Systematic Uncertainties in the Lattice Calculation of PDFs from Perturbative Improvements

Parton Distributions and Lattice Calculations
(PDF Lattice 2024)
Jefferson Lab, Nov 18, 2024

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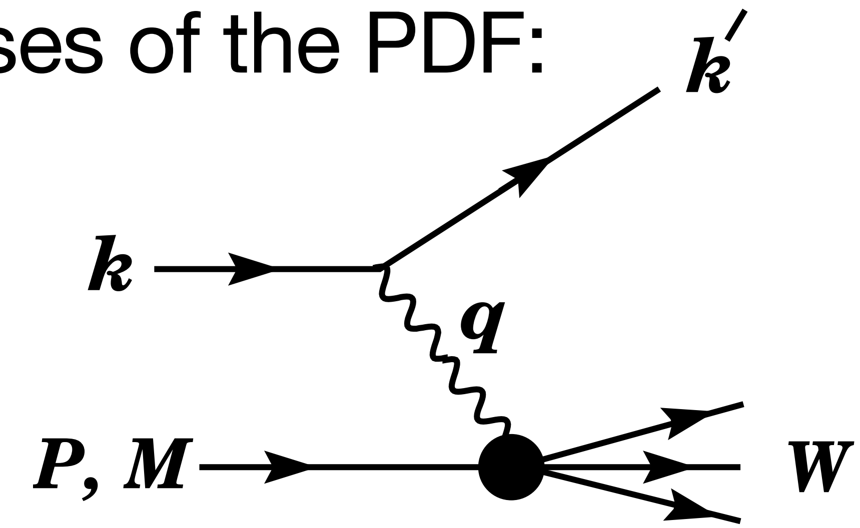
Outline

- **Introduction**
- **Large-momentum effective theory**
  - Leading renormalon subtraction
  - Renormalization group improvement
  - Threshold resummation
- **Short-distance factorization**
  - Renormalization group improvement
  - Threshold resummation
- **Summary**

Introduction

(DGLAP) evolution is essential for global analyses of the PDF:

$$\frac{d^2\sigma}{dxdy} \propto \int_{x_B}^{1} \frac{dy}{y}\, C\left(\alpha_s(\kappa Q), \frac{x_B}{y}\right) f(y, \kappa Q)$$



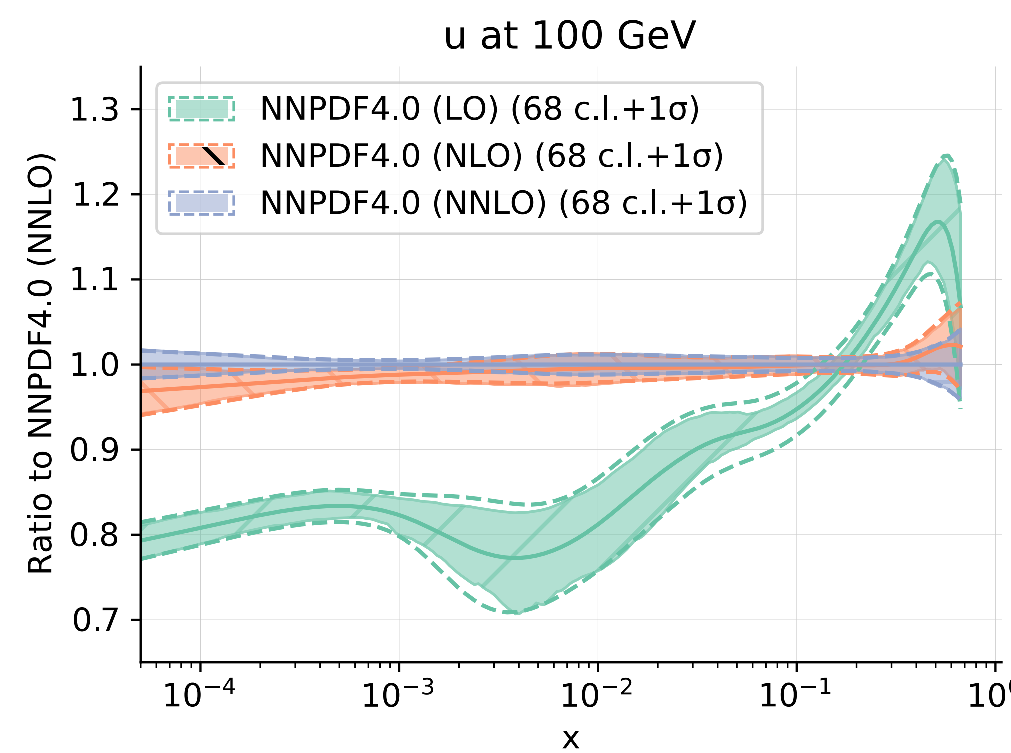
Particle Data Group

$$f(x, Q) = \int_x^1 dy\; U(x, y, Q, Q_0)\, f(y, Q_0) \qquad Q_0 \sim 1\ \mathrm{GeV}$$

$$\frac{\partial f_i(x,\mu)}{\partial \ln \mu^2} = \sum_j \int_x^1 \frac{dy}{y} P_{ij}\left(\alpha_s(\mu), \frac{x}{y}\right) f_j(y,\mu)$$

$$P_{ij} = \frac{\alpha_s}{2\pi} P_{ij}^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ij}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^3 P_{ij}^{(2)} + \ldots$$

