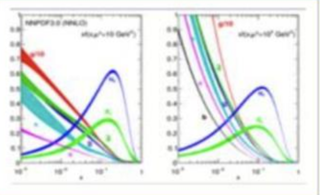

Parton Distributions and Lattice Calculations (PDFLattice 2024)

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# Extraction of PDFs from Lattice QCD Calculable Parton Correlation Functions


**Zheng-Yang Li**

**Theory Center, Jefferson Lab**


**In collaboration with Cheng, Huang, X. Li, Ma, Qiu, …**

Jefferson Lab

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# Introduction

## ❑ Parton Distribution Functions (PDFs):

$$q_{k/h}(x,\mu^2) = \int \frac{d\xi^-}{4\pi}\, e^{-ixp^+\xi^-} \langle h(p)|\bar{\psi}_k(\xi^-)\,\gamma^+\,\Phi^{(f)}(\xi^-,0)\,\psi_k(0)|h(p)\rangle\Big|_{\xi^+=0,\xi_T=0,\mu^2}$$

**Quark and/or gluon correlation functions on the light-cone – NOT direct physical observables!**

## ❑ QCD factorization – Match PDFs to experimental cross sections:

$$\sigma^{\text{phy}}_{\{h_i\}}(\{Q_i\}, 1/R_h \sim \Lambda_{\text{QCD}}) = \sum_{\{f_i\}} \widehat{H}_{\{f_i\}}(\{x_i\},\{Q_i\},\mu) \prod_i \otimes\, \phi_{f_i/h_i}(x_i,\mu,1/R_h) + \mathcal{O}(1/R_h Q)^n$$

Physical cross section

Calculable partonic "Probe"

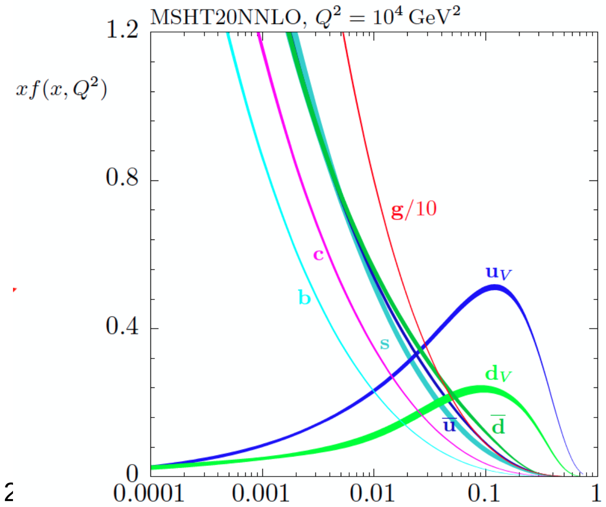
PDFs - Matching Parton to Hadron

Approximation "controllable?"

## ❑ QCD factorization works:

**MSHT20**

| Data set | $N_{\text{pts}}$ | NLO $\chi^2/N_{\text{pts}}$ | NNLO $\chi^2/N_{\text{pts}}$ |
|---|---|---|---|
| ATLAS 8 TeV s. diff $t\bar{t}$ | 25 | 1.56 | 0.98 |
| CMS 8 TeV d. diff $t\bar{t}$ | 15 | 2.19 | 1.50 |
| ATLAS 7 TeV $W$, $Z$ | 61 | 5.00 | 1.91 |
| ATLAS 8 TeV $W$ | 22 | 3.85 | 2.61 |
| ATLAS 8 TeV d. diff $Z$ | 59 | 2.67 | 1.45 |
| ATLAS 8 TeV $Z$ $p_T$ | 104 | 2.26 | 1.81 |
| ATLAS 8 TeV $W$ + jets | 39 | 1.13 | 0.60 |
| Total LHC data | 1328 | 1.79 | 1.33 |
| Total non-LHC data | 3035 | 1.13 | 1.10 |
| Total | 4363 | 1.33 | 1.17 |



# Introduction

## ❑ Equal Time Parton Correlation Functions (PCFs):

$$\mathcal{F}(h,\mathcal{O}_k^\nu,n;\omega,\xi;\mu^2) \equiv \frac{1}{2n\cdot p}\, n_\nu \langle h(p)|\mathcal{O}_k^{\nu,\mathrm{RS}}(\xi,\mu^2)|h(p)\rangle$$

$$\omega \equiv p\cdot\xi$$
$$\xi^\mu = (\xi_0,\xi_T,\xi_z) \to (0,0_T,\xi_z)$$
$$n^\mu = (n_0,0_T,0)$$
or $(0,0_T,n_z)$

$\Phi^{(f)}(\xi,0)$ Gauge-link

- **With bare quark correlation operator (similar for gluon):**

$$\mathcal{O}_k^{\nu,b}(\xi,\mu^2;\delta) = \bar\psi_k^b(\xi)\gamma^\nu\Phi^{(f)}(\xi,0)\psi_k^b(0)|_{\mu^2,\delta}$$

- **UV divergence as $\delta\to 0$, multiplicatively renormalizable:**

$$\mathcal{O}_k^{\nu,\mathrm{RS}}(\xi,\mu^2) = \mathcal{O}_k^{\nu,b}(\xi,\mu^2;\delta)/Z_{\mathcal{O}}^{\mathrm{RS}}(\xi,\mu^2;\delta)$$

- **NOT direct physical observables, but, Lattice QCD calculable**

Ji, Zhang, Zhao, 1706.08962
Ishikawa, Ma, Qiu, Yoshida,1707.03107
Green, Jansen, Steffens, 1707.07152
…

