Analyticity in $b \to s\ell\ell$ at two-loops

A paper with Christoph

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Christophest – October 25th, 2024

My relation to Christoph

- In May 2015 Christoph invited my to give a seminar at Bern. I gave a blackboard lunch seminar about $b \to s\gamma$, and a theory seminar about the (then recent) $b \to s\ell\ell$ anomalies.

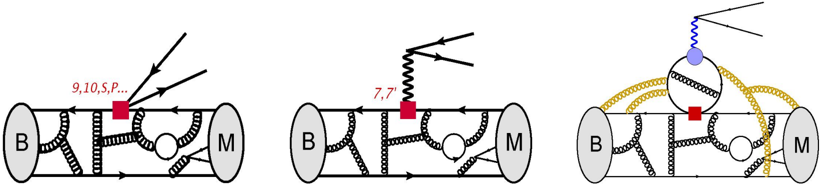
- In January 2016 Christoph offered me a postdoc at Bern. I accepted.

- I was a postdoc here from June 2016 to June 2017.

- I wrote two papers with Christoph: One together with Jason and Matteo, and the second one together with Hrachia on the two-loop calculation (finished in 2019). **This is the paper I will talk about today.**

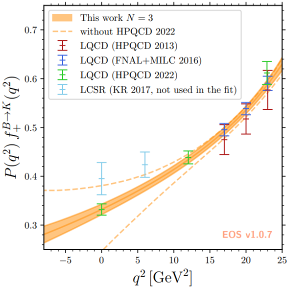
- My year in Bern was the peak of my career, my most prolific year. Bern had a true impact in my life, and in the life of my (2-y-o) son.

Anatomy of $B \to M_\lambda \ell^+ \ell^-$ EFT Amplitudes



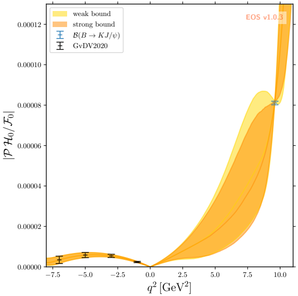
$$\mathcal{A}_\lambda^{L,R} = \mathcal{N}_\lambda \left\{ (C_9 \mp C_{10}) \mathcal{F}_\lambda(q^2) + \frac{2 m_b M_B}{q^2} \left[C_7 \mathcal{F}_\lambda^T(q^2) - 16\pi^2 \frac{M_B}{m_b} \mathcal{H}_\lambda(q^2) \right] \right\}$$

- Local (Form Factors) : $\mathcal{F}_\lambda^{(T)}(q^2) = \langle \bar{M}_\lambda(k) | \, \bar{s}\, \Gamma_\lambda^{(T)} \, b \, | \bar{B}(k+q) \rangle$
- Non-Local : $\mathcal{H}_\lambda(q^2) = i\, \mathcal{P}_\mu^\lambda \int d^4x \, e^{iq\cdot x} \, \langle \bar{M}_\lambda(k) | T\{ j^\mu_{\rm em}(x), \mathcal{C}_i\, \mathcal{O}_i(0) \} | \bar{B}(q+k) \rangle$



Local

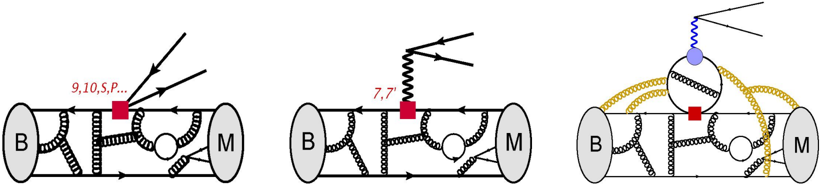
- Theory (LQCD / LCSRs)
- z-expansion (analyticity)
- Dispersive bounds (unitarity) [BGL/BCL]



Non-Local

- Theory (QCDF, (LC)OPE, models, ...)
- q^2-dependence
  - -z-expansion
  - -dispersion relations
  - -phenomenological
- Dispersive bounds
- Fits $\longrightarrow$ "data-driven" methods??

