Beam coupling impedance of resonant cavities in non-ultrarelativistic regime

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Thanks to: Giovanni Rumolo, Leonardo Sito, Michela Neroni

25.07.2024 | CEI Section Meeting

Outline

- **Introduction**
- **Simulations of a pillbox cavity**
  - Study of the behavior of the quality factor with β
- **Simulations of the PSB’s FINEMET cavities**
  - Model and challenges
  - Impedance: results
- **Simulations of a cubic cavity with `wakis`**
  - Use of `wakis`
  - Wake potential and impedance magnitude
  - Real and imaginary parts of the impedance
- **Conclusions**
- **Next steps**

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Beam coupling impedance

- The **beam coupling impedance** describes the **interaction of a** particle **beam with the** surrounding **environment.**
- For a device of length l, the beam coupling impedance is defined as

$$Z_{\parallel} = -\frac{1}{q_0}\int_0^l E_s\, e^{jks}\, ds$$

$$Z_{x,y} = \frac{j}{q_0}\int_0^l [E_{x,y} - \beta Z_0 H_{y,x}]\, e^{jks}\, ds$$

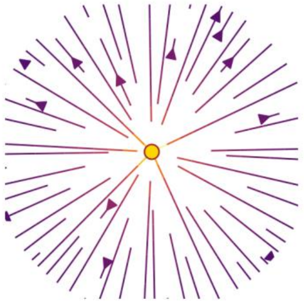
with $E_{s,x,y}$ and $H_{x,y}$ electric and magnetic induced fields in the frequency domain.

- This is the definition used by CST and `wakis`, but other definitions are possible, and they can include the normalization to the particle velocity (Sacherer definition).

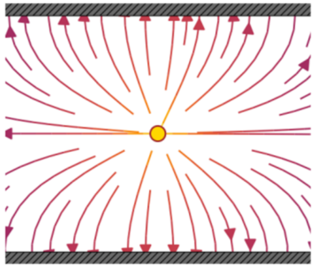
Space charge effects

- When $\beta < 1$, the **charged particles** of a beam **also create self-fields**, **that lead to space charge effects**:

  - **direct space charge (DSC):** interaction of the particles among each other in open space;



  - **indirect space charge (ISC)**: interaction of the particles among each other due to the external environment (e.g., perfectly conducting walls).



- While indirect space charge is typically taken into account directly in the impedance model, the **direct space charge impedance has to be removed**.

[Kevin Li, Collective effects – an introduction]

Electromagnetic simulations for non-ultrarelativistic beams

- For **ultrarelativistic beams**, the **reliability of CST** has been **extensively proved**.
- But **CST can't discriminate between the fields** induced by the beam, so the simulated beam coupling impedance of a device under test (DUT) is

$$\boldsymbol{Z^{tot}(\beta) = Z(\beta) + Z^{SC}(\beta)}$$

where

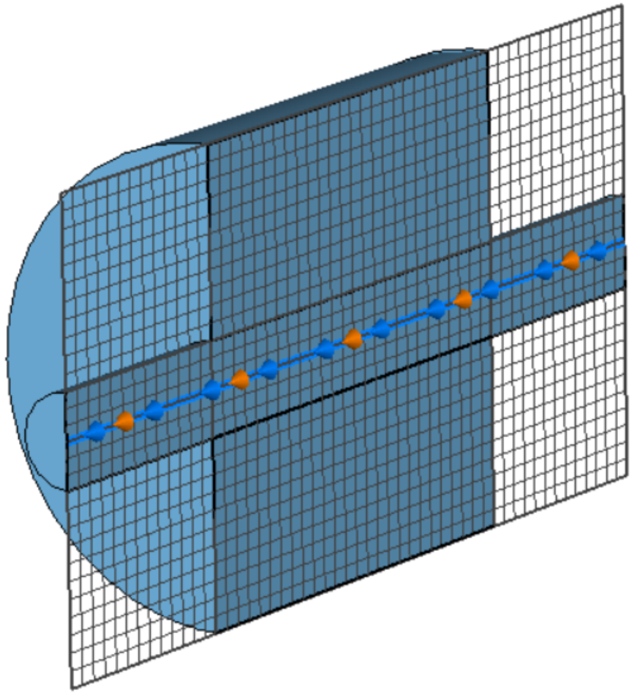
  - $\mathbf{Z(\beta)}$ is the DUT's impedance, which includes the indirect space charge impedance,
  - $\mathbf{Z^{SC}(\beta)}$ is the direct space charge impedance.

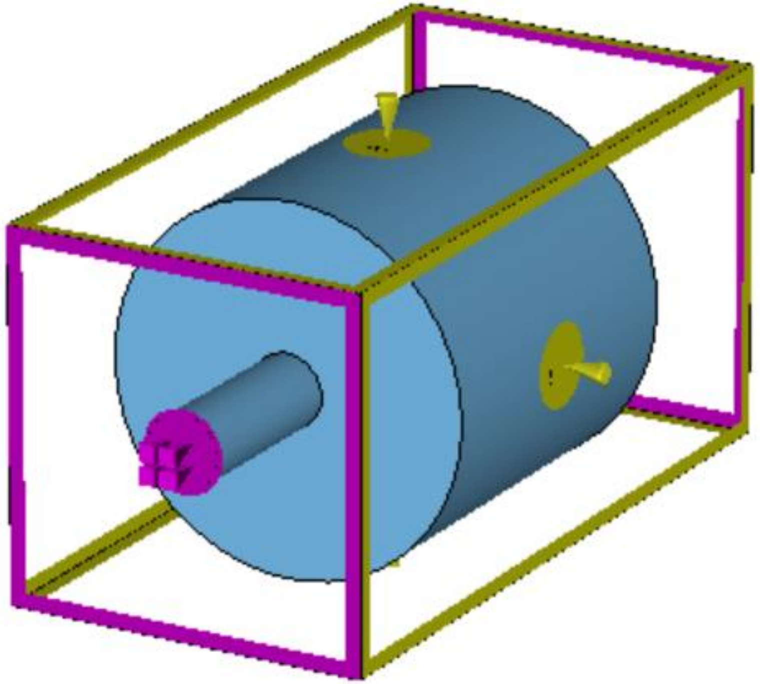
- For $\boldsymbol{\beta = 1}$ it results $Z^{SC}(\beta) = 0$.

- For **non-ultrarelativistic beams**, the main **complication** consists in **removing the contribution of the direct space charge** of the source bunch.

Simulations of the bounding box

- CST simulations take place within a delimited domain called ***bounding box***.
  - Since CST is a numerical solver, it discretizes the domain with a mesh grid.





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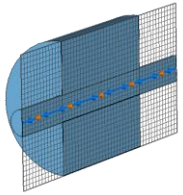
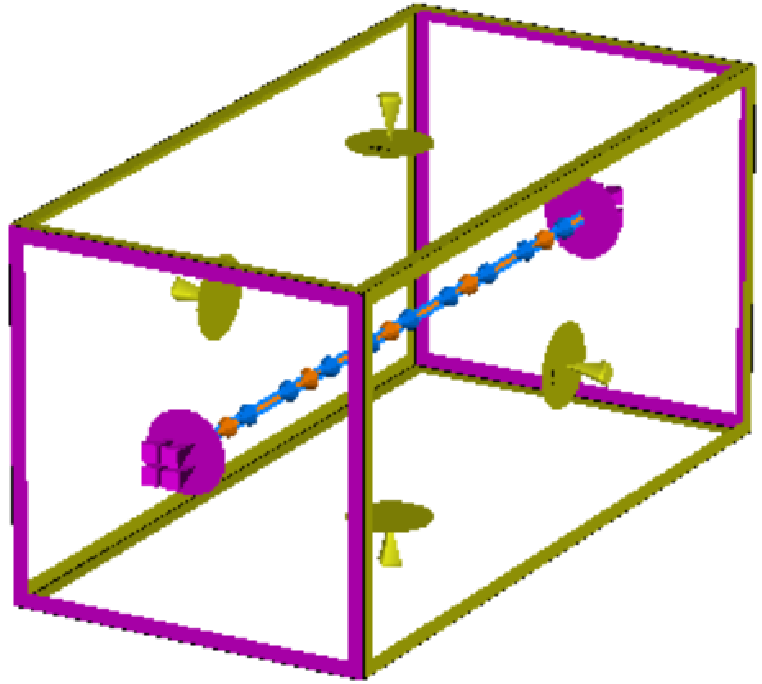




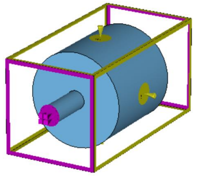
- The **bounding box** (bb) can be **simulated without changing its discretization**, by excluding all the elements of the DUT from the simulation.
- The resulting beam coupling impedance can be written as

$$Z_{bb}^{tot}(\beta) = Z_{bb}^{ISC}(\beta) + Z^{SC}(\beta)$$

where $Z_{bb}^{ISC}(\beta)$ is the indirect space charge impedance of the bounding box.

Numerical cancellation of $Z^{SC}(\beta)$ [1]

- Two simulations are run with the same mesh:
  1. Simulation of the device under test: $Z^{tot}(\beta) = Z(\beta) + Z^{SC}(\beta)$



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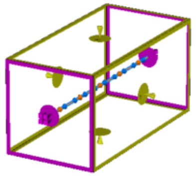
  2. Simulation of the bounding box: $Z_{bb}^{tot}(\beta) = Z_{bb}^{ISC}(\beta) + Z^{SC}(\beta)$



to **remove $Z^{SC}(\beta)$ directly from simulations**: $=$

$$Z^{tot}(\beta) - Z_{bb}^{tot}(\beta) = Z(\beta) - Z_{bb}^{ISC}(\beta)$$

- $Z_{bb}^{ISC}(\beta)$ can be analytically calculated and removed.
- This **technique** can **also** be applied **directly to the wake potential**.
- Examples of applications can be found in the presentation done during the ABP-CEI Section Meeting of 16 May 2024.

[1] C. Zannini *et al.*, “Electromagnetic simulations for non-ultrarelativistic beams and applications to the CERN low energy machines”

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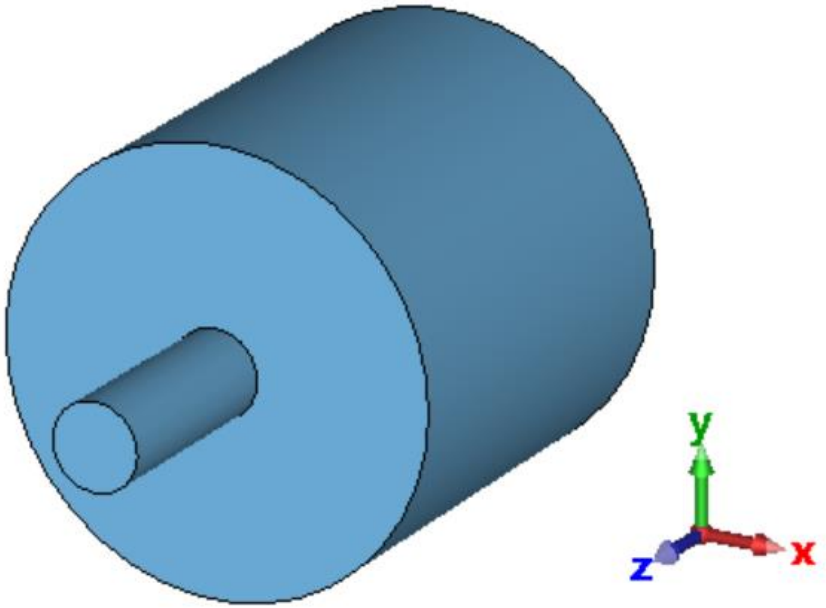
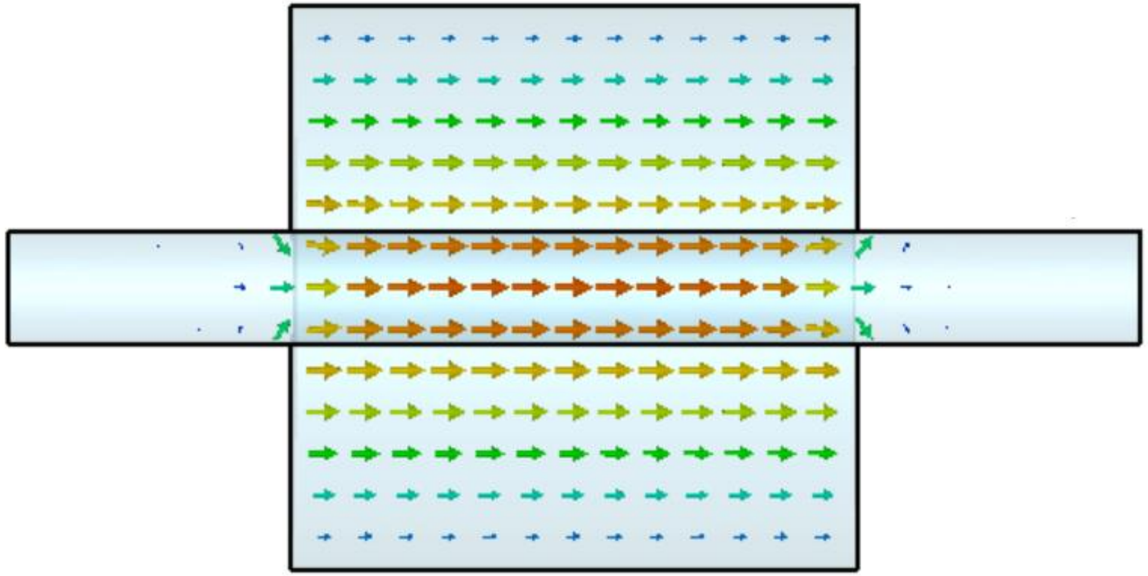
Pillbox cavity



Radius of the pipe	2 cm
Radius of the pillbox	10 cm
Length of the pipe	20 cm
Length of the pillbox	40 cm

Study of the first resonant mode: **TM010**



$f_{010} = 1.150\ GHz$

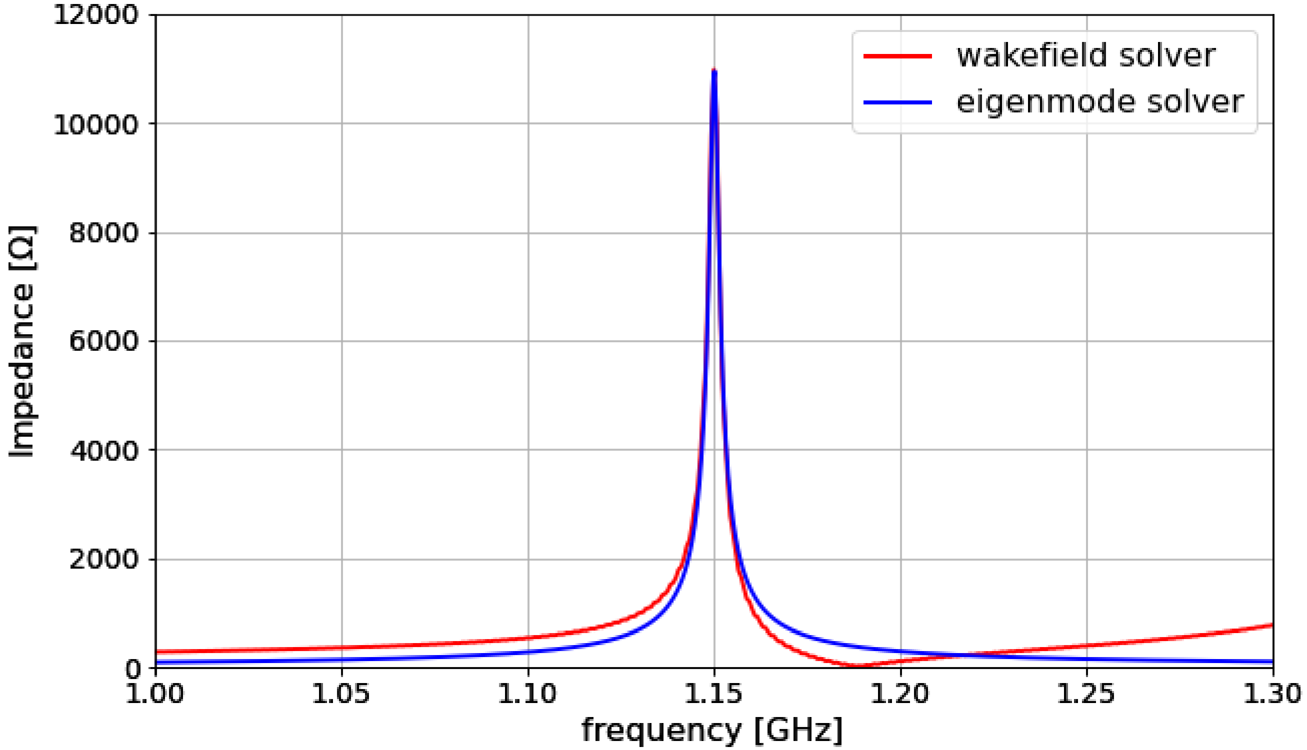
Eigenmode Solver vs Wakefield Solver

- Wakefield Solver (WF): directly provides the **impedance spectrum**.
- Eigenmode Solver (EM): provides three parameters.
  - **impedance spectrum reconstructed** based on the broad-band resonator model.

Eigenmode solver results	
Resonant frequency ω_r	**1.15 GHz**
Quality factor Q	**450**
Shunt impedance Rs	**21954 Ω**

$$Z(\omega) = \frac{R_s}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

broad-band resonator model

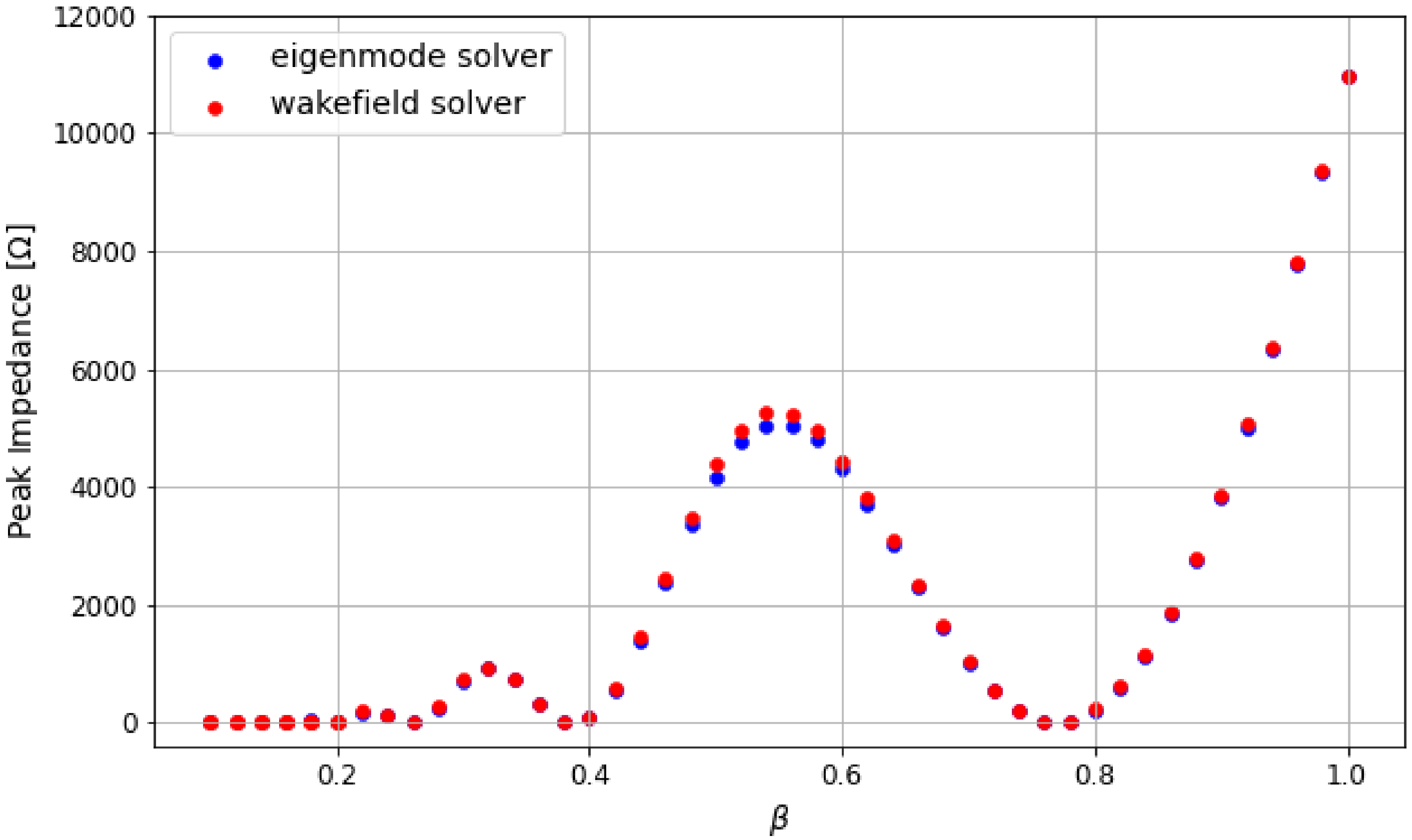


Longitudinal peak impedance varying β

- Parametric study of the real part of the impedance at f_{010} varying β.