## ❑ All-order QCD Factorization in term of PDFs:

$$\mathcal{F}(h,\mathcal{O}_k^\nu,n;\omega,\xi;\mu^2) = \sum_{f=q,\bar q,g}\int_0^1 \frac{dx}{x}\frac{1}{n\cdot p}\, n\cdot K_f(x\omega,\xi^2,\mu^2)\, f_h(x,\mu^2) + \mathcal{O}(\xi^2\Lambda_{\mathrm{QCD}}^2)$$

Equal-time Lattice PCFs

Calculable partonic “Probe”

Matching Parton to Hadron

Approximation “controllable?”

$$\equiv \sum_{f=q,\bar q,g} K_f(x\omega,\xi^2,\mu^2)\otimes f_h(x,\mu^2) + \mathcal{O}(\xi^2\Lambda_{\mathrm{QCD}}^2)$$

X. Ji, 1305.1539
Ma, Qiu, 1404.6860
Radyushkin, 1705.01488
Ma, Qiu, 1709.03018
Izubuchi, Ji, Jin, Stewart, Zhao, 1801. 03917

# Introduction

- **Calculation of the Matching Coefficients:** $K_f(x\omega, \xi^2, \mu^2)$
  - **Have to specify UV renormalization of the Equal-time PCFs**
    - **Not needed for the factorization of a physical cross section!**
  - **Apply the factorization formalism to an asymptotic parton state:** $h \to f = q, \bar{q}, g$

$$\mathcal{F}(h, \mathcal{O}_k^\nu, n; \omega, \xi; \mu^2) = \sum_{f=q,\bar{q},g} \int_0^1 \frac{dx}{x} \frac{1}{n\cdot p}\, n\cdot K_f(x\omega, \xi^2, \mu^2)\, f_h(x, \mu^2) + \mathcal{O}(\xi^2 \Lambda_{\rm QCD}^2)$$

  - **Calculate both $\mathcal{F}$ and $f$ order-by-order perturbatively with CO regularization & UV renormalization**
    - **TO derive the matching coefficients $K_f(x\omega, \xi^2, \mu^2)$ order-by-order**

- **Renormalization of Equal-time PCFs:**
  - **Renormalization constant $Z_{\mathcal{O}}^{\rm RS}$ should depend on the operator $\mathcal{O}_k^{\nu,b}$, but, NOT the state to define the correlation functions $\langle h(p)|\, \mathcal{O}_k^{\nu,b}\, |h(p)\rangle$ or $\langle h(0)|\, \mathcal{O}_k^{\nu,b}\, |h(0)\rangle$**

$Z_{\mathcal{O}}^{\rm RS}(\xi, \mu^2; \delta) \propto \langle q, g(p)|\mathcal{O}_k^{\nu,b}(\xi, \mu^2; \delta)|q, g(p)\rangle$ (e.g., Quasi-PDF approach – separation CO sensitivity)

$Z_{\mathcal{O}}^{\rm RS}(\xi, \mu^2; \delta) \propto \langle h(0)|\mathcal{O}_k^{\nu,b}(\xi, \mu^2; \delta)|h(0)\rangle$ (Pseudo-PDF approach – nonperturbative hadron state)

$Z_{\mathcal{O}}^{\rm RS}(\xi, \mu^2; \delta) \propto \langle 0|\mathcal{O}_k^{\nu,b}(\xi, \mu^2; \delta)|0\rangle$ (CO, IR safe – hard to calculate on lattice)

  - **How to match nonperturbative lattice renormalization to partonic calculation of $Z_{\mathcal{O}}^{\rm RS}$ ?**

# Renormalization Free Lattice Observable

- **Lattice Observable (or Lattice "cross section"):**

Ma, Qiu, 1404.6860

  - **is calculable in LQCD with an Euclidean time,**
  - **has a well-defined continuum limit as lattice spacing $a \to 0$, and**
  - **has the same and factorizable logarithmic collinear (CO) divergence as PDFs**

**FACT:**

**Any hadron matrix elements of quark/gluon correlation operators are not directly measured physical observables!**

**NEED:**

**Lattice calculable matrix elements with minimum sensitivities from their renormalization**

**e.g., matrix elements of current-current correlators, …**

Ma, Qiu, 1709.03018
Suffian, Karpie, Egerer, Orginos, Qiu, Richards, 1901.03921

- **Ratio of two matrix elements of the Same operator – free of renormalization:**

$$\frac{n_2\cdot p_2}{n_1\cdot p_1}\frac{\langle h_1(p_1)|n_1\cdot\mathcal{O}_k^{\rm RS}|h_1(p_1)\rangle}{\langle h_2(p_2)|n_2\cdot\mathcal{O}_k^{\rm RS}|h_2(p_2)\rangle}=\frac{\mathcal{F}(h_1,\mathcal{O}_k^{\nu,{\rm RS}},n_1;\omega_1,\xi)}{\mathcal{F}(h_2,\mathcal{O}_k^{\nu,{\rm RS}},n_2;\omega_2,\xi)}$$

$$=\frac{\sum_{i=q,\bar{q},g}K_{ki}(x\omega_1,\xi^2,\mu^2)\otimes f_{i/h_1}(x,\mu^2)+O(\xi^2\Lambda_{\rm QCD}^2)}{\sum_{j=q,\bar{q},g}K_{kj}(x\omega_2,\xi^2,\mu^2)\otimes f_{j/h_2}(x,\mu^2)+O(\xi^2\Lambda_{\rm QCD}^2)}$$

