Theory Seminar - CERN

The Tropical Geometry Of Subtraction Schemes

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Introduction

In this talk we will discuss **Euler Integrals**, i.e. of the form

Polynomials in α with coefficients $\mathbf{s}$

$$A(\mathbf{s},\epsilon) = \int_{\mathbb{R}^n_{\geq 0}} d\alpha_1 \ldots d\alpha_n \prod_{j=1}^{m} P_j(\alpha,\mathbf{s})^{c(\epsilon)}$$

Euler Integrals are ubiquitous in physics.

Feynman & Phase-Space Integrals, Region integrals, String Amplitudes, Cosmological correlators.

To some extent, all of these can be expressed in terms of Euler integrals

The goal

Our objective is to compute the expansion in the *"dimensional regulator"* ϵ

$$\tau_\epsilon A(\mathbf{s}, \epsilon) = \sum_{i=-N} \epsilon^i A_i(\mathbf{s})$$

With the coefficients $A_i(\mathbf{s})$ represented in terms of convergent integrals - to be then evaluated by other means

(E.g. Numerically, by direct integration or differential equations)

$$A_i(\mathbf{s}) = \int_{\mathbb{R}^{n-i}_{\geq 0}} \mathscr{J}(\mathbf{s})$$

Local Finiteness

An obvious idea is to expand in ϵ directly under sign of integration

$$\tau_\epsilon \int \mathscr{I}(\mathbf{s}, \epsilon) \stackrel{?}{=} \int \tau_\epsilon \mathscr{I}(\mathbf{s}, \epsilon)$$

In general this is incorrect.

For instance, it would not give poles in ϵ

We will say that the integrand $\mathscr{I}$ is **locally finite** when the above holds

Subtraction Schemes

The strategy we will use to compute $\tau_\epsilon A(\epsilon)$ is to build a *subtraction scheme*

That is we look for suitable *counter-terms* which allow us to re-write

$$\mathscr{I} = [\mathscr{I} - \mathscr{I}^{\text{ct}}] + \mathscr{I}^{\text{ct}}$$

In such a way that

1) The *renormalized* integrand $\mathscr{I}^{\text{ren}} := \mathscr{I} - \mathscr{I}^{\text{ct}}$ is locally finite

2) The counter-terms are easier, i.e. we can partly integrate them: $\displaystyle\int_{\mathbb{R}^n_{\geq 0}} \mathscr{I}^{\text{ct}} = \frac{1}{\epsilon^i} \int_{\mathbb{R}^{n-i}_{\geq 0}} \mathscr{J}$

Other approaches

Aside from techniques based on differential equations, there are other two important methods to be aware of

1) **Nilsson-Passare** analytical continuation

$$\int_{\mathbb{R}^n} \mathcal{I} = [\mathrm{IBP}] = \frac{1}{\epsilon} \int_{\mathbb{R}^n} \mathcal{I}'$$

With the remaining integrands locally finite. Drawback: no simplification in the computation of poles

2) **Sector Decomposition**

The *domain* is divided in pieces where subtraction is easier. Drawback: the artificial decomposition introduces spurious structures

Warming up I

Let us illustrate the challenges by means of simple toy examples

We begin with an example of a finite integral that fails to be *locally finite*

$$0 = \int_0^1 \frac{d\alpha}{\alpha} \left(\alpha^{\epsilon} - 2\alpha^{2\epsilon}\right) \neq \int_0^1 \frac{d\alpha}{\alpha} \left(-1 + \mathcal{O}(\epsilon)\right) = \infty$$

This simple example already points to the central issue:

In order for a combination of integrands to be locally finite their local behavior - as captured by a

series expansion around the boundary of the integration domain - must cancel

Warming up II

A natural guess for a counter-term for an integrand $\mathscr{I}$ is its own expansion around the boundary

$$\tau_\epsilon \int_0^1 \frac{d\alpha}{\alpha}[\alpha^\epsilon(1+\alpha)^\epsilon - \alpha^\epsilon] = \int_0^1 \frac{d\alpha}{\alpha}\tau_\epsilon[\alpha^\epsilon(1+\alpha)^\epsilon - \alpha^\epsilon] = \epsilon\int_0^1 \frac{d\alpha}{\alpha}\log(1+\alpha) + \ldots = \epsilon\frac{\zeta(2)}{2} + \ldots$$

However, a subtlety is that there may be singular behaviors “hidden“ at the boundary

$$(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)^\epsilon = [\alpha_2 \to \alpha_2\alpha_1] = \alpha_1^\epsilon(1 + \alpha_2 + \alpha_1\alpha_2)^\epsilon$$

Furthermore, the expansion of an integrand around a boundary introduces further singularities away from it

Warming up III

An even worse problem is **overcounting** of divergences.

Consider the integrand $\mathscr{I} = \alpha_1^\epsilon \alpha_2^\epsilon (\alpha_1 + \alpha_2 + \alpha_1\alpha_2)^\epsilon$, we tentatively subtract its divergences by

$$\int_0^1 \frac{d\alpha_1 d\alpha_2}{\alpha_1\alpha_2}[\alpha_1^\epsilon \alpha_2^\epsilon (\alpha_1 + \alpha_2 + \alpha_1\alpha_2)^\epsilon - \alpha_1^\epsilon \alpha_2^\epsilon (\alpha_2)^\epsilon - \alpha_1^\epsilon \alpha_2^\epsilon (\alpha_1)^\epsilon] = \int_0^1 \frac{d\alpha_1 d\alpha_2}{\alpha_1\alpha_2}[1 - \tau_{\alpha_1} - \tau_{\alpha_2}]\mathscr{I}$$

where we have introduced the operator τ_x that computes the series expansion in x (up to an appropriate order).

The above is not locally finite, because it is still singular as either variable goes to 0.

The issue is that we have subtracted twice the singular behavior at the origin.

We can try to compensate this with a further term $\tau_{\alpha_1}\tau_{\alpha_2}\mathscr{I}$

Warming up III

This could work **if** the operator τ_{α_i} commuted:

$$[1 - \tau_{\alpha_1} - \tau_{\alpha_2} + \tau_{\alpha_1}\tau_{\alpha_2}]\mathcal{I} = [1 - \tau_{\alpha_1}][1 - \tau_{\alpha_2}]\mathcal{I}$$

Which makes manifest that the renormalized integrand has no singular behavior for *neither* $\alpha_i \to 0$

However, in general the operators do **not commute**:

$$\tau_{\alpha_1}\tau_{\alpha_2}(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)^\epsilon = \alpha_1^\epsilon \neq \alpha_2^\epsilon = \tau_{\alpha_2}\tau_{\alpha_1}(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)^\epsilon$$

Wrapping up

We are now familiar with the basic issues in the construction of a *local* ”subtraction scheme“

1) Find all singular behaviors of the integrand

2) Cancel them locally with counter-terms, without introducing new divergences

3) Avoid double counting

In the rest of the talk, we will see how these can be addressed using elementary ideas from **Tropical Geometry**

As a concrete application we will develop a subtraction scheme that is applicable to Euler integrals satisfying a certain property

Outline

1. **Tropical Geometry 101**
2. A subtraction scheme
3. Some examples

What is Tropical Geometry?