Non-Local Form Factors



$$\mathcal{A}_\lambda^{L,R} = \mathcal{N}_\lambda \left\{ (C_9 \mp C_{10}) \mathcal{F}_\lambda(q^2) + \frac{2 m_b M_B}{q^2} \left[C_7 \mathcal{F}_\lambda^T(q^2) - 16\pi^2 \frac{M_B}{m_b} \mathcal{H}_\lambda(q^2) \right] \right\}$$

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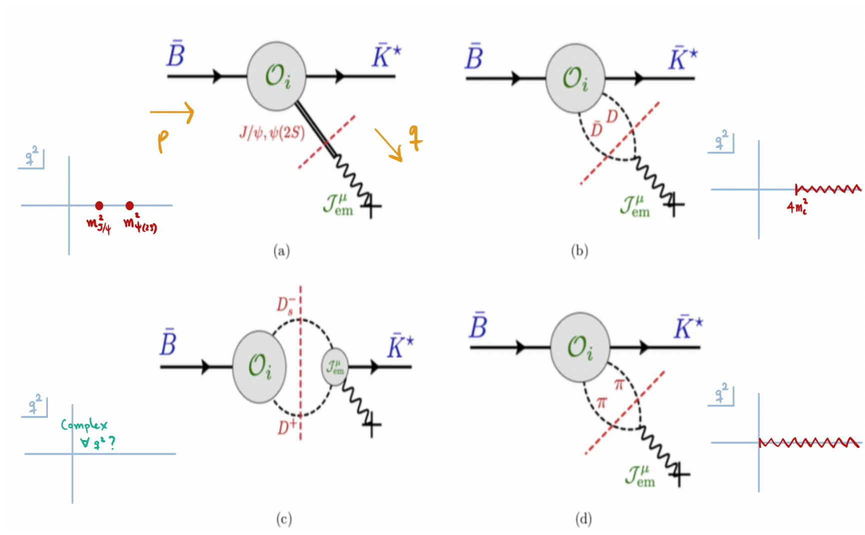
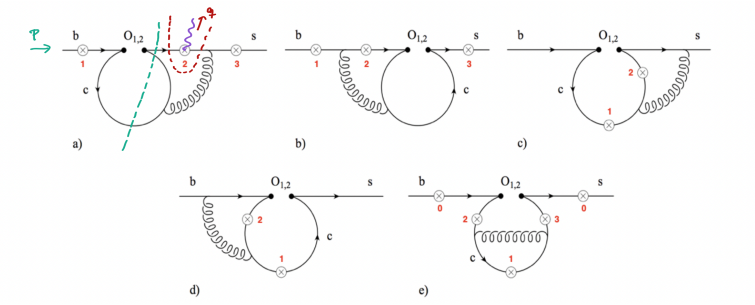
Non-Local Form Factors: Analytic structure



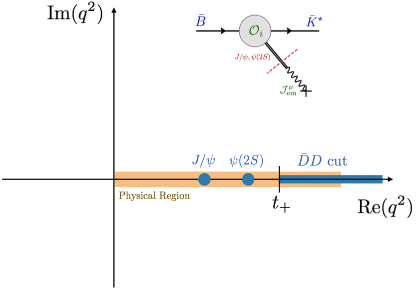
Non-Local Form Factors: Analytic structure

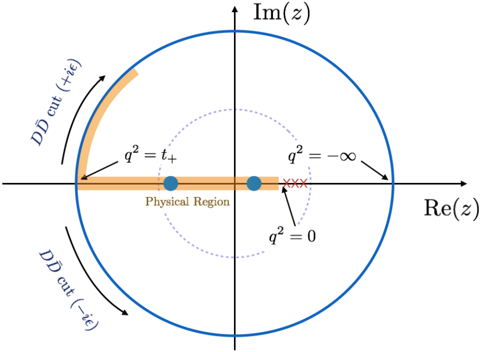
- There is a "light-hadron" cut for $q^2 > 0$, but it is OZI suppressed.
- p^2 cut makes $\mathcal{H}(q^2)$ complex everywhere, but does it affect q^2?
- Partonic calculation mimics all singularities (must be a Theorem)
- Two-loop partonic calculation confirms analytic structure

