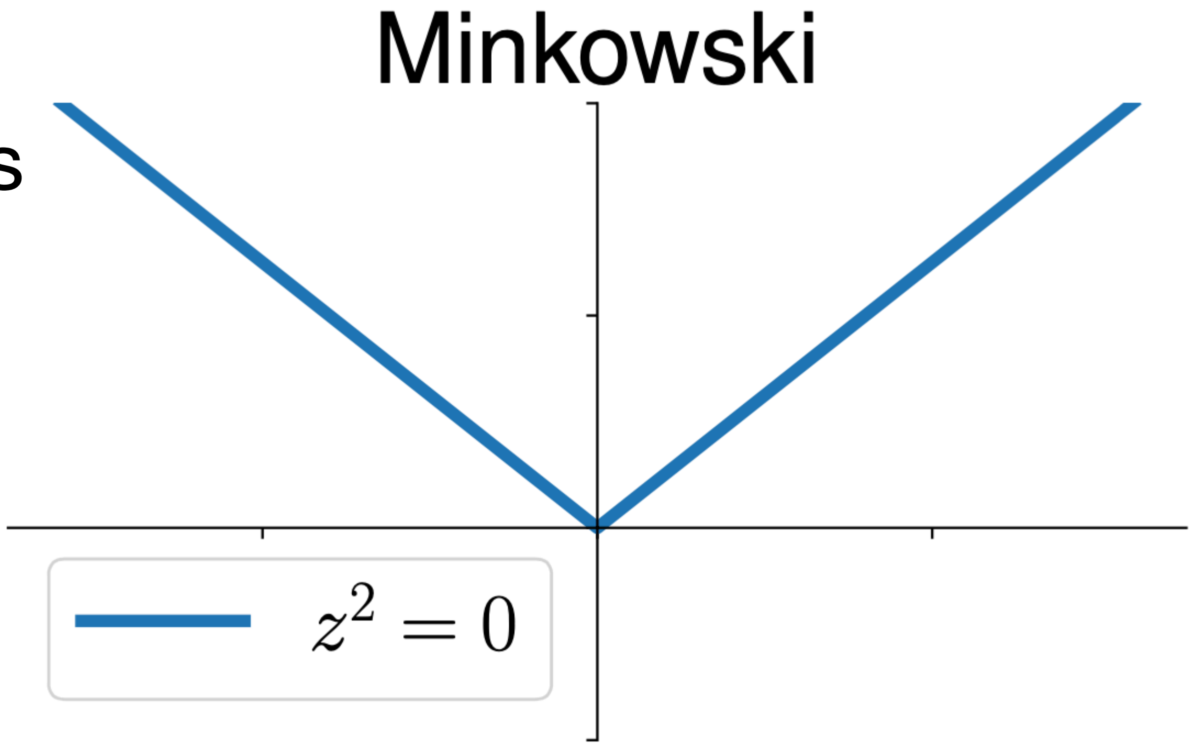
- Parton Distribution Functions

$$I_q(\nu, \mu^2) = \int_{-1}^{1} dx\, e^{ix\nu} f_q(x, \mu^2) \rightarrow f_q(x, \mu^2) = \int \frac{d\nu}{2\pi} e^{ix\nu} I_q(\nu, \mu^2)$$

$$I_g(\nu, \mu^2) = \int_0^1 dx\, \cos(x\nu)\, x f_g(x, \mu^2) \rightarrow x f_g(x, \mu^2) = \int \frac{d\nu}{2\pi} \cos(x\nu) I_g(\nu, \mu^2)$$

Parton Distributions and the Lattice

- Parton Distributions are defined by operators with light-like separations



- Use space-like separations

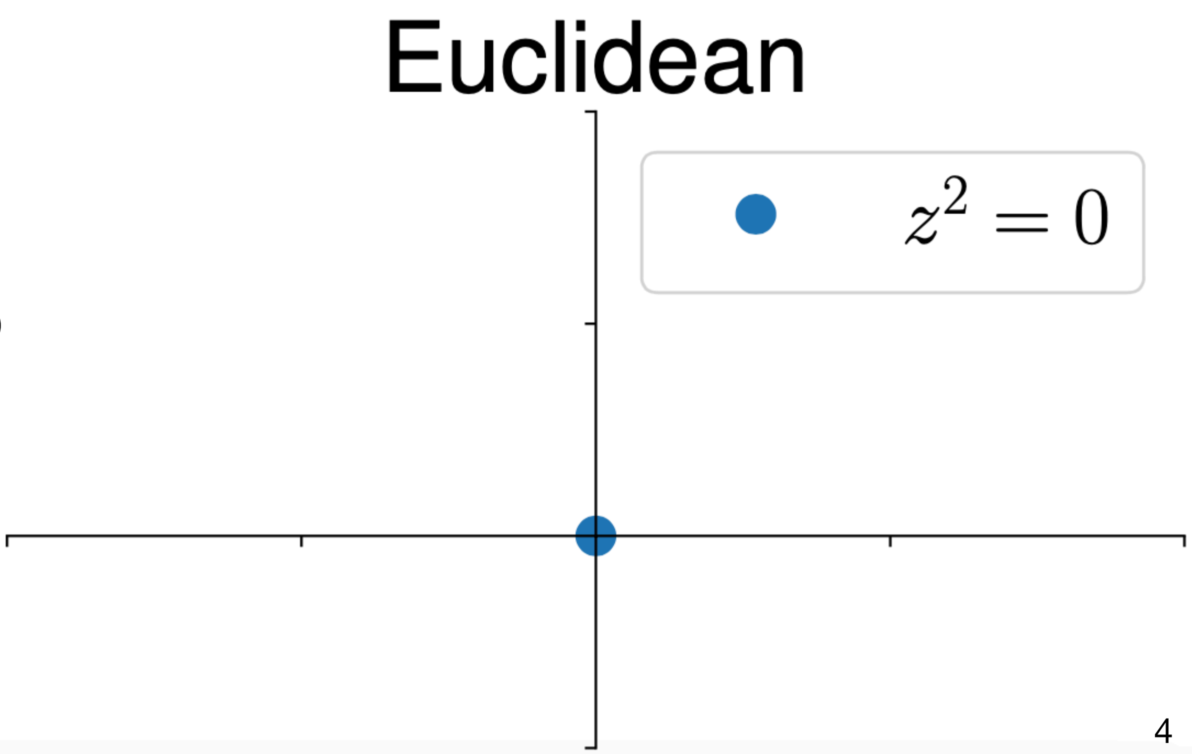
  X. Ji *Phys Rev Lett* 110 (2013) 262002

  - Wilson line operators

$$O_\Gamma^{\mathrm{WL}}(z) = \bar{\psi}(z)\Gamma W(z;0)\psi(0)$$

$$z^2 \neq 0$$

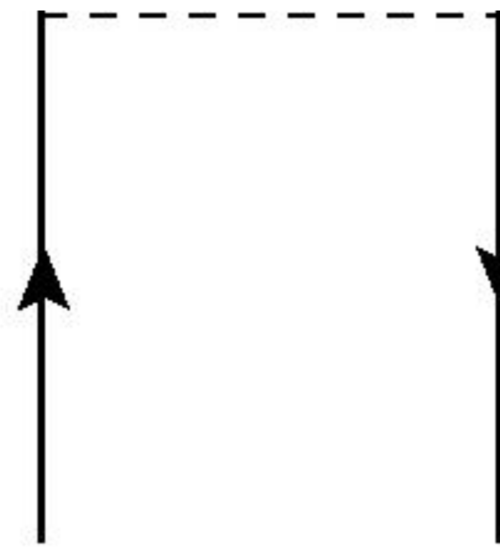
  - Factorizations exist analogous to cross sections



Many approaches

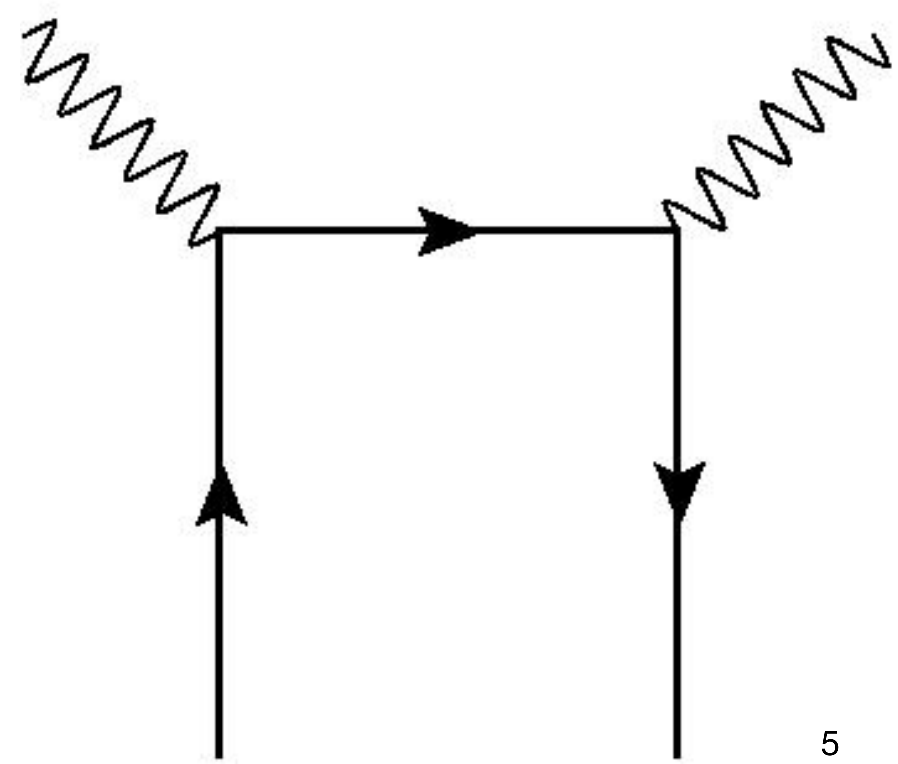
- **Wilson line operators**

$$O_{WL}(x; z) = \bar{\psi}(x+z)\Gamma W(x+z; x)\psi(x)$$



  - LaMET X. Ji *Phys. Rev. Lett.* 110 (2013) 262002
  - Pseudo-PDF A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025
- Two current correlators

$$O_{CC}(x, y) = J_{\Gamma}(x)J_{\Gamma'}(y)$$



  - Hadronic Tensor
    K.-F. Liu et al *Phys. Rev. Lett.* 72 1790 (1994)
    Phys. Rev. D 62 (2000) 074501
  - HOPE
    W. Detmold and C.-J. D. Lin, *Phys. Rev. D* 73 (2006) 014501
  - Short distance OPE
    V. Braun and D. Muller *Eur. Phys. J. C* 55 (2008) 349
  - OPE-without-OPE
    A. Chambers et al, *Phys. Rev. Lett.* 118 (2017) 242001
  - Good Lattice Cross Sections
    Y.-Q. Ma and J.-W. Qiu *Phys. Rev. Lett.* 120 (2018) 2, 022003