- **Good agreement** between the two solvers:
  - Relative error < 5%

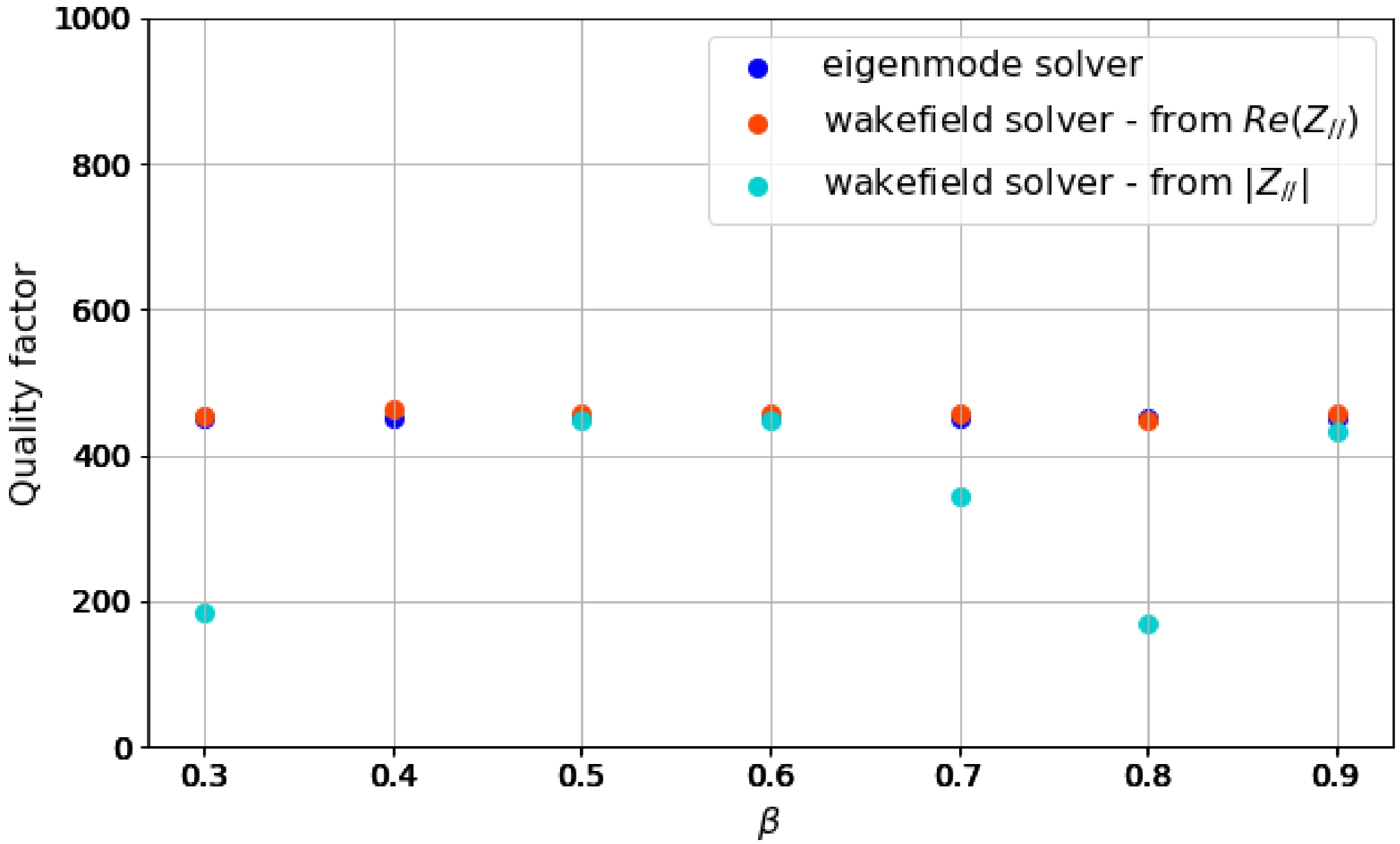
- This agreement was **not obvious** because in the **EM solver** the **particle velocity** is taken into account **only in post-processing**.



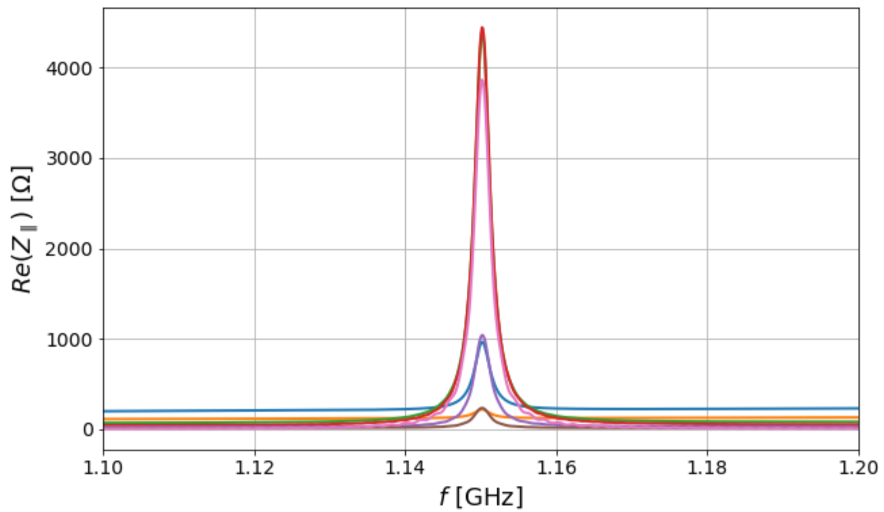
Quality factor varying β: computation from the real part of the longitudinal impedance

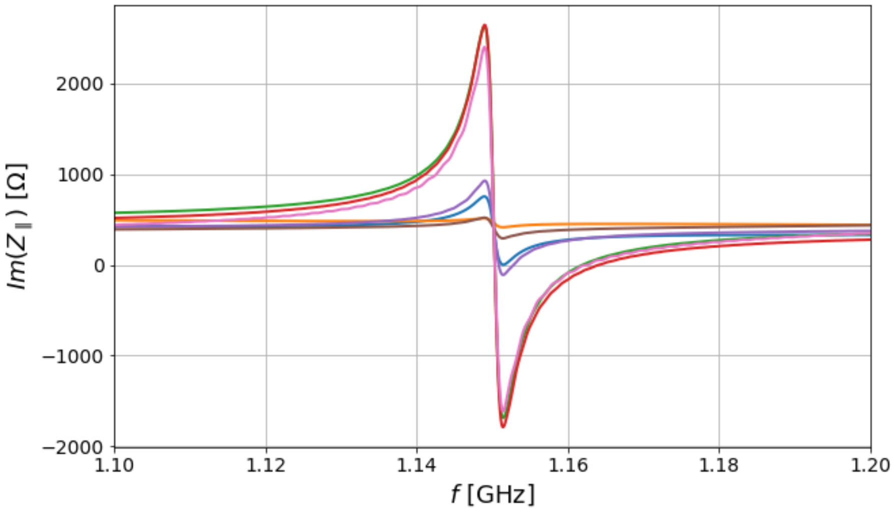
The quality factor of the beam coupling impedance can be computed from its real part or from its magnitude.

- **Eigenmode solver**: **Q is constant with β**.
- **Q** computed **from the real part**: **good agreement** between the two solvers.
- **Q** computed **from the magnitude**: **lower values**.

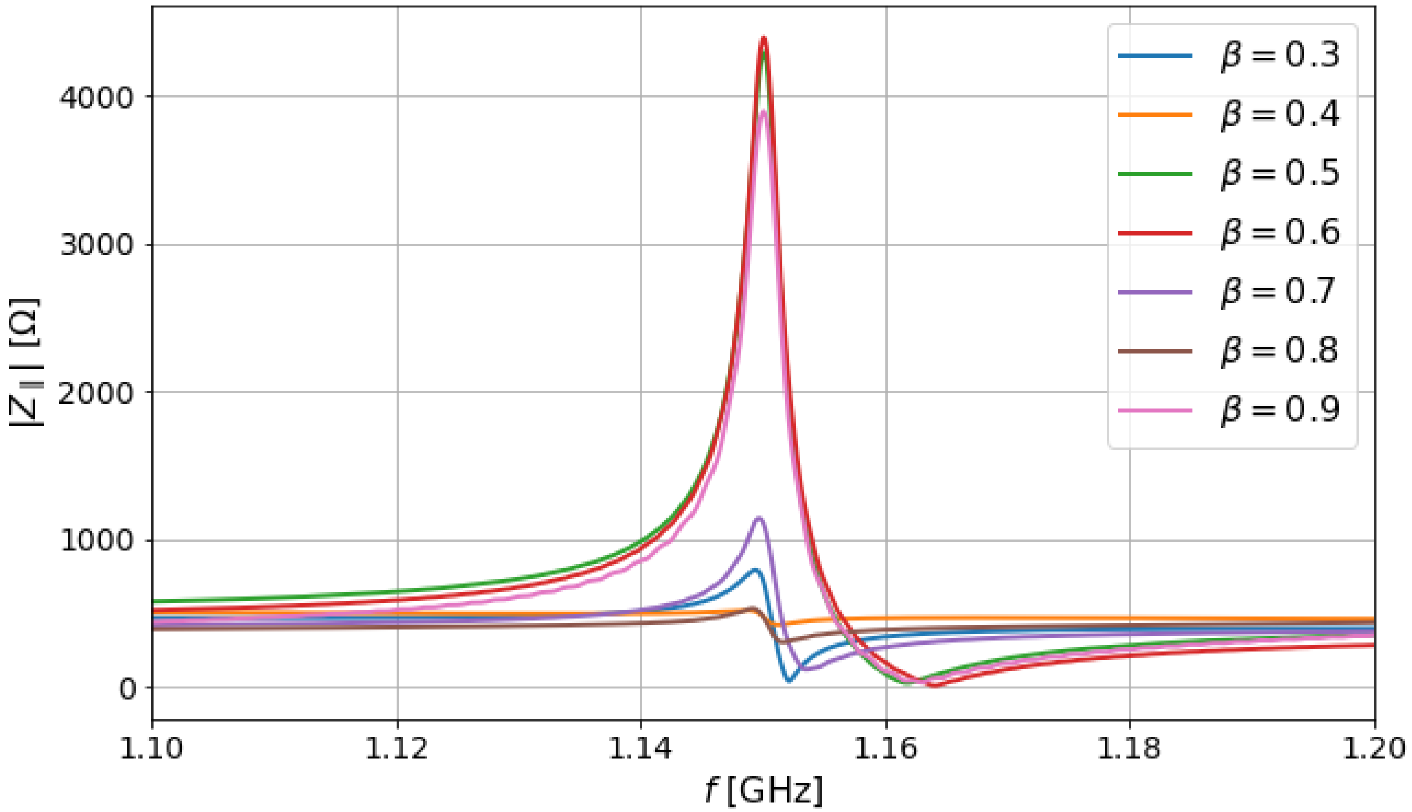


Wakefield solver: impedance spectrum varying β

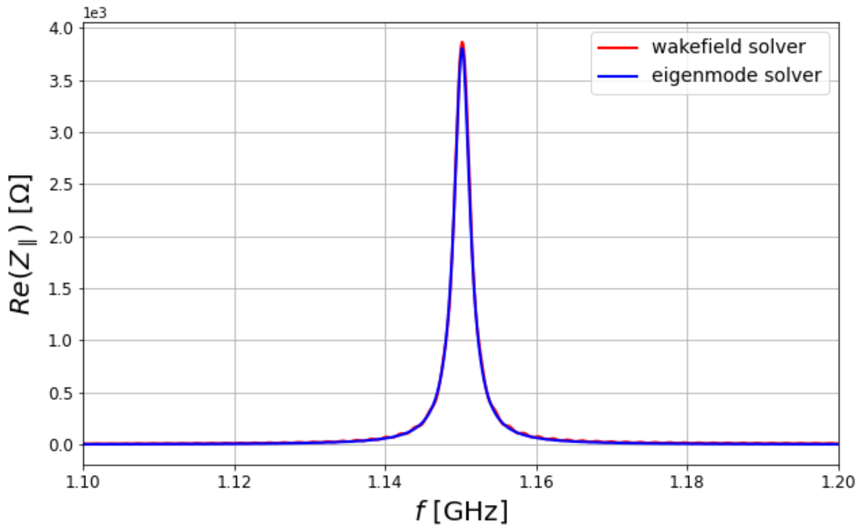


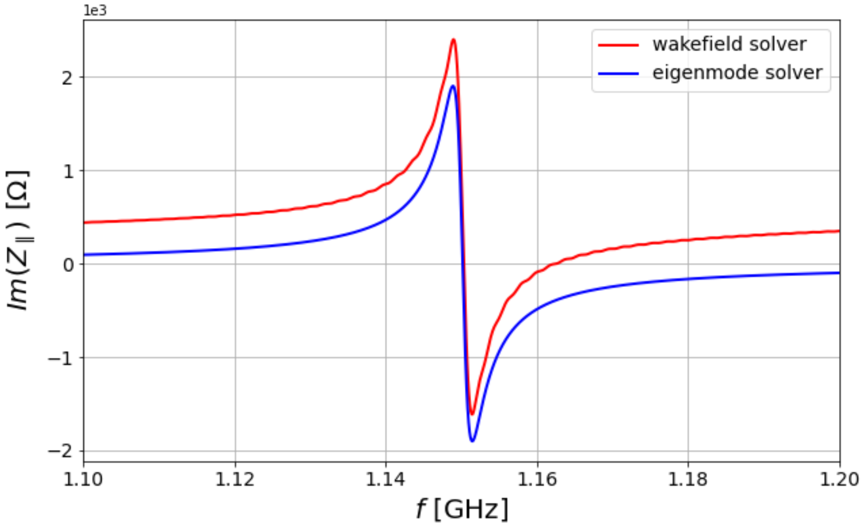


Indirect space charge is still present and **affects the imaginary part** of the impedance, thus altering its magnitude.

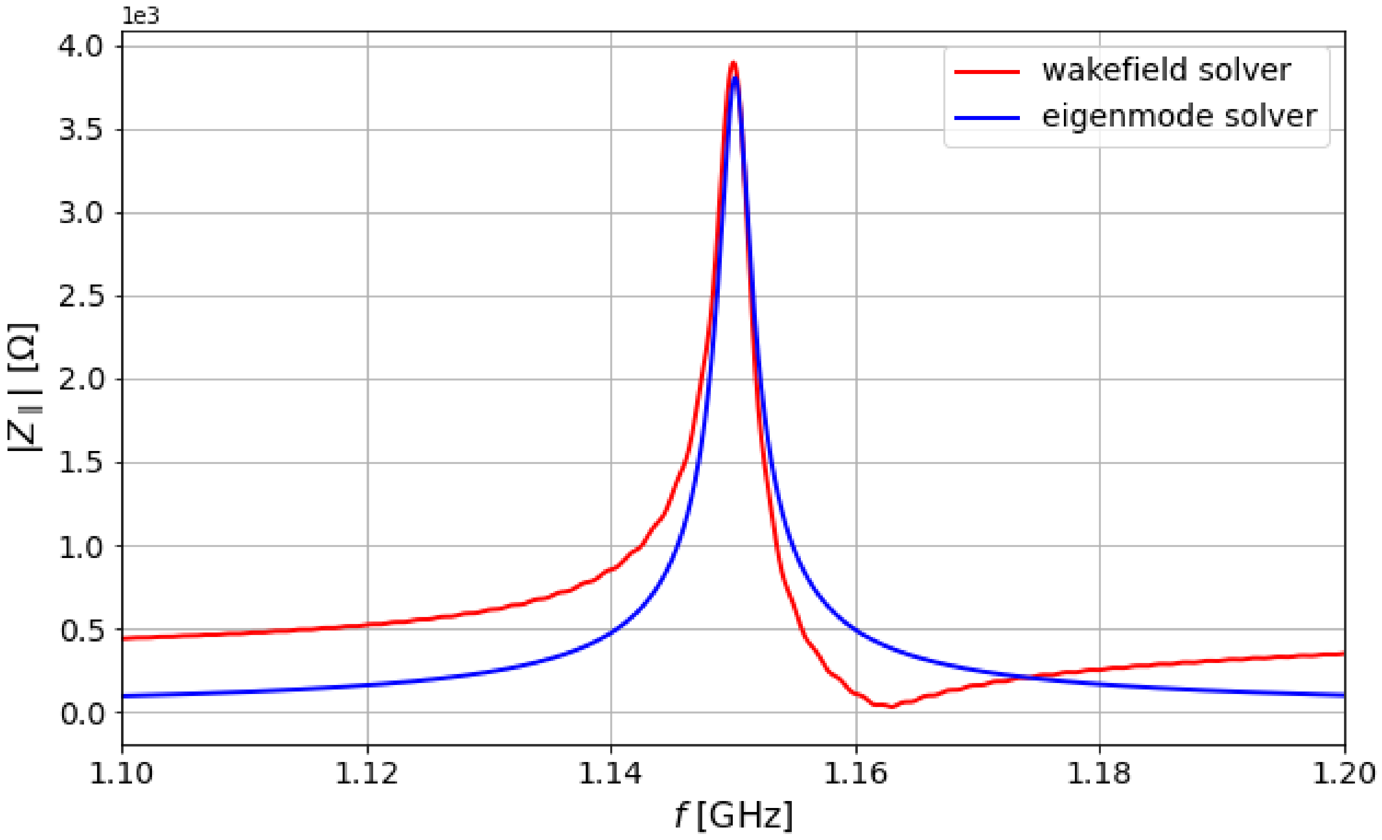


Impedance spectrum comparison: β=0.9

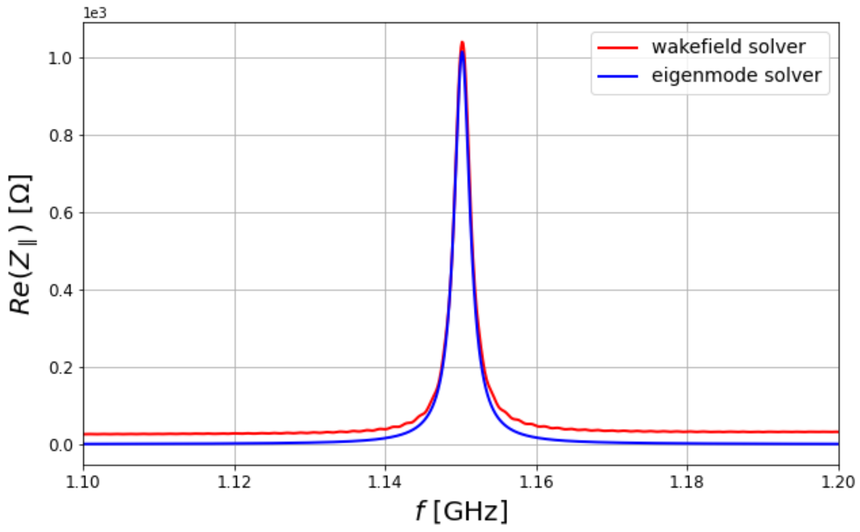


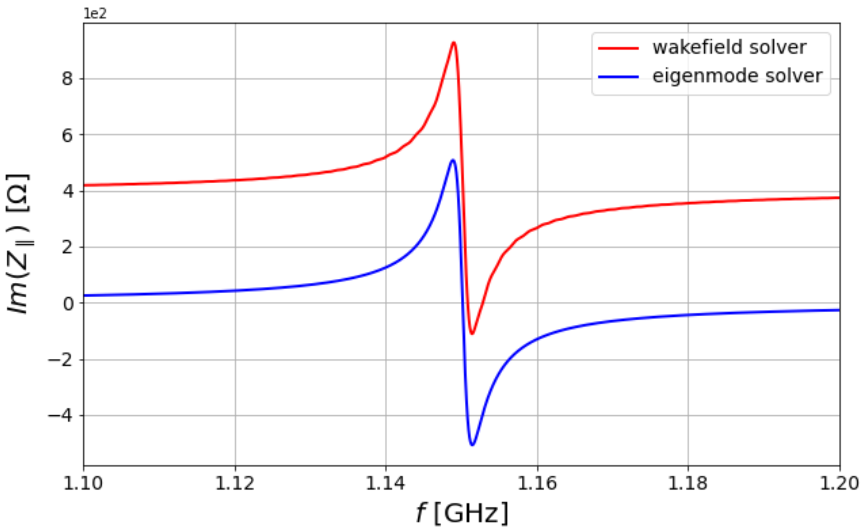


ISC is **low**: **agreement** between the two solvers.



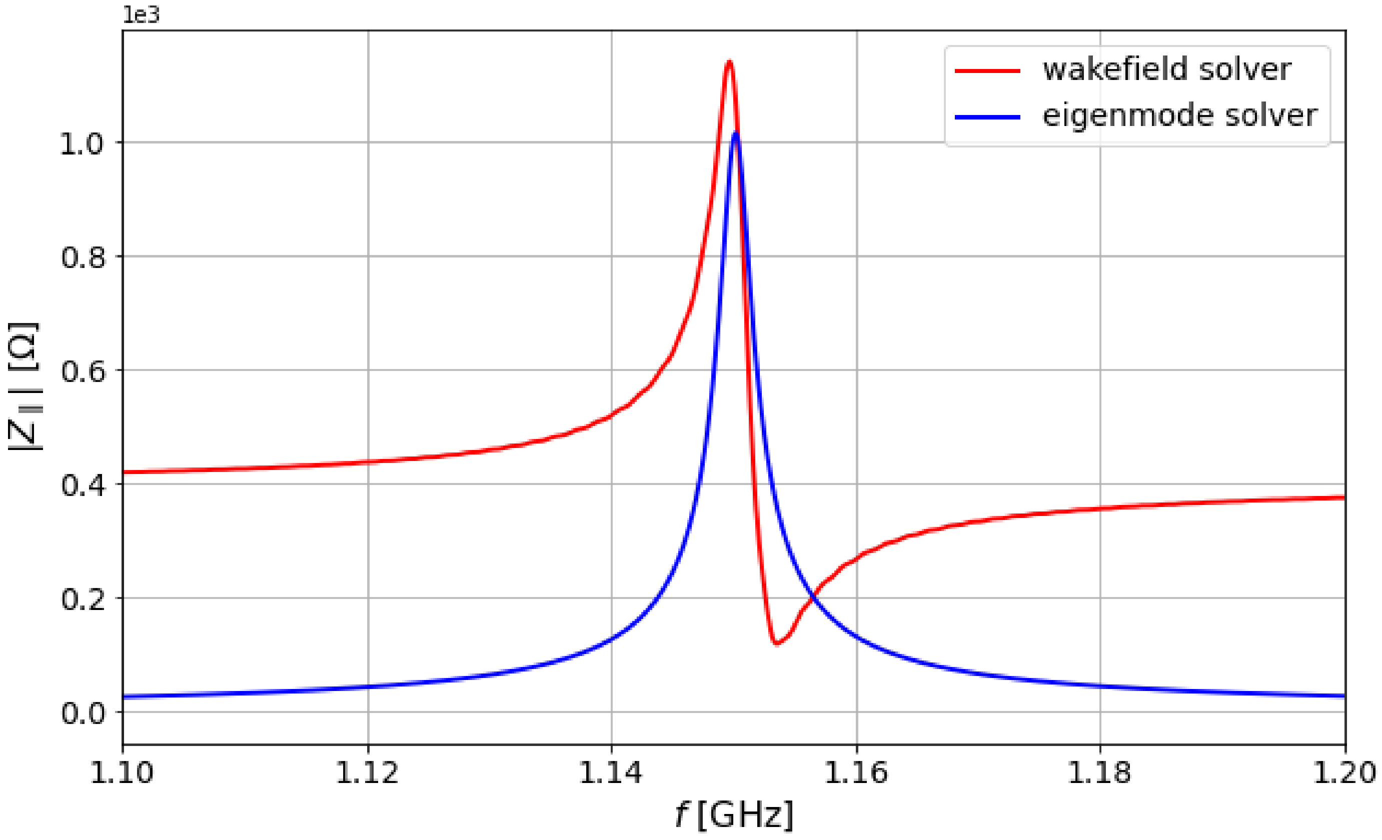
Impedance spectrum comparison: β=0.7





ISC is **higher**, the **shape** of the magnitude's curve is **corrupted**.