LO **NLO** **NNLO**



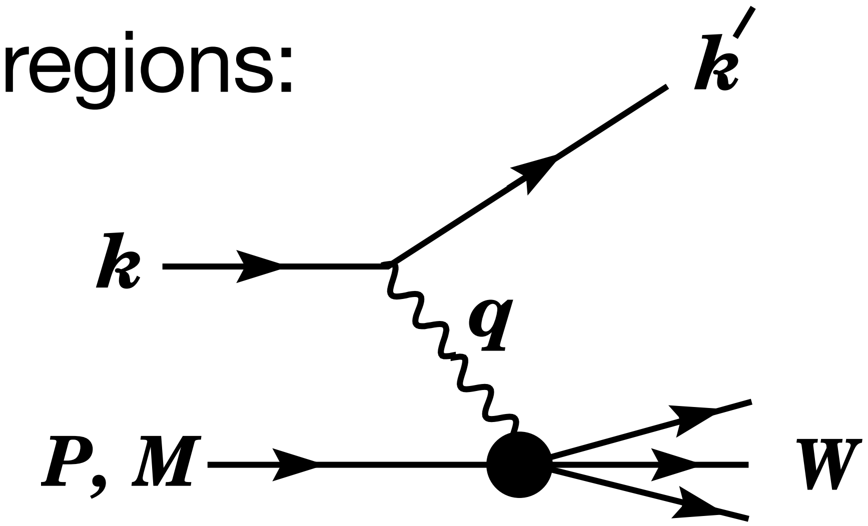
R. D. Ball et al., **Eur.Phys.J.C** 82 (2022) 5.

Scale variation typically chosen as $2^{-1} < \kappa < 2$

Introduction

Resummations are relevant for the end-point regions:

$$\frac{d^2\sigma}{dxdy} \propto \int_{x_B}^{1} \frac{dy}{y}\, C\left(\alpha_s(\kappa Q),\, \frac{x_B}{y}\right) f(y, \kappa Q)$$



Particle Data Group

- Small-*x*

$$\alpha_s^n \ln^n(\xi) \in C\left(\alpha_s(Q),\ \xi\right)$$

Probably necessary for sufficient precision for $x<10^{-3}$ at low Q.

Particle Data Group, **Phys.Rev.D** 110 (2024) 3

- Large-*x* (threshold)

$$\alpha_s^n \frac{\ln^n(1-\xi)}{1-\xi} \in C\left(\alpha_s(Q),\ \xi\right)$$

Dependence on the resummation scheme?

P. Barry et al. (JAM), **Phys.Rev.Lett. 127** (2021) 23

Introduction

Lattice QCD calculations have not yet widely adopted evolution and resummations.

- Most of the considerations are just practical:

  Perturbation theory uncertainty is still smaller than the lattice systematic errors. (Caveat: in the region where pQCD is supposed to work.)

- But without the perturbative improvements lattice QCD always underestimate the systematics in the end-point regions.

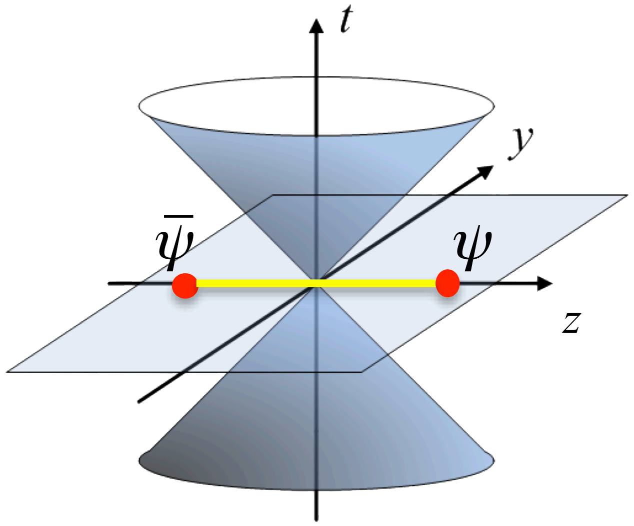
- Moreover, high-precision calculations may become reality with new ideas, thus making the perturbative improvements necessary.
  - Lancosz algorithm for analysis of lattice correlation functions

    M. Wagman, 2406.20009; M. Wagman and D. Hackett, 2407.21777.
  - Better hadron interpolation operators at large boost momentum

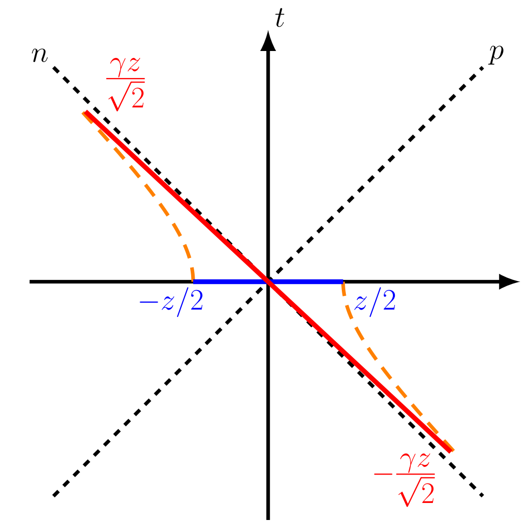
    See Rui Zhang's poster presentation
  - Lattice simulation without Wilson lines (mainly for the TMDs)
    - X. Gao, W.-Y. Liu and YZ, Phys.Rev.D 109 (2024) 9;
    - YZ, 2311.01391, to appear in PRL.