Tropical geometry studies *algebraic varieties* by approximating them via *piecewise linear* geometries, e.g.

$$V = \{x, y \in \mathbb{C}, x + y + 1 = 0\}$$

This is done by solving the equations defining the variety over the *Puiseaux series field*

$$\mathbb{C}((t)) = \{\phi = t^a \left(\sum_i c_i t^i\right), a \in \mathbb{C}\}$$

And studying the image under the evaluation map, which returns the leading term of the series $\mathrm{Trop} : \phi \mapsto a$

$$\mathrm{Trop}\ V = \mathrm{Trop}\{x, y \in \mathbb{C}((t)), x + y + 1 = 0\}$$

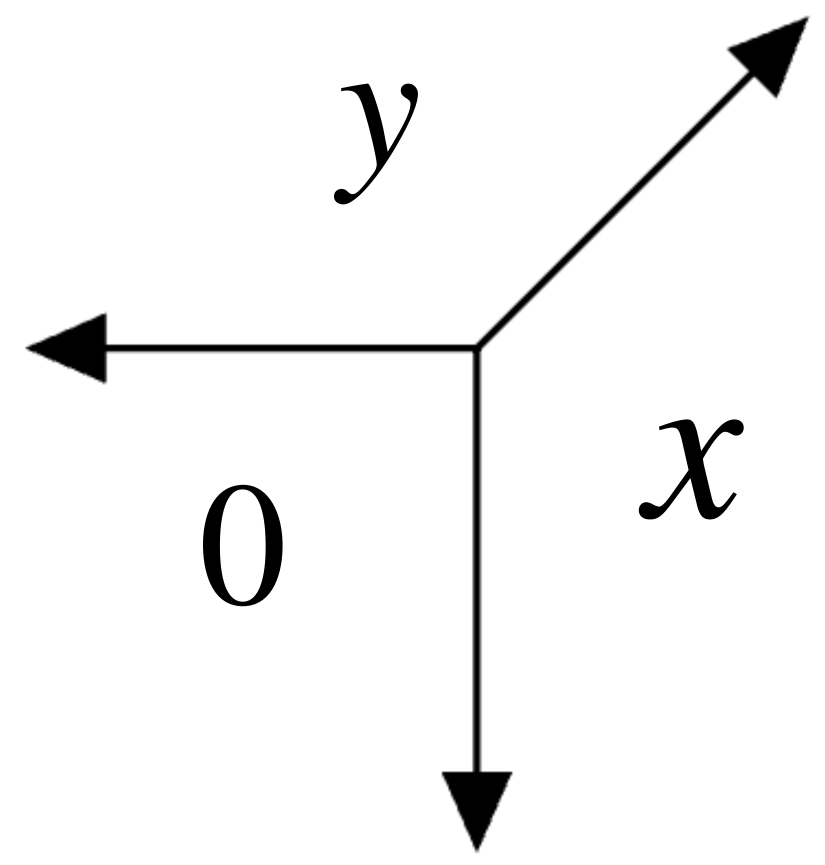
What is Tropical Geometry?

For the equation $x(t) + y(t) + 1 = 0$ to be satisfied, the leading terms must cancel.

This requires $\max(0, \text{Trop}(x), \text{Trop}(y))$ to be attained twice in $(v_x, v_y) = (\text{Trop } x, \text{Trop } y)$ space

This yield the tropical line

$$\text{Trop } V = \text{Trop}\{x, y \in \mathbb{C}((t)), x + y + 1 = 0\} =$$



Tropicalization made simple

Consider a function $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ and let us define its tropicalization Trop $f : \mathbb{R}^n \to \mathbb{R}$ as

$$\text{Trop } f(\rho) = \lim_{\alpha' \to 0^+} \alpha' \log f(\alpha = \exp(-\rho/\alpha'))$$

For simple functions, such as polynomials, we compute the tropicalization by "replacing (*,+) with (+,max)", for instance:

$$\text{Trop } (ax^2 + by + c) = \max(2x, y, 0)$$

The tropicalization of a function captures it behavior at the boundary where $\alpha_i \sim \lambda^{-\rho_i},\ \lambda \to 0$

Finiteness

The tropicalization of an Euler integrand $\mathcal{J} = \prod_j P_j(\alpha, \mathbf{s})^{\nu_j}$ is computed as $\text{Trop } \mathcal{J} = \max(\nu_j \text{Trop } P_j)$

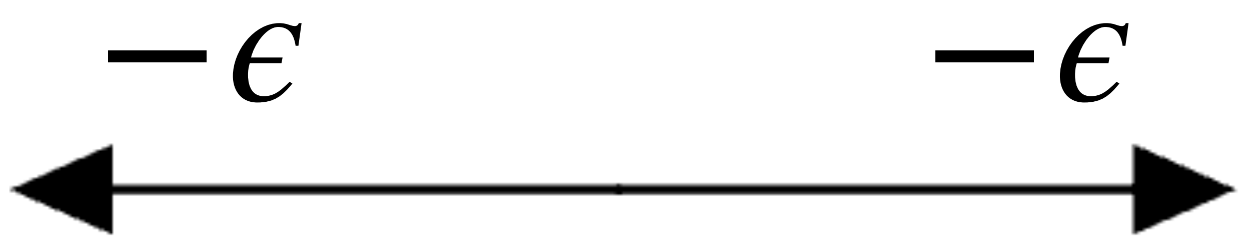
Our interest in the tropical integrand is due to the following result [Arkani-Hamed et al, 2202.12296]

If $\mathbf{s} \geq \mathbf{0}$, the Euler integral $\displaystyle\int_{\mathbb{R}^n_{\geq 0}} \frac{d\alpha}{\alpha} \mathcal{J}$ is finite if $\text{Trop } \mathcal{J} < 0$

Example:

$$\mathcal{J} = \left(\frac{\alpha}{1+\alpha}\right)^{\epsilon} \left(\frac{1}{1+\alpha}\right)^{\epsilon}$$

$\text{Trop } \mathcal{J} =$



So the integral is finite for $\epsilon > 0$ and diverges for $\epsilon \to 0$

Local Finiteness

In fact a stronger result holds [GS, 2406.14606]:

A *combination* $\mathcal{I} = \sum_i \mathcal{I}_i$ of Euler integrands is *locally* finite if $\text{Trop } \mathcal{I}(\rho) = a + \mathcal{O}(\epsilon),\ a < 0$ on all vectors ρ

I will not go through the proof of the above in this talk

But I will explain the basic notions of tropical geometry on which the proof builds

This will also give us the necessary intuition to understand the construction of the subtraction scheme

Newton Polytopes

The previous result shows the importance of understanding the tropical integrand

For a single Euler integrand it turns out that it is entirely characterized by a geometrical object: the *Newton Polytope*

Consider a polynomial $P = \sum_{\mathbf{m}\in\mathbb{Z}^n} s_{\mathbf{m}}\alpha^{\mathbf{m}}, \alpha^{\mathbf{m}} := \prod_i \alpha_i^{m_i}$, then we define Newt $P = \mathrm{Conv}\{\mathbf{m}\ s.t.\ \mathbf{s_m} \neq 0\}$