- Is Q computed from the magnitude still meaningful?



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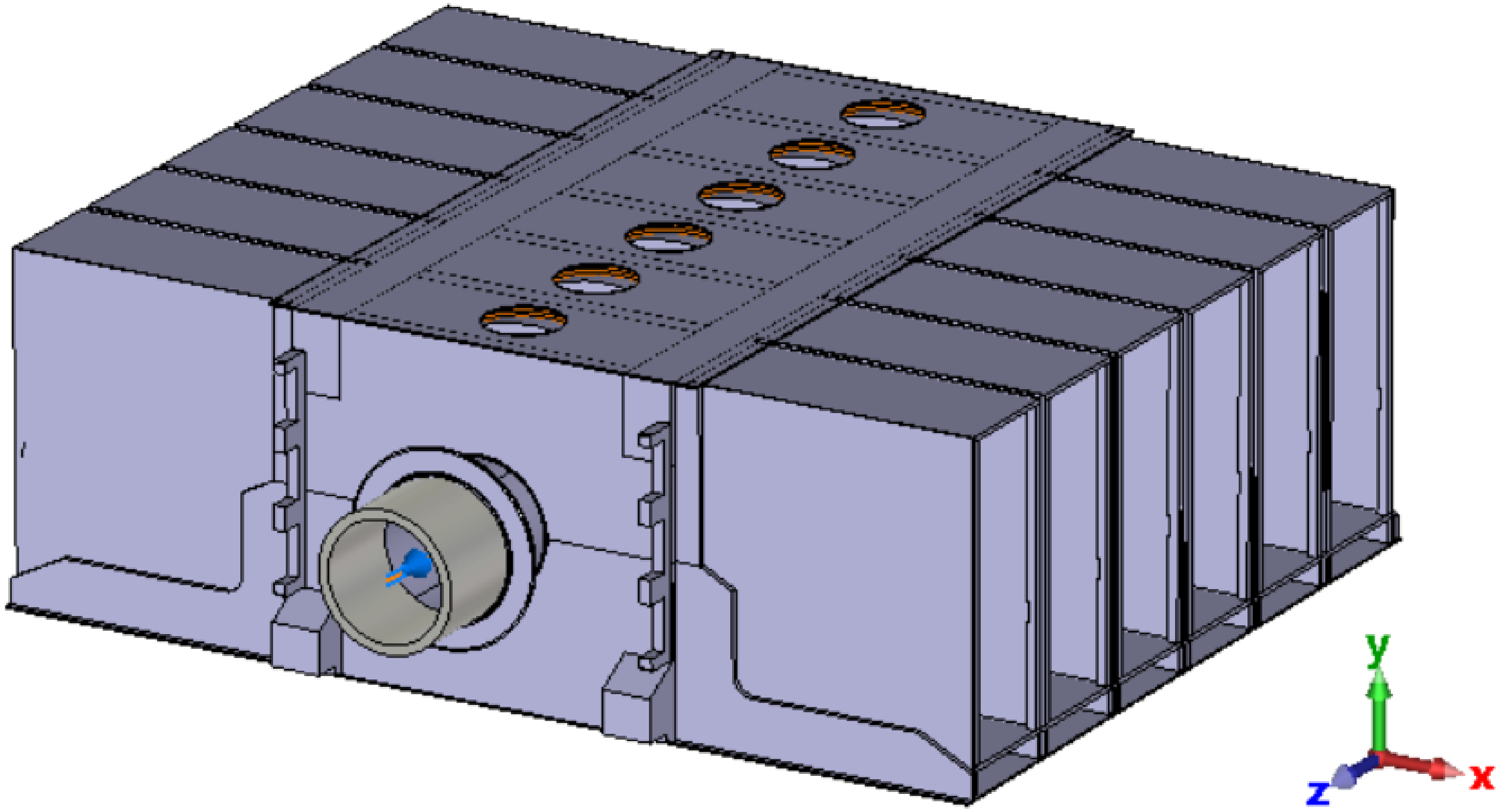
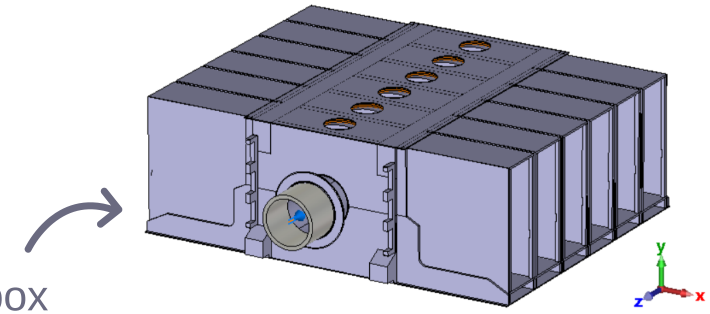
The PSB's FINEMET cavities

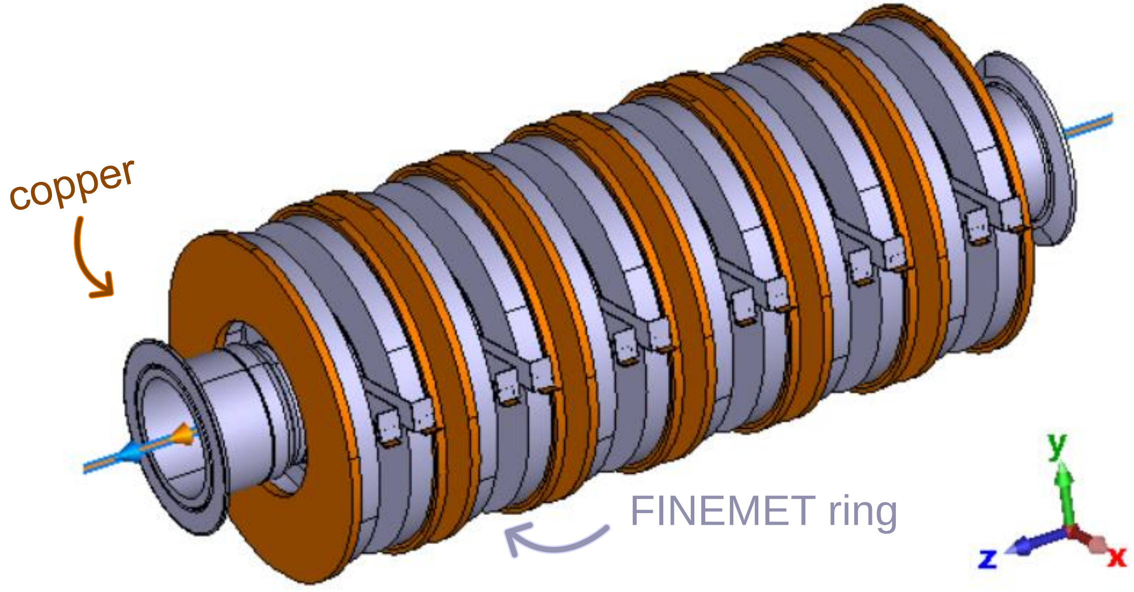
Study on the FINEMET cavities' **realistic 3D model**, imported from CATIA and simplified for electromagnetic simulations:



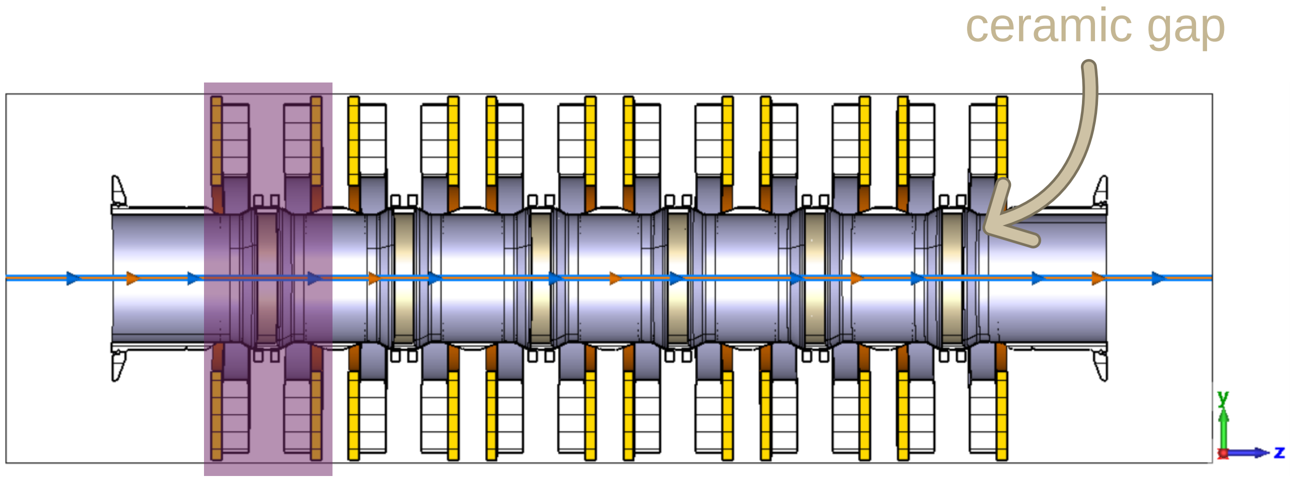
The PSB's FINEMET cavities

What is inside the metallic box:





One **cell**: vacuum chamber with ceramic gap at the center and a FINEMET ring on either side



One cavity = 6 cells

2 cavities in each accelerating station
3 accelerating stations in each of the 4 rings of the PSB [2]

[2] M. Neroni *et al.*, "Recent updates in the impedance characterization of the CERN PS Booster FINEMET RF system"

Challenges

- To have convergence of the results: $\cong$ **60 million mesh cells**
- For $\boldsymbol{\beta = 1}$: total simulation time $\cong$ 3 hours
- For $\boldsymbol{\beta < 1}$ the simulations of the bounding box are **very long** $\rightarrow$ first to be tried

Messages

Unable to solve the electrostatic problem for the interface region. Memory for 226217046 mesh points is required, but the maximum number of allowed points is 20000000. More...

Unable to solve the electrostatic problem for the interface region. Memory for XXX mesh points is required, but the maximum number of allowed points is XXX.

Particle beams with a beta value below or equal 0.999 are treated differently than beams with a beta larger than 0.999. If beta was chosen to be less than 0.999 a full 3D electrostatic problem must be solved before the wake-simulation is started. This is necessary to imprint the eigenfield of the moving charge at the calculation domain's boundaries. Depending on the beta value and the maximum stable time step of the wakefield simulation, this can be a very time- and memory consuming operation.

For ultra-relativistic beams with a beta value above 0.999 the eigenfield can be determined by solving a 2D electrostatic problem which requires a very small amount of memory only.

In case this error message appears several actions can be taken:

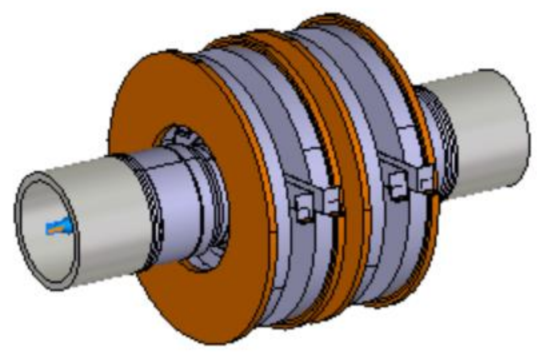
- Relax the mesh to increase the maximum stable timestep of the wakefield simulation. ✓
- Reduce the number of mesh points in the interface region in order to downsize the 3D electrostatic problem.
- Use transversal symmetry conditions in order to downsize the 3D electrostatic problem. ✗
- Choose a beta value larger than 0.999 if applicable. ✗
- Increase the maximum number of allowed points with the VBA command: `Solver.MaxPointsBeamInterfaceCalc "XXX"` ✓

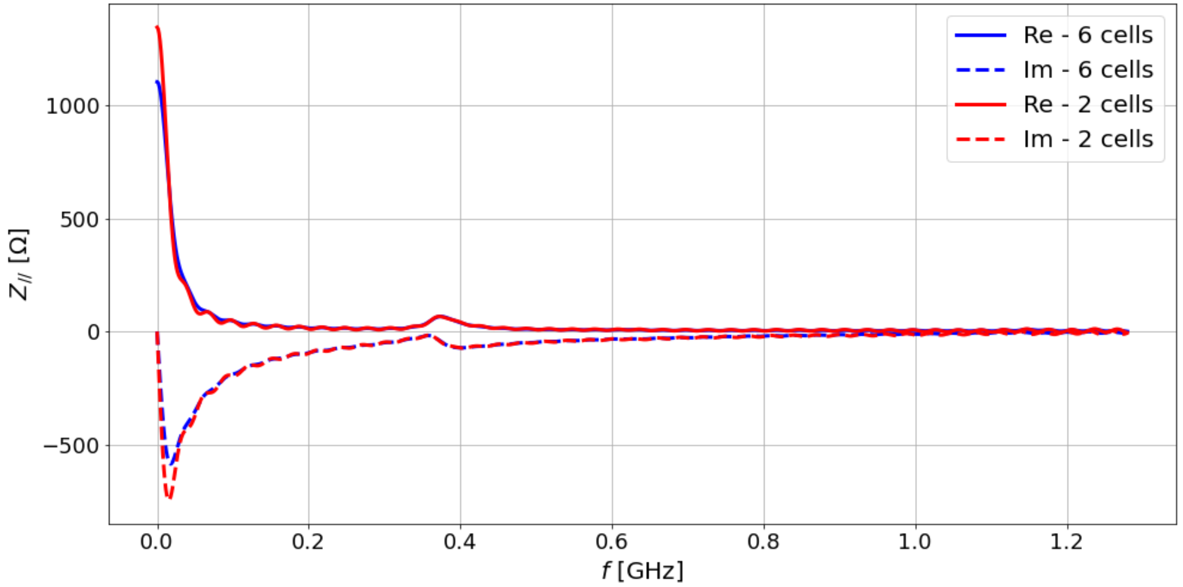
Solutions

- Number of allowed mesh points increased → simulation runs but then stops again: "**Memory allocation failed**", even on high-capacity RAM computers.

- **Model simplified** → removal of negligible parts and simpler beam pipe: ≅ 50 million cells.
  - 10 million cells less, but **still too heavy**.

- Final **solution: 2 cells model → 7÷16 million cells** depending on β





$\beta = 1$ ✓

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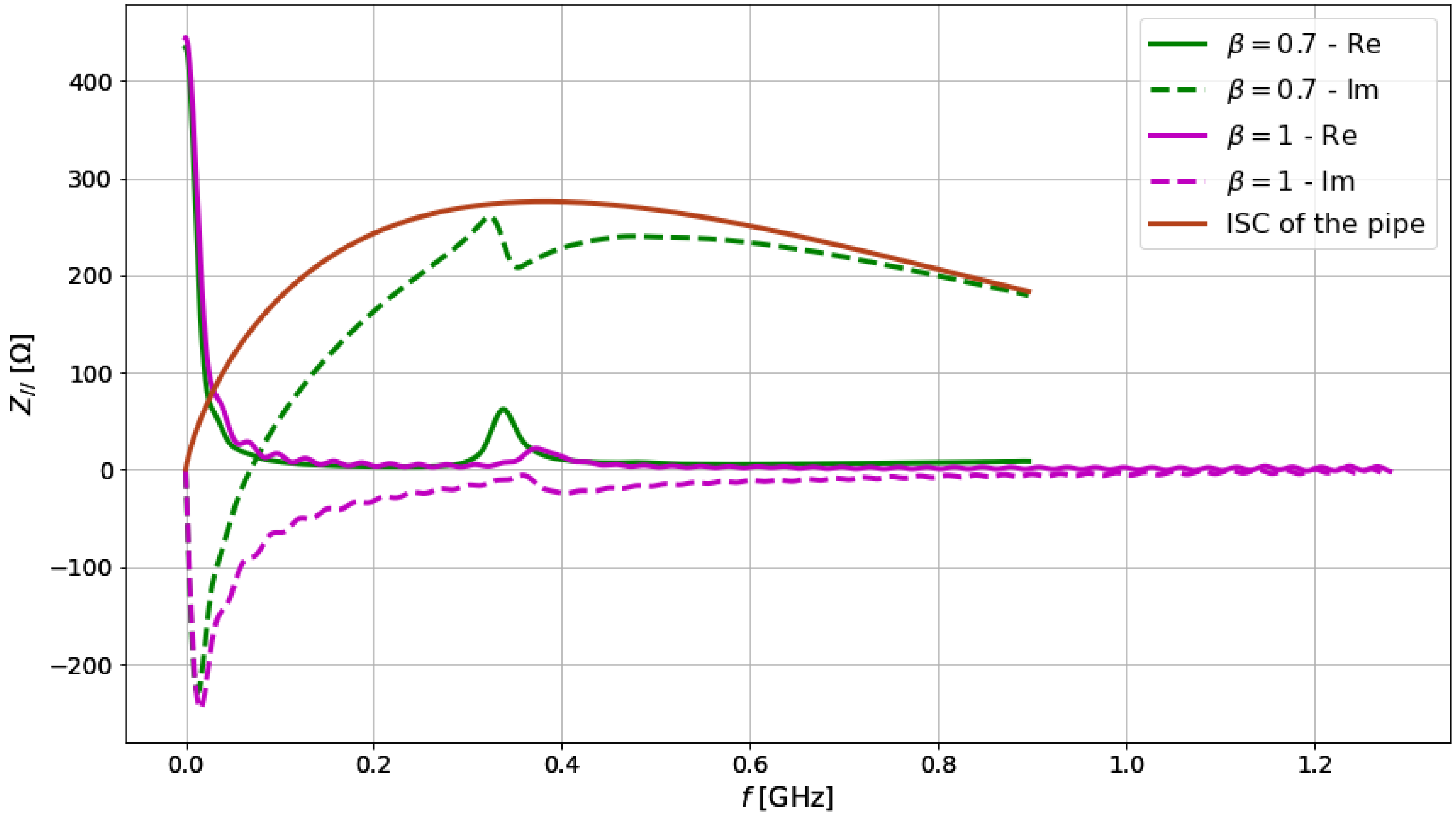
Longitudinal impedance: β=1 vs β=0.7

The shape of the imaginary part for $\boldsymbol{\beta = 0.7}$ is due to the **presence of the ISC**.