Wilson Line Matrix Elements

- More generic element $$M^{\alpha}(p,z) = \langle p \,|\, \bar{\psi}(z)\gamma^{\alpha}W(z;0)\psi(0)\,|\, p\rangle$$ $$= 2p^{\alpha}\mathcal{M}(\nu, z^2) + 2z^{\alpha}\mathcal{N}(\nu, z^2)$$
- PDF (given collinear divergence handled): $$f_q(x,\mu^2) = \int d\nu e^{ix\nu}\mathcal{M}(\nu,0)$$
- Quasi-PDF: $\tilde{q}(y,p_z^2) = \int d\nu e^{i\nu y}\mathcal{M}(\nu, \frac{\nu^2}{p_z^2})$ $\quad z^2 < 0$
- Large Momentum Effective Theory: X. Ji *Phys. Rev. Lett.* 110 (2013) 262002
- $$\tilde{q}(y,p_z^2) = \int \frac{dx}{|x|} K\left(\frac{y}{x}, \frac{\mu^2}{(xp_z)^2}\right) q(x,\mu^2) + O\left(\frac{\Lambda_{\mathrm{QCD}}^2}{(yp_z)^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{((1-y)p_z)^2}\right)$$
- Pseudo-PDF: A. Radyushkin *Phys. Rev. D* 96 (2017) 3, 034025

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} P(x, z^2) = \int_{-1}^{1} dx\, C(x\nu, \mu^2 z^2) q(x,\mu^2) + O(\Lambda_{\mathrm{QCD}}^2 z^2)$$

$$= \int_{-1}^{1} du C'(u, \mu^2 z^2) I_q(u\nu, \mu^2) + O(\Lambda_{\mathrm{QCD}}^2 z^2)$$

The Role of Separation and Momentum

- In Structure Functions, quasi-PDF, and pseudo-PDF, variables have common roles

Scale:

$Q^2 \,/\, p_z^2 \,/\, z^2$

Dynamical variable:

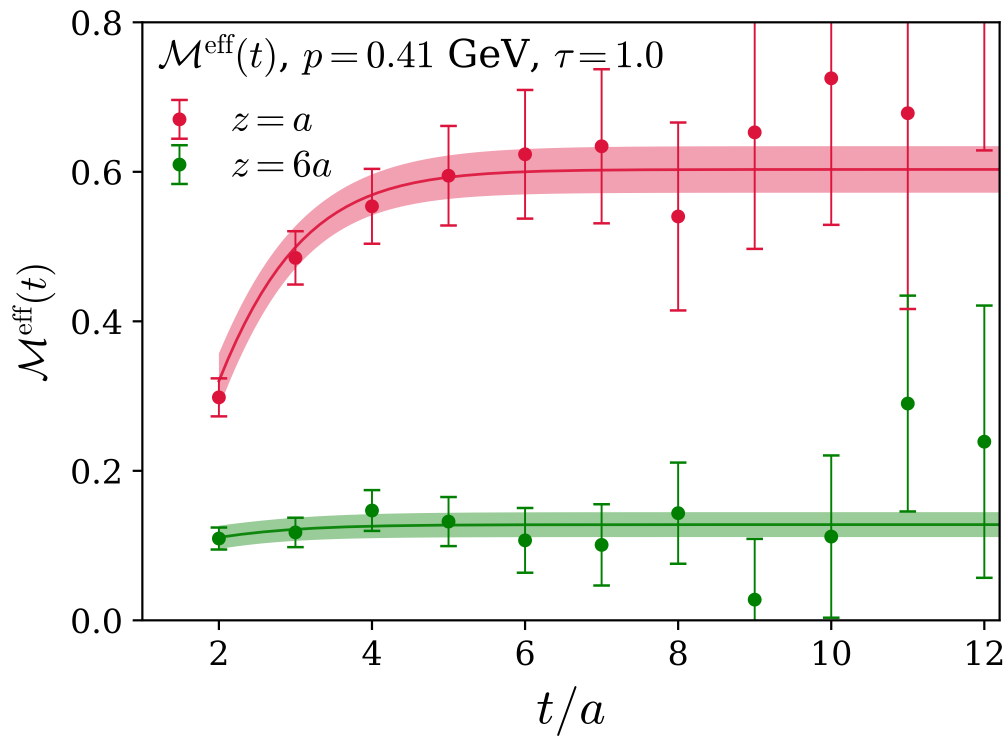
$x_B \,/\, z \,/\, p_z$, or $\nu = p \cdot z$

- Scale for factorization to PDF
- Scale in power expansion
- Keep away from Λ_{QCD}^2
- Technically only requires single value, use many to study systematics

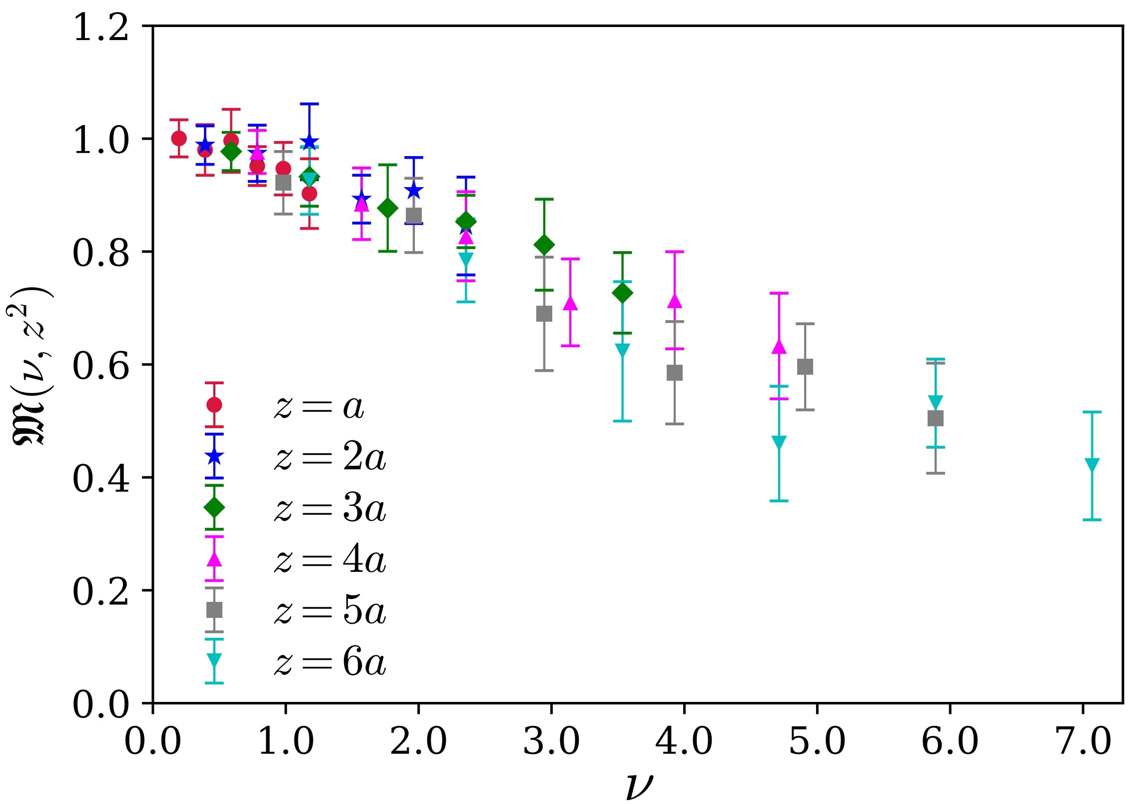
- Variable describes non-perturbative dynamics
- Can take large or small value
- Want as many as are available
- Wider range improves the inverse problem

From Lattice QCD to PDFs

Lattice Correlation Functions



Hadron Matrix Elements



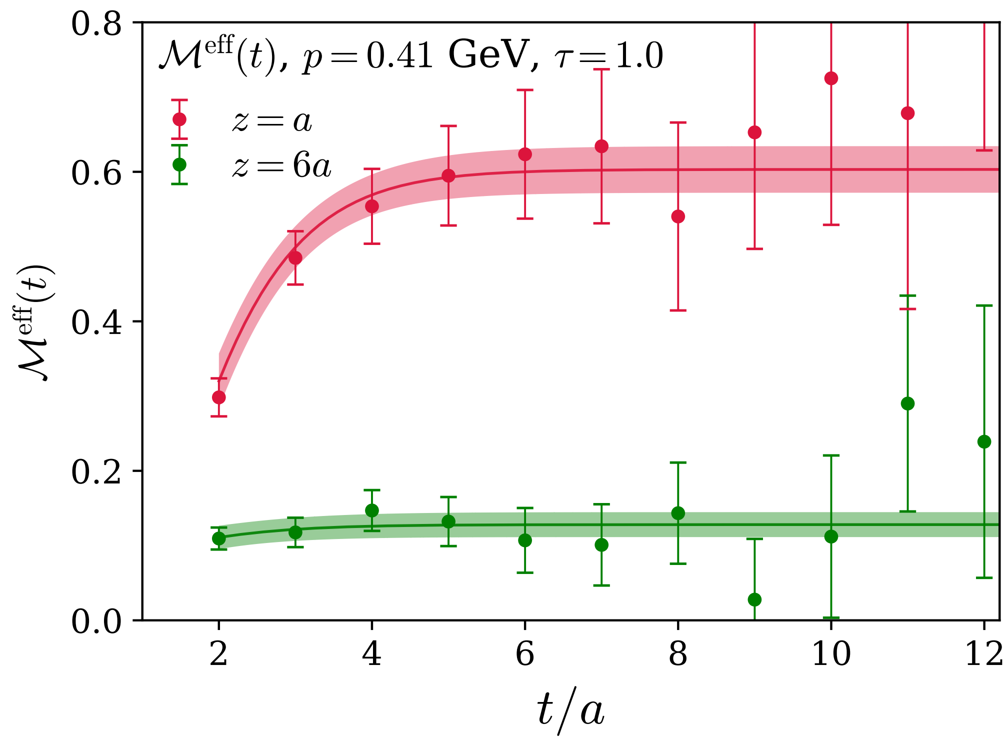
- Correlators (vacuum expectation values of time separated operators) are described as sums over an exponential for each energy eigenstate.
- Coefficients are matrix elements and exponential rates are energy levels
- Model and/or remove subdominant states by using large time but noise grows exponentially