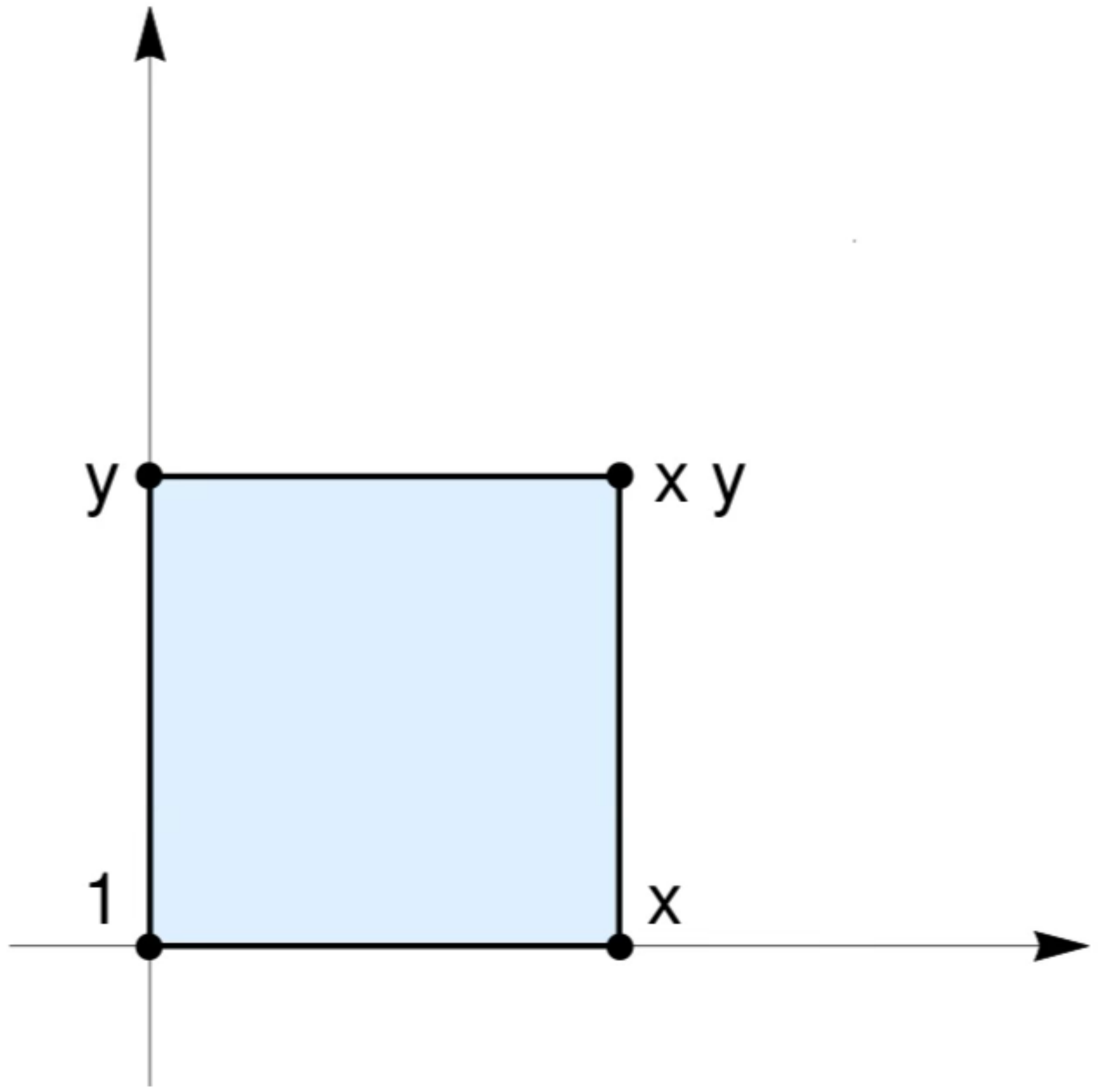
Like all polytopes, the Newton polytope admits a dual presentation as Newt $\mathrm{P} = \{z, d_\rho - \rho \cdot z \geq 0\}$

We have that Trop $P(\rho) = d_\rho$

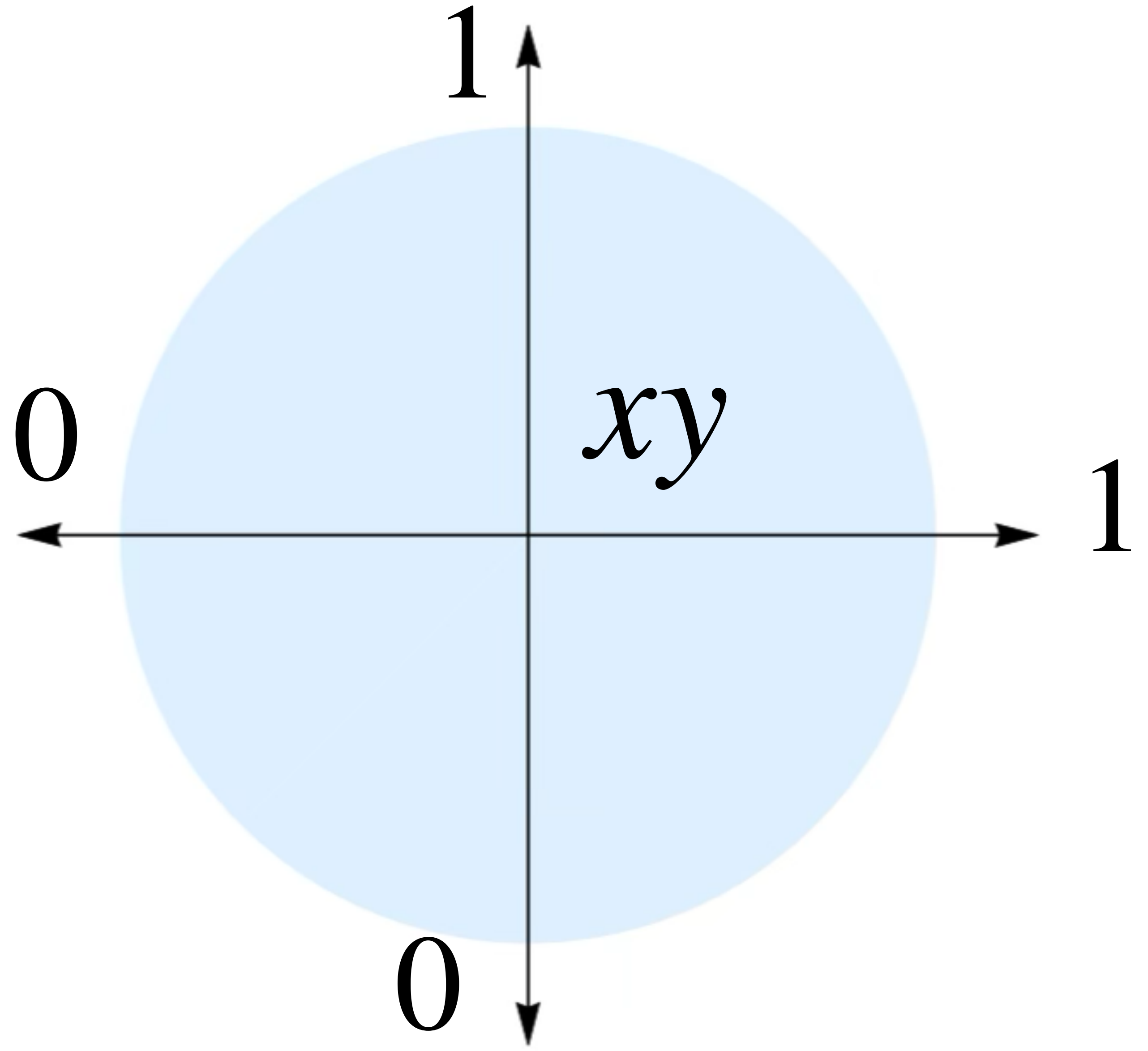
Newton Polytopes

$P = 1 + x + y + xy \Rightarrow \text{Newt } P = \{x \geq 0,\ 1 - x \geq 0,\ y \geq 0,\ 1 - y \geq 0\}$, $\text{Trop } P = \max(0, x, y, x + y)$

$\text{Newt } P =$



$\text{Trop } P =$



It turns out that Trop P is piecewise linear on the *normal fan of* Newt P

This is the collection of cones σ formed by linear functionals $\rho \cdot z$ extremized at a common face $F_\sigma \subset \partial$ Newt P