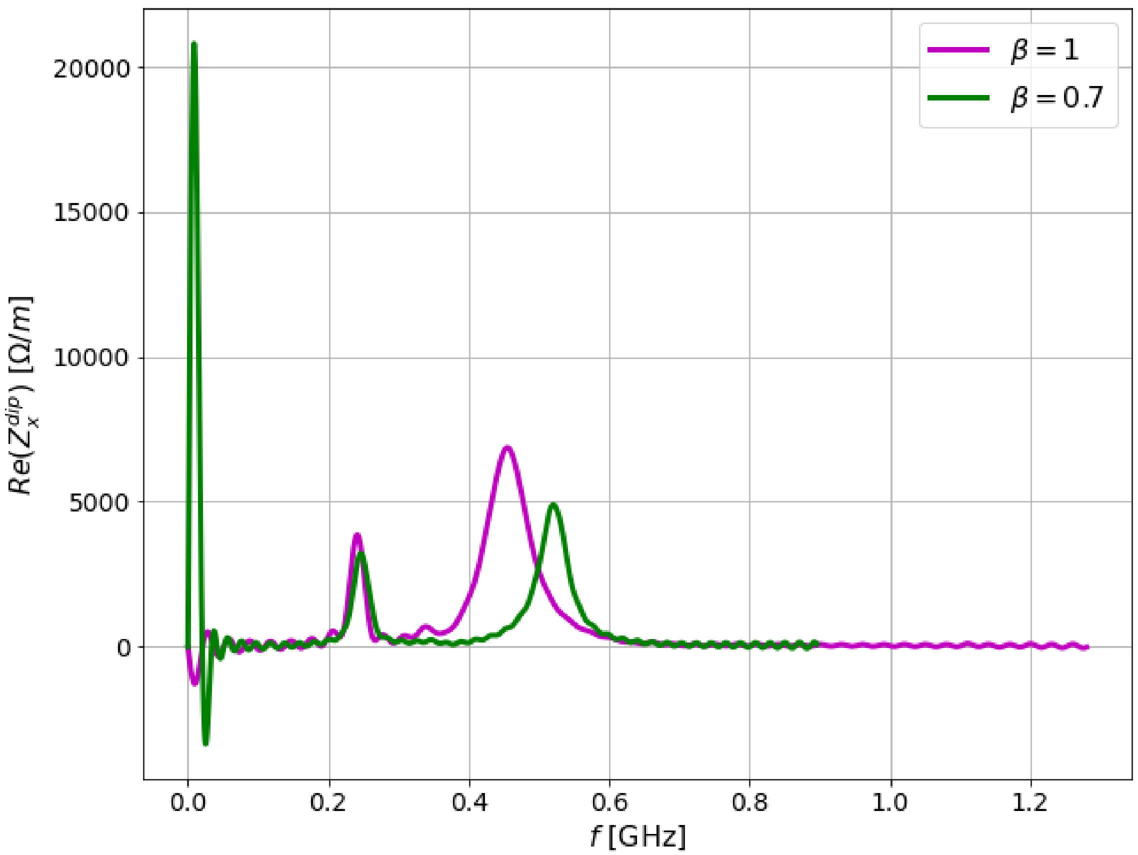


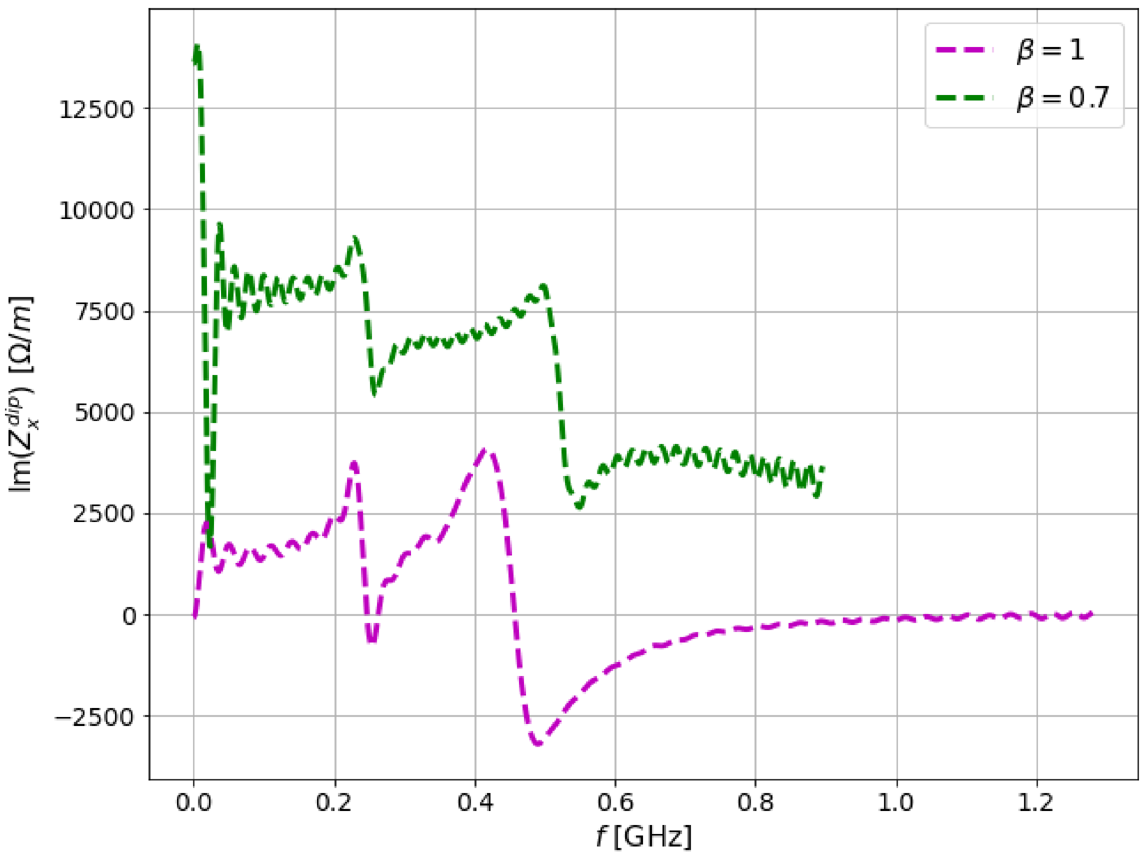
For the *simplified beam pipe*:

$$Z_{||}(\omega) = \frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^2}\frac{K_0\left(\frac{kb}{\gamma}\right)}{I_0\left(\frac{kb}{\gamma}\right)}$$

Transverse horizontal dipolar impedance: β=1 vs β=0.7

The higher value of the imaginary part for $\boldsymbol{\beta = 0.7}$ is due to the **presence of the ISC**.





Transverse horizontal dipolar impedance: β=1 vs β=0.7

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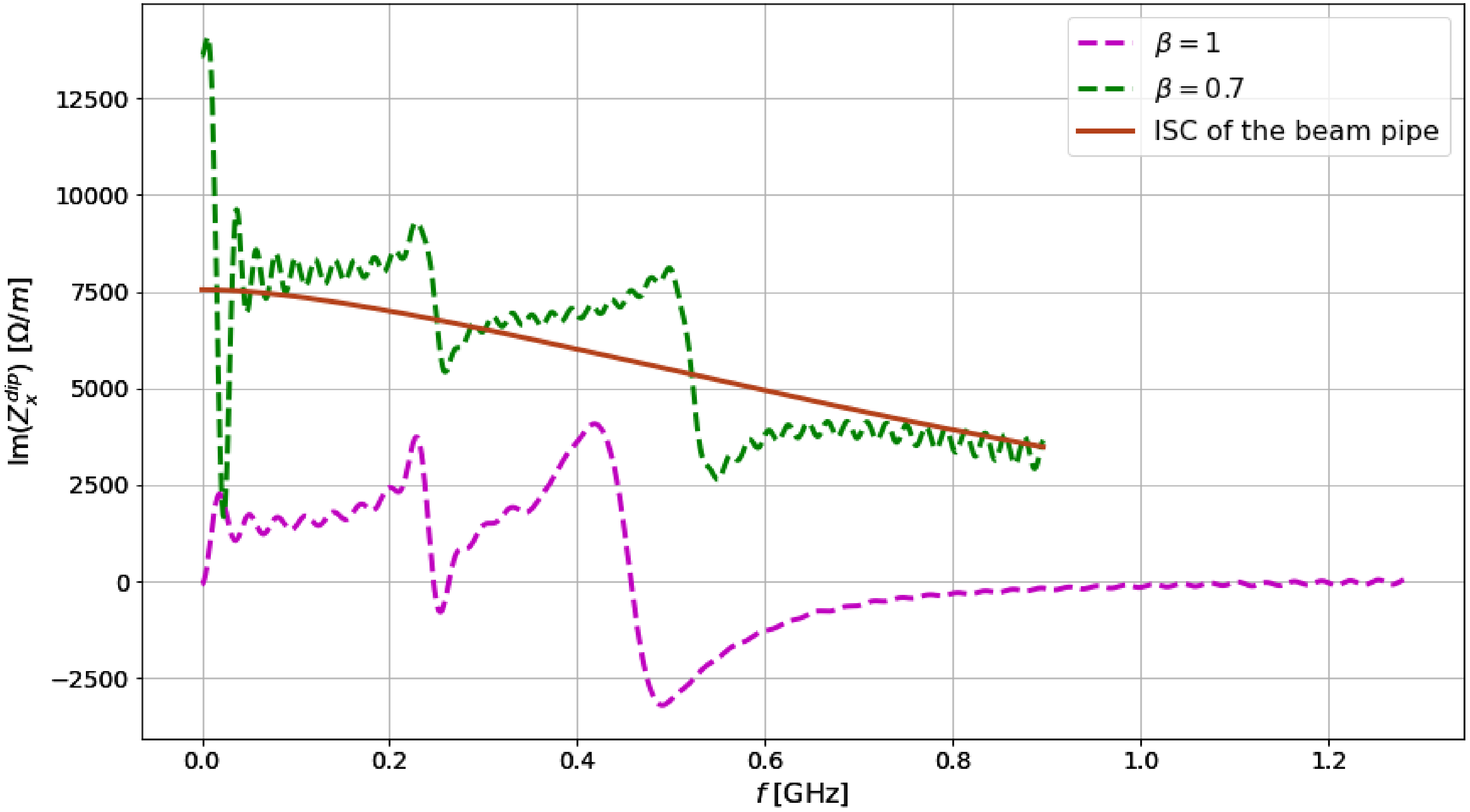
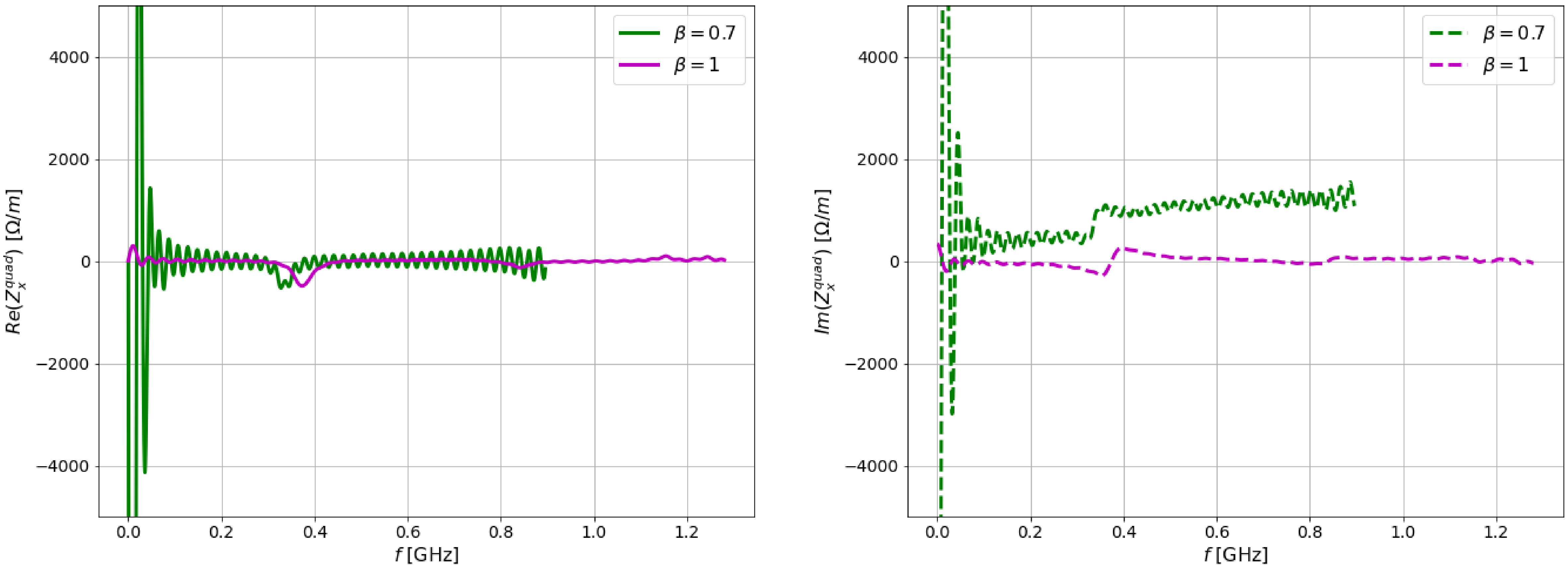


For the *simplified beam pipe*:

$$Z_{\perp}^{dip}(\omega) = \frac{jk^2 Z_0 L}{4\pi\beta\gamma^4} \frac{K_1\left(\frac{kb}{\gamma}\right)}{I_1\left(\frac{kb}{\gamma}\right)}$$

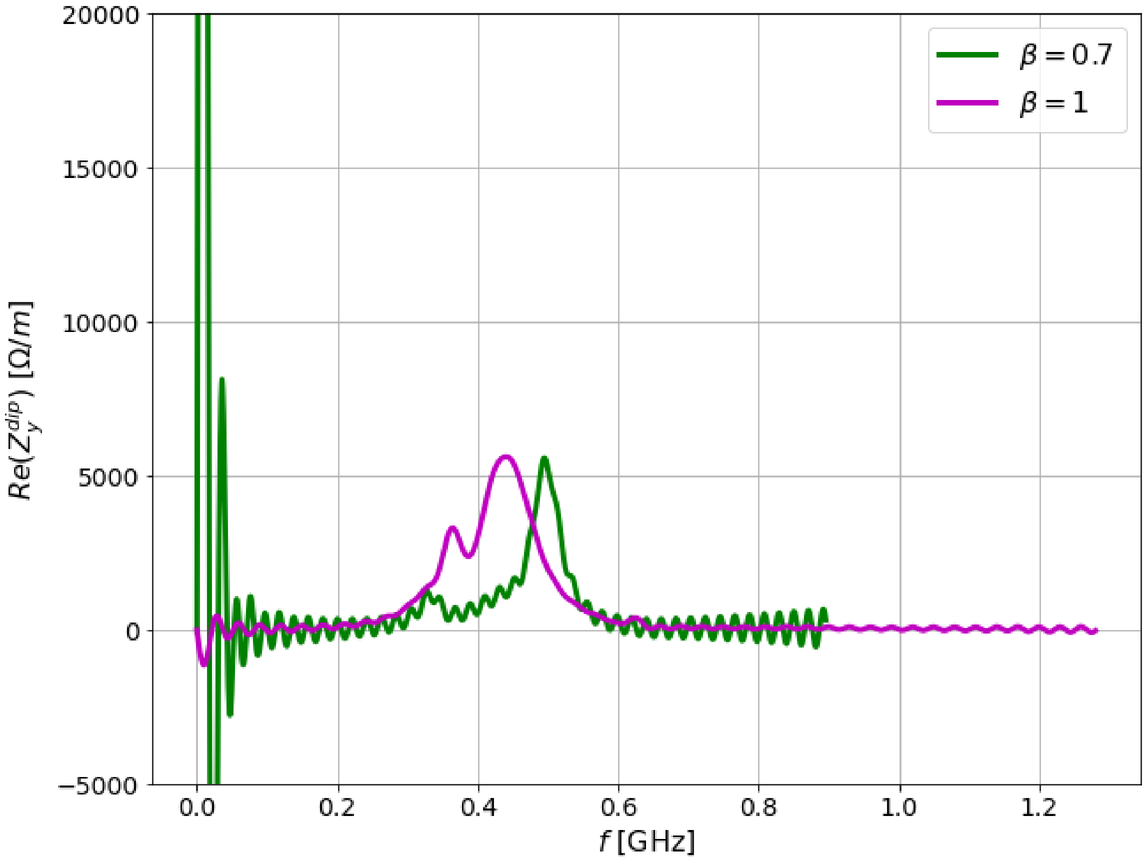
Transverse horizontal quadrupolar impedance: β=1 vs β=0.7

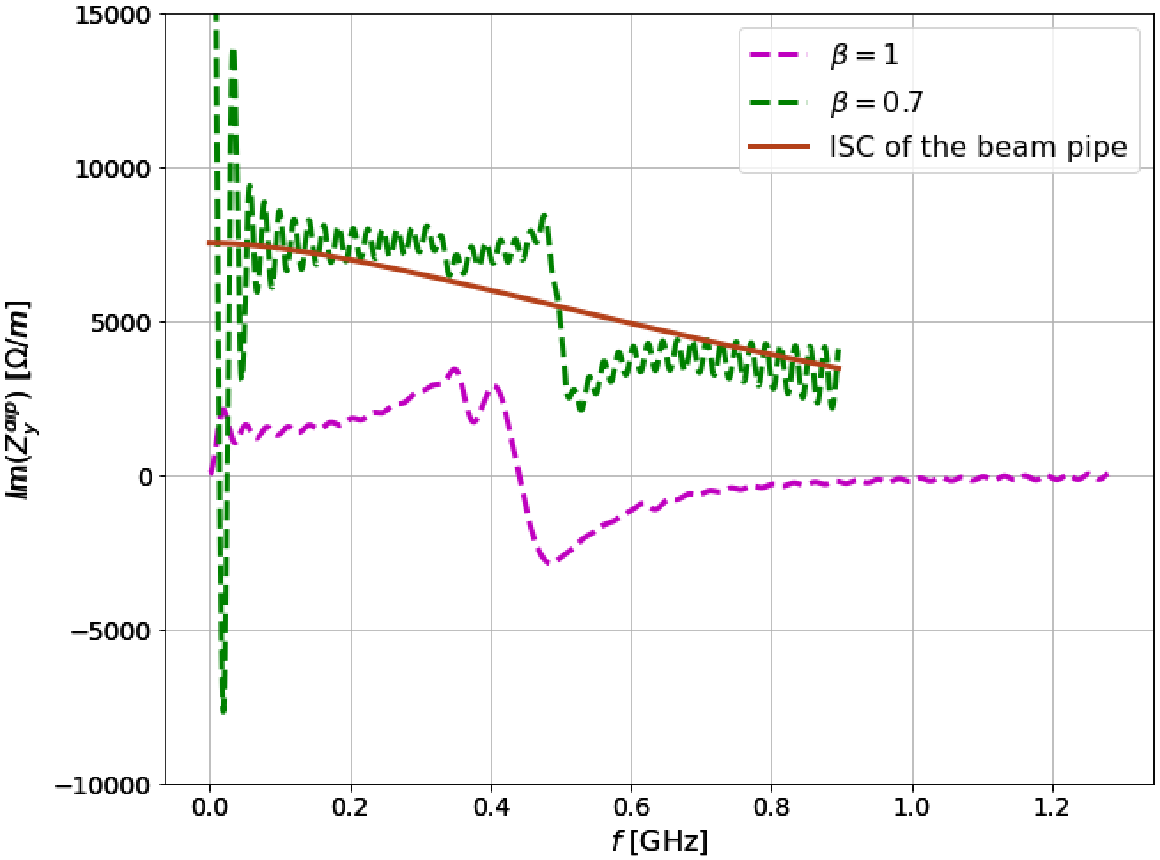
- For $\beta = 1$, $Z^{quad} \cong 0$ because there is almost a circular symmetry.
- For $\boldsymbol{\beta < 1}$ there is a **radial field dependence**: $\boldsymbol{Im\{Z^{quad}\}}$ gets **higher**.



Transverse vertical dipolar impedance: β=1 vs β=0.7

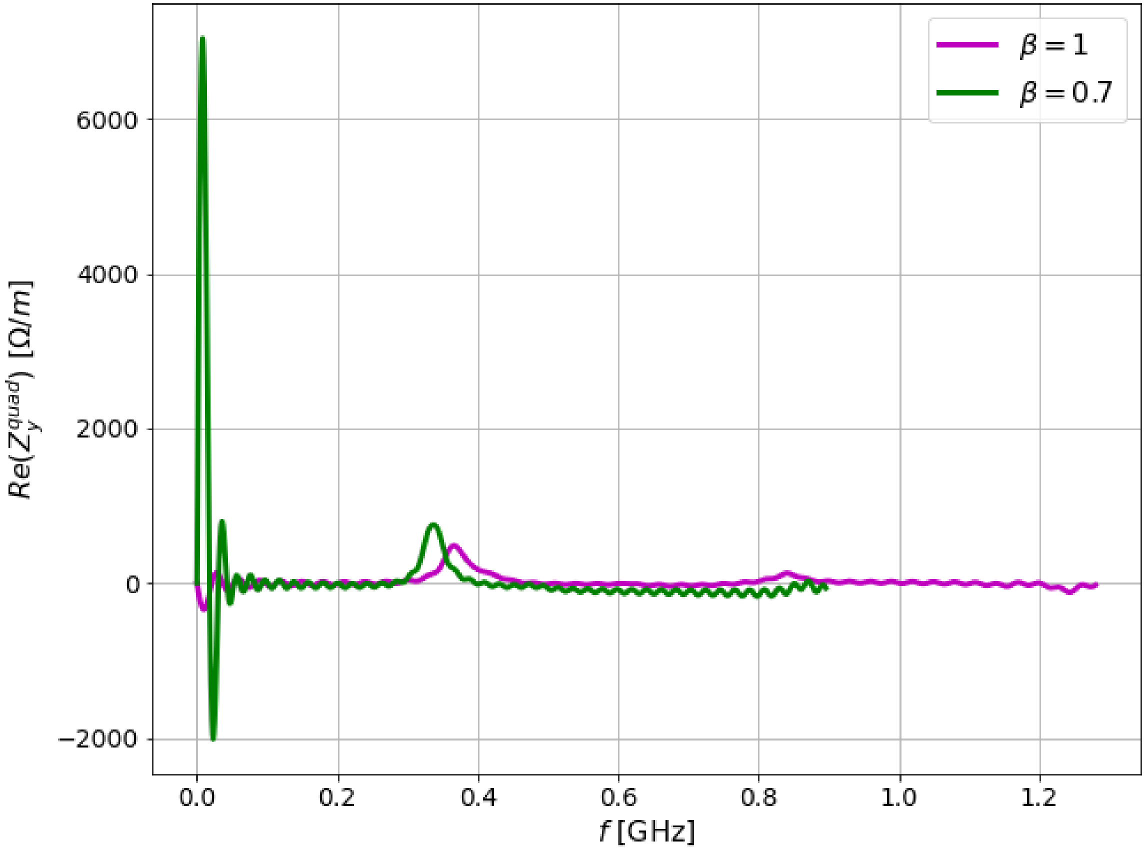
As before, the higher value of the imaginary part for $\beta = 0.7$ is due to the **presence of the ISC**.