Unpolarized Gluon PDF

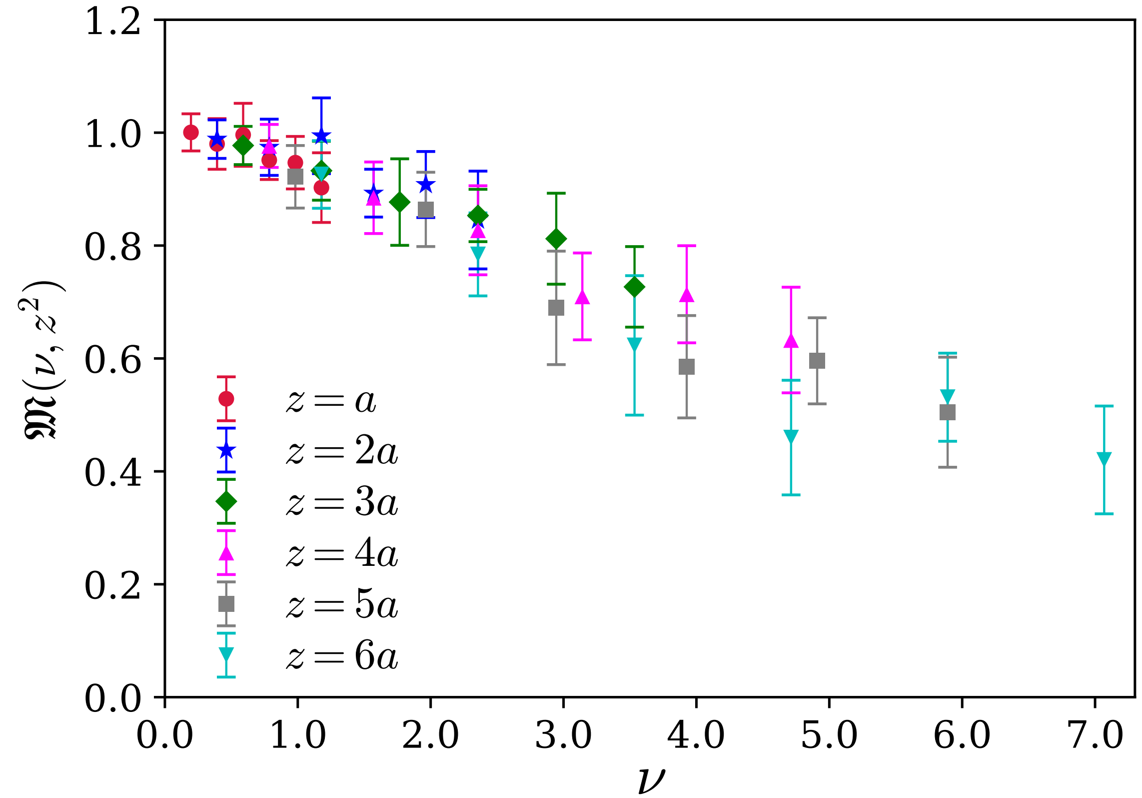
T. Khan, R. Sufian, JK, C. Monahan, C. Egerer, B. Joo, W. Morris, K. Orginos, A. Radyushkin, D. Richards, E. Romero, S. Zafeiropoulos
PRD 104 (2021) 9, 094516

From Lattice QCD to PDFs

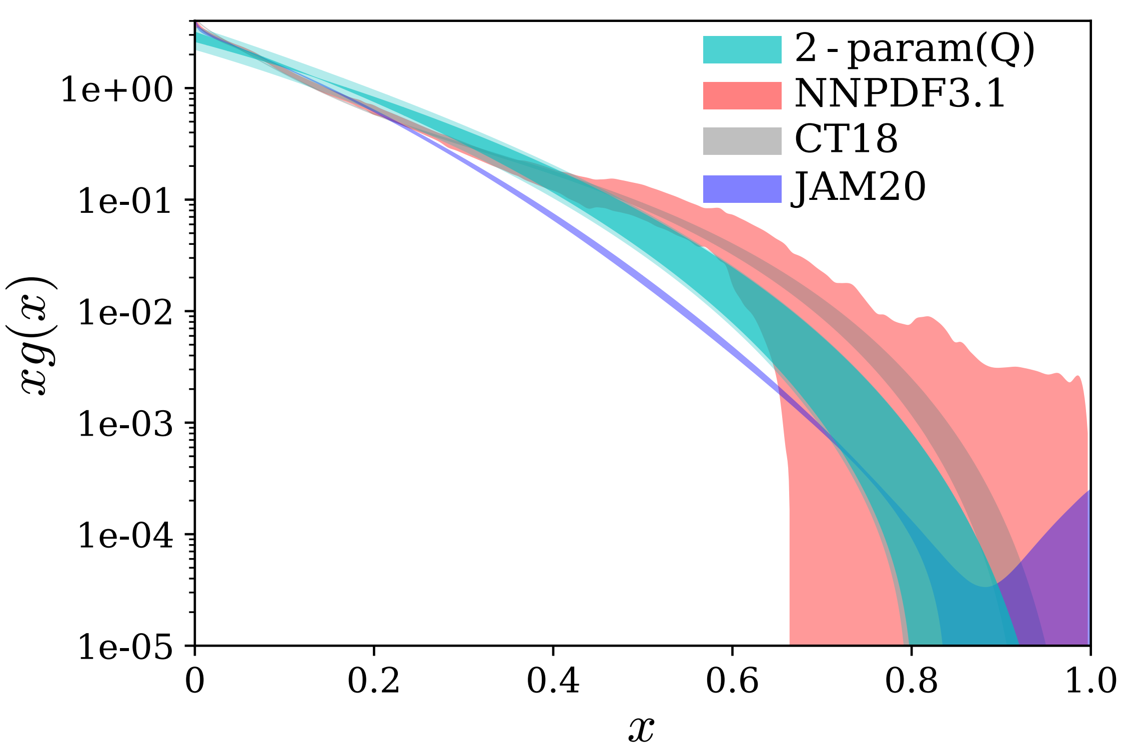
Lattice Correlation Functions



Hadron Matrix Elements



Parton Distributions



- Incomplete information gives integral inverse problem

$$M(\nu) = \int dx C(x\nu) xg(x)$$

$$xg(x) = x^a (1-x)^b / B(a+1, b+1)$$

- To more accurately infer PDF, we need larger ν

Unpolarized Gluon PDF

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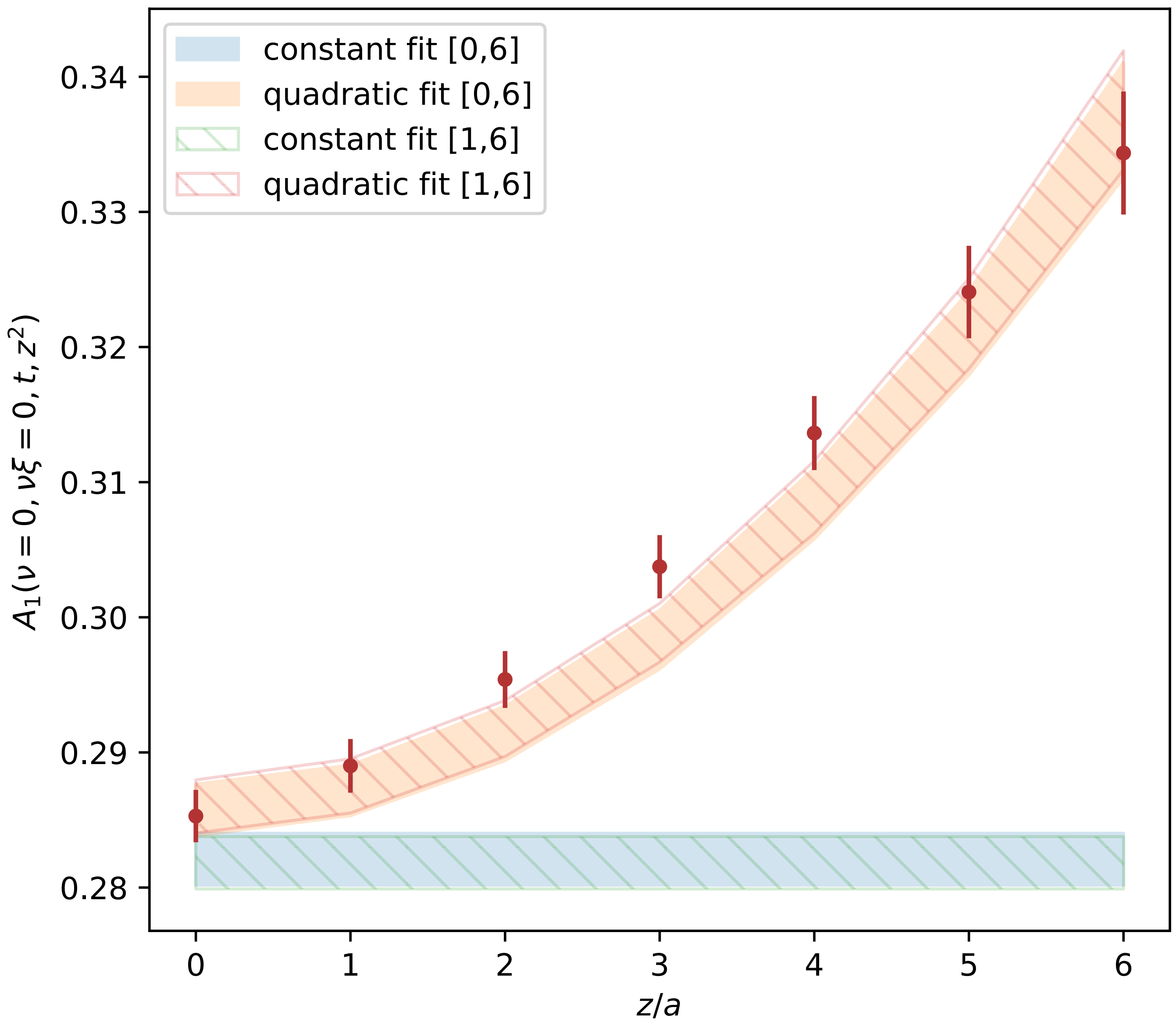
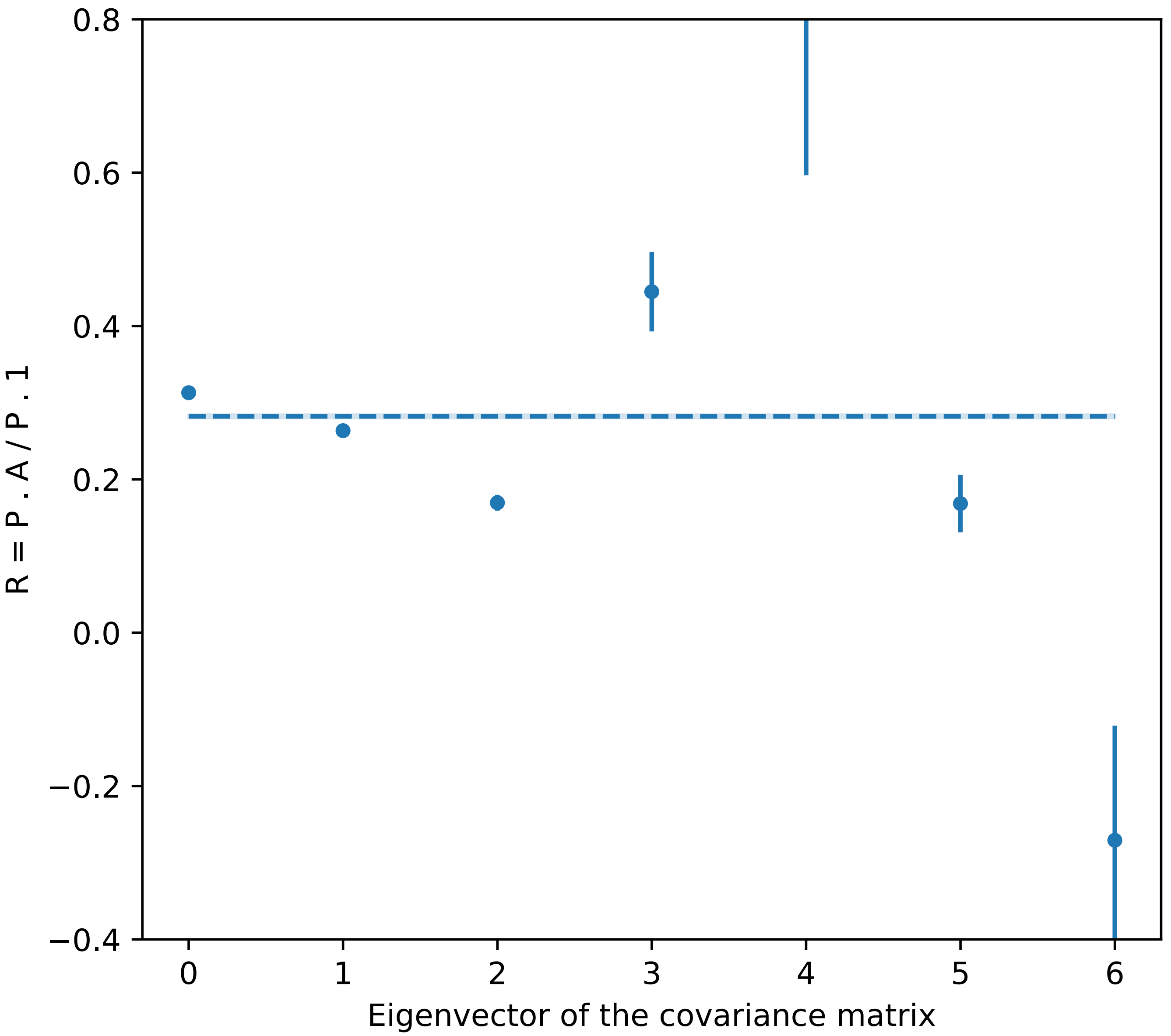
There is less than you think