Newton Polytope & Local Behaviour

We will introduce a further notion that will play an important role in the construction of counter-terms

$$P = \sum_{\mathbf{m} \in \text{Newt P}} s_{\mathbf{m}} \alpha^{\mathbf{m}} \Rightarrow P|_\rho = \sum_{\mathbf{m} \in (\text{Newt P})|_\rho} s_{\mathbf{m}} \alpha^{\mathbf{m}}$$

Boundary of P where $\rho \cdot y$ is extremized

In other words, $P|_\rho$ is obtained by keeping only the monomials of P that lie on the face F_ρ of the polytope where ρ is extremized

The restriction (or *initial form*) of P captures its behaviour around the boundary $\alpha_i \sim \lambda^{-\rho_i}$, $\lambda \to 0$

$$\boxed{P(\alpha_i \lambda^{-\rho_i}) = \lambda^{-\text{Trop } P(\rho)} P|_\rho (1 + \mathcal{O}(\lambda))}$$

Back to tropicalization

Understanding the tropicalization of a *combination* of integrands $\mathcal{I} = \sum_i \mathcal{I}_i$ is considerably harder

This is because, in general, we can only say that $\text{Trop }(f - g) \leq \max(\text{Trop } f, \text{Trop } g)$

To compute $\text{Trop } \mathcal{I}$ we need to introduce the operator τ_ρ which returns a Puiseux series via

$$\tau_\rho \mathcal{I}(\alpha\lambda^{-\rho}) = \tau_\rho \lambda^{-\text{Trop } \mathcal{I}(\rho)} \mathcal{I}\,|_\rho (1 + \mathcal{O}(\lambda)) = \lambda^{-\text{Trop } \mathcal{I}(\rho)} \mathcal{I}\,|_\rho \tau_\lambda (1 + \mathcal{O}(\lambda))$$

We compute $\text{Trop } \sum_i \mathcal{I}_i$ by applying τ_ρ to each integrand and collecting the leading term in the resulting series

Commutativity of Series

The operators τ_ρ satisfy an important property

We have that $\tau_\rho$and $\tau_{\rho'}$ commute when acting on $\mathscr{I}$ if and only if the vectors ρ, ρ' are *compatible with respect to* Newt $\mathscr{I}$

(Reminder: two vectors are compatible if they are extremized at a common face)

This is essentially follows from the same property being satisfied by the operation $\mathscr{I} \to \mathscr{I}|_\rho$

This trivially follows from the fact that "restricting on a face common to several facets" does not depend on the order of the facets

Note: the above property is crucial in ***Sector Decomposition,*** and will be in our subtraction scheme

Outline

Setup

Let $\mathcal{I}$ be an Euler integrand and suppose that it has only *logarithmic divergences,* i.e. Trop $\mathcal{I} = a + \mathcal{O}(\epsilon),\ a \leq 0$

By applying Nilsson-Passare analytical continuation we can always reduce to this case

Let $\Sigma^{\text{div}}(1)$ be the rays which are normal to Newt $\mathcal{I}$ on which $a = 0$, i.e. the divergent directions

We will denote by Σ^{div} the collection of cones formed by these divergent directions

Let us assume that for all $\rho \in \Sigma^{\text{div}}(1)$ we can find a vector w_ρ such that $w_\rho \cdot \rho' = -\delta_{\rho,\rho'}$ for all $\rho' \in \Sigma^{\text{div}}(1)$ compatible with ρ

In general this is not the case, our subtraction scheme only applies to integrands satisfying this *geometric property*

Counter-terms

As a first guess, let us consider the counter-term $\mathcal{I}_\rho = \mathcal{I}|_\rho$ to fix a divergence associated with $\rho \in \Sigma^{\text{div}}(1)$

Accordingly, we set $\mathcal{I}^{\text{ren}} = \mathcal{I} - \mathcal{I}_\rho$, and ask ourselves if Trop $\mathcal{I}^{\text{ren}} = a + \mathcal{O}(\epsilon)$ with $a < 0$

While the condition holds on ρ, the counterterm is divergent on new directions which would require further counter-terms...

Let us modify the counter-term to $\mathcal{I}_\rho = \left(1 + \alpha_\rho^w\right)^{-1+\epsilon} \mathcal{I}|_\rho$

The extra factor $v_\rho = (1 + \alpha^{w_\rho})$ guarantees that the counter-term is divergent only on rays which are *compatible with* ρ

This we can fix by adding further counter-terms that corresponds to *higher dimensional cones of* Σ^{div}

The renormalization map

Given an Euler integrand $\mathscr{I}$ satisfying the geometric property, we define

$$\mathscr{I}^{\mathrm{ren}} = \sum_{\sigma\in\Sigma^{\mathrm{div}}} (-1)\left(\prod_{\rho\in\mathrm{Rays}\ \sigma} v_\rho^{1+\epsilon}\right)\mathscr{I}\,|_\sigma$$

The first main result is that the above is *locally finite*

A sketch of the proof

As a toy example consider a case with $\Sigma^{\text{div}} = \{0, \rho_1, \rho_2, (\rho_1, \rho_2)\}$

$$\mathscr{I}^{\text{ren}} = \mathscr{I} - \mathscr{I}_{\rho_1} - \mathscr{I}_{\rho_2} + \mathscr{I}_{(\rho_1,\rho_2)}$$

Computing Trop $\mathscr{I}^{\text{ren}}$ on any ray $\rho \in (\rho_1, \rho_2)$ requires applying τ_ρ collecting the leading term of the resulting series

The terms of order $\mathcal{O}(\epsilon)$ cancel, because the integrands all have same initial form: $\mathscr{I}\,|_{(\rho_1,\rho_2)}$

It follows that the leading term is of order $a + \mathcal{O}(\epsilon)$, with $a < 0$

Integrating the counter-terms

The counter-terms are simpler than the original integrand, which allows to integrate them exposing the poles in ϵ

This is due to the fact that their Newton polytopes are *lower dimensional,* by construction

This implies that $\mathcal{I}\,|_{\rho}$ is homogeneous under rescaling $\alpha \to \alpha\lambda^{-\rho}$

The integral in the λ direction can be performed trivially

Final Result

Theorem 6 (Subtraction Formula)**.** *Consider a logarithmically divergent Euler integrand $\mathcal{I}$ satisfying the geometric property 1 with vectors w_ρ. Let $\Sigma = \text{Fan Newt } \mathcal{I}$ be its normal fan, and Σ^{div} be the fan formed by taking positive spans of those compatible rays of Σ on which $\text{Trop } \mathcal{I} = \mathcal{O}(\epsilon)$. For each cone $\sigma \in \Sigma^{\text{div}}$, including $\mathbf{0}$, define*

$$v_{\sigma,w} = \prod_{\rho \in \text{Rays } \sigma} \left(\frac{1}{1 + \alpha^{w_\rho}} \right)^{1 - \text{Trop } \mathcal{I}(\rho)},$$

and

$$\mathcal{I}_\sigma^{\text{ren}} = v_\sigma^{-1} \sum_{\sigma' \in \Sigma^{\text{div}}|_\sigma} (-1)^{\dim(\sigma')} v_{\sigma'} \mathcal{I}|_{\sigma'}.$$

List the vectors $\text{Rays } \sigma$ as columns of a matrix ρ^σ, and choose an index $J_\sigma \subseteq (1, \ldots, n)$ of cardinality $n - m$, such that its complement, $J_\sigma^c = (1, \ldots, n) \setminus J_\sigma$, labels a non-zero minor $\det(\rho^\sigma_{J_\sigma^c})$ of ρ^σ. Let $\pi_\sigma : \mathbb{R}^n \to \mathbb{R}^{n-m}$ be the linear map that keeps only the entries labelled by J_σ. Let $u_\sigma = \sum_{\rho \in \text{Rays } \sigma} \text{Trop } \mathcal{I}(\rho)\ \pi_\sigma(w_\rho)$. Set $\text{Vol}(\sigma) := \det(\rho^\sigma_{J_\sigma^c}) \prod_{\rho \in \text{Rays } \sigma} (-\text{Trop } \mathcal{I}(\rho))^{-1}$. Then