    See Xiang Gao's talk

Large-Momentum Effective Theory



Lorentz boost



$$f(x,\mu) = \int_{-\infty}^{\infty} \frac{dy}{|y|} \bar{C}\left(\frac{x}{y}, \frac{\mu}{yP^z}, \frac{\tilde{\mu}}{\mu}\right) \tilde{f}(y, P^z, \tilde{\mu}) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{(xP^z)^2}, \frac{\Lambda_{\rm QCD}^2}{((1-x)P^z)^2}\right)$$

- Power expansion in parton momentum
- Valid for a moderate range of *x* at finite P^z
- No parameterization from global analyses
- Matching kernel available at N3LO now

See Zheng-Yang Li's talk.

- X. Ji, PRL **110** (2013); SCPMA **57** (2014).
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, Rev.Mod.Phys. **93** (2021).

NNLO

- Chen, Zhu and Wang, PRL. **126** (2021);
- Li, Ma and Qiu, PRL **126** (2021);

N3LO

- Cheng, Huang, Li, Li and Ma, 2410.05141.

Large-Momentum Effective Theory

Lattice renormalization

"Hybrid scheme renormalization" X. Ji, YZ, et al., NPB **964** (2021).

$$|z| \leq z_s\,, \quad \frac{h(z, P^z, a)}{h(z, 0, a)} \qquad |z| > z_s\,, \quad e^{\delta m(a)|z|}\frac{h(z, P^z, a)}{h(z_s, 0, a)}$$

$$\delta m(a) = \frac{m_{-1}}{a} + \mathcal{O}(\Lambda_{\rm QCD})$$

Wilson-line mass correction

Y. Huo, et al. (LPC), NPB **969** (2021).

X. Gao, YZ, et al., PRL **128** (2022).

Continuum limit $a \to 0$ ↓ $z_s \ll \Lambda_{\rm QCD}^{-1}$

$$|z| \leq z_s\,, \quad \frac{h^{\overline{\rm MS}}(z, P^z, \mu)}{C_0^{\overline{\rm MS}}(z, \mu)} \qquad |z| > z_s\,, \quad e^{-\bar{m}_0|z|}\frac{h^{\overline{\rm MS}}(z, P^z, \mu)}{C_0^{\overline{\rm MS}}(z_s, \mu)}$$