- Lattice data are highly correlated and you have less information than you think
- What is the average of these 7 data points?





H. Dutrieux et al (HadStruc) JHEP 08 (2024) 162

What can we do beyond looking at nice PDF fits?

If PDFs are universal….

*If the **same** PDFs are factorizable from lattice and experiment,*
and if power corrections can be controlled for both

Why not analyze both simultaneously?

- Factorization of hadronic cross sections

$$d\sigma_h = d\sigma_q \otimes f_{h/q} + P.C.$$

- Factorization of Lattice observables

$$M_h = M_q \otimes f_{h/q} + P.C.$$

Consider Lattice data as a theoretical prior
to the experimental Global Fit

Complementarity in Lattice and Experiment

EXPERIMENT

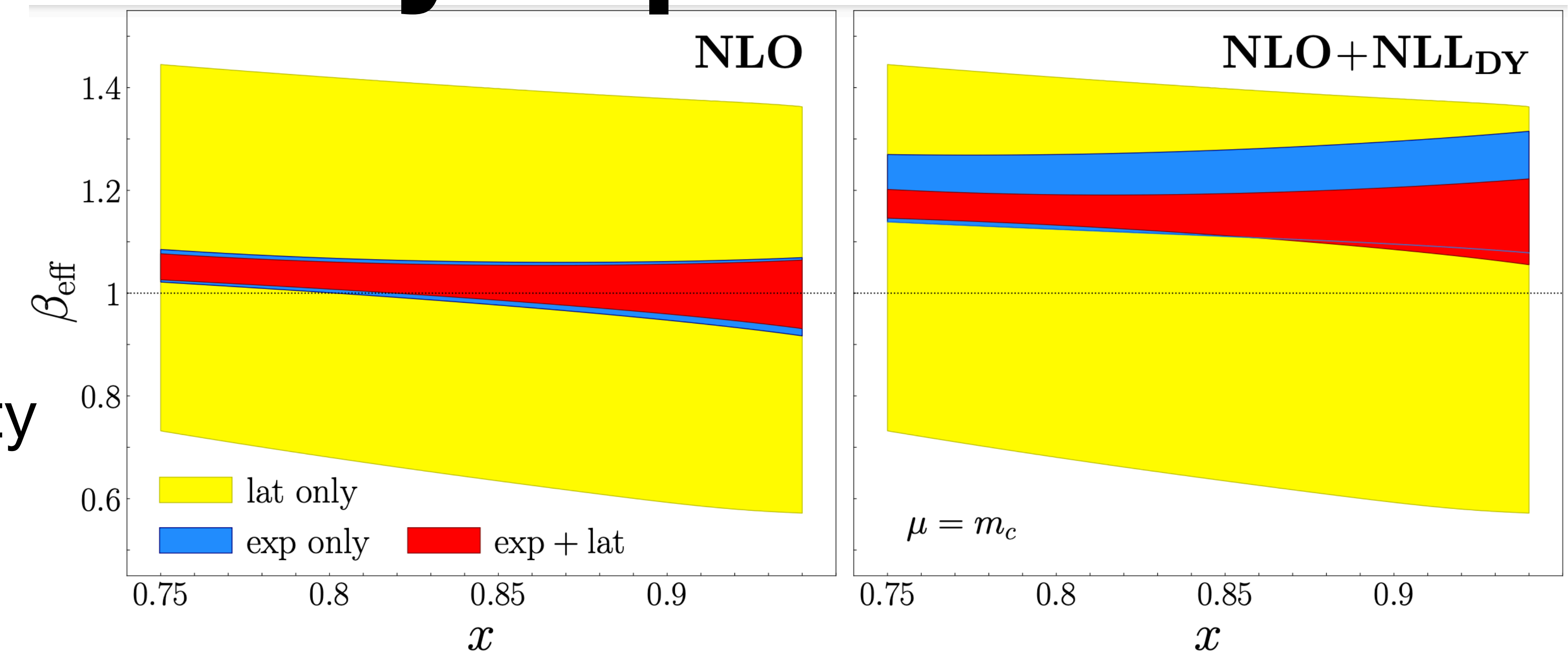
- Cross Sections limited to specific max but can reach low x_B
- Cross Section matching is integral from x_B to 1
  - Creates sensitivity of large x_B data to hard kernel in large x region
- Wealth of decades of experimental data outweigh modern lattice in both number and systematic error control

LATTICE

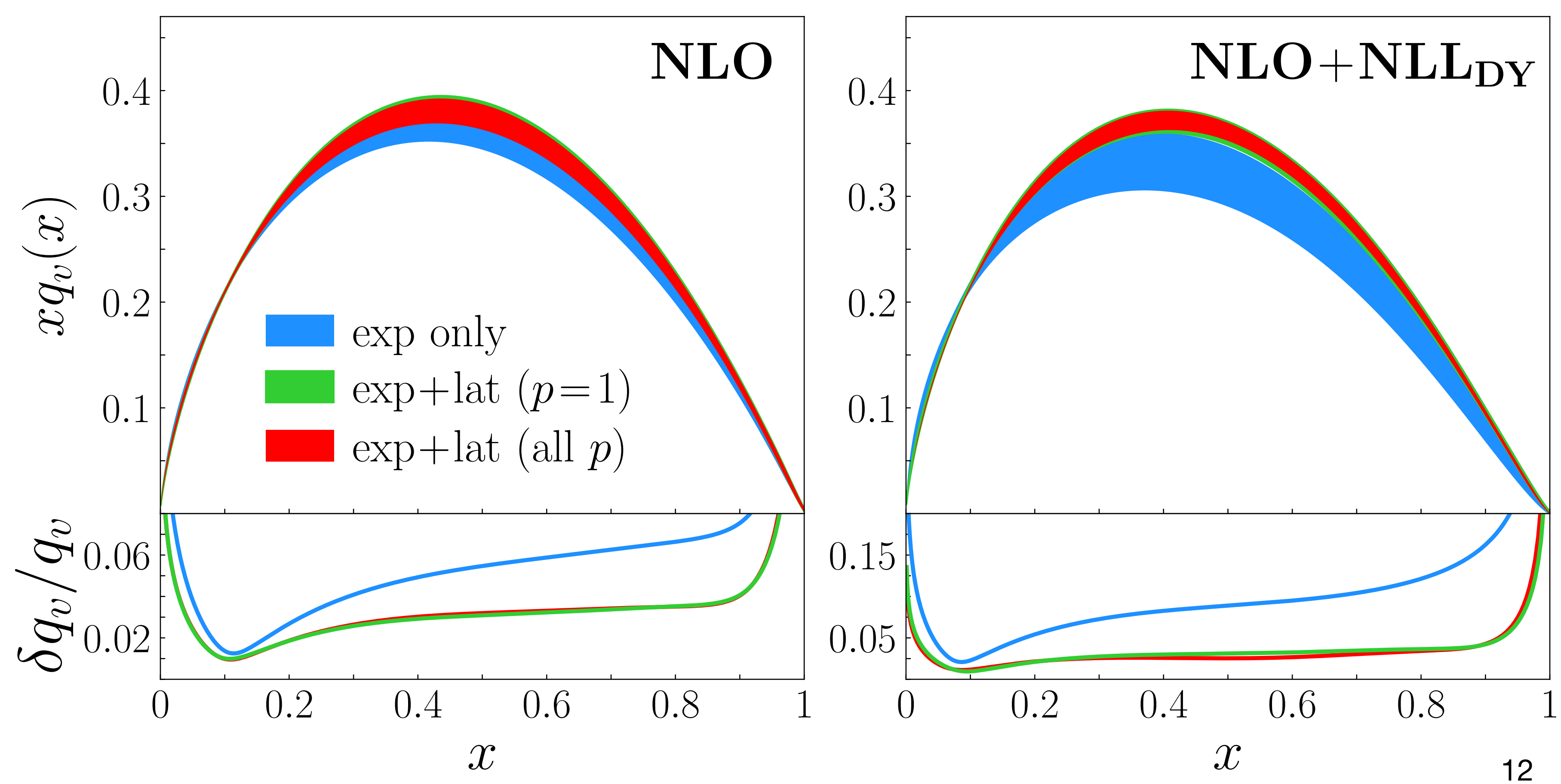
- Lattice limited to low ν, inverse Fourier gives to $x \gtrsim 0.2$, but higher sensitivity to larger x
- Lattice matching relation is integral over all x
- Low p_z data can reach high signal-to-noise compared to experiment
- Lattice can evaluate independently each spin, flavor, and hadron

Complementarity in pion PDF

- Lattice can readily access different hadrons
- Lattice lacks sensitivity to threshold logs and can be used to study theoretical kernels
- Improves precision in large x where experimental data does not exist
- Low momentum pion data are extremely precise