*Factorization for both numerator and denominator when $\xi^2$ is small*

- **The $Z_{\mathcal{O}}^{\rm RS}(\xi,\mu^2;\delta)$ in numerator and denominator are canceled point-by-point at all value of $\xi$.**
- **$K_f(x\omega,\xi^2,\mu^2)$ – calculated in any preferred renormalization scheme**

# Renormalization Free Lattice Observable

## ❑ Pseudo-PDF approach:

$$\widetilde{\mathcal{M}}_h(\mathcal{O}_q^{\rm RS},\omega,\xi) = \frac{\mathcal{F}(h,\mathcal{O}_q^{\nu,{\rm RS}},n;\omega,\xi)}{\lim_{\omega\to 0}\mathcal{F}(h,\mathcal{O}_q^{\nu,{\rm RS}},n;\omega,\xi)} \Rightarrow \frac{K_{ns}(x\omega,\xi^2,\mu^2)\otimes q_v(x,\mu^2)}{K_{ns}(0,\xi^2,\mu^2)\int dx\, q_v(x,\mu^2)}$$

PDFs are boost invariant

Matching coefficient, $K_{ns}(x\omega,\xi^2,\mu^2)$, can be calculated in any scheme, and

$$K_{ns}(0,\xi^2,\mu^2) = 1 + \sum_{n=1}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{i=0}^{n}\sum_{j=1}^{n^2-1} a_{ij0}\, L^i(\xi^2\mu^2) + \text{terms} \propto \xi\cdot n$$

with $L(\xi^2\mu^2) = \ln(-\xi^2\mu^2/4) + 2\gamma_E$

## ❑ Minimum impact from lattice Renormalization – "Physical observables":

$$\frac{\mathcal{F}(h_1,\mathcal{O}_k^{\nu,{\rm RS}},n_1;\omega_1,\xi)}{\mathcal{F}(h_2,\mathcal{O}_k^{\nu,{\rm RS}},n_2;\omega_2,\xi)} = \frac{\sum_{i=q,\bar{q},g} K_{ki}(x,\omega_1,\xi^2,\mu^2)\otimes f_{i/h_1}(x,\mu^2)}{\sum_{j=q,\bar{q},g} K_{kj}(x,\omega_2,\xi^2,\mu^2)\otimes f_{j/h_2}(x,\mu^2)}$$

*Like the spin asymmetries with QCD factorization in both numerator and denominator, …*

- Lattice observable for each operator $\mathcal{O}_k^{\nu,{\rm RS}}$, e.g., $k = q, g, \ldots$
- New observable for different combination of hadron types, e.g., $h_1 = \text{proton}, \text{neutron}, \pi, \ldots$ $h_2 = \text{proton}$
- New information with different ranges of hadron momentum, …

# Renormalization Free Lattice Observable

- Go beyond the non-singlet case:

$$\mathcal{F}(h,\mathcal{O}_k^{\nu,\overline{\mathrm{MS}}},n;\omega,\xi) = \mathrm{Re}\Big\{K_{ns}(x\omega,\xi^2,\mu^2)\otimes\Big[f_{k/h}(x,\mu^2)-f_{\bar{k}/h}(x,\mu^2)\Big]\Big\}$$

$$+\,i\,\mathrm{Im}\Big\{K_{ns}(x\omega,\xi^2,\mu^2)\otimes\Big[f_{k/h}(x,\mu^2)+f_{\bar{k}/h}(x,\mu^2)\Big]+K_{kg}(x\omega,\xi^2,\mu^2)\otimes f_{g/h}(x,\mu^2)$$

$$+\sum_{j'=q,\bar{q}}K_{ps}(x\omega,\xi^2,\mu^2)\otimes f_{j'/h}(x,\mu^2)\Big\}+O(\xi^2\Lambda_{\mathrm{QCD}}^2)$$

$K_{ps}\equiv K_{kk'}$ **with** $k'\neq k,\bar{k}$

$K_{ns}\equiv K_{kk}-K_{ps}$

Assuming $n\cdot\xi=0$

$$\frac{1}{n\cdot p}n\cdot K_{ns}(x\omega,\xi^2,\mu^2) = 2xe^{ix\omega}+x\frac{\alpha_s C_F}{\pi}\Bigg[\left(\frac{3}{2}L+\frac{5}{2}\right)e^{ix\omega}$$

$$+\int_0^1 dz\,\frac{(z^2+1)L-z^2+4z-1+4\ln(1-z)}{z-1}\left(e^{ixz\omega}-e^{ix\omega}\right)\Bigg]+O(\alpha_s^2)$$

$$\frac{1}{n\cdot p}n\cdot K_{kg}(x\omega,\xi^2,\mu^2) = x\frac{\alpha_s T_F}{\pi}\int_0^1 dz\,L(-4z^2+4z-2)\,[i\sin(xz\omega)]+O(\alpha_s^2)$$

$$\frac{1}{n\cdot p}n\cdot K_{ps}(x\omega,\xi^2,\mu^2) = O(\alpha_s^2)$$

# Renormalization Free Lattice Observable

- Go beyond the non-singlet case:

$$\mathcal{F}(h, \mathcal{O}_k^{\nu,\overline{\rm MS}}, n; \omega, \xi) = \mathrm{Re}\Big\{K_{ns}(x\omega, \xi^2, \mu^2) \otimes \Big[f_{k/h}(x,\mu^2) - f_{\bar{k}/h}(x,\mu^2)\Big]\Big\}$$
$$+ i\,\mathrm{Im}\Big\{K_{ns}(x\omega, \xi^2, \mu^2) \otimes \Big[f_{k/h}(x,\mu^2) + f_{\bar{k}/h}(x,\mu^2)\Big] + K_{kg}(x\omega, \xi^2, \mu^2) \otimes f_{g/h}(x,\mu^2)$$
$$+ \sum_{j'=q,\bar{q}} K_{ps}(x\omega, \xi^2, \mu^2) \otimes f_{j'/h}(x,\mu^2)\Big\} + O(\xi^2\Lambda_{\rm QCD}^2)$$