$$\bar{m}_0 \sim \mathcal{O}(\Lambda_{\rm QCD})$$

Linear renormalon

Large-Momentum Effective Theory

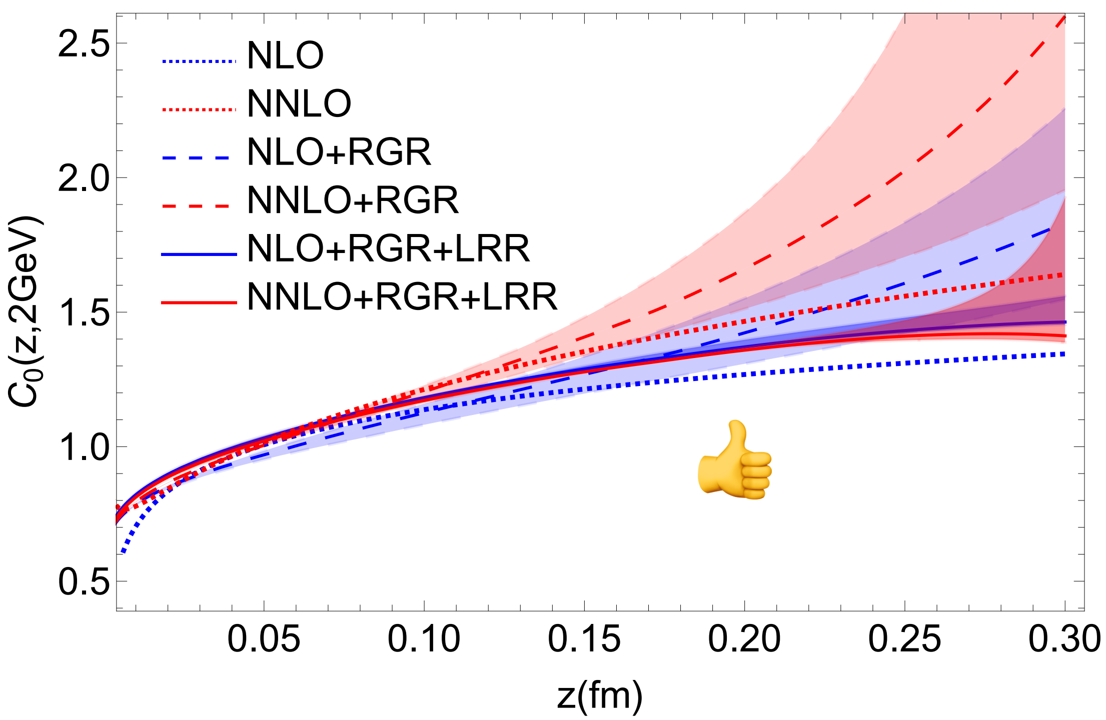
- Leading-renormalon subtraction

$$f(x,\mu) = \int_{-\infty}^{\infty} \frac{dy}{|y|} \bar{C}\left(\frac{x}{y}, \frac{\mu}{yP^z}, \frac{\tilde{\mu}}{\mu}\right) \tilde{f}(y, P^z, \tilde{\mu}) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{(xP^z)}\right) + \mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{(xP^z)^2}, \frac{\Lambda^2_{\rm QCD}}{((1-x)P^z)^2}\right)$$

$$C_0^{\overline{\rm MS}}(1/z, z) = 1 + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}, \qquad c_n \sim \left(\frac{\beta_0}{2\pi}\right)^n n!$$

Borel-sum with principle value prescription

$$C_0^{\rm LRR}(1/z, z) = 1 + \sum_{n=0}^{N} \left[c_n - \left(\frac{\beta_0}{2\pi}\right)^n n!\right] \alpha_s^{n+1} + \sum_{n=0}^{\infty} \left(\frac{\beta_0}{2\pi}\right)^n n! \alpha_s^{n+1}$$



Subtraction of leading renormalon ambiguity

Leading-renormalon resummation (LRR)

- Holligan, Ji, Lin, Su and Zhang, NPB **993** (2023);
- Zhang, Ji, Holligan and Su (ZJHS23), PLB **844** (2023).

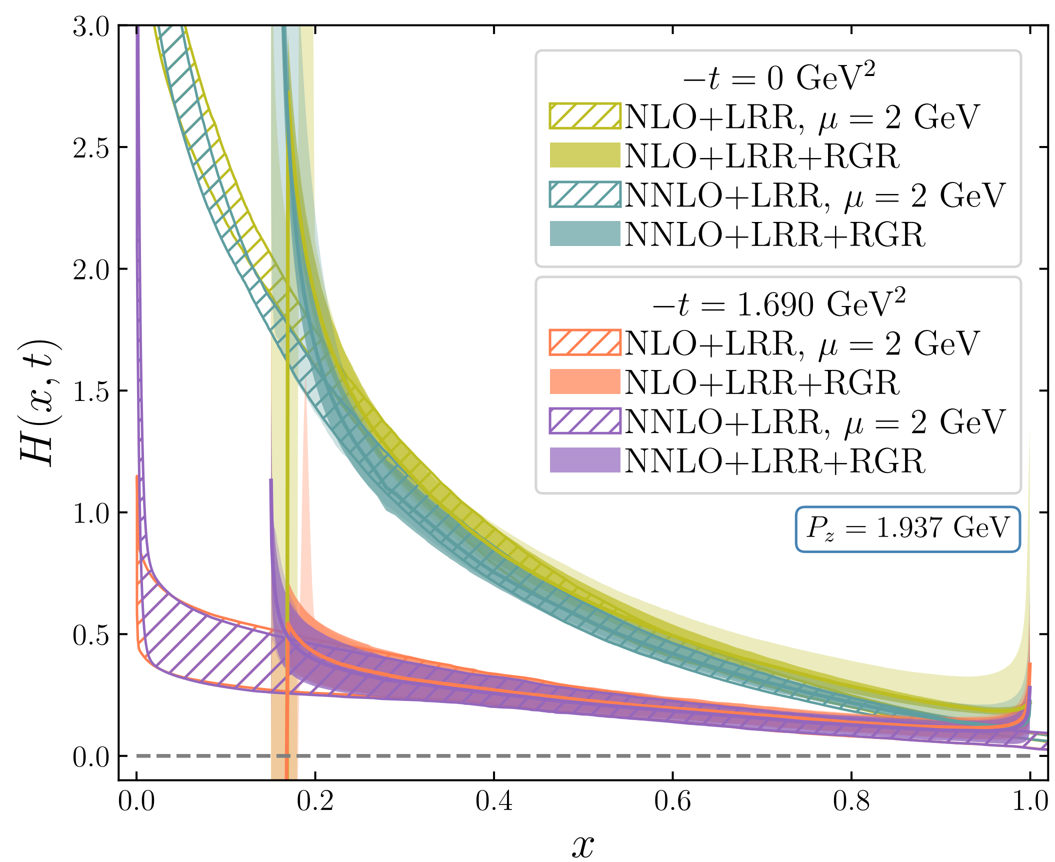
Large-Momentum Effective Theory

Renormalization group improvement (DGLAP evolution)

$$f(x,\mu) = \int_{-\infty}^{\infty} \frac{dy}{|y|} \bar{C}_{\rm LRR}\left(\frac{x}{y}, \frac{2x}{y}\frac{\mu}{2xP^z}, \frac{\tilde{\mu}}{\mu}\right) \tilde{f}(y, P^z, \tilde{\mu}) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{(xP^z)^2}, \frac{\Lambda_{\rm QCD}^2}{((1-x)P^z)^2}\right)$$

Inverse matching at $\mu = \kappa \cdot 2xP^z$ → DGLAP evolution from $\kappa \cdot 2xP^z$ to $\mu = 2$ GeV → Vary $\kappa \sim 1$ to estimate scale uncertainty

- Matching out of control at small *x* as $\alpha_s(2\kappa x P^z) \sim 1$
- Singular behavior at $x \sim 1$ due to DGLAP evolution
- No noticeable change at moderate *x*.