P. Barry et al, *Phys. Rev. D* 105 (2022) 11, 114051



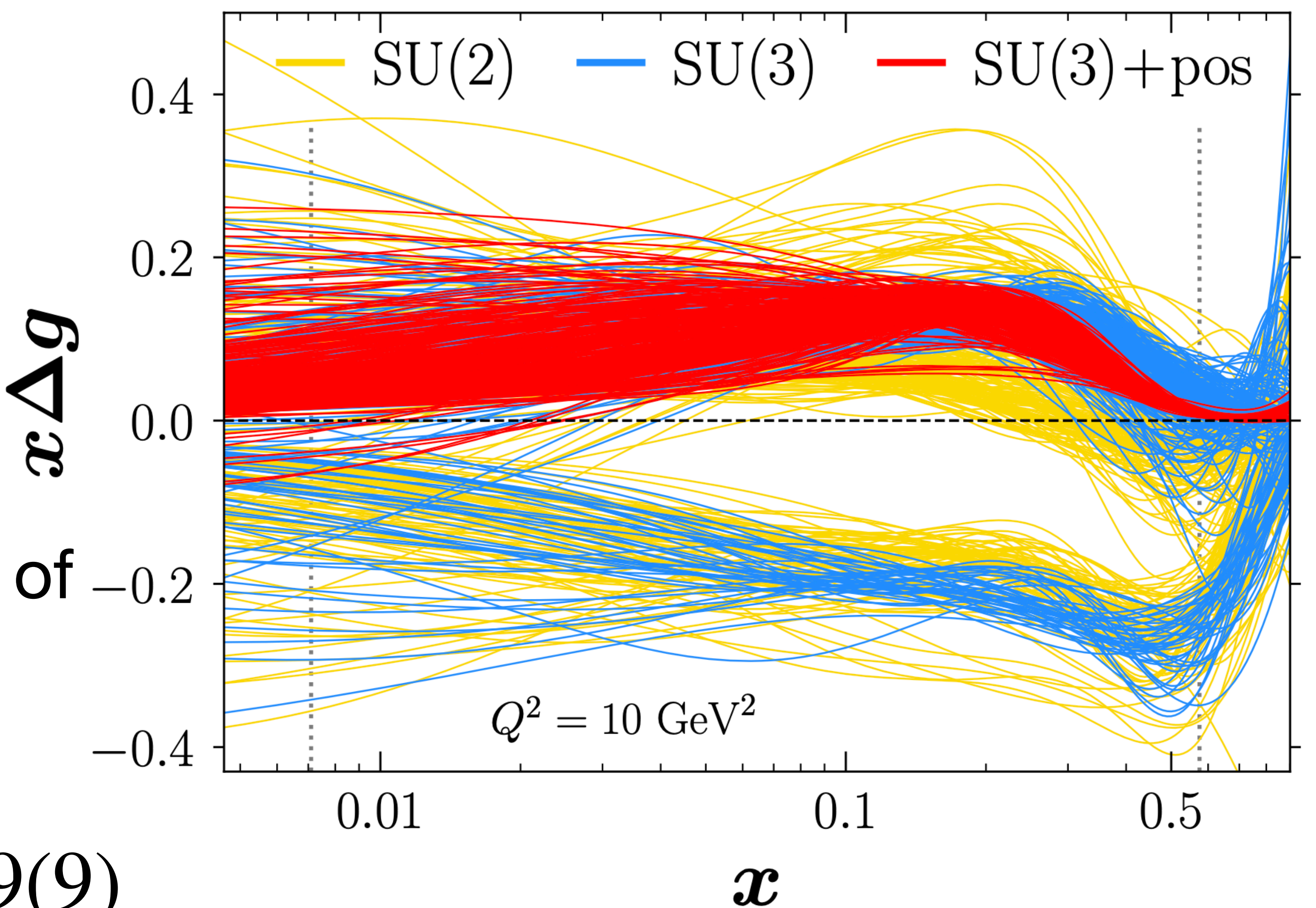
Spinning gluons

Y. Zhou et al (JAM) Phys. Rev. D 105, 074022 (2022)

- Positivity assumed in many analyses

$$|\Delta g| \leq g(x) \qquad g_\uparrow = \frac{1}{2}(g + \Delta g) \qquad g_\downarrow = \frac{1}{2}(g - \Delta g)$$

  - Removing reveals new band of solutions



- With constraint: $\Delta G = 0.39(9)$
- Without constraint: $\Delta G = 0.3(5)$
- Lattice: $\Delta G = 0.251(47)(16)$

R. Jaffe and A. Manohar, Nucl. Phys. B 337, 509 (1990)

$$J = \frac{1}{2} = \frac{1}{2}\Delta\Sigma + L_q + L_G + \Delta G$$

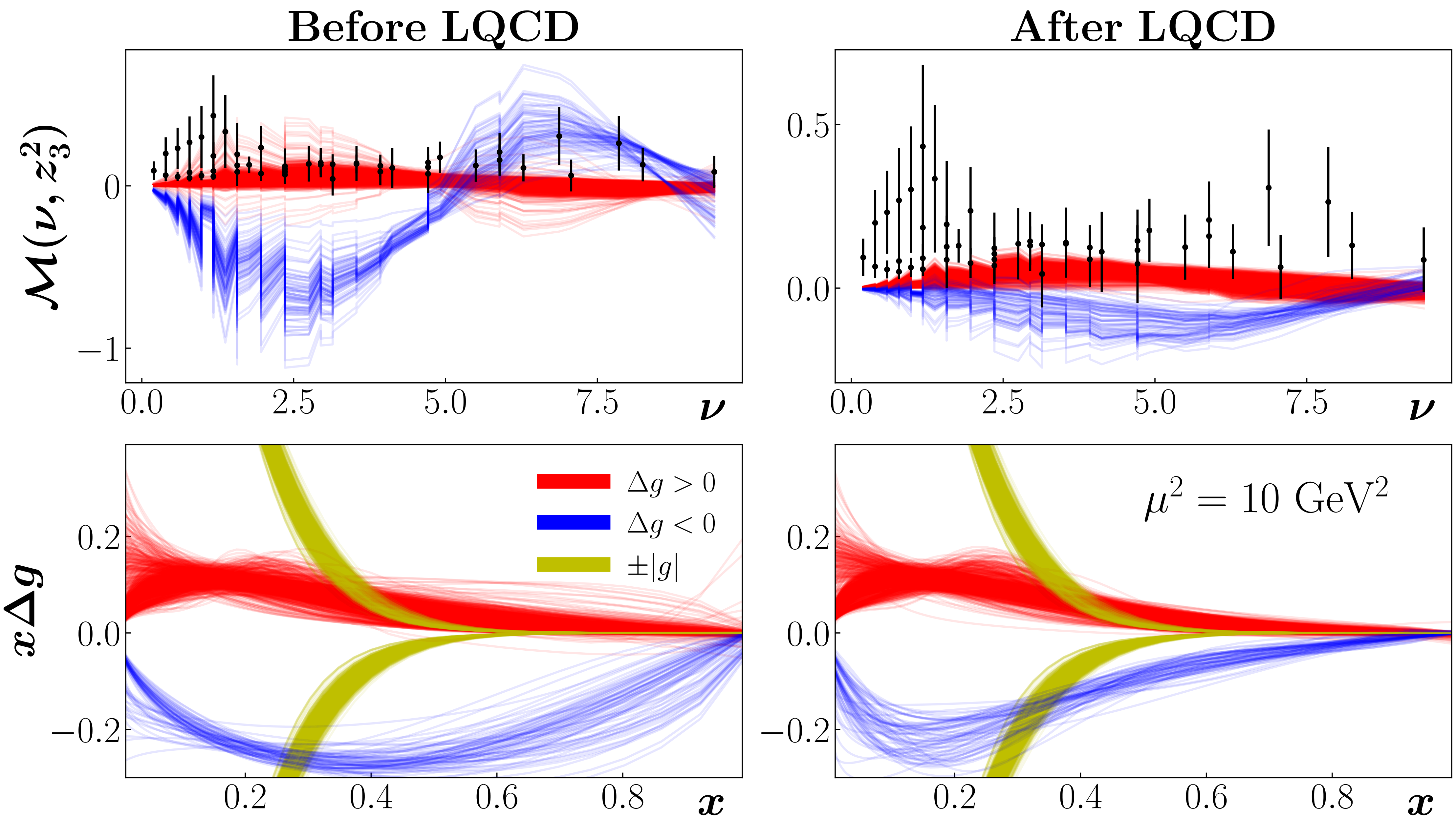
$$\Delta G = \int dx\, \Delta g(x)$$

Y-B. Yang et al (χ-QCD) Phys. Rev. Lett. 118, 102001 (2017)
K-F. Liu arXiv: 2112.08416

Spinning gluons

C. Egerer et al (HadStruc) arXiv:2207.08733
JK et al arXiv:2310.18179

Can lattice data affect phenomenological polarized gluon analysis?

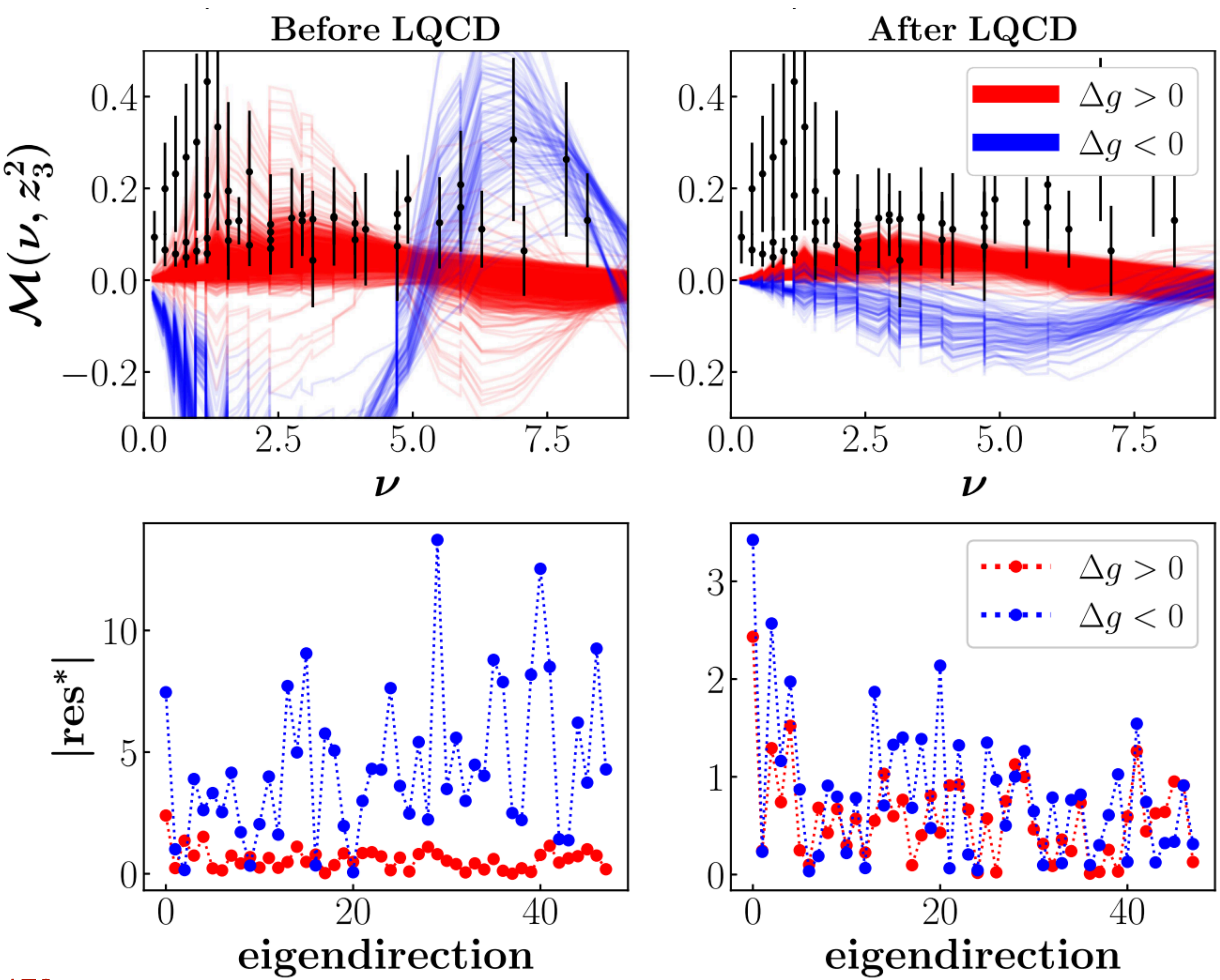


- The positive and negative solutions without positivity constraints

- Only positive band “consistent” with lattice data, but is too noisy to constrain.