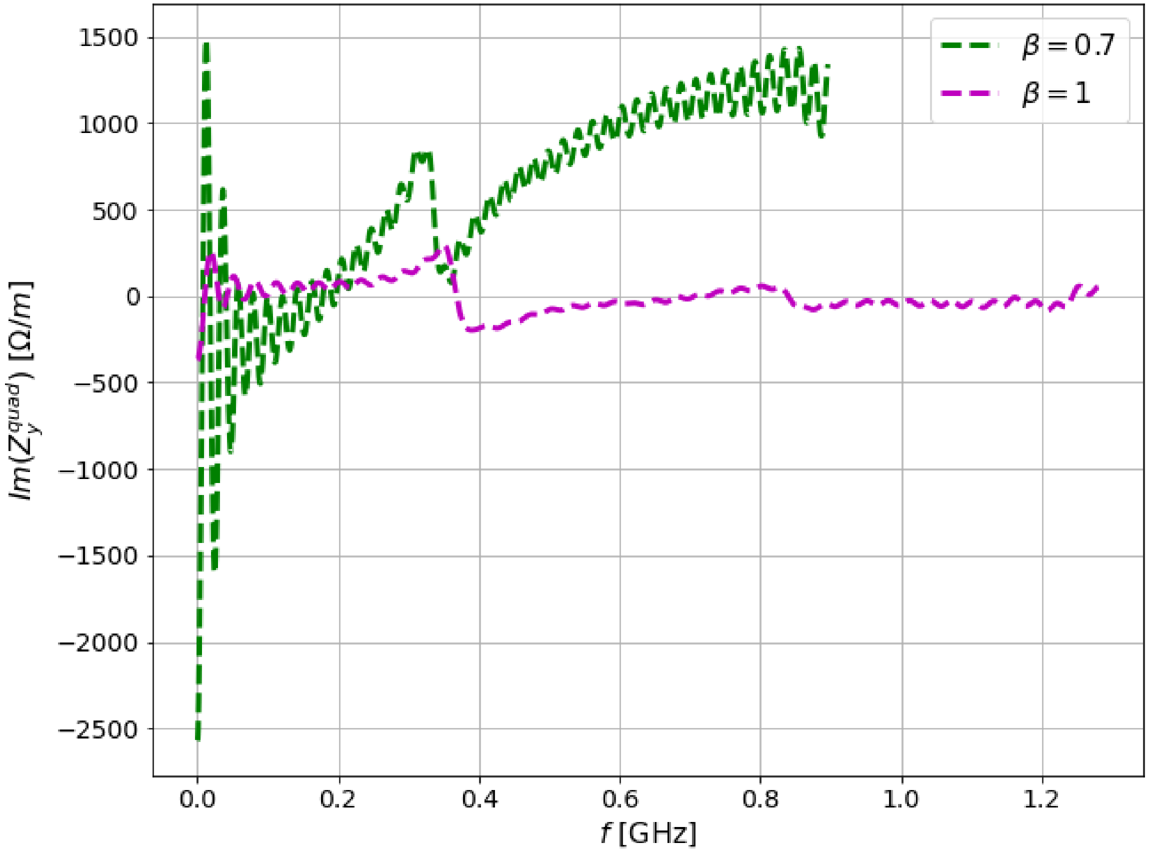




Transverse vertical quadrupolar impedance: β=1 vs β=0.7

- For $\beta = 1$, $Z^{quad} \cong 0$ because there is almost a circular symmetry.
- For $\boldsymbol{\beta} < \mathbf{1}$ there is a **radial field dependence**: $\boldsymbol{Im\{Z^{quad}\}}$ gets **higher.**





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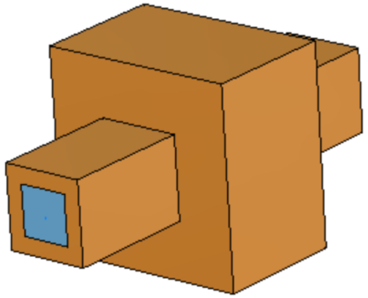
Use of wakis: cubic cavity

```python
from wakis import GridFIT3D, SolverFIT3D, WakeSolver
import pyvista as pv

# ---------- Domain setup ---------
# Number of mesh cells
Nx = 49+20
Ny = 49+20
Nz = 94+20
dt = 1.181512253e-12 # CST

# Embedded boundaries
stl_cavity = 'cavity.stl'
stl_shell = 'shell.stl'
surf = pv.read(stl_shell)

stl_solids = {'cavity': stl_cavity, 'shell': stl_shell}
stl_materials = {'cavity': 'vacuum', 'shell': [10, 1.0, 10]}

# Domain bounds
xmin, xmax, ymin, ymax, zmin, zmax = surf.bounds
Lx, Ly, Lz = (xmax-xmin), (ymax-ymin), (zmax-zmin)

# set grid and geometry
grid = GridFIT3D(xmin, xmax, ymin, ymax, zmin, zmax, Nx, Ny, Nz,
                stl_solids=stl_solids,
                stl_materials=stl_materials)

# ------------ Beam source ----------------
# Beam parameters
beta = 0.9              # beam beta
sigmaz = beta*18.5e-3    #[m] -> 5.53 GHz
q = 1e-9                #[C]
xs = 0.                 # x source position [m]
ys = 0.                 # y source position [m]
xt = 0.                 # x test position [m]
yt = 0.                 # y test position [m]
# [DEFAULT] tinj = 8.53*sigmaz/c_light  # injection time offset [s]
```



```python
# Simulation
wakelength = 1. #[m]
add_space = 8   # no. cells

wake = WakeSolver(q=q, sigmaz=sigmaz, beta=beta,
            xsource=xs, ysource=ys, xtest=xt, ytest=yt,
            add_space=add_space, save=True, logfile=True)

# ----------- Solver & Simulation ----------
# boundary conditions``
bc_low=['pec', 'pec', 'abc']
bc_high=['pec', 'pec', 'abc']

solver = SolverFIT3D(grid, wake, dt=dt, bc_low=bc_low, bc_high=bc_high,
                     use_stl=True, bg='pec')

# Run wakefield time-domain simulation
run = True
if run:
    solver.wakesolve(wakelength=wakelength, add_space=add_space,
                    plot=False, plot_every=30, save_J=True, **plotkw)
```

Bounding box simulations:

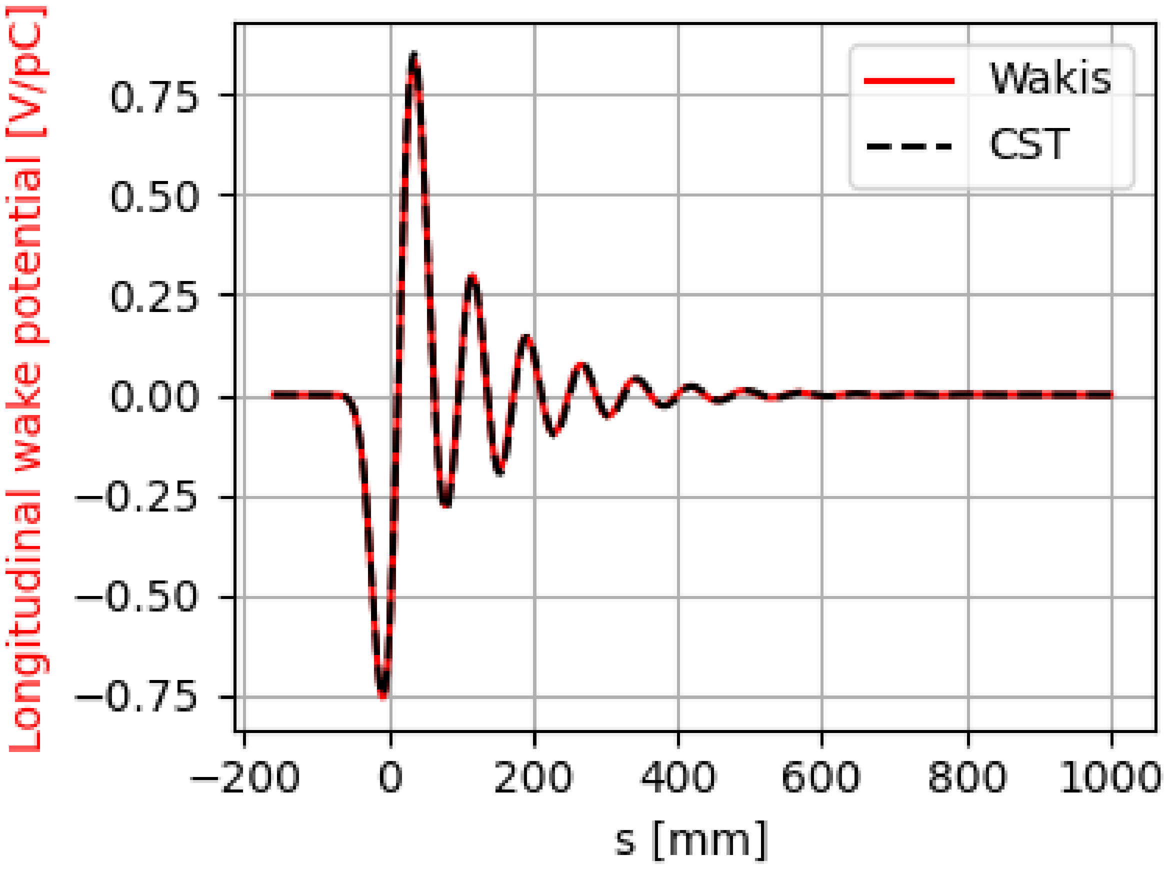
```python
solver = SolverFIT3D(grid, wake, dt=dt, bc_low=bc_low, bc_high=bc_high,
                     use_stl=False, bg='vacuum')
```

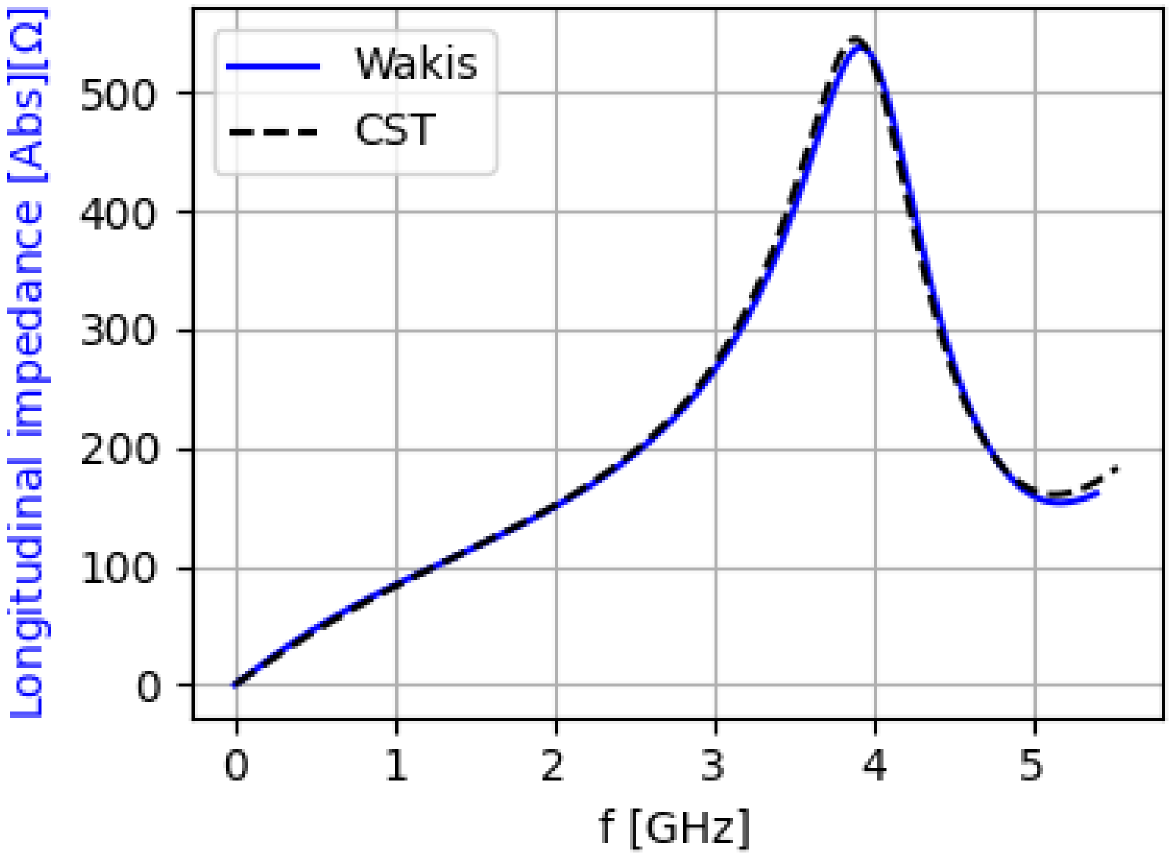
Thanks to Elena de la Fuente García

GitHub wakis repository

Wake potential and |Z|: β=1

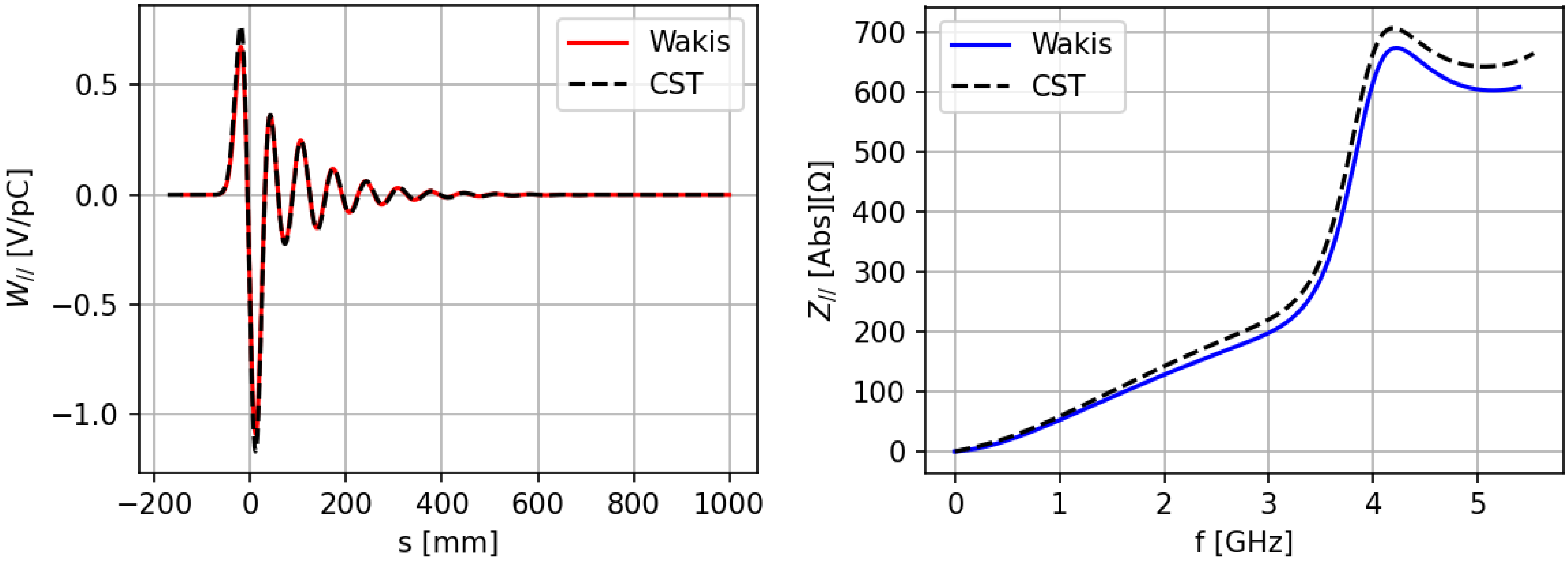
For $\boldsymbol{\beta} = \mathbf{1}$ there is **good agreement** between CST and `wakis`.





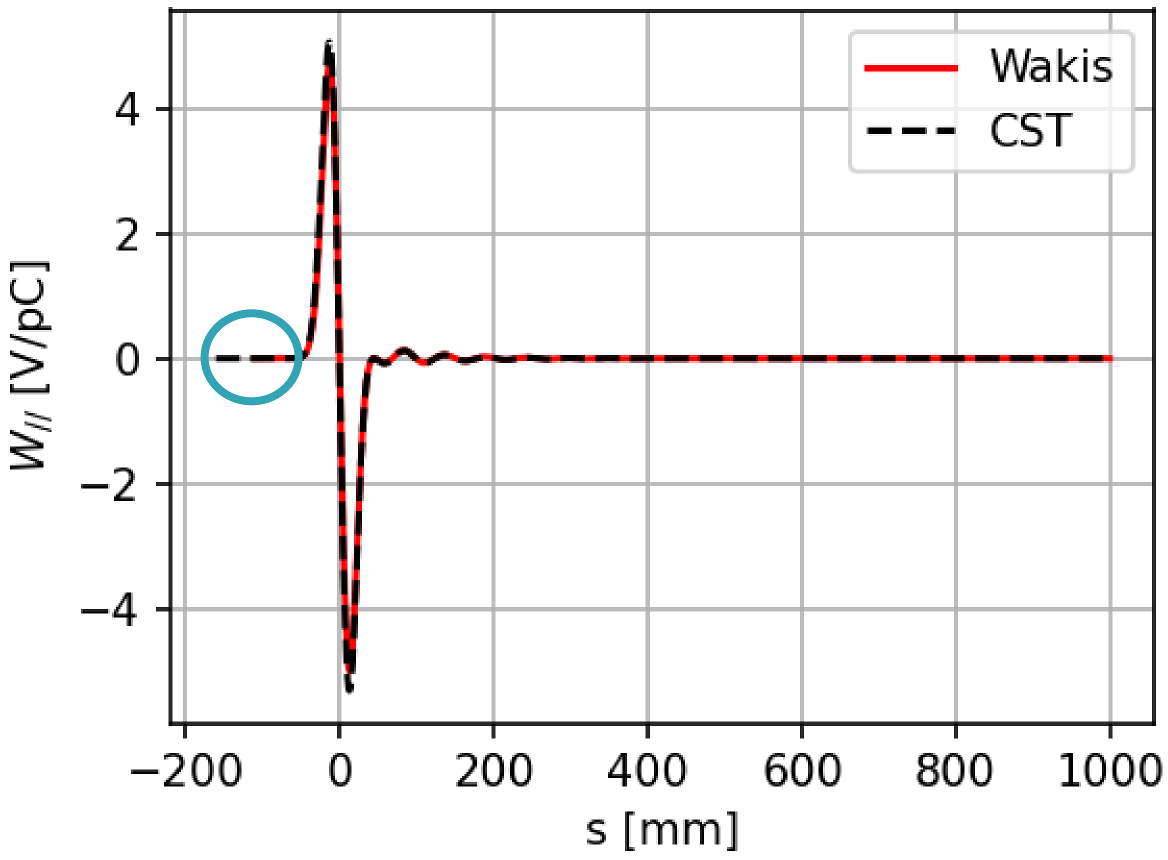
Wake potential and |Z| varying β: β=0.9

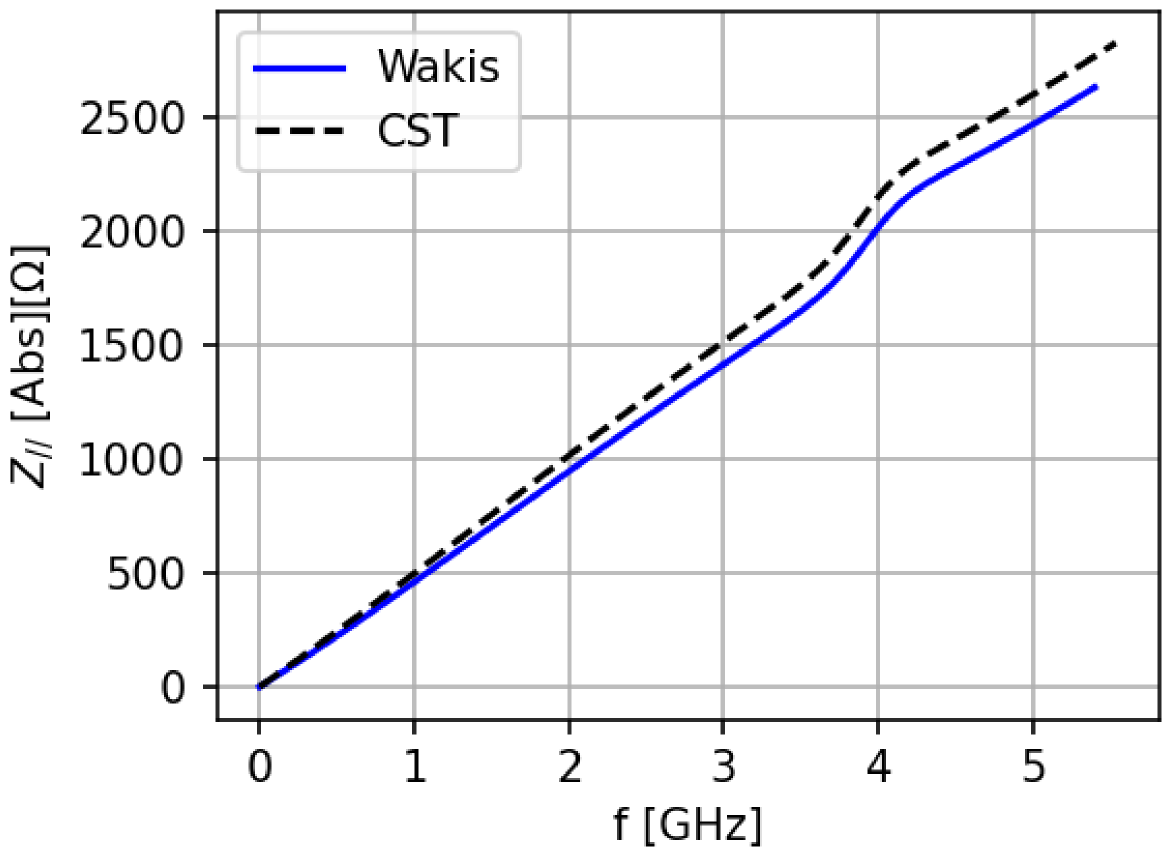
As β decreases, there is still good agreement between the two solvers.