  Asatrian, Greub, Virto 2019

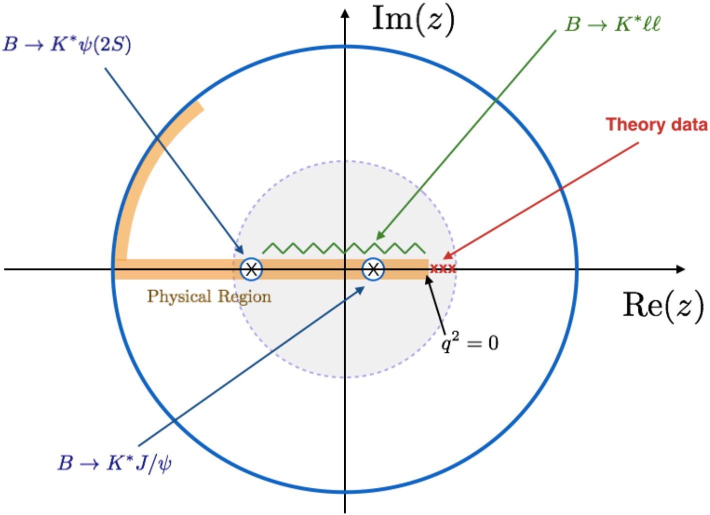


z-parametrisation for $\mathcal{H}_\lambda(q^2)$





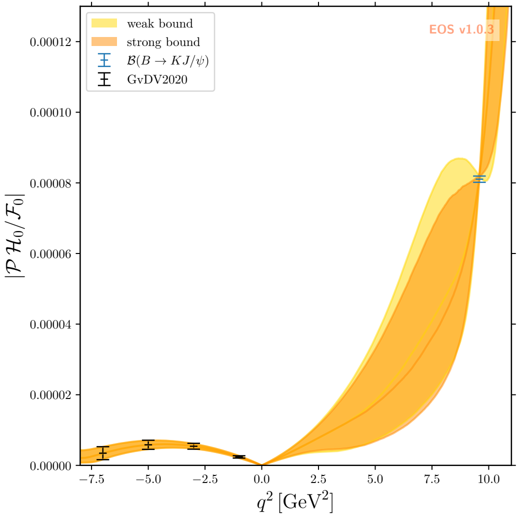
- $\hat{\mathcal{H}}_\lambda(q^2(z)) = (q^2 - M^2_{J/\psi})(q^2 - M^2_{\psi(2S)})\,\mathcal{H}_\lambda(q^2)$ is analytic in $|z| < 1$
- Taylor expand $\hat{\mathcal{H}}_\lambda(z)$ around $z = 0$: $\hat{\mathcal{H}}_\lambda(z) = \left[\sum_{k=0}^{K} \alpha_k^{(\lambda)} z^k\right] \mathcal{H}_\lambda(z)$
- Expansion needed for $|z| < 0.52$ ($-7\,\text{GeV}^2 \le q^2 \le 14\text{GeV}^2$)



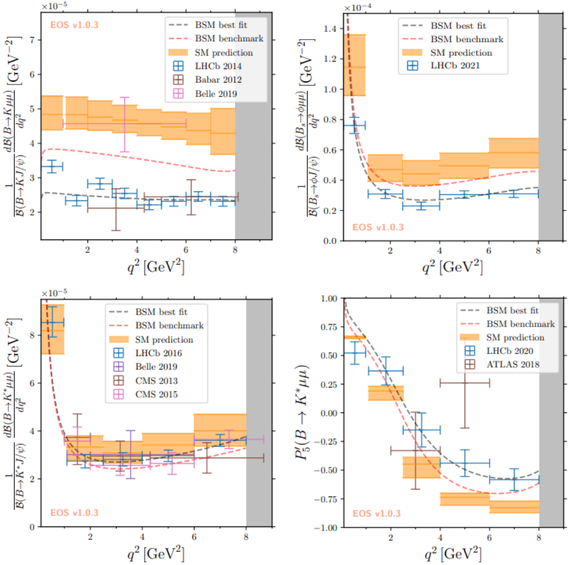
Im(z)
$B \to K^* \ell\ell$
$B \to K^* \psi(2S)$
Theory data
Physical Region
Re(z)
$q^2 = 0$
$B \to K^* J/\psi$

Results

Gubernari, Reboud, van Dyk, Virto 2022



Gubernari, Reboud, van Dyk, Virto 2022



Issues:

1. Is the **theory data** reliable?

2. Validity of z-expansion: Do we understand the **analytic structure**?

3. Truncation of z-expansion $\rightarrow$ **dispersive bound**

4. Technical aspects of **fits** (convergence, interpretation, ...)

Any **concern** should be linked to one of these points **clearly**.