Helicity Gluon PDF with LQCD

- Can this Lattice QCD data discriminate the red and blue solutions?



- Before LQCD:

$\chi^2/n \sim 0.65\ 30$

- After LQCD:

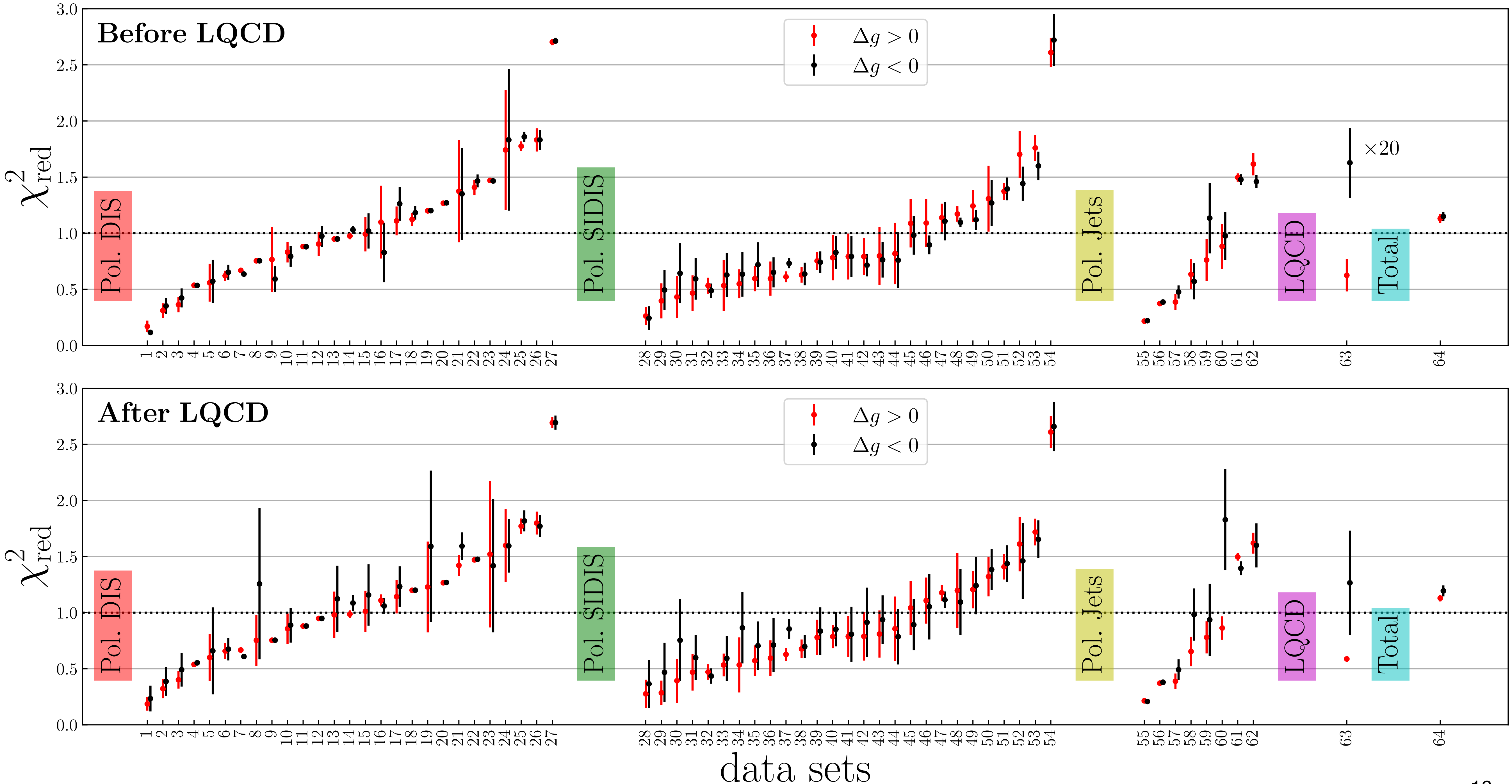
$\chi^2/n \sim 0.65\ 1.5$

JK et al arXiv:2310.18179

Helicity Gluon PDF with LQCD

JK et al arXiv:2310.18179

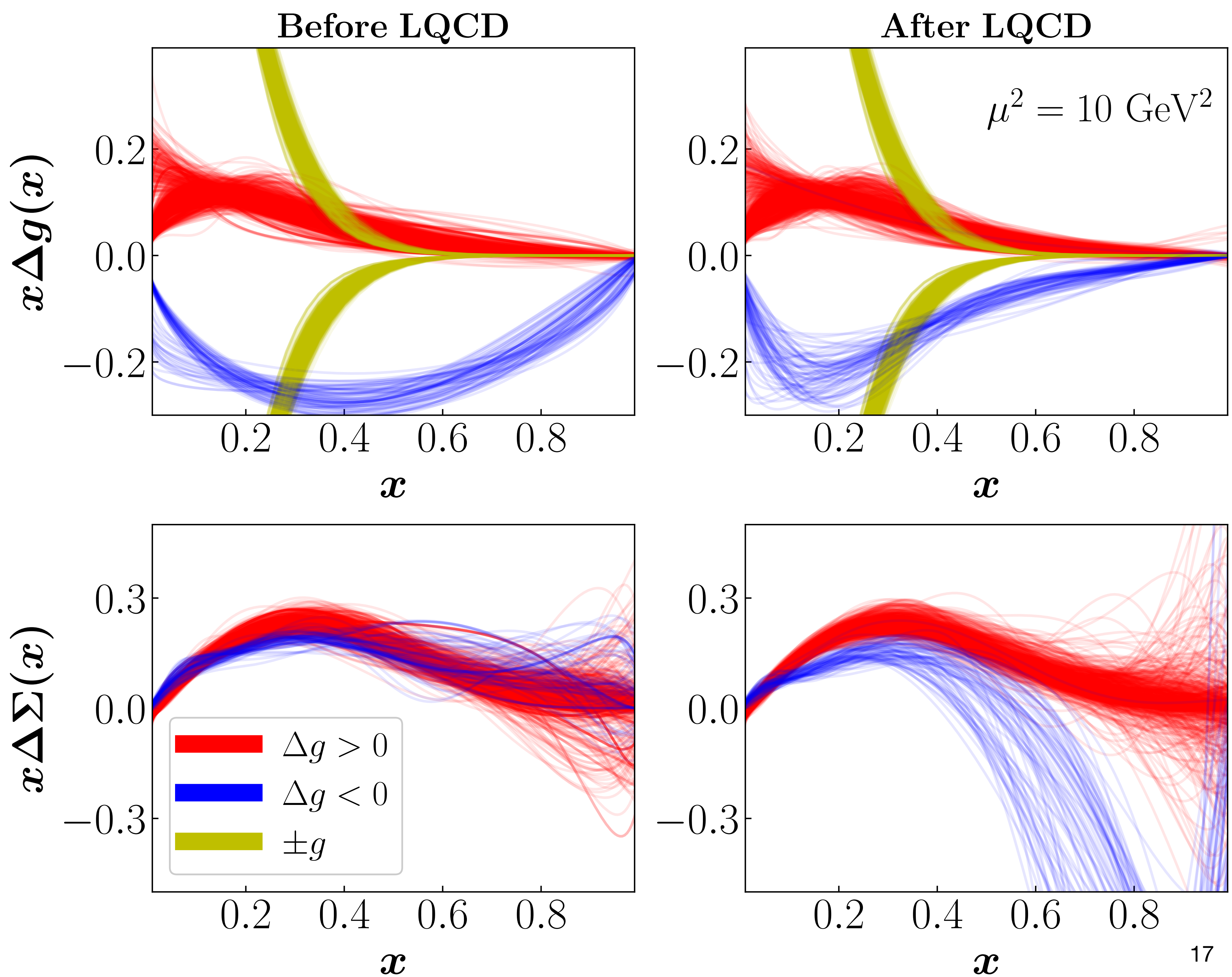
- Negative and positive Δg appear consistent with experiment and lattice



Lattice gluon data impacts quarks

C. Egerer et al (HadStruc) arXiv:2207.08733
JK et al arXiv:2310.18179

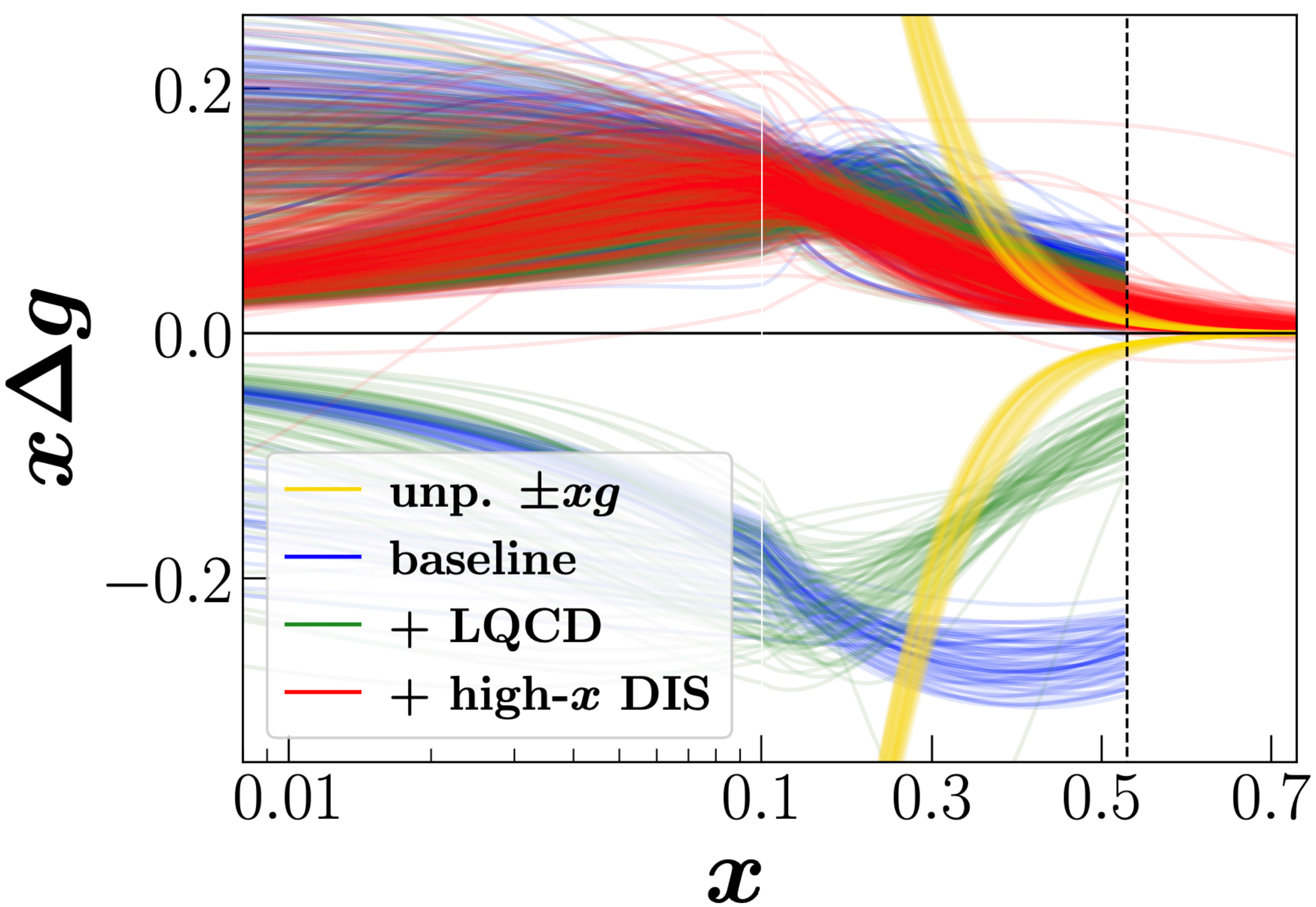
- Quark gluon mixing leads to impact on singlet
- Unexpected change in extrapolation region
- Compensates reduced magnitude of Δg in relation to cross sections