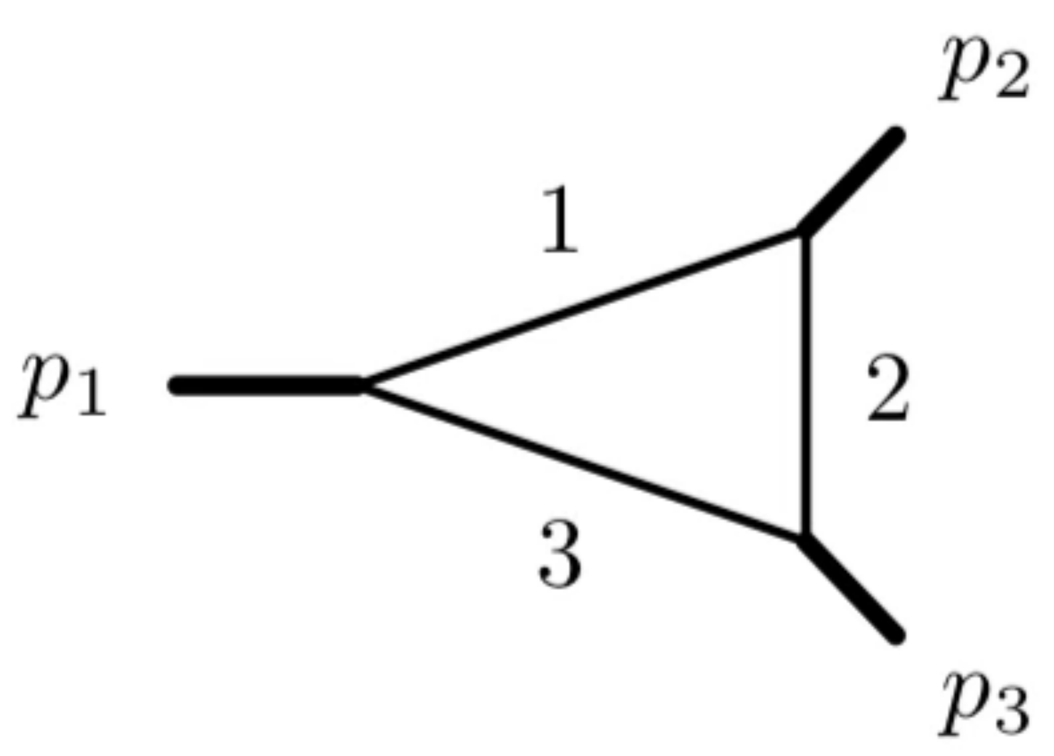
$$\int_{\mathbb{R}_+^n} \frac{d\alpha}{\alpha} \mathcal{I} = \sum_{\sigma \in \Sigma^{\text{div}}} \text{Vol}(\sigma) \int_{V_\sigma} \frac{d\alpha}{\alpha} \alpha^{u_\sigma} \mathcal{I}_\sigma^{\text{ren}}, \tag{6.19}$$

where $V_\sigma = \{\alpha_j \geq 0\}_{j \in J_\sigma} \cap \{\alpha_i = 1\}_{i \in J_\sigma^c}$ and all integrands are locally finite.

Outline

1. Tropical Geometry 101
2. A subtraction scheme
3. **Some examples**

Simplest Example



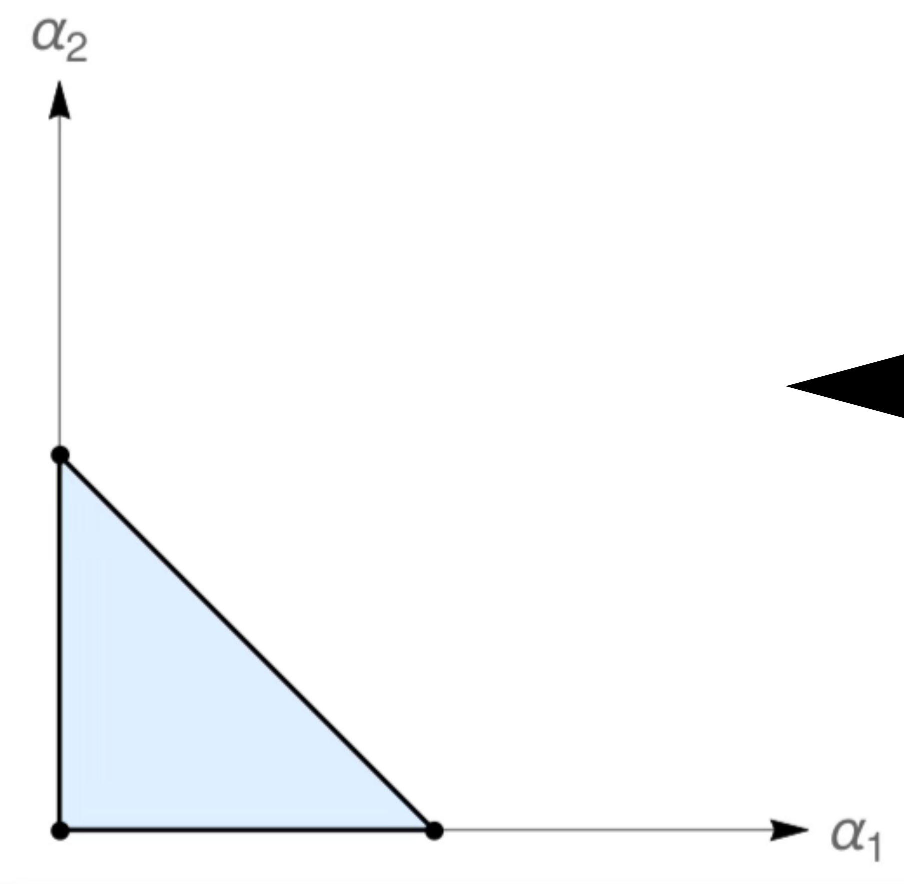
$$I_G(p_1^2,p_2^2,p_3^2;\epsilon) = \Gamma(d_G)\int_0^\infty \frac{d\alpha}{\alpha}\mathcal{I}_G(p_1^2,p_2^2,p_3^2;\epsilon)$$