Non-local form factors: Operator Product Expansion

$$\mathcal{H}^\mu(q,k) = i \int d^4x\, e^{iq\cdot x}\, \langle \bar{M}_\lambda(k)| \mathcal{T}\{\mathcal{J}^\mu_{\rm em}(x), \mathcal{C}_i\, \mathcal{O}_i(0)\} |\bar{B}(q+k)\rangle$$

- Large-q^2: Dominated by $x \sim 0$ (short-distance dominance - OPE)

Grinstein, Pirjol; Beylich, Buchalla, Feldmann

- Low-q^2: Dominated by $x^2 \sim 0$ (light-cone dominance - LCOPE)

Khodjamirian, Mannel, Pivovarov, Wang

+ analytically-continue from OPE region to physical region

Non-local form factors: Operator Product Expansion

We write

$$\mathcal{H}^\mu(q,k) = \langle \bar{M}_\lambda(k)|\mathcal{K}^\mu(q)|\bar{B}(q+k)\rangle$$

With the operator $\mathcal{K}^\mu(q)$ given by

$$\mathcal{K}^\mu(q) = i\int d^4x\, e^{iq\cdot x}\, \mathcal{T}\{\mathcal{J}^\mu_{\rm em}(x), \mathcal{C}_i\, \mathcal{O}_i(0)\}$$

It turns out that: **Leading-order OPE = Leading order LCOPE**

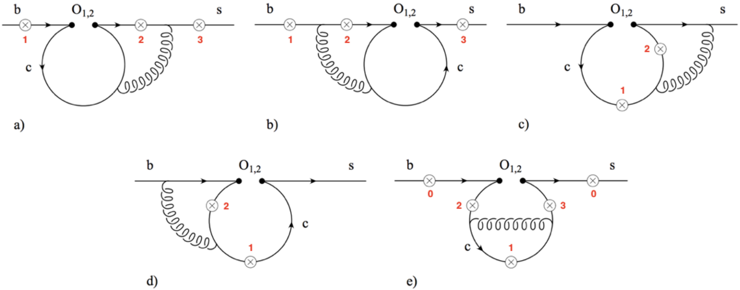
$$\mathcal{K}^\mu_{\rm OPE}(q) = \Delta C_9(q^2)\big(q^\mu q^\nu - q^2 g^{\mu\nu}\big)\,\bar{s}\gamma_\nu P_L b + \Delta C_7(q^2)\,2im_b\,\bar{s}\sigma^{\mu\nu}q_\nu P_R b + \cdots$$

With this we have:

$$\mathcal{H}^\mu_{\rm OPE}(q,k) = \Delta C_9(q^2)\big(q^\mu q^\nu - q^2 g^{\mu\nu}\big)\mathcal{F}_\nu + 2im_b\,\Delta C_7(q^2)\mathcal{F}^{T\mu} + \cdots$$

Asatrian, Greub, Virto 2019

Objective: Fully analytical calculation in two variables: q^2 and m_c.



$$\Delta C_7 = \mathcal{O}(\alpha_s) \quad \text{and} \quad \Delta C_9 \sim \log\left(4m_c^2 - q^2\right) + \mathcal{O}(\alpha_s)$$

Asatrian, Greub, Virto 2019

Two-loop Master Integrals

$$J_i(q^2, m_c) = (2\pi)^{-2d} \int \frac{(m_b^2)^{N_i - 4} (\tilde{\mu}^2)^{2\epsilon}\; d^d\ell\; d^d r}{P_{i_1}^{n_{i_1}} P_{i_2}^{n_{i_2}} P_{i_3}^{n_{i_3}} P_{i_4}^{n_{i_4}} P_{i_5}^{n_{i_5}} P_{i_6}^{n_{i_6}} P_{i_7}^{n_{i_7}}}$$

$$\begin{aligned}
P_1 &= (\ell + q)^2 - m_c^2 & P_5 &= (r + p - q)^2 & P_9 &= \ell \cdot q \\
P_2 &= \ell^2 - m_c^2 & P_6 &= r \cdot q & P_{10} &= (r + p - q)^2 - m_b^2 \\
P_3 &= (\ell + r)^2 - m_c^2 & P_7 &= \ell \cdot (p - q) & P_{11} &= (r + p)^2 - m_b^2 \\
P_4 &= r^2 & P_8 &= (r + p)^2 & P_{12} &= (\ell + r + q)^2 - m_c^2 \\
P_{13} &= r \cdot (p - q) & & & &
\end{aligned}$$

Differential Equations in Canonical Form

Henn 2013

$$J_i(q^2, m_c) = (2\pi)^{-2d} \int \frac{(m_b^2)^{N_i-4} (\tilde{\mu}^2)^{2\epsilon}\ d^d\ell\ d^d r}{P_{i_1}^{n_{i_1}} P_{i_2}^{n_{i_2}} P_{i_3}^{n_{i_3}} P_{i_4}^{n_{i_4}} P_{i_5}^{n_{i_5}} P_{i_6}^{n_{i_6}} P_{i_7}^{n_{i_7}}}$$

$$\partial_x J_{i,k}(\epsilon, x, y) = a_{i,x}^{k\ell}(\epsilon, x, y)\, J_{i,\ell}(\epsilon, x, y)\ , \quad \partial_y J_{i,k}(\epsilon, x, y) = a_{i,y}^{k\ell}(\epsilon, x, y)\, J_{i,\ell}(\epsilon, x, y)\ ,$$