$K_{ps} \equiv K_{kk'}$ **with** $k' \neq k, \bar{k}$

$K_{ns} \equiv K_{kk} - K_{ps}$

- **Re-part: valence quark**
- **Im-part: valence quark + sea quark + gluon**

$$\mathrm{Im}\left[\lim_{\omega\to 0} \mathcal{F}(h, \mathcal{O}_k^{\nu,\rm RS}, n; \omega, \xi)\right] = 0$$

$$\Rightarrow \quad \mathrm{Im}\left[\frac{\mathcal{F}(h_1, \mathcal{O}_k^{\nu,\rm RS}, n_1; \omega_1, \xi)}{\mathcal{F}(h, \mathcal{O}_k^{\nu,\rm RS}, n; 0, \xi)}\right] / \mathrm{Im}\left[\frac{\mathcal{F}(h_2, \mathcal{O}_k^{\nu,\rm RS}, n_2; \omega_2, \xi)}{\mathcal{F}(h, \mathcal{O}_k^{\nu,\rm RS}, n; 0, \xi)}\right] = \frac{\mathrm{Im}\left[\mathcal{F}(h_1, \mathcal{O}_k^{\nu,\rm RS}, n_1; \omega_1, \xi)\right]}{\mathrm{Im}\left[\mathcal{F}(h_2, \mathcal{O}_k^{\nu,\rm RS}, n_2; \omega_2, \xi)\right]}$$

- **Ratios of Re-part and Im-part can be handled separately**

# Multi-loop Matching Coefficients

- **Go beyond the non-singlet case:**

$$\mathcal{F}(h, \mathcal{O}_k^{\nu,\overline{\rm MS}}, n; \omega, \xi) = {\rm Re}\Big\{K_{ns}(x\omega, \xi^2, \mu^2) \otimes \Big[f_{k/h}(x, \mu^2) - f_{\bar{k}/h}(x, \mu^2)\Big]\Big\}$$
$$+ i\,{\rm Im}\Big\{K_{ns}(x\omega, \xi^2, \mu^2) \otimes \Big[f_{k/h}(x, \mu^2) + f_{\bar{k}/h}(x, \mu^2)\Big] + K_{kg}(x\omega, \xi^2, \mu^2) \otimes f_{g/h}(x, \mu^2)$$
$$+ \sum_{j'=q,\bar{q}} K_{ps}(x\omega, \xi^2, \mu^2) \otimes f_{j'/h}(x, \mu^2)\Big\} + O(\xi^2 \Lambda_{\rm QCD}^2)$$

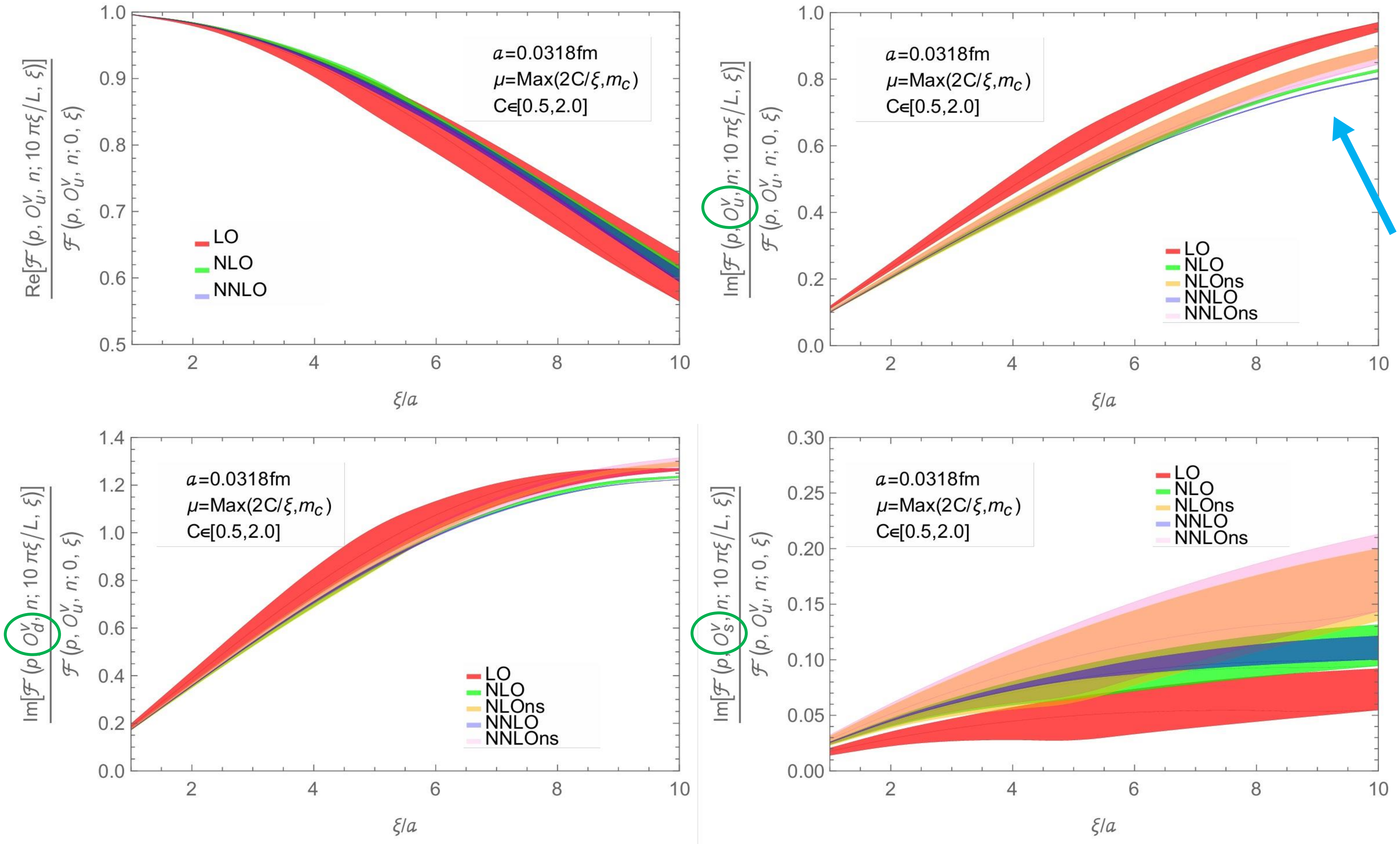
$K_{ps} \equiv K_{kk'}$ **with** $k' \neq k, \bar{k}$