Wake potential and |Z| varying β: β=0.7

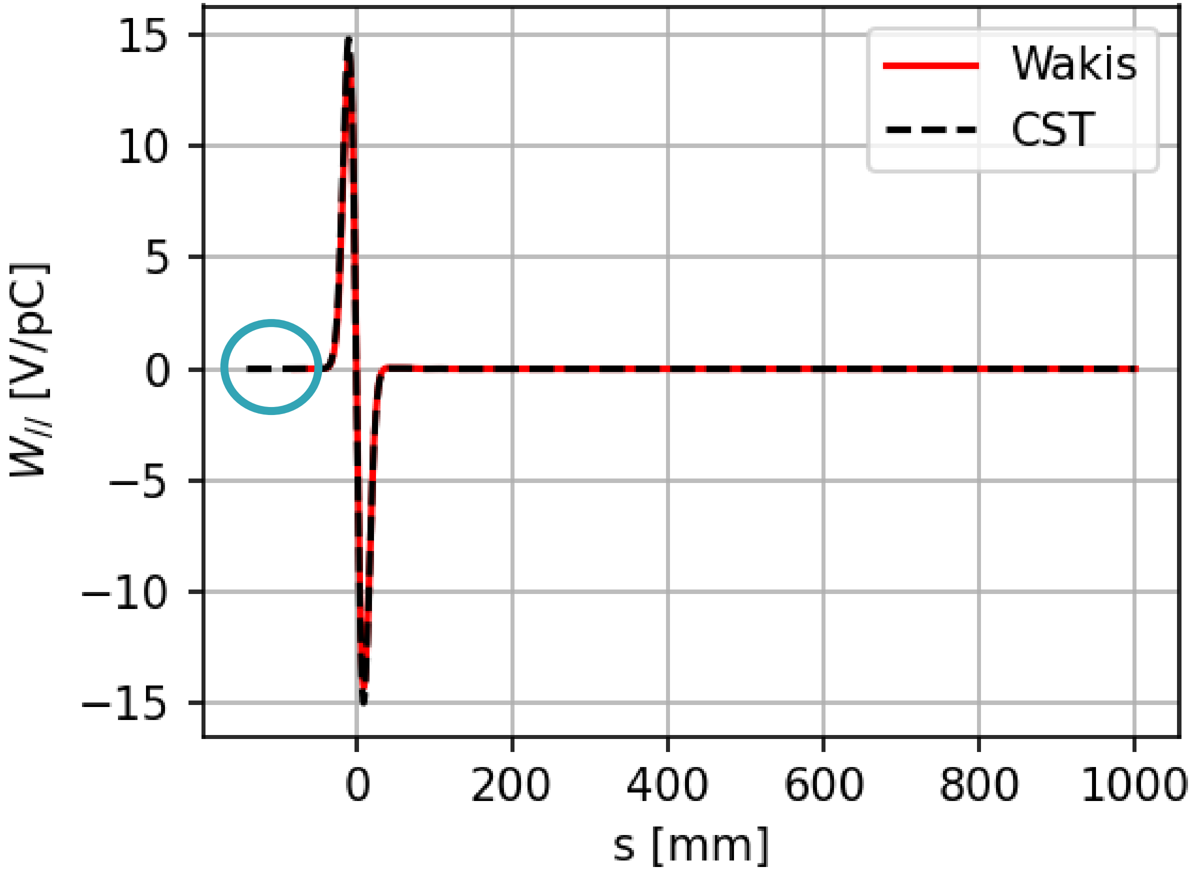
As β decreases, there is still good agreement between the two solvers.
From the wake potential, we can notice that CST **injects** the beam **after or slower**.

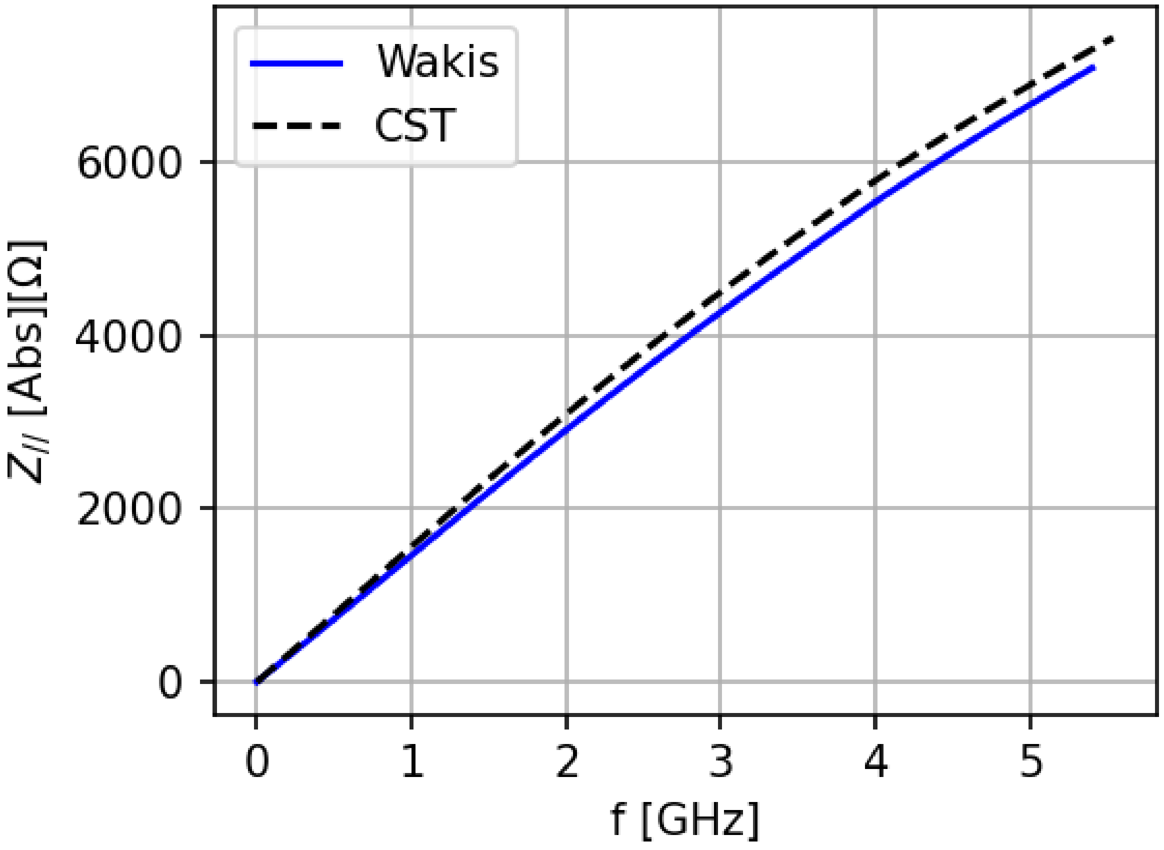




Wake potential and |Z| varying β: β=0.5

As β decreases, there is still good agreement between the two solvers.
From the wake potential, we can notice that CST **injects** the beam **after or slower**.





Outline

- **Introduction**
- **Simulations of a pillbox cavity**
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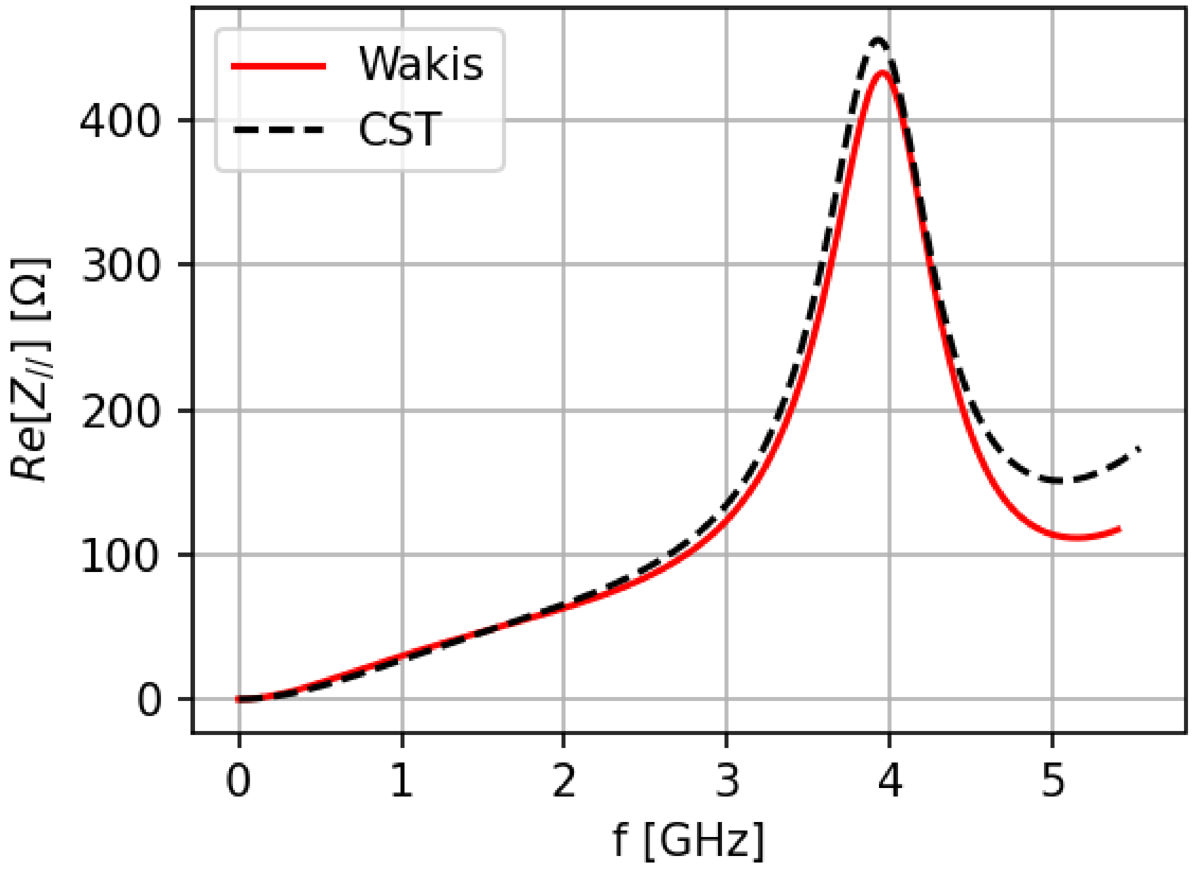
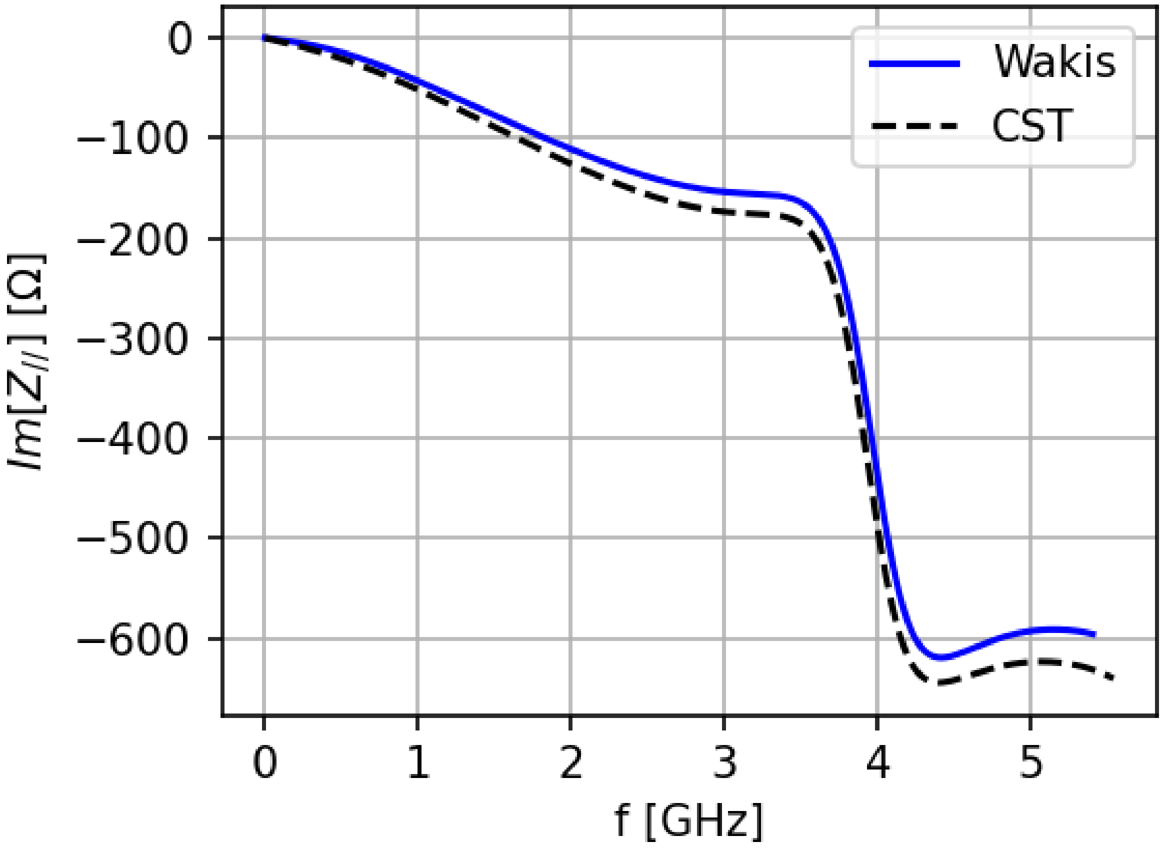
Real and imaginary parts of Z varying β: β=0.9

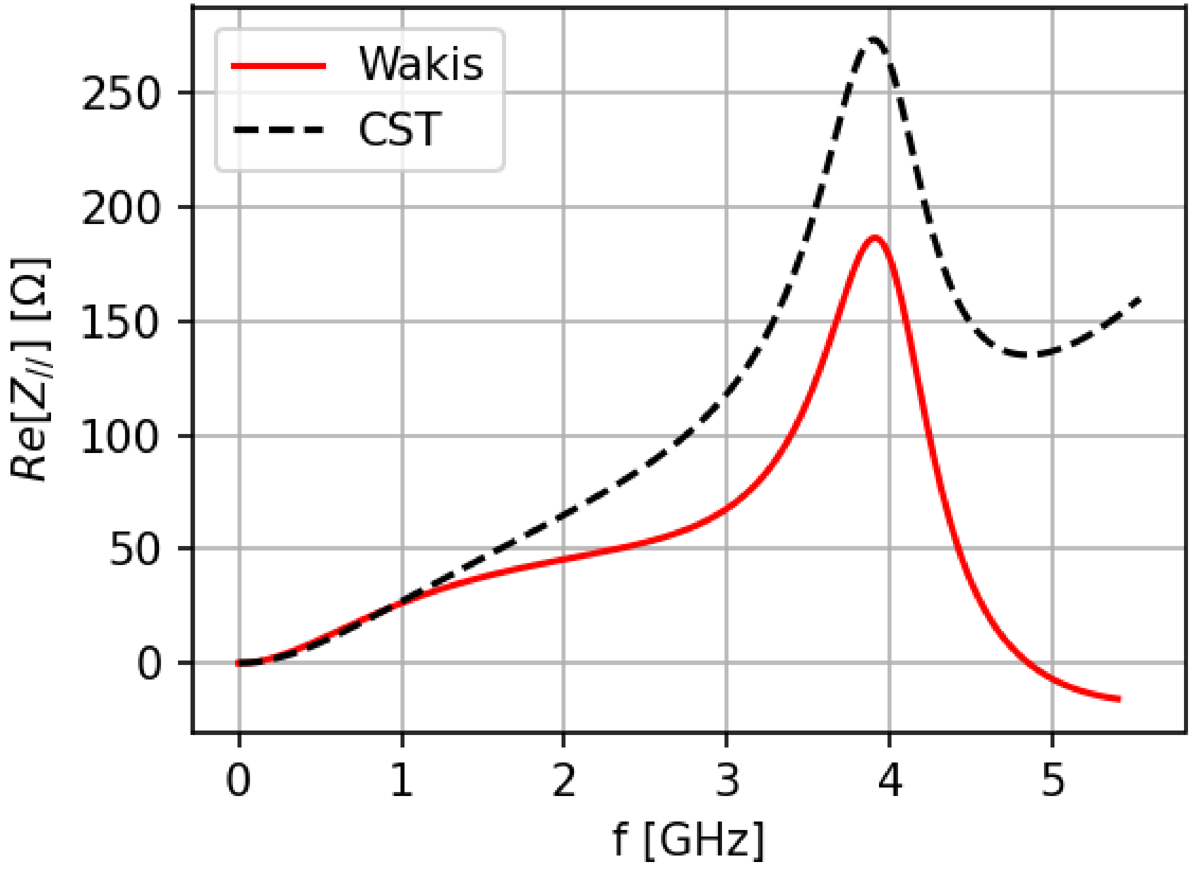
As β decreases, the real parts of Z in the two solvers start to drift apart.

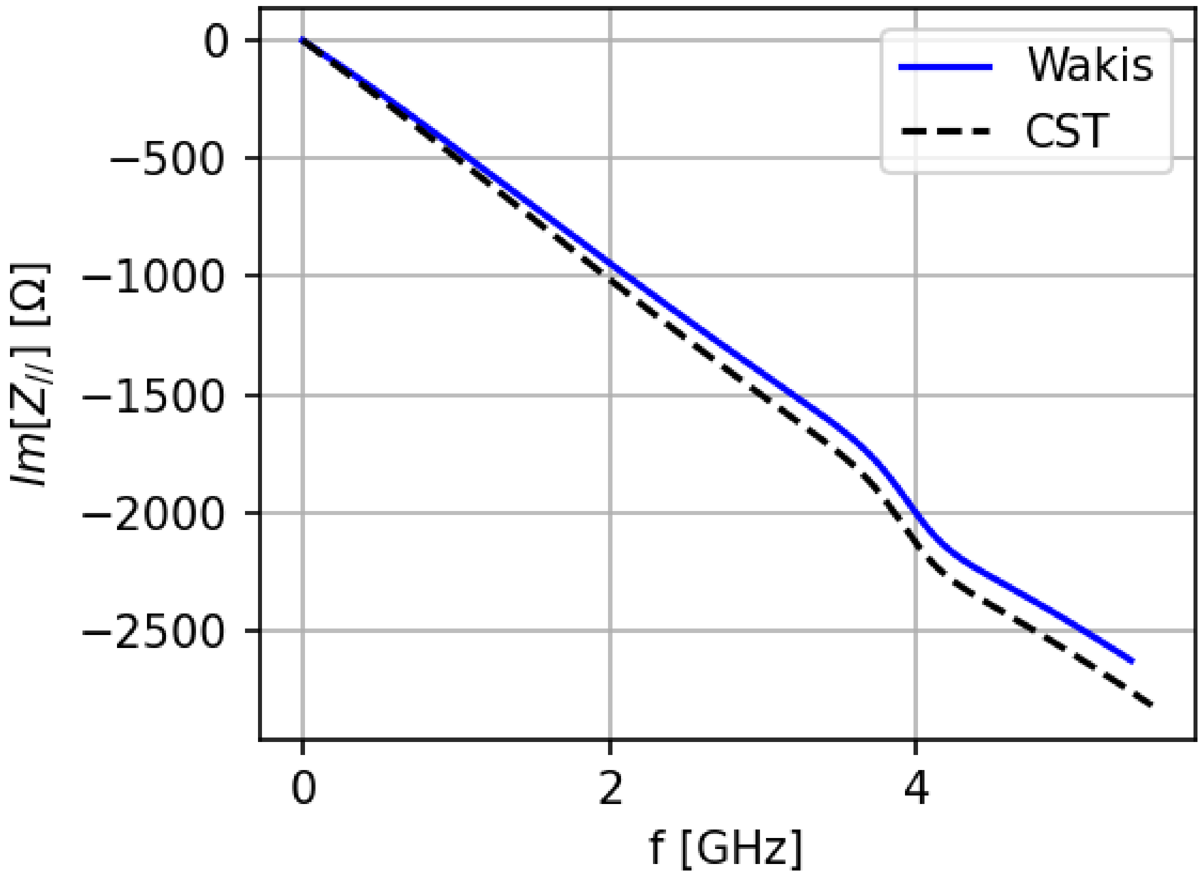




Real and imaginary parts of Z varying β: β=0.7

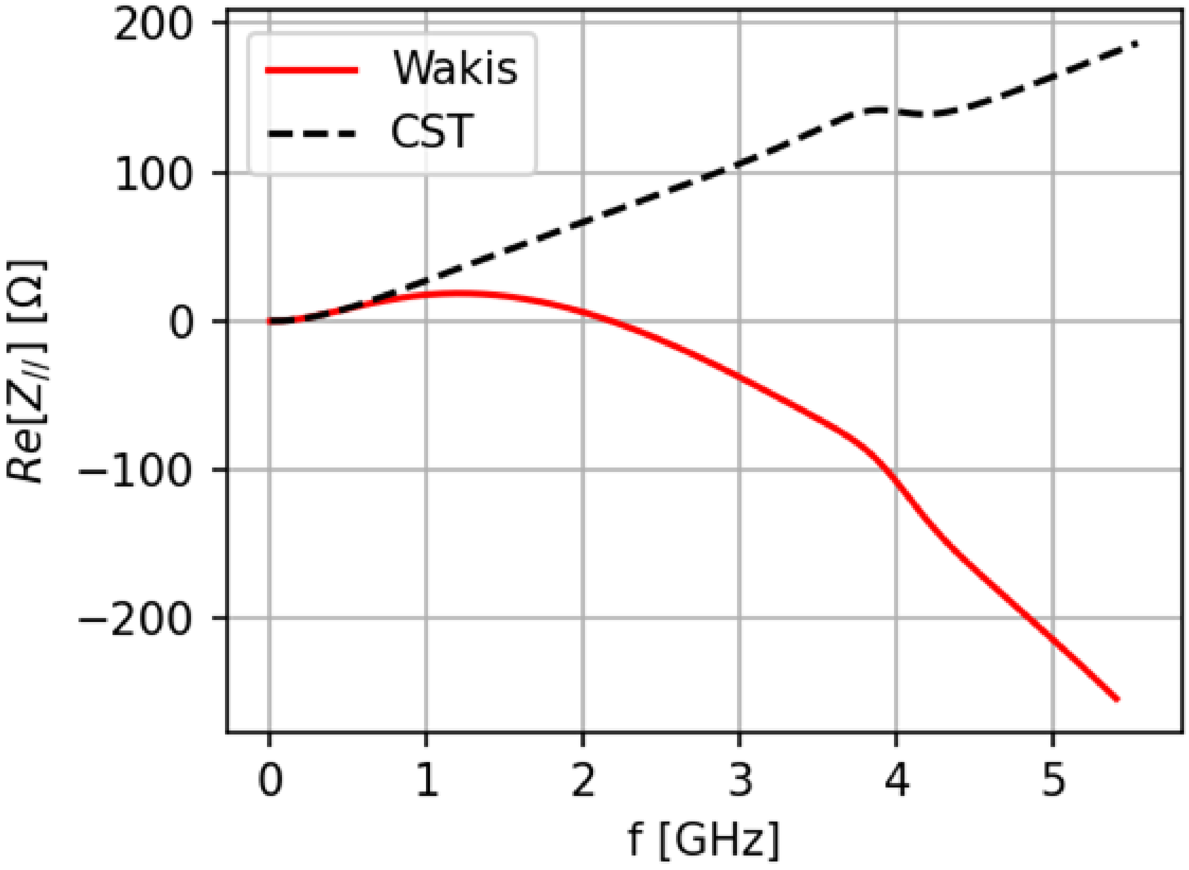
As β decreases, the real parts of Z in the two solvers start to drift apart.