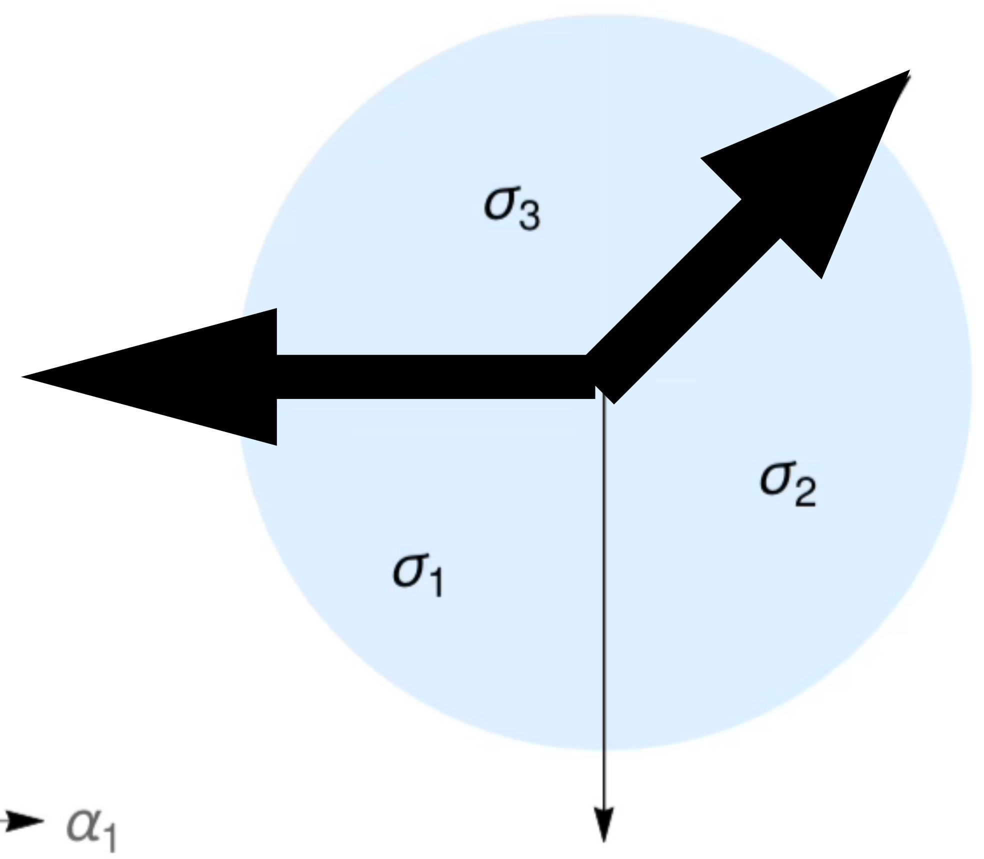
$$= \Gamma(d_G)\int_0^\infty \frac{d\alpha}{\alpha}\alpha\,(1+\alpha_1+\alpha_2)^{d_G-\frac{D}{2}}\,\left(\alpha_1 p_1^2+\alpha_1\alpha_2 p_2^2+\alpha_2 p_3^2\right)^{-d_G}$$

$$p_2^2 = p_3^2 = 0$$

$$w_{\rho_1} = (-1,1) \quad \Rightarrow \quad v_{\rho_1} = \frac{1}{1+\alpha_2/\alpha_1}$$

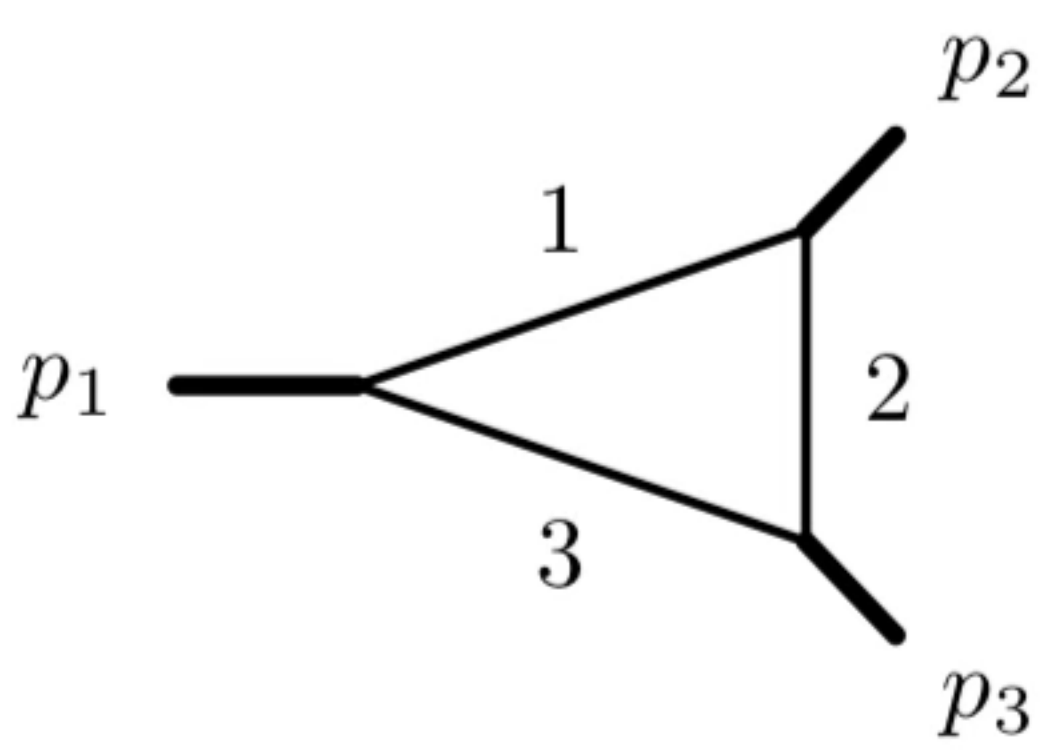
$$w_{\rho_3} = (0,1) \quad \Rightarrow \quad v_{\rho_1} = \frac{1}{1+\alpha_2}$$





$\sigma \in \Sigma^{\mathrm{div}}$	$\mathrm{Vol}(\sigma)$	$\mathcal{I}\vert_\sigma$	V_σ	u_σ
$\mathbf{0}$	1	$(p_1^2)^{-(1+\epsilon)}\alpha_1^{-\epsilon}\alpha_2(1+\alpha_1+\alpha_2)^{-1+2\epsilon}$	(α_1,α_2)	(0,0)
ρ_1	ϵ	$(p_1^2)^{-(1+\epsilon)}\alpha_1^{-\epsilon}\alpha_2(1+\alpha_2)^{-1+2\epsilon}$	$(1,\alpha_2)$	(0,1)
ρ_3	ϵ	$(p_1^2)^{-(1+\epsilon)}\alpha_1^{-\epsilon}\alpha_2(\alpha_1+\alpha_2)^{-1+2\epsilon}$	$(\alpha_1,1)$	(1,0)
$\mathrm{Span}_+\{\rho_1,\rho_3\}$	ϵ	$(p_1^2)^{-(1+\epsilon)}\alpha_1^{-\epsilon}\alpha_2^{2\epsilon}$	$(1,1)$	(0,0)

Simplest Example



$$I_G(p_1^2,p_2^2,p_3^2;\epsilon) = \Gamma(d_G)\int_0^\infty \frac{d\alpha}{\alpha}\mathcal{I}_G(p_1^2,p_2^2,p_3^2;\epsilon)$$
$$= \Gamma(d_G)\int_0^\infty \frac{d\alpha}{\alpha}\alpha\,(1+\alpha_1+\alpha_2)^{d_G-\frac{D}{2}}\left(\alpha_1 p_1^2+\alpha_1\alpha_2 p_2^2+\alpha_2 p_3^2\right)^{-d_G}$$