$K_{ns} \equiv K_{kk} - K_{ps}$

- **NNLO** $K_{ns}$ **:** Li, Ma, Qiu, arXiv:2006.12370; Chen, Wang, Zhu, arXiv:2006.14825
- **N3LO** $K_{ns}$**:** Cheng, Huang, Li, Li, Ma, arXiv:2410.05141
- **NNLO and N3LO** $K_{kg}$ **and** $K_{ps}$ **:** Li, Ma, Qiu, work in preparation
- **Unpolarized valence PDFs extracted from LQCD calculation, utilizing NNLO** $K_{ns}$ **:**

Gao, Hanlon, Mukherjee, Petreczky, Scior, Syritsyn, Zhao, arXiv:2112.02208
Gao, Hanlon, Karthik, Mukherjee, Petreczky, Scior, Shi, Syritsyn, Zhao, Zhou, arXiv:2208.02297
Bhat, Chomicki, Cichy, Constantinou, Green, Scapellato, arXiv:2205.07585
Gao, Hanlon, Holligan, Karthik, Mukherjee, Petreczky, Syritsyn, Zhao, arXiv:2212.12569

# Numerical Predictions




**NLOns, NNLOns**
**= ignoring $K_{kg}$ and $K_{ps}$, simultaneously.**

**Significant difference between LO and higher-order predictions:**
**actual LQCD computations are needed**

Proton PDFs use CT18LO, CT18NLO, and CT18NNLO



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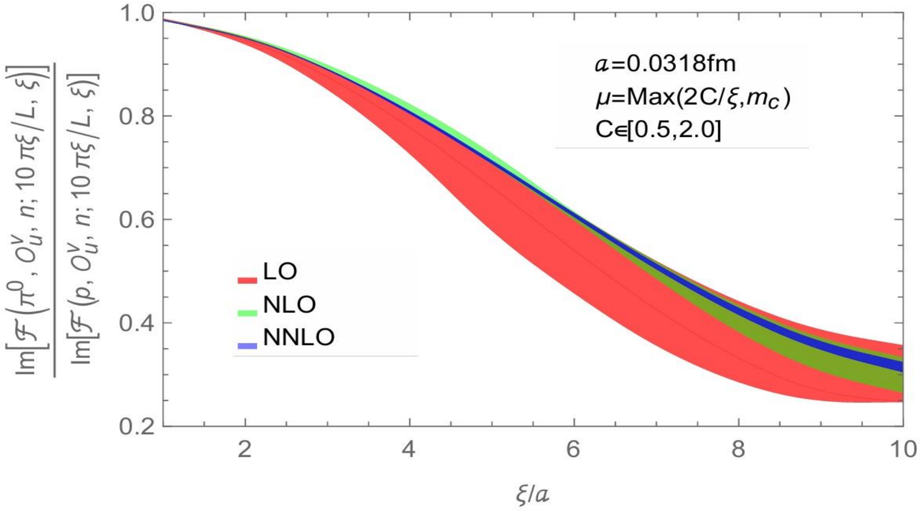
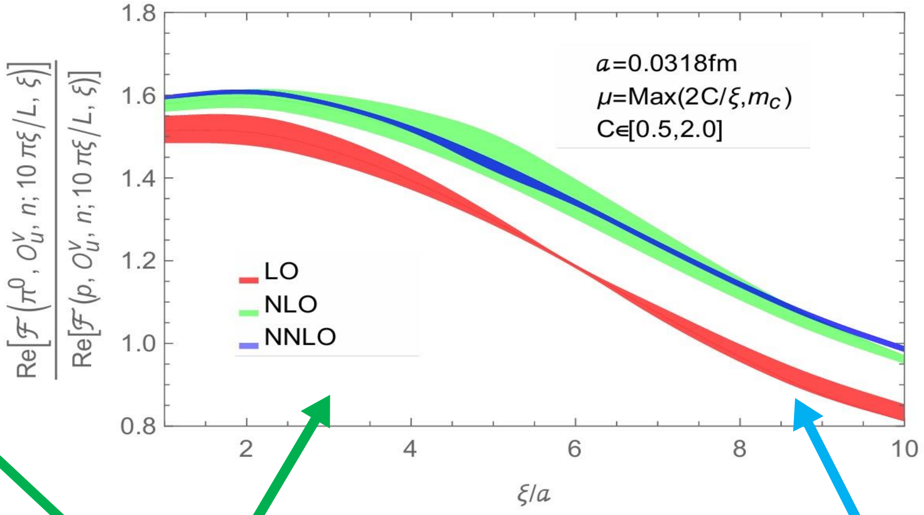
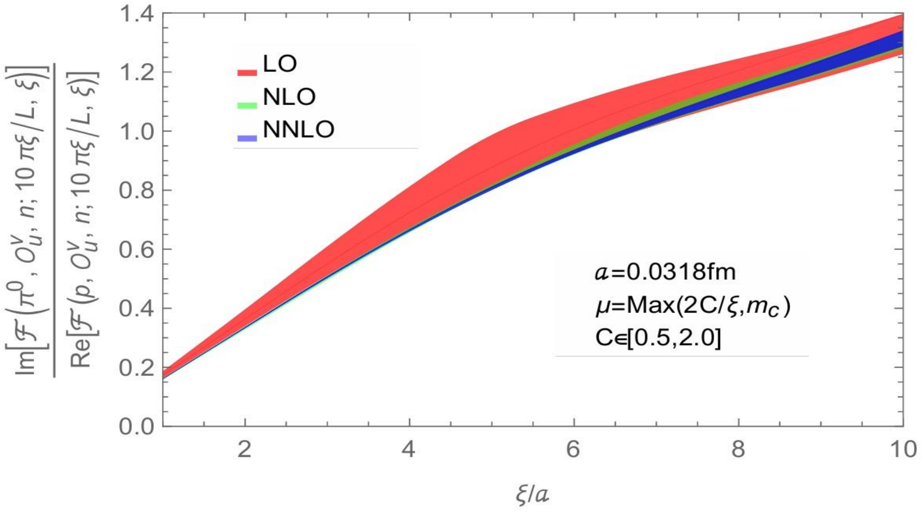
# Numerical Predictions