Pion valence quark GPD at zero skewness P^z=1.94 GeV

X. Gao, Q. Shi, YZ et al., arXiv: 2407.03516.

See Qi Shi's talk

Large-Momentum Effective Theory

Threshold resummation (TR)

- X. Ji, Y. Liu and Y. Su, **JHEP** 08 (2023) 037;
- Y. Liu and Y. Su, **JHEP** 2024 (2024) 204;
- X. Ji, Y. Liu, Y. Su and R. Zhang, 2410.02910.

$$\bar{C}\left(\frac{x}{y}, \frac{\mu}{2xP^z}\right)\Bigg|_{x\to y} = H\left(\frac{\mu}{2xP^z}\right)|xP^z|\, S\left(\frac{(y-x)P^z}{\mu}, \frac{\mu}{2xP^z}\right)$$

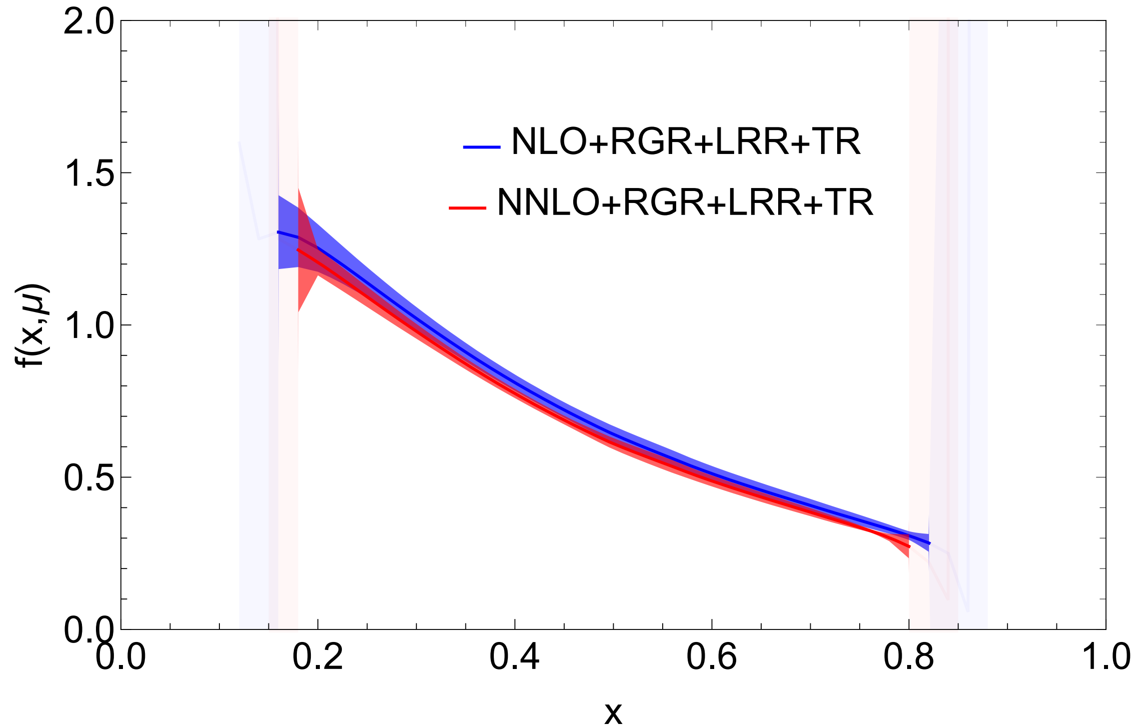
$$H: \quad \alpha_s \ln^2 \frac{2xP^z}{\mu}, \; \alpha_s \ln \frac{2xP^z}{\mu} \qquad \mu_h \propto 2xP^z$$

$$S: \quad \alpha_s \frac{1}{|x-y|} \ln \frac{|x-y|P^z}{\mu} \qquad \mu_s \propto (1-x)P^z$$

- Matching out of control near the end points where $\alpha_s(\mu_h), \alpha_s(\mu_s) \sim 1$
- Improved perturbative convergence at moderate *x*

See Yushan Su's poster presentation

Pion valence quark PDF at P^z=1.94 GeV



X. Ji, Y. Liu, Y. Su and R. Zhang, 2410.02910.

Short-distance factorization

- Operator product expansion (OPE):

- A. Radyushkin, Phys.Rev.D **96** (2017);
- K. Orginos et al., Phys.Rev.D **96** (2017);
- Braun and Müller, Eur.Phys.J.C **55** (2008);
- Ma and Qiu, Phys.Rev.Lett. **120** (2018).