$$p_2^2 = p_3^2 = 0$$

$$w_{\rho_1} = (-1,1) \quad \Rightarrow \quad v_{\rho_1} = \frac{1}{1+\alpha_2/\alpha_1}$$
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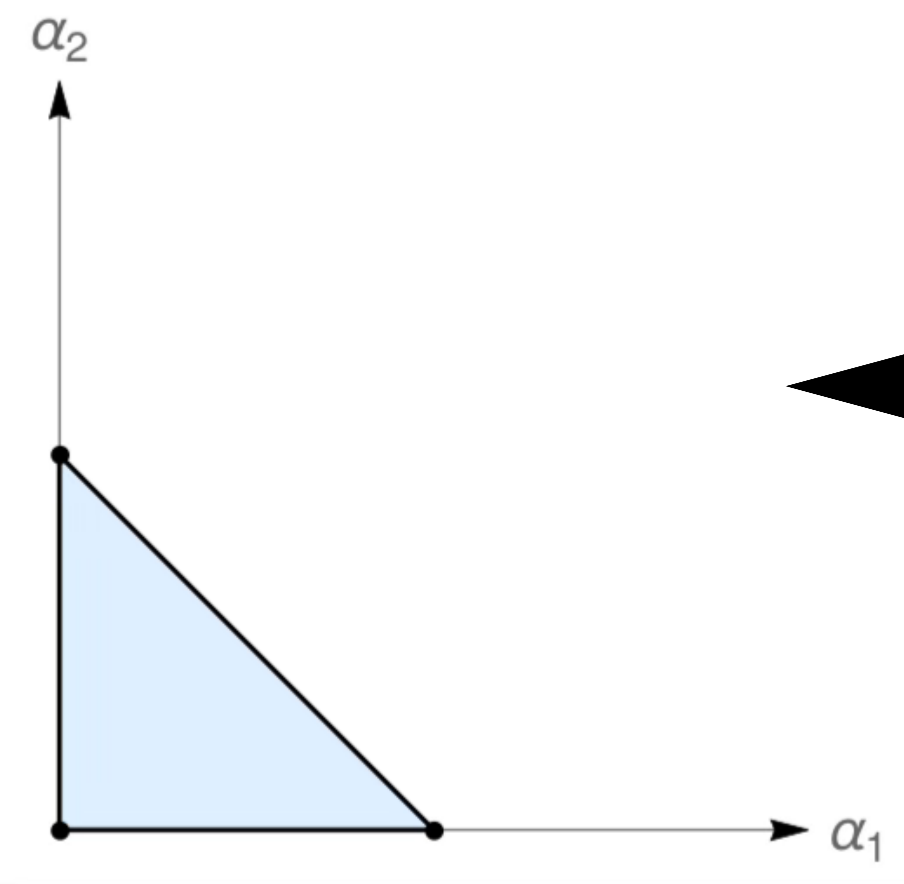
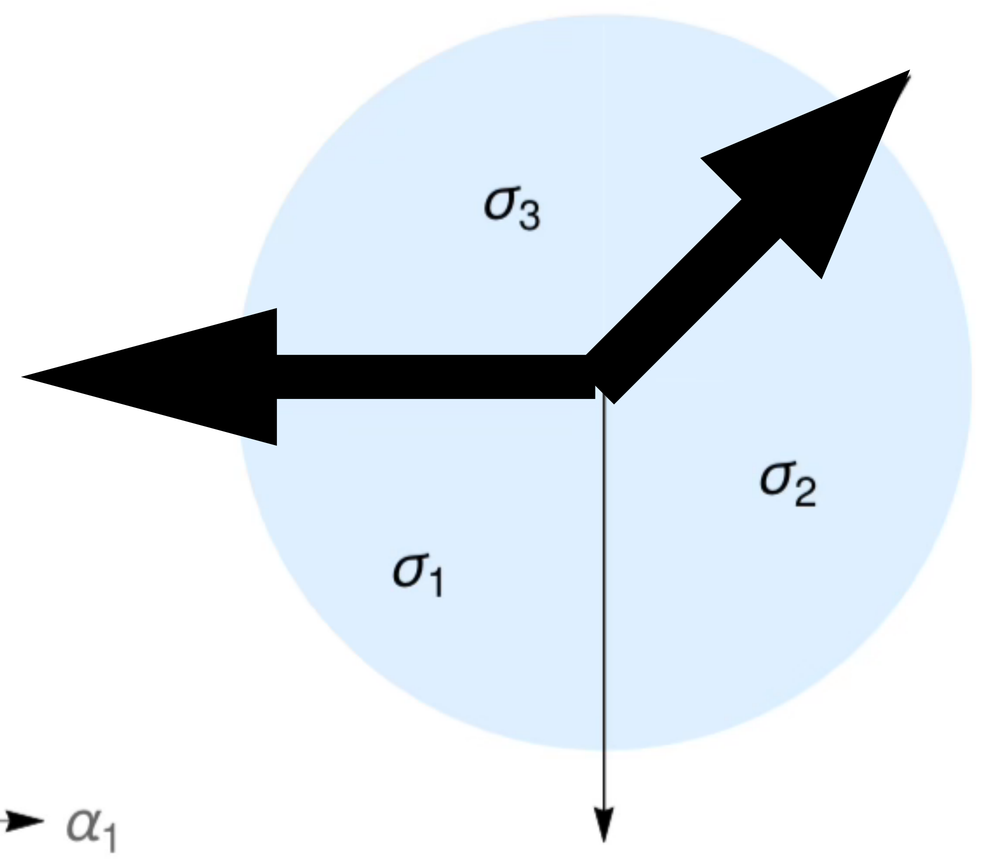




$$\frac{1}{\Gamma(d_G)}I_G(p_1^2,0,0;\epsilon) = \int_{V_0}\frac{d\alpha}{\alpha}\left(\mathcal{I}-(v_{\rho_1})^{1+\epsilon}\mathcal{I}|_{\rho_1}-(v_{\rho_3})^{1+\epsilon}\mathcal{I}|_{\rho_3}+(v_{\rho_1})^{1+\epsilon}(v_{\rho_3})^{1+\epsilon}\mathcal{I}|_{\text{Span }\{\rho_1,\rho_3\}}\right)$$
$$+\frac{1}{\epsilon}\int_{V_{\rho_1}}\frac{d\alpha}{\alpha}\alpha^{u_{\rho_1}}\left(\mathcal{I}|_{\rho_1}-(v_{\rho_3})^{1+\epsilon}\mathcal{I}_{\text{Span}_+\{\rho_1,\rho_3\}}\right)$$
$$+\frac{1}{\epsilon}\int_{V_{\rho_3}}\frac{d\alpha}{\alpha}\alpha^{u_{\rho_3}}\left(\mathcal{I}|_{\rho_3}-(v_{\rho_1})^{1+\epsilon}\mathcal{I}_{\text{Span}_+\{\rho_1,\rho_3\}}\right)$$
$$+\frac{1}{\epsilon^2}(p_1^2)^{-(1+\epsilon)}.$$

An angular integral

Angular integrals are important building blocks often appearing in phase-space integrations

In 2405.13120 was presented a parametrization in terms of Euler integrals. Here's a non-trivial example (4 propagators, one mass)

$$I(\mathbf{v};\epsilon) = \mathcal{N}\int_{\mathbb{R}_+^4} \frac{dt}{t} t^{(1,1,1,5/2)}(1+t_2)(1+t_3)^2(1+t_4)^{\epsilon+1/2}\mathcal{Q}^{-2} \qquad \mathcal{N} = \frac{2\pi}{1-2\epsilon}\frac{\Gamma(4)\Gamma(\frac{3}{2}-\epsilon)}{\Gamma(\frac{5}{2})\Gamma(-\epsilon-1)} = 8\pi\epsilon + \mathcal{O}(\epsilon^2)$$

$$\begin{aligned}\mathcal{Q} =& (t_1+1)^2(t_2+1)^2(t_3+1)^2+\\ & t_4\left[t_3v_{34} + 2t_1(t_2+1)(t_3+1)(t_2(t_3+1)v_{12} + t_3v_{13} + v_{14}) + 2t_2(t_3+1)(t_3v_{23}+v_{24}) + v_{44}\right]\end{aligned}$$

While the above does not immediately satisfies the geometric property, it can reduced by Nilsson-Passare to one that does.