- ❑ **Direct nonperturbative comparison from LQCD calculations:**




**QCFs of external pion decreases more rapidly: Pion PDFs < Proton PDFs in small x region**

**Significant difference between LO and higher-order predictions:**
**actual LQCD computations are needed**

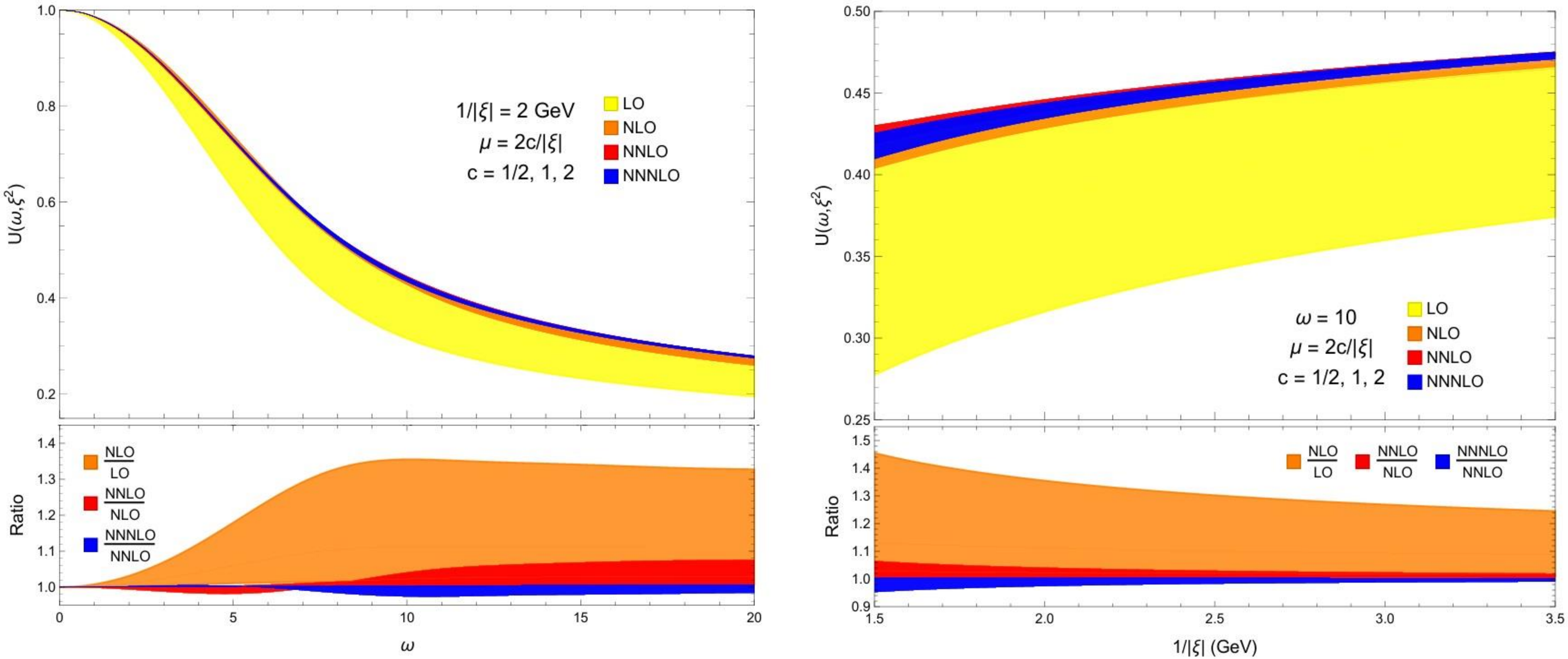
Proton PDFs use CT18LO, CT18NLO, and CT18NNLO
Pion PDFs use JAM21PionPDFnlo