$$\tilde{h}(\lambda = zP^z, z^2) = \sum_{n=0}^{\infty} C_n(z^2\mu^2)\frac{(-i\lambda)^n}{n!}a_n(\mu) + \mathcal{O}(z^2\Lambda_{\mathrm{QCD}}^2),$$

$$a_n(\mu) = \int_0^1 dx\ x^n f(x,\mu)$$

- Factorization formula:

T. Izubuchi, X. Ji, L. Jin, I. Stewart, and YZ, Phys.Rev.D **98** (2018)

$$\tilde{h}(\lambda = zP^z, z^2) = \int_0^1 d\omega\ \ C(\omega, z^2\mu^2)h(\omega\lambda, \mu) + \mathcal{O}(z^2\Lambda_{\mathrm{QCD}}^2),$$

$$h(\lambda) = \int_0^1 dx\ e^{ix\lambda}f(x,\mu)$$

- Power expansion in the distance
- Valid up to a maximum zP^z at finite P^z
- Determining the lowest few moments or fitting the *x*-dependence with a model
- Matching kernel available at N3LO now

Complementing with LaMET
X. Ji, 2209.09332

Cheng, Huang, Li, Li and Ma, 2410.05141.

Short-distance factorization

Renormalization group improvement

X. Gao, YZ et al., *Phys.Rev.D* 103 (2021) 9

$$\tilde{h}(\lambda = zP^z, z^2) = \sum_{n=0}^{\infty} C_n(\alpha_s(\kappa/z_0), \kappa^2)\frac{(-i\lambda)^n}{n!}a_n(\kappa/z_0) \ + \ \mathcal{O}(z^2\Lambda_{\rm QCD}^2)\,,$$

$$z_0 = ze^{\gamma_E}/2 \approx 0.89z \qquad a_n(\mu) = a_n(\mu_0)\exp[\int_{\mu_0}^{\mu} d\mu' \ \gamma_n(\alpha_s(\mu'))]$$

$$\tilde{h}(\lambda = zP^z, z^2) = \int_0^1 d\omega \ \ C(\alpha_s(\kappa/z_0), \omega, \kappa^2)\int_0^1 dx\ e^{ix\lambda}\, f(x, \kappa/z_0) \ + \ \mathcal{O}(z^2\Lambda_{\rm QCD}^2)\,,$$

Q: what is the value of κ? Or, what is the physical scale for OPE?

- Conventionally one would choose $\kappa = 1$ so that the apparent logarithms vanish;
- In practice, κ could be different from 1, but one should still expect $\kappa \sim 1$;
- With $\kappa = 1$, perturbation theory will only be reliable for $z_0 \lesssim 0.25$ fm;
- After all, we need to estimate the theory uncertainty associated with κ variation.

Short-distance factorization

- Threshold resummation

X. Gao, YZ et al., *Phys.Rev.D* 103 (2021) 9

$$\tilde{h}(\lambda = zP^z, z^2) = \sum_{n=0}^{\infty} C_n(\alpha_s(\kappa/z_0), \kappa^2) \frac{(-i\lambda)^n}{n!} a_n(\kappa/z_0)\,,$$

$$\lim_{N\to\infty} C_N = \frac{\alpha_s C_F}{2\pi}\left[-2\ln^2 N' + 2\ln N' - \frac{\pi^2}{3}\right], \quad N' = Ne^{\gamma_E}$$

$$\tilde{h}(\lambda, z^2) = \int_0^1 d\omega \;\; C(\alpha_s(\kappa/z_0), \omega, \kappa^2) \int_0^1 dx\; e^{ix\lambda} f(x, \kappa/z_0)$$