Resolution of the helicity sign

- Rejection of negative helicity gluon PDF requires 3 datasets
  - **RHIC Spin Asymmetries**
    - Linear and quadratic in Δg
  - **Lattice QCD matrix element**
    - Linear in Δg
  - **JLab high-x DIS from relaxing cuts on Final state mass**
    - Linear in Δg
    - $W^2 > 10\,\mathrm{GeV}^2 \rightarrow W^2 > 4\,\mathrm{GeV}^2$

N.T. Hunt-Smith et al arXiv:2403.08117



Conclusions

- Lattice matrix elements can be related to PDFs and their calculation have matured over the decade

- Both Lattice and Perturbative observables deserve to be described by ν not $p_z z$ or $p^+ z^-$

- Adding Lattice data into global fits give better results than either could do alone

- Including Lattice correlations are fundamental to correct error analysis and hypothesis testing

Back up slides

Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) arXiv:2207.08733

- Helicity Gluon Matrix Element:

$$\widetilde{M}_{\mu\alpha;\nu\beta}(z,p,s) = \frac{1}{2}\epsilon_{\nu\beta\rho\sigma}M_{\mu\alpha;\rho\sigma} = \langle p,s\,|\,\mathrm{Tr}\left[F^{\mu\alpha}(z)W(z;0)\widetilde{F}^{\nu\beta}(0)\right]|\,p,s\rangle$$

- Useful Combination $\widetilde{\mathscr{M}}(z,p) = \left[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\right]$
    - Gives **two** amplitudes, one has no leading twist contribution

Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
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- Use ratio with finite continuum limit

$$\widetilde{\mathfrak{M}}(\nu,z^2) = i\frac{\left[\widetilde{\mathscr{M}}(z,p)/p_z p_0\right]/Z_L(z/a)}{\mathscr{M}(0,z^2)/m^2}$$

Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
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- Relation to gluon and quark singlet ITD

$$\langle x\rangle_g \widetilde{\mathfrak{M}}(\nu,z^2) = \int_0^1 \widetilde{C}^{gg}(u,\mu^2z^2)\widetilde{I}_g(u\nu,\mu^2) + \widetilde{C}^{qg}(u,\mu^2z^2)\widetilde{I}_s(u\nu,\mu^2)$$

Helicity Gluon matrix element

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Pol Gluon Lorentz decomposition

I. Balitsky, W. Morris, A. Radyushkin
JHEP 02 (2022) 193

$$\widetilde{M}^{(2)}_{\mu\alpha;\lambda\beta}(z,p) = (sz)\left(g_{\mu\lambda}p_\alpha p_\beta - g_{\mu\beta}p_\alpha p_\lambda - g_{\alpha\lambda}p_\mu p_\beta + g_{\alpha\beta}p_\mu p_\lambda\right)\widetilde{\mathcal{M}}_{pp}$$
$$+(sz)\left(g_{\mu\lambda}z_\alpha z_\beta - g_{\mu\beta}z_\alpha z_\lambda - g_{\alpha\lambda}z_\mu z_\beta + g_{\alpha\beta}z_\mu z_\lambda\right)\widetilde{\mathcal{M}}_{zz}$$
$$+(sz)\left(g_{\mu\lambda}z_\alpha p_\beta - g_{\mu\beta}z_\alpha p_\lambda - g_{\alpha\lambda}z_\mu p_\beta + g_{\alpha\beta}z_\mu p_\lambda\right)\widetilde{\mathcal{M}}_{zp}$$
$$+(sz)\left(g_{\mu\lambda}p_\alpha z_\beta - g_{\mu\beta}p_\alpha z_\lambda - g_{\alpha\lambda}p_\mu z_\beta + g_{\alpha\beta}p_\mu z_\lambda\right)\widetilde{\mathcal{M}}_{pz}$$
$$+(sz)\left(p_\mu z_\alpha - p_\alpha z_\mu\right)\left(p_\lambda z_\beta - p_\beta z_\lambda\right)\widetilde{\mathcal{M}}_{ppzz}$$
$$+(sz)\left(g_{\mu\lambda}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\lambda}\right)\widetilde{\mathcal{M}}_{gg}$$

$$\widetilde{M}^{(1)}_{\mu\alpha;\lambda\beta}(z,p) = \left(g_{\mu\lambda}s_\alpha p_\beta - g_{\mu\beta}s_\alpha p_\lambda - g_{\alpha\lambda}s_\mu p_\beta + g_{\alpha\beta}s_\mu p_\lambda\right)\widetilde{\mathcal{M}}_{sp}$$
$$+\left(g_{\mu\lambda}p_\alpha s_\beta - g_{\mu\beta}p_\alpha s_\lambda - g_{\alpha\lambda}p_\mu s_\beta + g_{\alpha\beta}p_\mu s_\lambda\right)\widetilde{\mathcal{M}}_{ps}$$
$$+\left(g_{\mu\lambda}s_\alpha z_\beta - g_{\mu\beta}s_\alpha z_\lambda - g_{\alpha\lambda}s_\mu z_\beta + g_{\alpha\beta}s_\mu z_\lambda\right)\widetilde{\mathcal{M}}_{sz}$$
$$+\left(g_{\mu\lambda}z_\alpha s_\beta - g_{\mu\beta}z_\alpha s_\lambda - g_{\alpha\lambda}z_\mu s_\beta + g_{\alpha\beta}z_\mu s_\lambda\right)\widetilde{\mathcal{M}}_{zs}$$
$$+(p_\mu s_\alpha - p_\alpha s_\mu)(p_\lambda z_\beta - p_\beta z_\lambda)\widetilde{\mathcal{M}}_{pspz}$$
$$+(p_\mu z_\alpha - p_\alpha z_\mu)(p_\lambda s_\beta - p_\beta s_\lambda)\widetilde{\mathcal{M}}_{pzps}$$
$$+(s_\mu z_\alpha - s_\alpha z_\mu)(p_\lambda z_\beta - p_\beta z_\lambda)\widetilde{\mathcal{M}}_{szpz}$$
$$+(p_\mu z_\alpha - p_\alpha z_\mu)(s_\lambda z_\beta - s_\beta z_\lambda)\widetilde{\mathcal{M}}_{pzsz}$$

Want: $M_{\Delta g}(\nu, z^2) = \left[\widetilde{\mathcal{M}}^{(+)}_{sp} - \nu\,\widetilde{\mathcal{M}}_{pp}\right]$

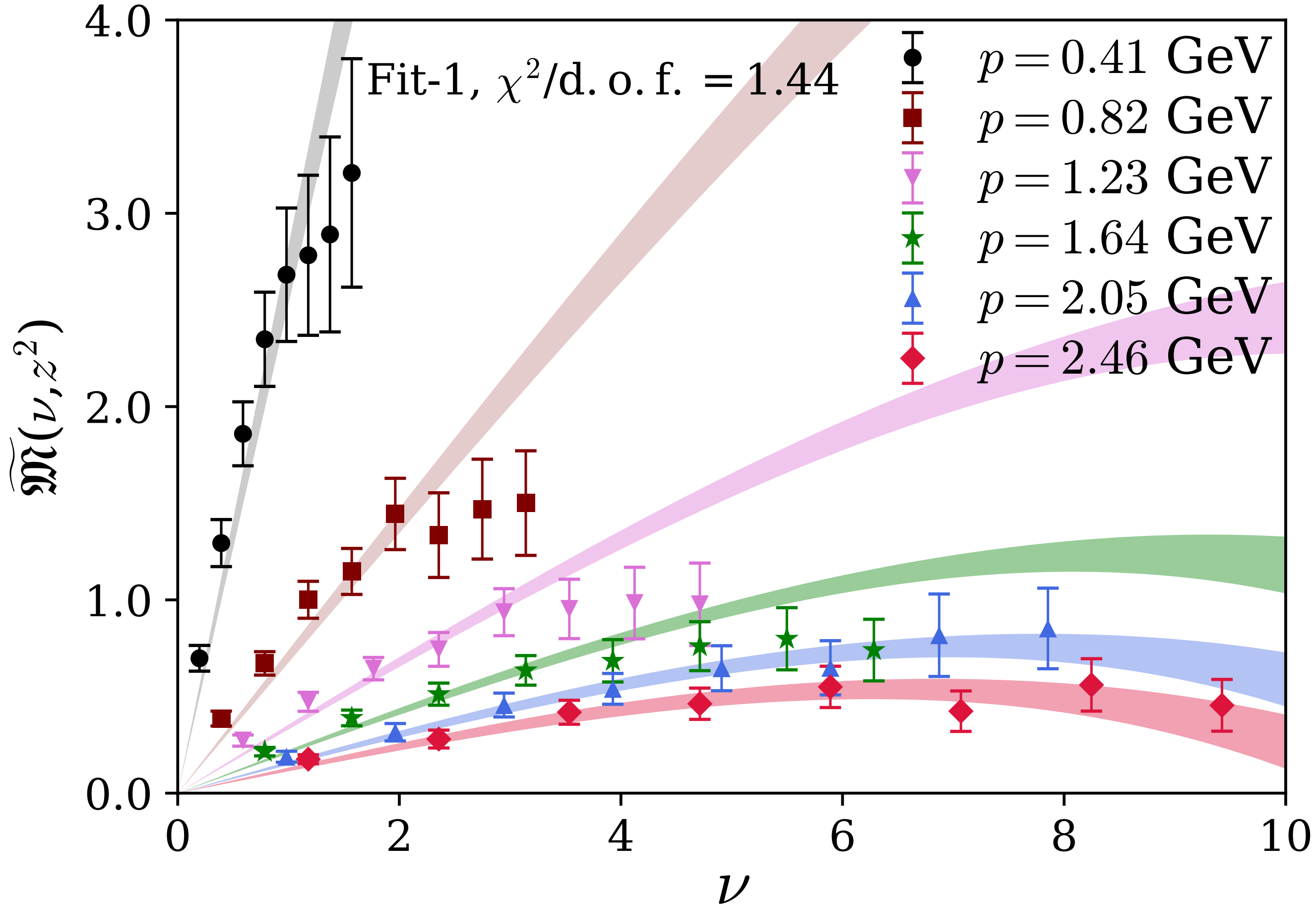
Can get: $\widetilde{\mathcal{M}}(z,p) = \left[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}\right]$