# Numerical Predictions

- **N3LO predictions:**

PDFs use CT18NNLO

Cheng, Huang, Li, Li, Ma, arXiv:2410.05141

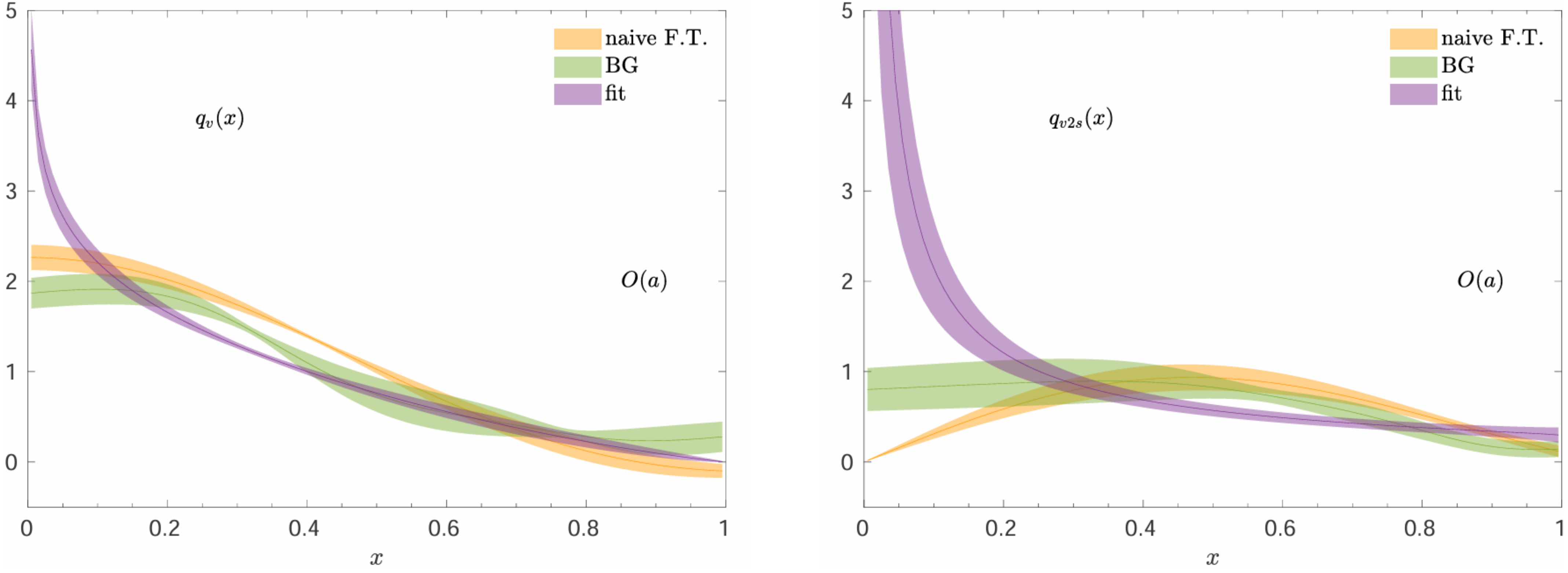



**Precision may still be improved by utilizing the N3LO results**

# Extracting PDFs from Lattice QCD calculations

- **Three approaches to extract PDFs from LQCD data:**
  - **naive Fourier transform**
  - **Backus-Gilbert method**
  - **ansatz for PDFs:** $f_{j/h}(x, \mu^2) = N x^{\alpha}(1-x)^{\beta}$, or more fitting parameters