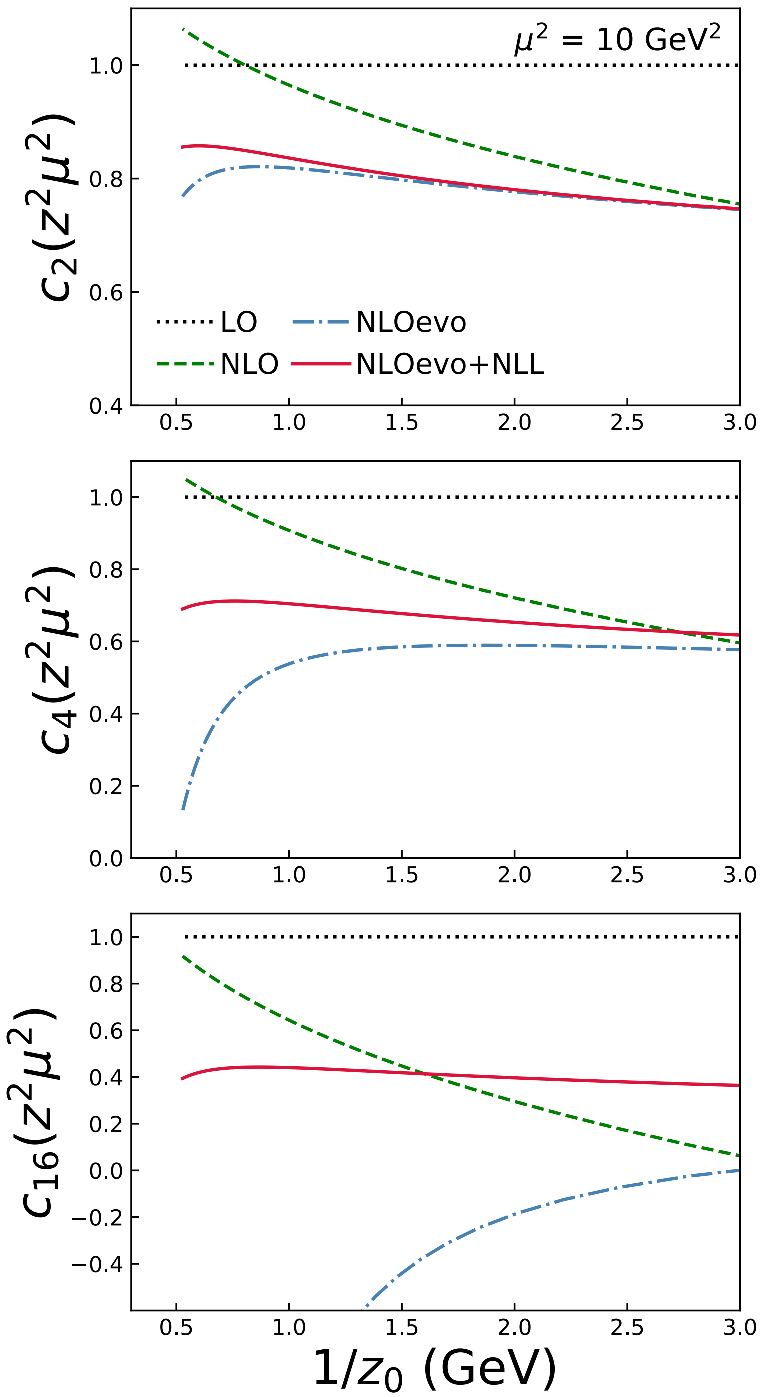
$$\lim_{\omega\to 1} C(\omega) = -\frac{\alpha_s C_F}{2\pi}\frac{4\ln(1-\omega)}{1-\omega}$$

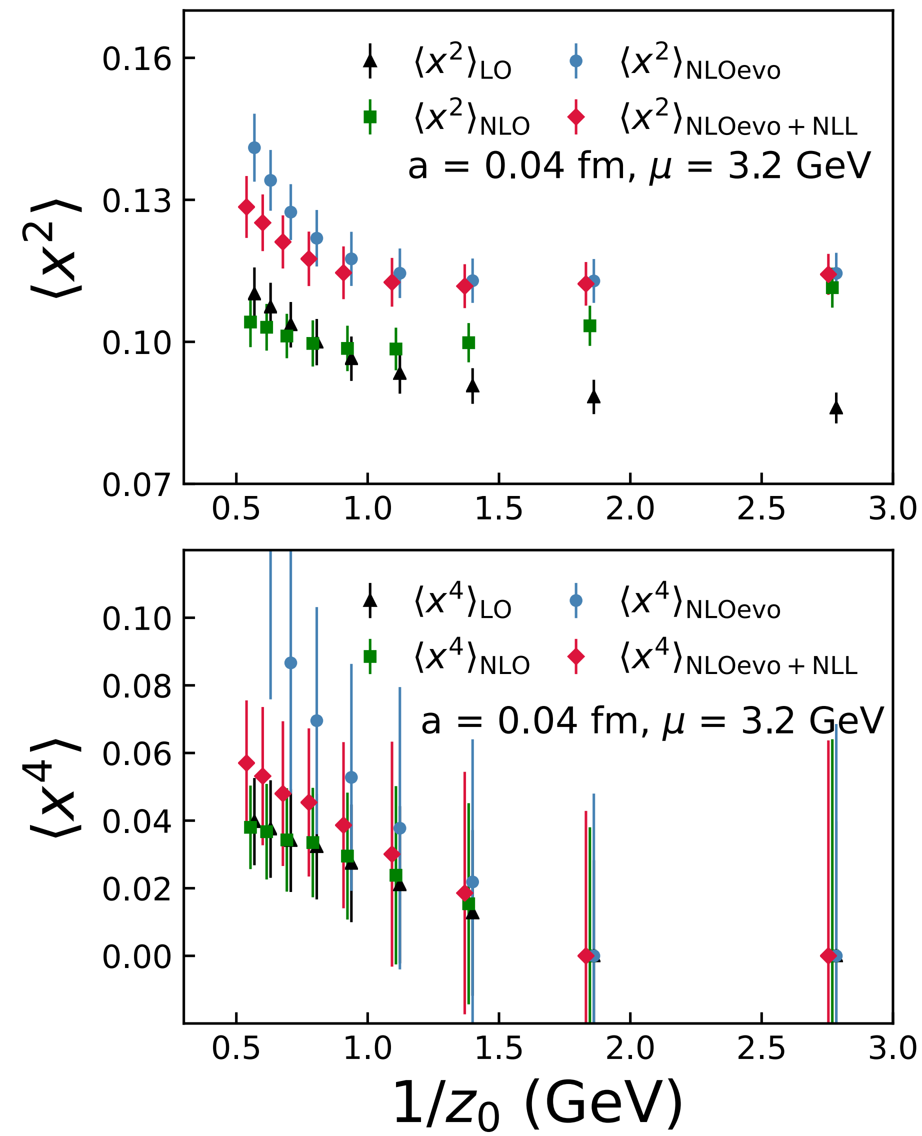
- In the Mellin moment space, threshold resummation only impacts high moments;
- In the corrrelation function, it mainly affects large λ;