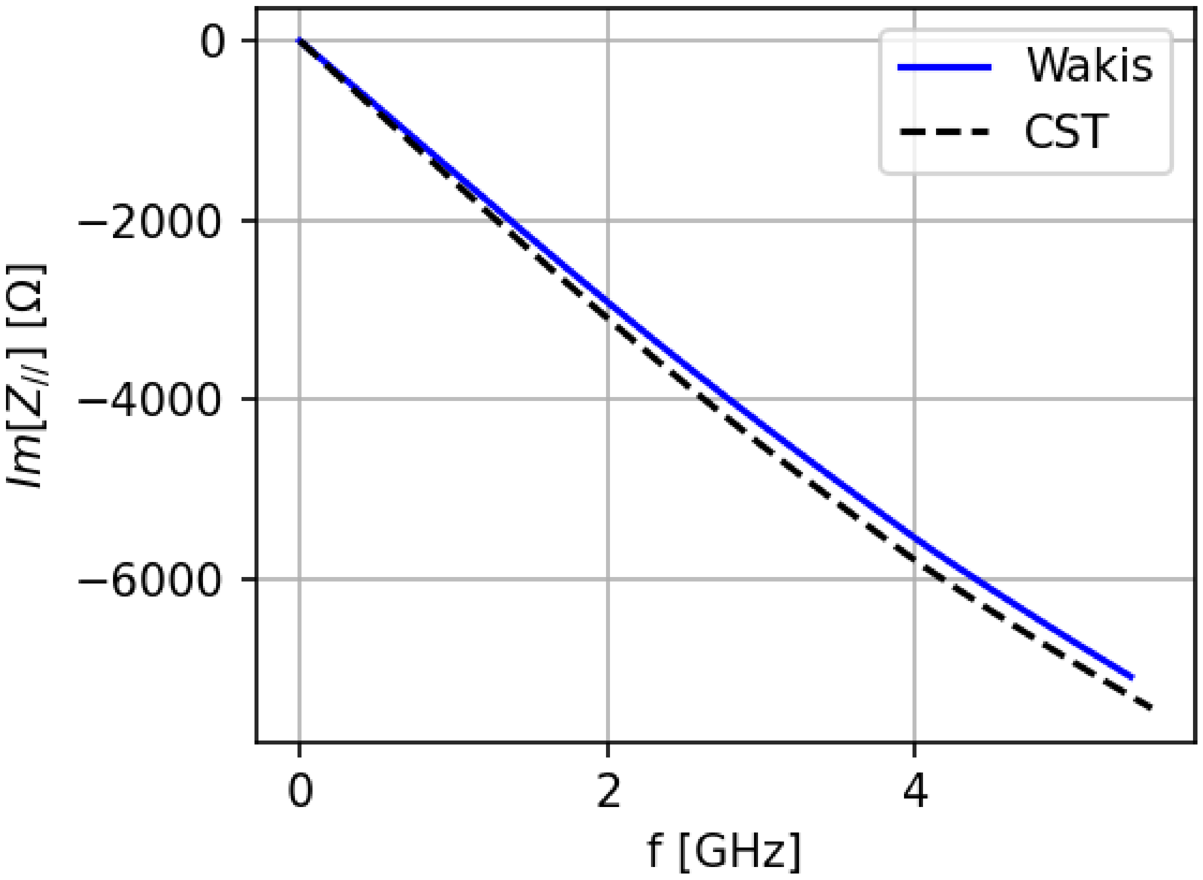




Real and imaginary parts of Z varying β: β=0.5

As β decreases, the real parts of Z in the two solvers start to drift apart.





Real and imaginary parts of Z varying β: β=0.5

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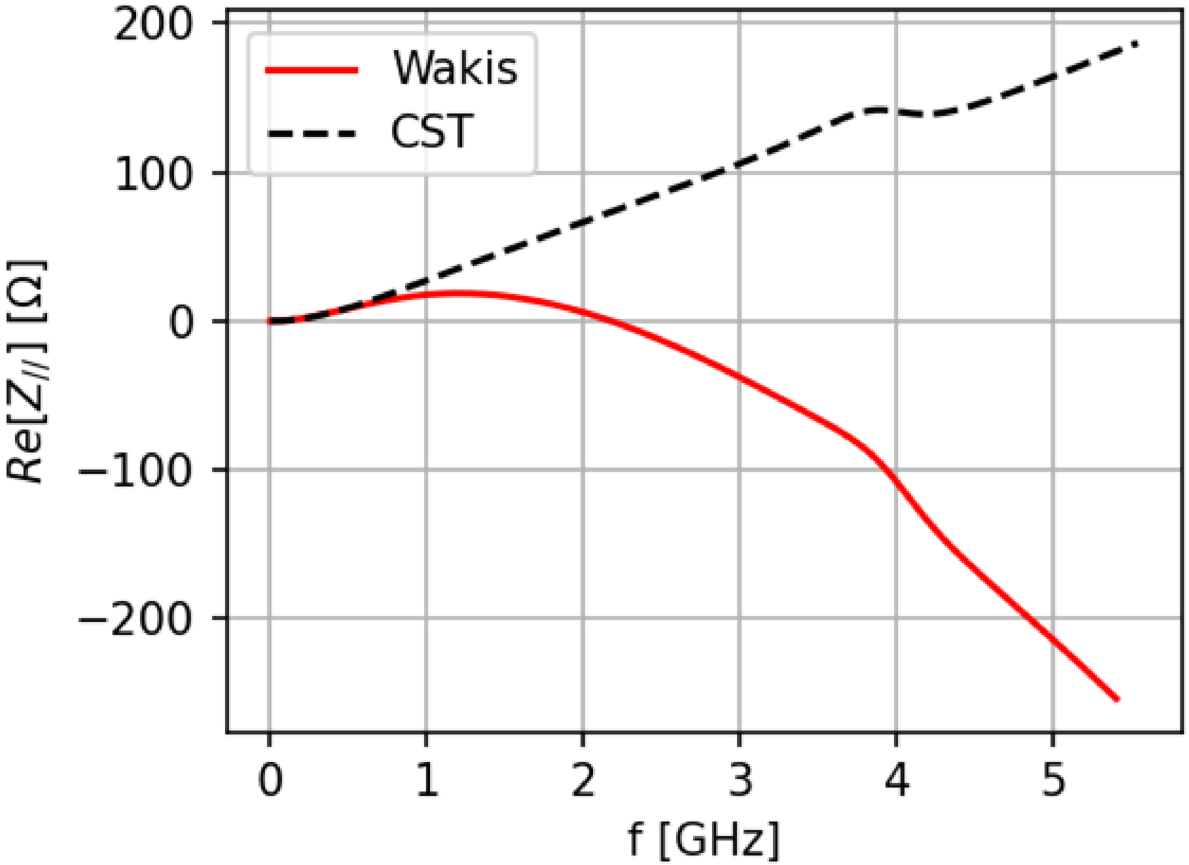
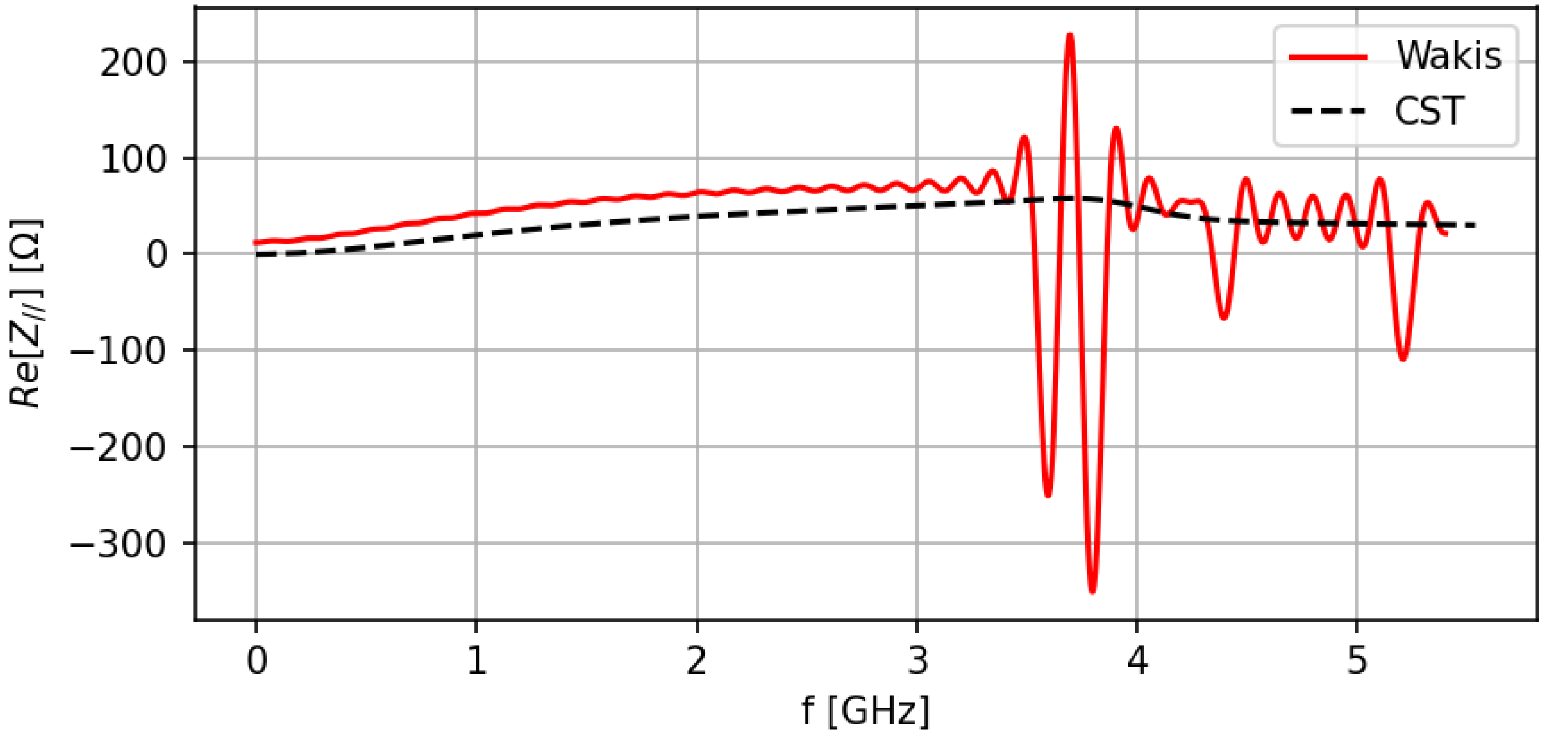


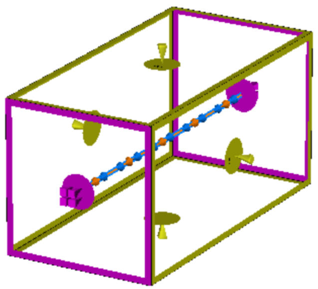
$$Z_{||}(\omega) = -\frac{\int_{-\infty}^{\infty} W_{||}(s)\, e^{-j\omega s} ds}{\int_{-\infty}^{\infty} \beta c \lambda(s)\, e^{-j\omega s} ds}$$

Possible solution: adjustment of the deconvolution algorithm in `wakis` to account for $\beta < 1$.

Numerical cancellation of DSC on $\mathrm{Re}(Z_{||})$ for β=0.5

- Performing the numerical cancellation* of the DSC on both, the two curves appear to get closer.
- The presence of resonances in `wakis` is due to the fact that PML boundary conditions are still under development.
  - PEC was used instead: the bounding box acts as a resonator.





*Simulations of the bounding box are needed.

Numerical cancellation of DSC on $\mathrm{Re}(Z_{||})$ for β=0.5

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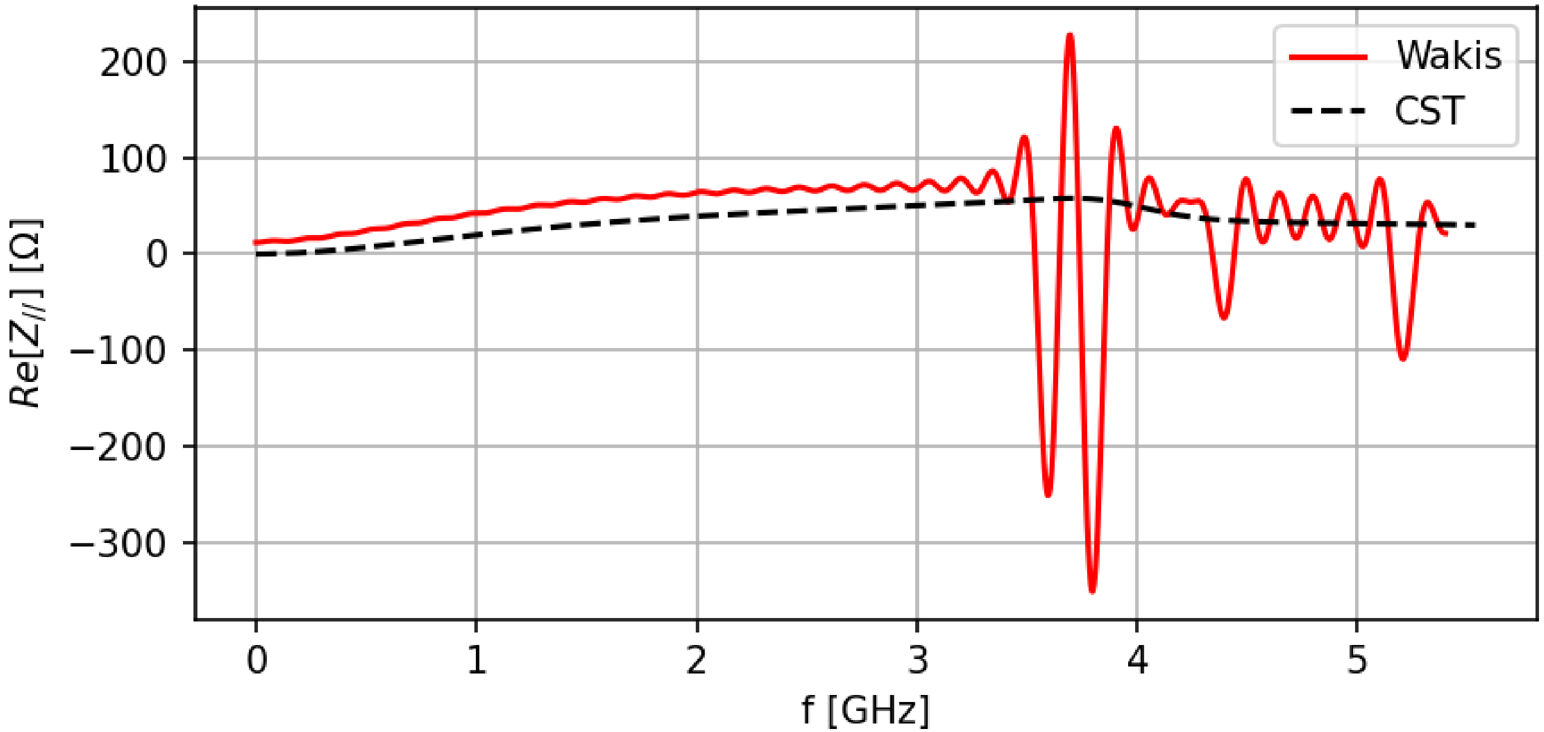

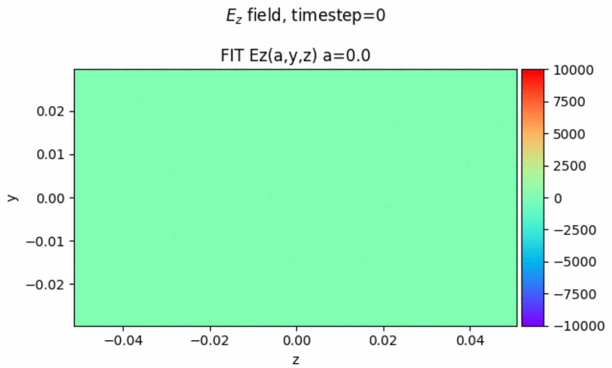

The E field is trapped inside the bounding box and resonates.

Outline

Conclusions (with recap of past studies[3])

- **Low-beta** simulations are **extremely challenging** due to **a series of factors** (mesh convergence, direct integration method, removal of direct space charge, etc.).
- The numerical cancellation technique for the **removal of the DSC** contribution was **benchmarked** with a **resistive wall beam chamber:**
  - $Z_{||}$ **doesn't change with β**, as expected;
  - $Z_{\perp}$ **scales with β**, as expected.
- Simulations of a **pillbox cavity**:
  - **Good agreement between** the *peak impedance* computed with the **EM Solver and** the **WF Solver**:
    - The non-ultrarelativistic WF simulations are accurate.
    - The **EM Solver approximation** of adding particle velocity only in post-processing with the transit time factor **has been found to be accurate**.
  - Comparison between the *quality factor* computed with the **EM Solver and** the **WF Solver**:
    - Good agreement when Q is computed from the real part of the impedance;
    - Computation of Q from the magnitude is energy-dependent due to the impact of ISC. Is it still meaningful?

[3] ABP-CEI Section Meeting of 16 May 2024

Conclusions

- Low-beta simulations of **complex structures** like the FINEMET cavities are **extremely demanding** due to **computational challenges** posed by the simulations of the DSC.
  - For $\beta = 1$, the simulation of the 6-cell model is ~ 3 hours long.
  - For $\beta = 0.7$, the simulations of the simplified 2-cell model are ~ 3 hours long and the simulations of the bounding box are $\sim 24 \div 28$ hours long.
- While the real part has a similar amplitude and behavior, the **imaginary part** of the impedance when $\beta < 1$ show higher values with respect to the results for $\beta = 1$, consistently with the **presence of indirect space charge**.
- **`Wakis`** is **easy to use** after a brief introduction.
  - Despite the lack of time, it was possible to conduct a **full analysis varying β**.
- It was possible to confirm that **`wakis`** simulations **correctly follow the behavior** of the impedance **with β**.
- Differences are visible disentangling real and imaginary parts.
  - **Investigations ongoing** with `wakis`.

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Next steps

- **Simulations** on the FINEMET cavities **at different energies** are running.

- Assessment of the **impact of the non-ultrarelativistic FINEMET impedance model on the total PSB impedance**.
    - Impact on the **coherent tune shift vs chromaticity**.
    - **Consistency** of the model **with tune shift measurements**.
    - Impact on instability growth rate and benchmark with measurements could also be performed.

- **Tailored studies** to **optimize low-beta** simulations **in `wakis`**.