Bhat, Chomicki, Cichy, Constantinou, Green, Scapellato, arXiv:2205.07585

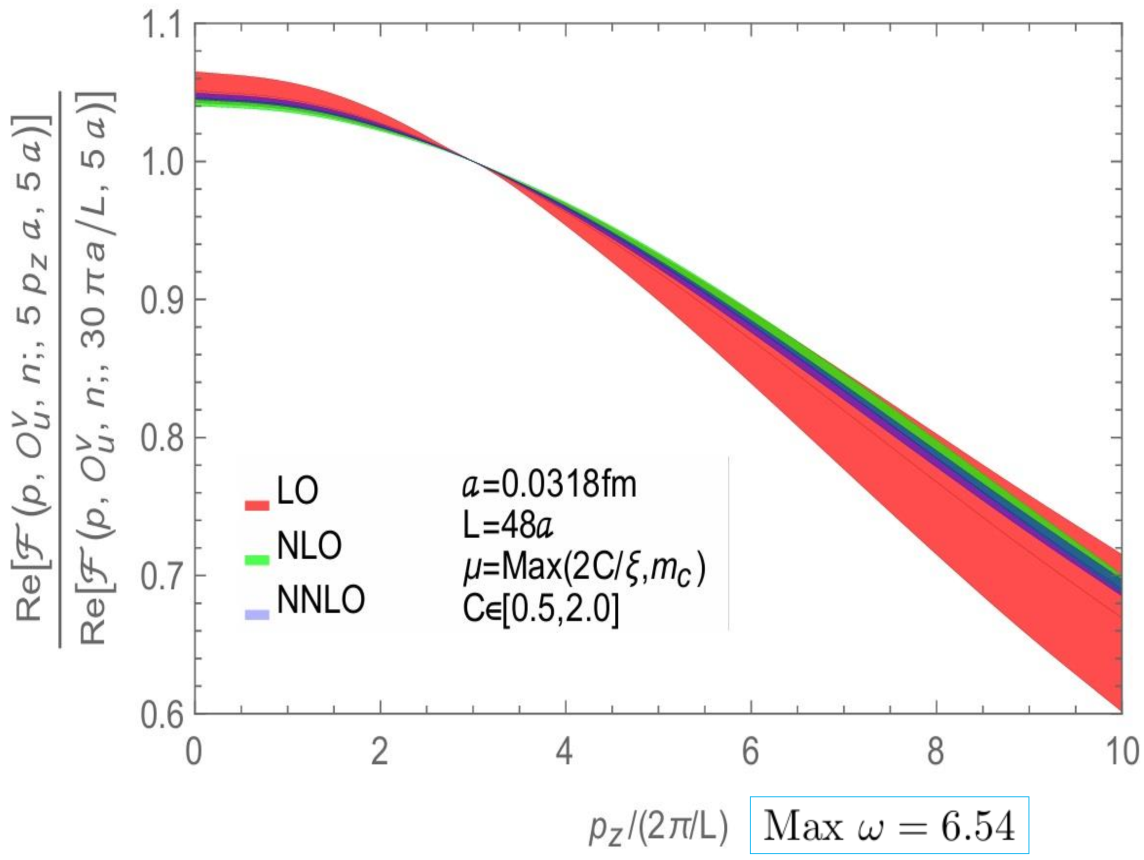



**Lack of data from large $\omega$ leads to the errors of naive F.T. and BG in small $x$**

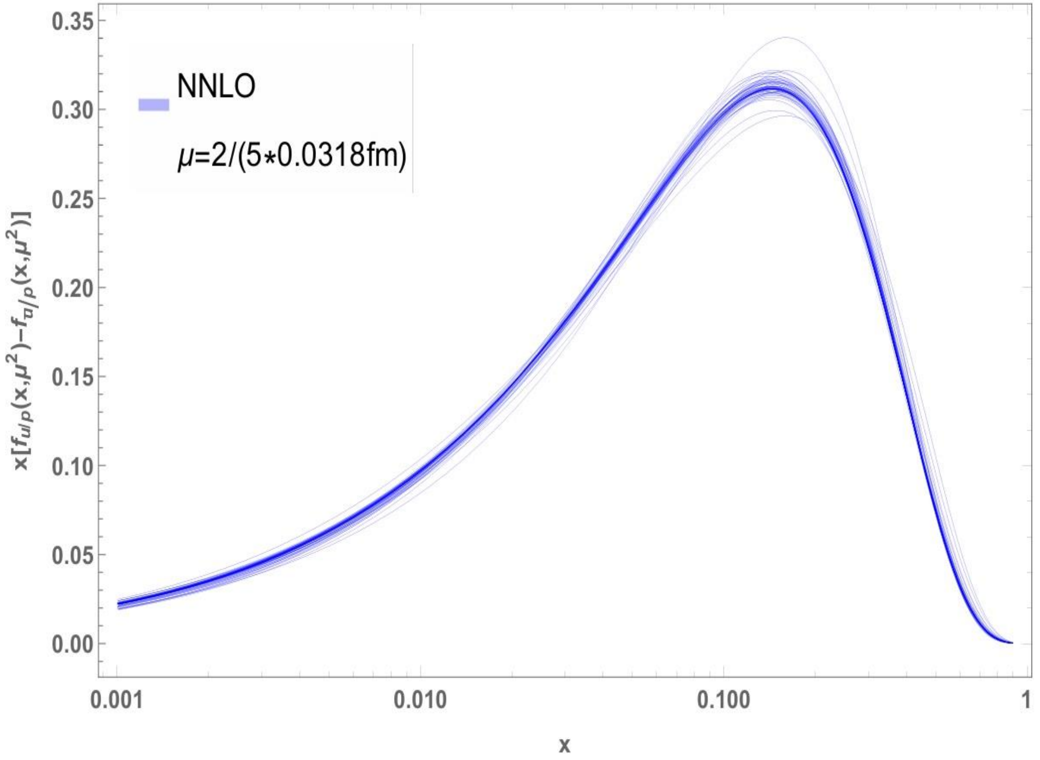
# Extracting PDFs from Lattice QCD calculations

## ❑ Extract Valence PDFs from ratio of matrix elements:

- **ansatz for PDFs:** $f_{j/h}(x,\mu^2) = N x^{\alpha}(1-x)^{\beta}$




**fit**



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# Summary and Outlook

- **We provide a renormalization "free" schedule for extracting PDFs from Lattice QCD calculations**
- **Valence quark, sea quark, and gluon PDFs can all be extracted from the matrix elements of quark (and gluon) correlation operators – "Observable"**
- **The Ratio of LQCD calculations of matrix elements of the same operator, but different hadrons provide a direct nonperturbative comparison of hadron structure between different hadrons.**
- **Extracting higher-precision PDFs using N3LO matching coefficients should be feasible**

**Thank you!**