Short-distance factorization

Numerical results:

X. Gao, YZ et al., *Phys.Rev.D* 103 (2021) 9





Lattice data not sensitive to the higher moments (or large λ) which require threshold resummation.

Summary

- In LaMET
  - Resummations provide quantified estimates of the theory uncertainty, which is important for **not** underestimating the end-point regions.
  - Because $P^z \sim 2$ GeV is not very large compared to $\mu = 2$ GeV, there is no significant improvement at moderate *x*.
- In short-distance factorization
  - Resummations crucial for determining $z_{\rm max}$
  - Because the typical lattice spacing $a \gtrsim (4\ {\rm GeV})^{-1}$ is not very small, there is no significant impact on the lowest few moments or large *x*

Summary

LaMET	Observable	Kinematic range	Accuracy
Zhang, Ji, Holligan and Su, PLB 844 (2023).	Pion valence PDF	Pz=1.94 GeV, k=0.75, 1.0, 1.5	NNLO+RGR+LRR
Holligan, Ji, Lin, Su and Zhang, NPB 993 (2023);	Pion DA	Pz=1.73 GeV, k=0.8, 1.0, 1.2	NLO+RGR+LRR
Su, Ji, Yao et al., Nucl.Phys.B 991 (2023)	Pion valence PDF	Pz=1.94 GeV, k=0.8, 1.0, 1.2	NNLO+RGR
R. Zhang et all, JHEP 07 (2024) 211 & 2407.00206	Pion and kaon DAs	Pz=1.8 GeV, k=0.707, 1, 1.414	NLO+RGR+LRR+TR
X. Gao, Q. Shi, YZ et al., Phys.Rev.D 109 (2024)	Proton isovector transversity PDF	Pz=1.53 GeV, k=0.707, 1, 1.414	NLO+RGR+LRR
J. Holligan and H.-W. Lin, PRD 110 (2024)	Proton isovector unpolarized GPD	Pz=2 GeV, k=0.75, 1, 1.5	NNLO+RGR+LRR
J. Holligan and H.-W. Lin, J.Phys.G 51 (2024) 6	Pion valence PDF	Pz=1.72 GeV, k=0.75, 1, 1.5	NNLO+RGR+LRR
J. Holligan and H.-W. Lin, PLB 854 (2024)	Nucleon isovector helicity PDF	Pz=1.75 GeV, k=0.75, 1, 1.5	NLO+RGR+LRR
Q. Shi, X. Gao, YZ et al., 2407.03516	Pion valence GPD	Pz=1.94 GeV, k=1, 1.414, 2	NNLO+RGR+LRR
X. Ji, Y. Liu, Y. Su and R. Zhang, 2410.12910	Pion valence PDF	Pz=1.94 GeV, k=0.8,1,1.2	NNLO+RGR+LRR+TR
SDF	**Observable**	**Kinematic range**	**Accuracy**
X. Gao, YZ et al., Phys.Rev.D 103 (2021) 9	Pion valence PDF 2nd and 4th moments	$a < z < 0.35$ fm	NLO+RGR+TR
Su, Ji, Yao et al., Nucl.Phys.B 991 (2023)	Pion valence PDF 2nd and 4th moments	$a < z < 0.35$ fm, k=0.8, 1, 1.2	NLO+RGR
S. Bhattacharya, X. Gao et al., PRD 108 (2023)	Proton GPD 2nd to 6th moments*	$a < z < 0.3$ fm	NNLO+RGR*
S. Bhattacharya, X. Gao et al., 2410.03539	Proton axial GPD 2nd, 4th, 6th moments*	$a < z < 0.3$ fm	NNLO+RGR*
H. Dutrieux et al. (HadStruc), JHEP 08 (2024) 162	Proton GPD, 2nd to 4th moments	$z < 0.57$ fm	LO+RGR*

Outlook

- **Comparing** lattice with phenomenology:
  - PDF within a moderate range of *x* from LaMET

    A $P^z \to \infty$ extrapolation is still necessary to demonstrate power convergence
  - Lowest Mellin moments from OPE
- **Combining** lattice with phenomenology:
  - Short-distance factorization
    - Natural to treat lattice matrix elements as experimental data
    - Perturbative accuracy better be consistent
    - Converge on the kinematic cuts, i.e., $z_{\rm max}$ (and $z_{\rm min}$) and range of κ
  - LaMET
    - Weighted χ^2 with the *x*-dependent results? How to select points from a continuous range of *x* (with correlation)?