$$= M_{\Delta g} - \frac{m^2 z^2}{\nu}\widetilde{\mathcal{M}}_{pp}$$

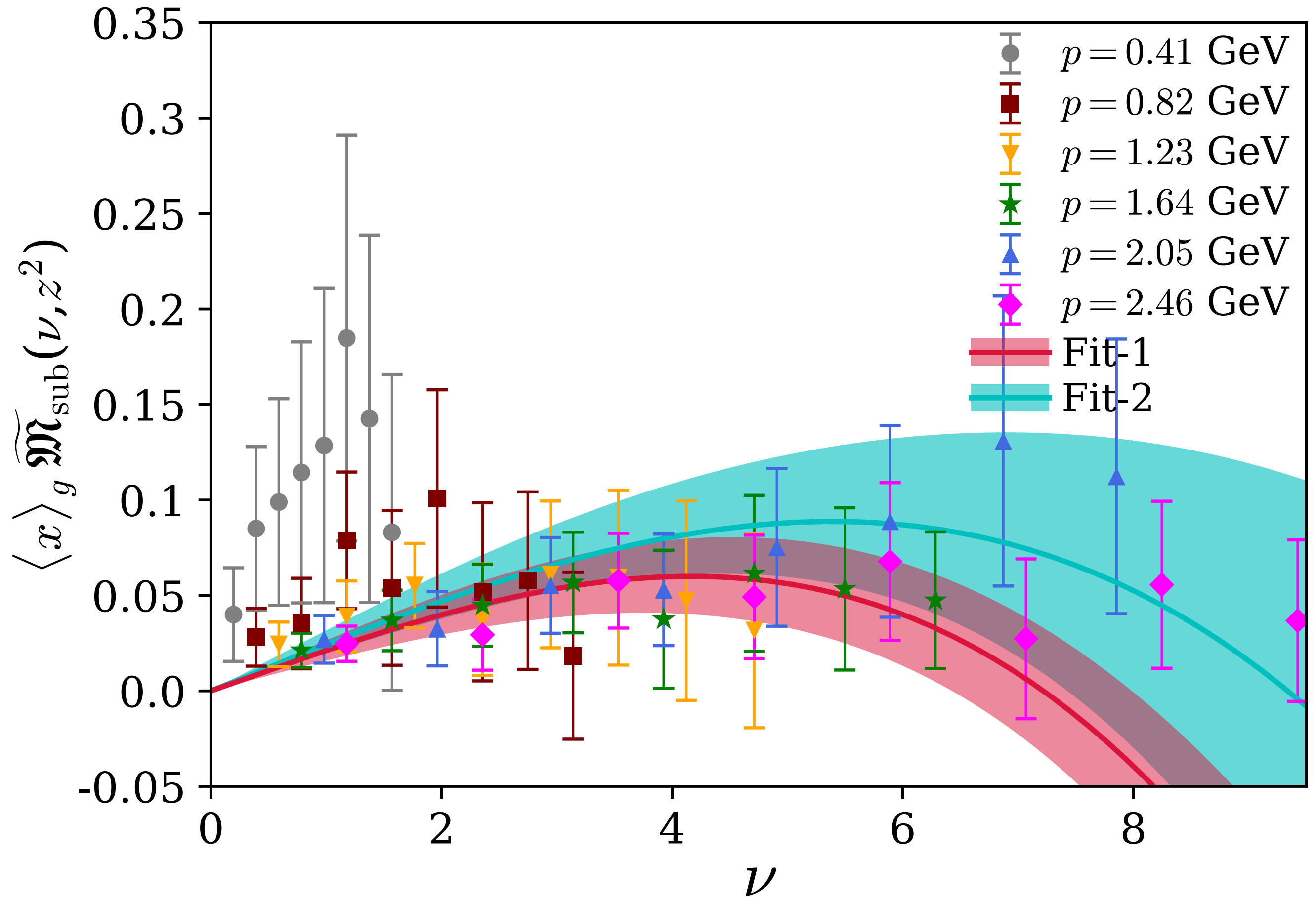
$$= M_{\Delta g} - \frac{m^2}{p_z^2}\nu\,\widetilde{\mathcal{M}}_{pp}$$

Helicity Gluon PDF

- Model both terms



- Subtract rest frame



$a = 0.094$ fm

$m_\pi = 358$ MeV

C. Egerer et al (HadStruc) arXiv: 2207.08733

What is this workshop about?

The focus of this workshop is on Uncertainty Quantification

A **list of Key Questions** is attached to the Indico page of the workshop

1. Accessing PDFs: global analyses and lattice computations

   $\longrightarrow$ How does PDF determination work in global analyses and lattice QCD?

2. Global QCD analyses: inverse problem and objective functions

   $\longrightarrow$ How is the inverse problem entailed by PDF determination addresed?

3. Lattice QCD: considerations on the validity of the perturbative matching

   $\longrightarrow$ How is the equivalence between zP_z and $\xi^- P^+$ defined?

4. Setting up a common language: definitions and benchmarks

   $\longrightarrow$ How to benchmark lattice moments and quasi-/pseudo-PDFs with global analyses?

5. Combining lattice and experimental data to determine PDFs

   $\longrightarrow$ What are the efforts/limitations to incorporate lattice data in PDF determinations?

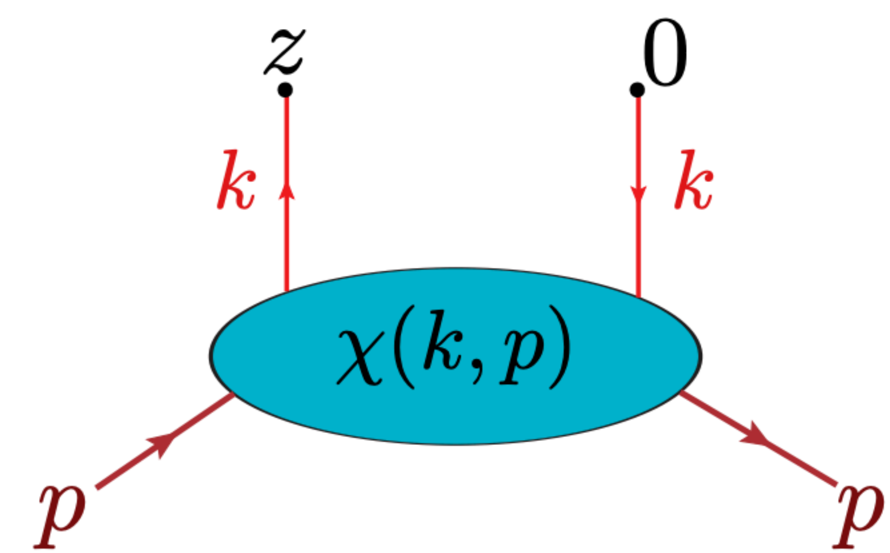
6. Uncertainty quantification and bias/variance trade-off

   $\longrightarrow$ How are aleatoric and epistemic uncertainties combined? How is a model chosen?

Other Faces of WL Matrix Element

- Some are Lorentz invariant interpretations
- These interpretations nor the functions' bounds require small z^2 , only relation to light cone PDF with $z^2 = 0$ and some other regulation

Review: A. Radyushkin (2019) 1912.04244



$$i\chi_{d_i}(k,p) = i^l \frac{P(\text{c.c.})}{(4\pi i)^{2L}} \int_0^\infty \prod_{j=1}^{l} d\alpha_j [D(\alpha)]^{-2}$$
$$\times \exp\left\{ ik^2 \frac{A(\alpha)}{D(\alpha)} + i \frac{(p-k)^2 B_s(\alpha) + (p+k)^2 B_u(\alpha)}{D(\alpha)} \right\}$$
$$\times \exp\left\{ ip^2 \frac{C(\alpha)}{D(\alpha)} - i \sum_j \alpha_j (m_j^2 - i\epsilon) \right\} ,$$

$$i\chi(k,p) = \int_0^\infty d\sigma \int_{-1}^{1} dx e^{i\sigma\left[k^2 - 2x(k\cdot p) + i\epsilon\right]} V(x,\sigma)$$

$$\sigma_{d_i} = \frac{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}{D_{d_i}(\alpha)}$$

$$x_{d_i} = \frac{B_{s_{d_i}}(\alpha) - B_{u_{d_i}}(\alpha)}{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}$$

α_j are positive numbers and A, B_u, B_s, C, D are sums of products of α_j

Fourier transform to position space

$$\mathcal{M}(\nu, z^2) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot z} \chi(k,p) = \int_{-1}^{1} dx e^{i\nu x} \int_0^\infty e^{-i\sigma(z^2-\epsilon)} V(x,\sigma)$$

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Virtuality Distribution Function
Lorentz invariant picture
σ^{-1} pole gives log z^2
Limits from nature of Feynman diagrams

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} \int_0^{\infty} d\sigma e^{-i\sigma(z^2-\epsilon)} V(x, \sigma)$$

pseudo-PDF
Lorentz invariant picture
log z^2 divergence from poles of TMD/VDF

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} P(x, z^2)$$

$$\tilde{q}(y, p_z^2) = \int dz \int_{-1}^{1} dx e^{ip_z z(x-y)} P(x, z^2)$$

Musch, Hagler, Negele, Schafer PRD 83 (2011) 094507

Straight Link / Primordial TMD
Frame dependent picture with nice interpretation
$1/k_T^2$ pole gives log z^2

$$z = (0, z^-, z_T) \qquad p = (p^+, \frac{m^2}{p^+}, 0_T)$$

$$\mathcal{M}(\nu, z^2) = \int_{-1}^{1} dx e^{i\nu x} \int d^2k_T e^{ik_T \cdot z_T} F(x, k_T^2)$$

Light cone PDF from regulated integral of TMD
Relate to the Lorentz invariant VDF
$1/k_T^2$ or σ^{-1} poles generate $\log \mu^2$ divergence

$$f(x, \mu^2) = \int^{\mu^2} d^2k_T F(x, k_T^2) = \int_0^{\infty} d\sigma \left[1 - e^{-\frac{i}{\sigma}(\mu^2 - i\epsilon)}\right] V(x, \sigma)$$