The resulting integrals can be performed with HyperInt. For the most complicated part we find (full result available upon request)

$$\frac{1}{4\,(\mathrm{r1}-\mathrm{r2})\,\mathrm{v13}\,\mathrm{v23}}\Bigg(\mathrm{Log}[-\mathrm{r1}]\,\mathrm{Log}\Big[\frac{\mathrm{v14}}{\mathrm{v12}}\Big] - \mathrm{Log}[-\mathrm{r2}]\,\mathrm{Log}\Big[\frac{\mathrm{v14}}{\mathrm{v12}}\Big] - \mathrm{Log}[-\mathrm{r1}]\,\mathrm{Log}\Big[\frac{\mathrm{v44}}{2\,\mathrm{v24}}\Big] + \mathrm{Log}[-\mathrm{r2}]\,\mathrm{Log}\Big[\frac{\mathrm{v44}}{2\,\mathrm{v24}}\Big] - \mathrm{Mpl}[\{2\},\{1+\mathrm{r1}\}] +$$
$$\mathrm{Mpl}[\{2\},\{1+\mathrm{r2}\}] - \mathrm{Mpl}\Big[\{1,1\},\Big\{\frac{(1+\mathrm{r1})\,\mathrm{v13}}{\mathrm{v13}-\mathrm{v14}},\frac{\mathrm{v13}-\mathrm{v14}}{\mathrm{v13}}\Big\}\Big] + \mathrm{Mpl}\Big[\{1,1\},\Big\{\frac{(1+\mathrm{r2})\,\mathrm{v13}}{\mathrm{v13}-\mathrm{v14}},\frac{\mathrm{v13}-\mathrm{v14}}{\mathrm{v13}}\Big\}\Big] - \mathrm{Mpl}\Big[\{1,1\},\Big\{\frac{(1+\mathrm{r1})\,\mathrm{v23}}{\mathrm{v23}-\mathrm{v24}},\frac{\mathrm{v23}-\mathrm{v24}}{\mathrm{v23}}\Big\}\Big] +$$
$$\mathrm{Mpl}\Big[\{1,1\},\Big\{\frac{(1+\mathrm{r2})\,\mathrm{v23}}{\mathrm{v23}-\mathrm{v24}},\frac{\mathrm{v23}-\mathrm{v24}}{\mathrm{v23}}\Big\}\Big] + \mathrm{Mpl}\Big[\{1,1\},\Big\{\frac{2\,(1+\mathrm{r1})\,\mathrm{v34}}{2\,\mathrm{v34}-\mathrm{v44}},\frac{2\,\mathrm{v34}-\mathrm{v44}}{2\,\mathrm{v34}}\Big\}\Big] - \mathrm{Mpl}\Big[\{1,1\},\Big\{\frac{2\,(1+\mathrm{r2})\,\mathrm{v34}}{2\,\mathrm{v34}-\mathrm{v44}},\frac{2\,\mathrm{v34}-\mathrm{v44}}{2\,\mathrm{v34}}\Big\}\Big]\Bigg)$$

(Thanks to V. Smirnov and F.Wunder for spotting a few mistakes numerically)

Other Examples

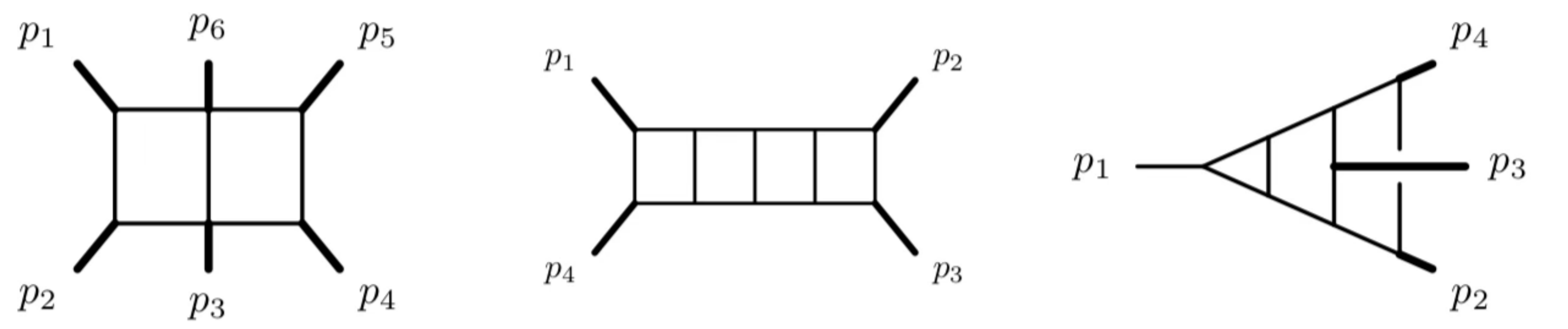
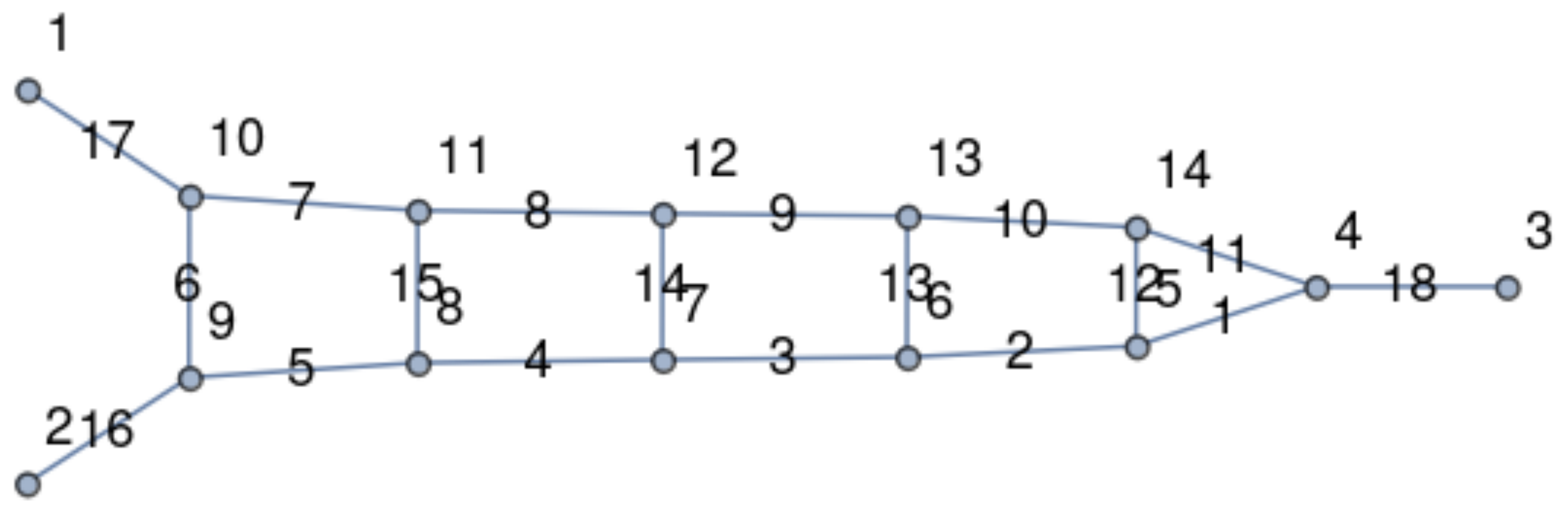


Figure 8: A collection of Feynman integrals which can be treated by our subtraction scheme. At least one of the solid lines must carry a momentum with $p_i^2 \neq 0$.



Future Directions

1. Beyond “Geometrical Property“?
2. Expansion in multiple regulators (disentangle logs in Method of Regions)?
3. How to compute the integrals? Tropical Sampling [Borinsky] / Differential Equations
4. Interaction with Curve Integral formalism (2309.15913): remove divergences locally without Feynman diagrams?

Thanks for your attention!