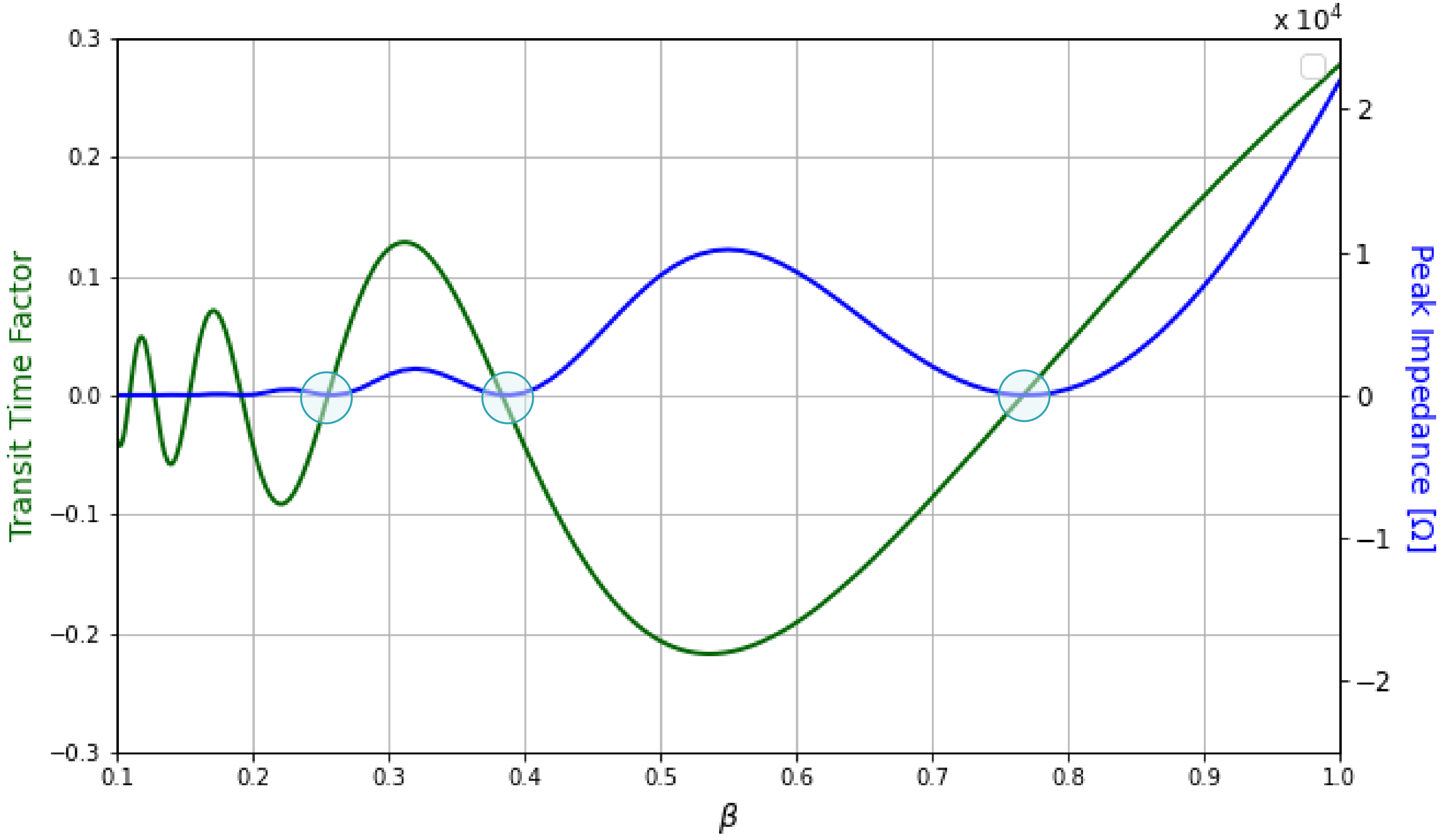
Thank you for your attention

Backup slides

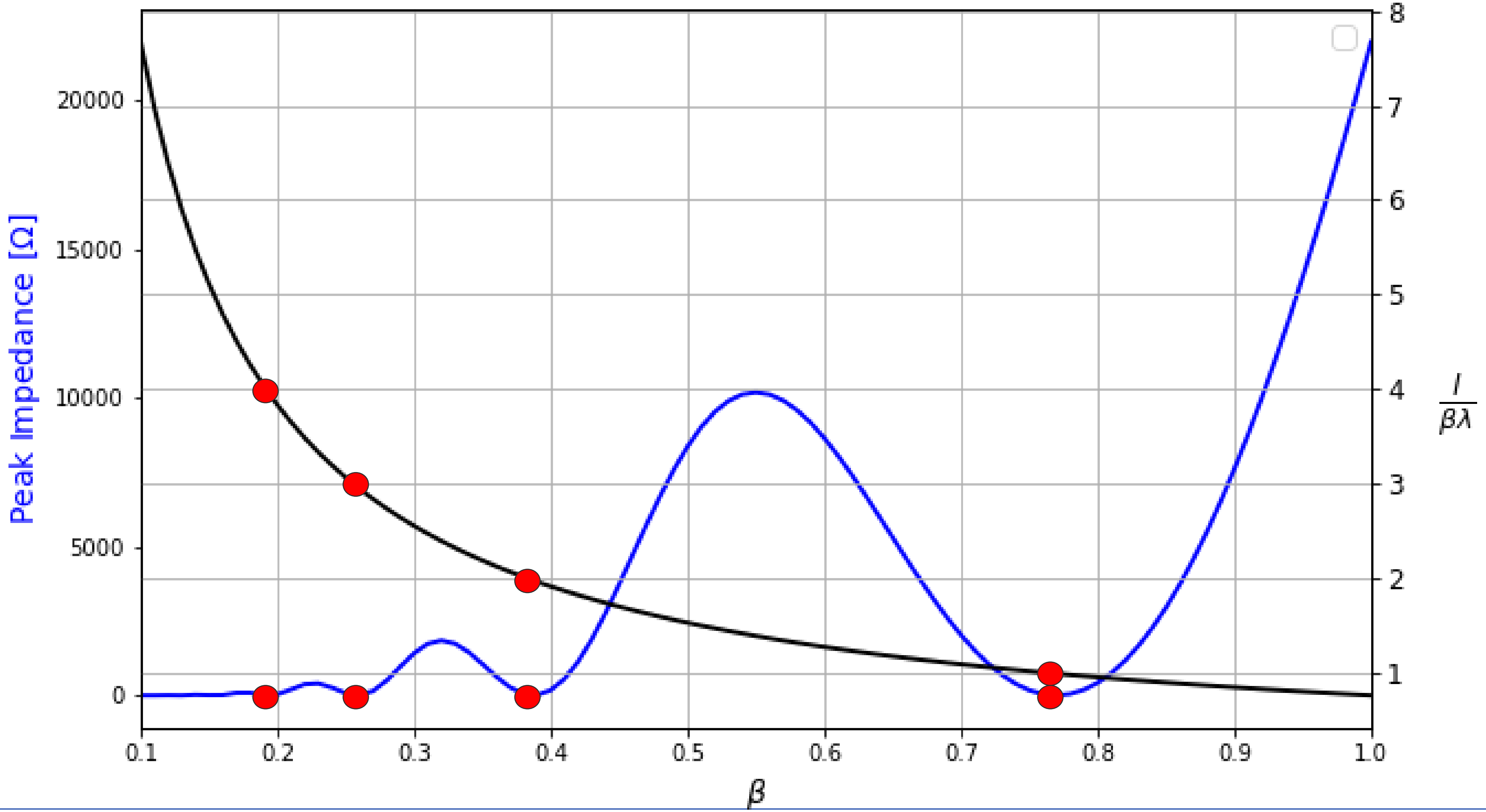
Beam coupling impedance varying β: relationship with the Transit Time Factor

- Study to understand the shape of the curve
  - in particular *values of **β** for which the **peak impedance goes to 0***.
- It can be explained analytically looking at the transit time factor for the fundamental mode:

$$T \propto \frac{\sin\left(\frac{\pi l}{\beta \lambda}\right)}{\frac{\pi l}{\beta \lambda}}$$



Peak impedance and $\frac{l}{\beta\lambda}$ varying β for TM_{010} mode



Generalized transverse beam impedance varying β and role of the quadrupolar component

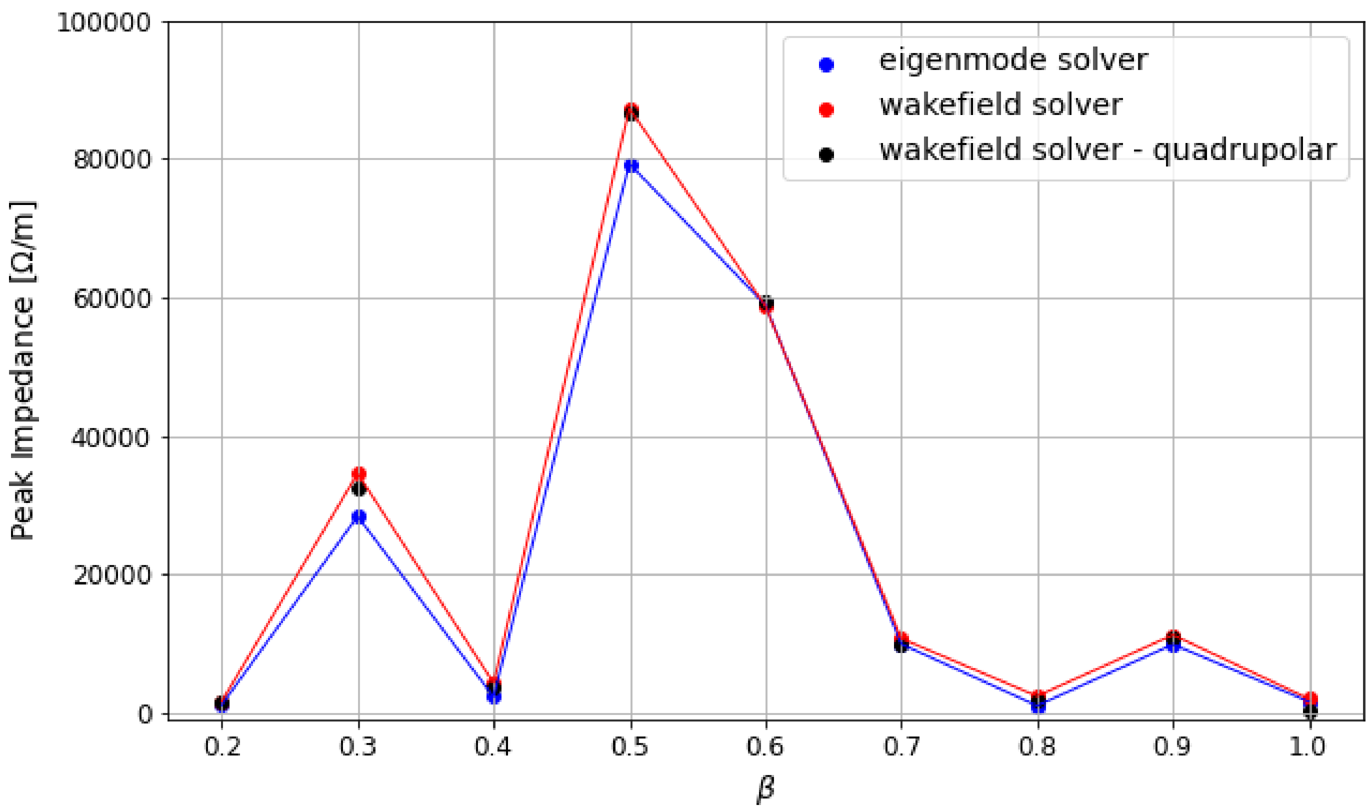
$$E_s(r,\phi) = Q\sum_m A_m I_m\left(\frac{k_0}{\beta\gamma}r\right)\cos(m\phi)$$ [4]

$$H_s(r,\phi) = \frac{Q}{Z_0}\sum_m B_m I_m\left(\frac{k_0}{\beta\gamma}r\right)\sin(m\phi)$$

with $Q = j\frac{q_0 k_0 Z_0}{2\pi\beta^2\gamma^2}$

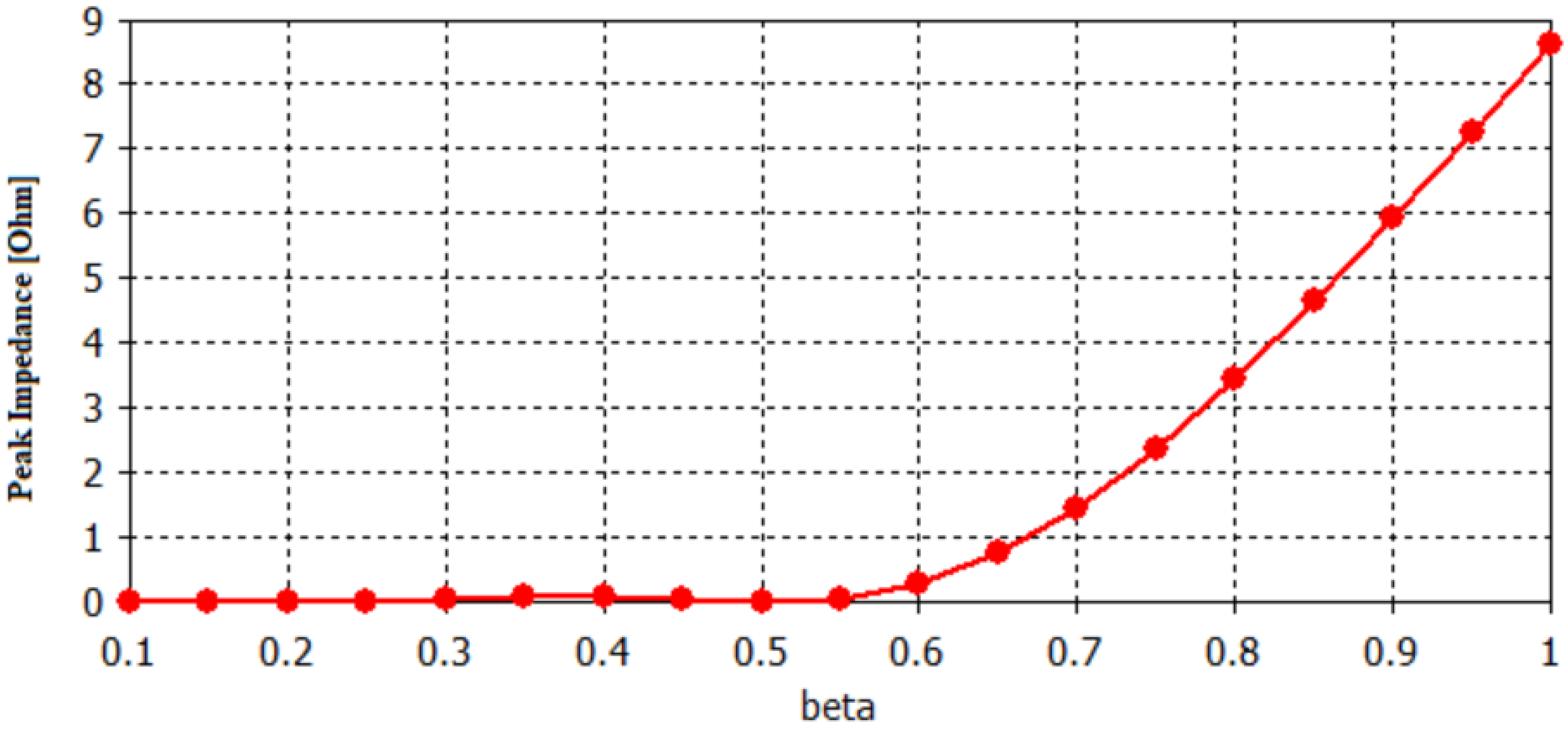
The **mode** is mainly **quadrupolar**:

- **$\beta = 1$: no radial field dependence**
  - $Z_{x,y}^{quad} = 0 \rightarrow Z_{x,y}^{gen}$ small
- **$\beta < 1$: radial field dependance**
  - $Z_{x,y}^{quad} \neq 0 \rightarrow Z_{x,y}^{gen}$ higher



[4] C. Zannini, "Electromagnetic simulations of a CERN accelerator component and experimental applications"

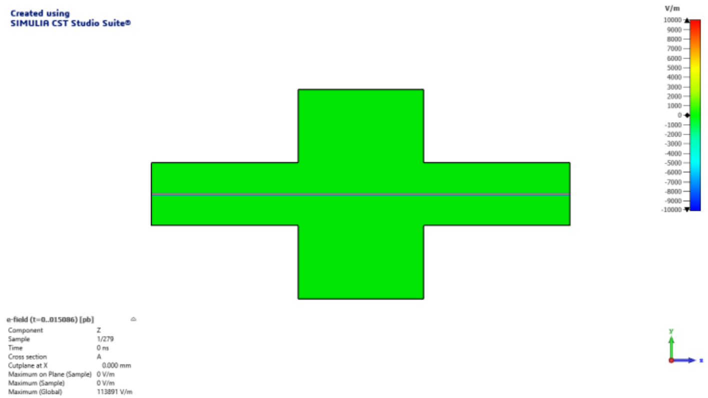
Longitudinal peak impedance varying β
for the cubic cavity used to benchmark `wakis`

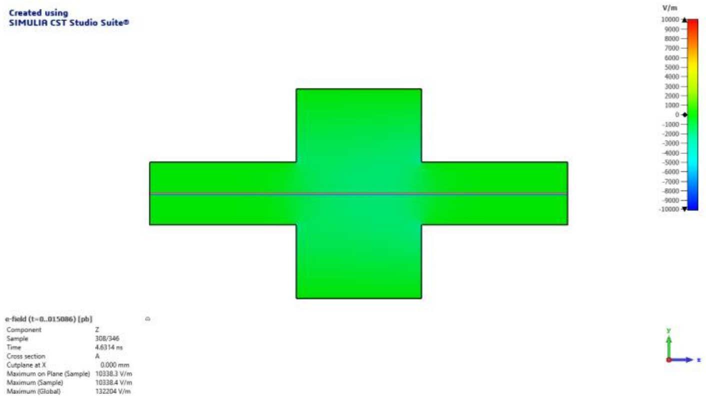


Time-domain field monitors

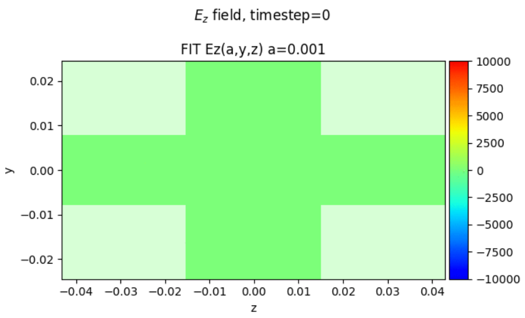
$\beta = 1$ | $\beta = 0.8$

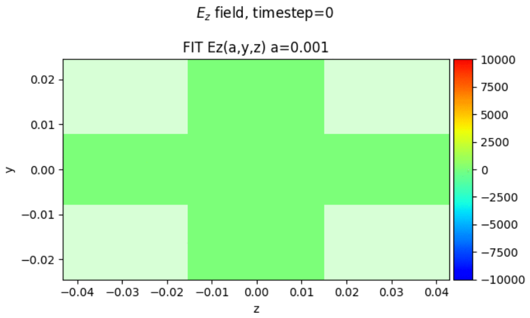
CST





wakis



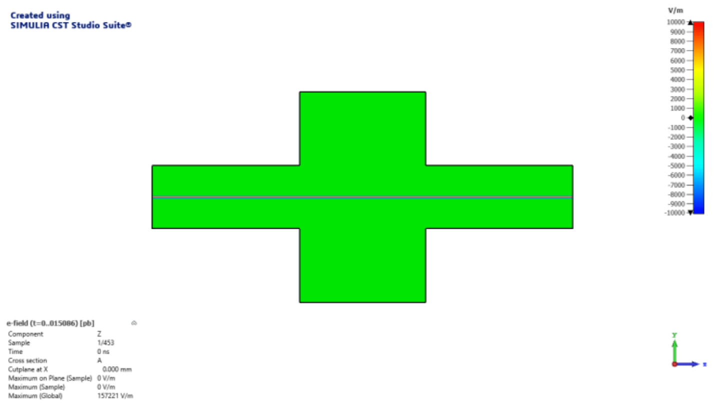


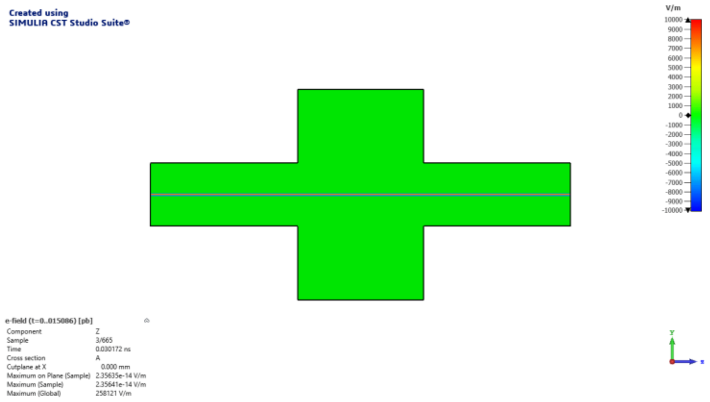
Courtesy of Elena de la Fuente García

Time-domain field monitors

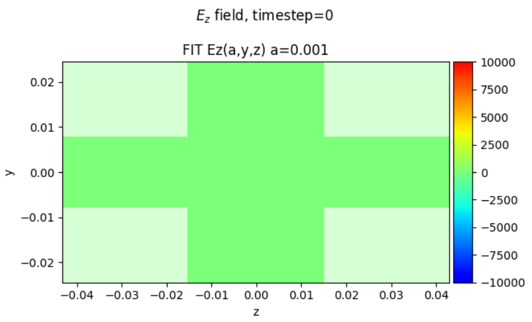
$\beta = 0.6$ $\beta = 0.4$

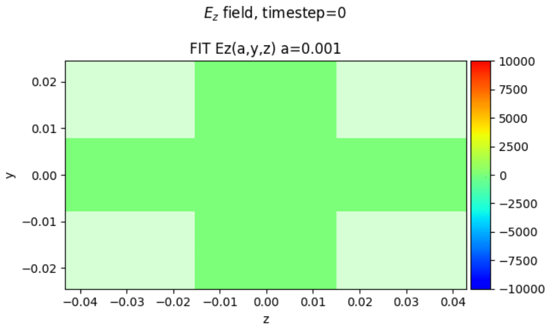
CST





wakis





Courtesy of Elena de la Fuente García

Wake potential scaling with β

Simulations on a resistive wall beam chamber show that the **wake potential**, both longitudinal and transverse, **scales with $\boldsymbol{\beta^{3/2}}$**, even though it should show the same behaviour as the impedance ($Z_{||}$ doesn't change with β and $Z_{\perp}$ scales with β).[3]

This is due to the **different dependence on time** in the definitions of the longitudinal and transverse wake potentials:

$$W_{||}(\mathrm{t}) = -\frac{\mathrm{L}}{4\pi^{3/2}b}\sqrt{\frac{Z_0}{c\sigma}}\cdot\frac{1}{t^{3/2}}$$

$$W_{\perp}(\mathrm{t}) = -\frac{\mathrm{L}}{\pi^{3/2}b^3}\sqrt{\frac{cZ_0}{\sigma}}\cdot\frac{1}{t^{1/2}}$$

$Z_{||}$ **constant with β** $\quad$ $Z_{\perp} \propto \beta$

$$t \propto \beta^{-1}$$

$W_{||} \propto \beta^{3/2}$ $\quad$ $W_{\perp} \propto \beta^{3/2}$

[3] ABP-CEI Section Meeting of 16 May 2024