$\rightarrow$ Transformation to “Canonical” Basis: $\vec{M}(x, y) = T(\epsilon, x, y) \cdot \vec{J}(x, y)$

$$\partial_x \vec{M}(\epsilon, x, y) = \epsilon\, A_x(x, y)\, \vec{M}(\epsilon, x, y) \quad ; \quad \partial_y \vec{M}(\epsilon, x, y) = \epsilon\, A_y(x, y)\, \vec{M}(\epsilon, x, y)$$

Need to find the right variables x, y

Asatrian, Greub, Virto 2019

Differential Equations in Canonical Form

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→ Transformation to "Canonical" Basis: $\vec{M}(x,y) = T(\epsilon,x,y)\cdot\vec{J}(x,y)$

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Need to find the right variables x, y, functions of $s = q^2/m_b^2$, $z = m_c^2/m_b^2$.

$$x_a = x_c = x_e = \frac{1}{\sqrt{1-4z}}\ , \quad x_b = x_d = \sqrt{4z} - \sqrt{4z-1}\ ,$$

$$y_a = \frac{1}{\sqrt{1-\frac{4z}{1-s}}}\ , \quad y_b = \frac{1}{\sqrt{1-\frac{4}{s}}}\ , \quad y_c = y_d = y_e = \frac{1}{\sqrt{1-\frac{4z}{s}}}\ .$$

$$t_b = \frac{-4x_b^2 + 4x_b^2 y_b + 2\sqrt{2}x_b^2(1+y_b)\sqrt{\frac{2x_b^4 - x_b^2 y_b + 2x_b^4 y_b - x_b^6 y_b + x_b^2 y_b^2 + 4x_b^4 y_b^2 + x_b^6 y_b^2}{x_b^4(1+y_b)^2}}}{-1 + 6x_b^2 - x_b^4 + y_b + 2x_b^2 y_b + x_b^4 y_b}\ ,$$

$$v_b = \frac{-4x_b^2 - 4x_b^2 y_b + 4\sqrt{2}x_b^2(1-y_b)\sqrt{\frac{2x_b^4 + x_b^2 y_b - 2x_b^4 y_b + x_b^6 y_b + x_b^2 y_b^2 + 4x_b^4 y_b^2 + x_b^6 y_b^2}{x_b^4(1-y_b)^2}}}{1 - 6x_b^2 + x_b^4 + y_b + 2x_b^2 y_b + x_b^4 y_b}\ .$$

similar to Huber, Bell 2014

Matching calculation at NLO

Asatrian, Greub, Virto 2019

Iterative solution of DEs

$$\partial_x \vec{M}(\epsilon, x, y) = \epsilon\, A_x(x, y)\, \vec{M}(\epsilon, x, y) \quad ; \quad \partial_y \vec{M}(\epsilon, x, y) = \epsilon\, A_y(x, y)\, \vec{M}(\epsilon, x, y)$$

$$\vec{M}(\epsilon, x, y) = \sum_{n=0}^{\infty} \epsilon^n\, \vec{M}_n(x, y)$$

$$\partial_{x,y} \vec{M}_n(x, y) = A_{x,y}(x, y) \vec{M}_{n-1}(x, y)$$

Iterative solution of DEs First y dependence, then x:

$$
\begin{aligned}
\vec{M}_0(x,y) &= \vec{C}_0(x)\ , \\
\vec{M}_1(x,y) &= \sum_{j_1} \left[A_y^{j_1}\, G(w_{j_1}(x);y)\right] \vec{C}_0(x) + \vec{C}_1(x)\ , \\
\vec{M}_2(x,y) &= \sum_{j_2,j_1} \left[A_y^{j_2}\, A_y^{j_1}\, G(w_{j_2}(x),w_{j_1}(x);y)\right] \vec{C}_0(x) \\
&\quad + \sum_{j_2} \left[A_y^{j_2}\, G(w_{j_2}(x);y)\right] \vec{C}_1(x) + \vec{C}_2(x)\ , \\
\vec{M}_3(x,y) &= \cdots
\end{aligned}
\qquad (1)
$$

Iterative solution of DEs

Solutions in terms of **Generalized Polylogarithms (GPLs)** Goncharov 1998

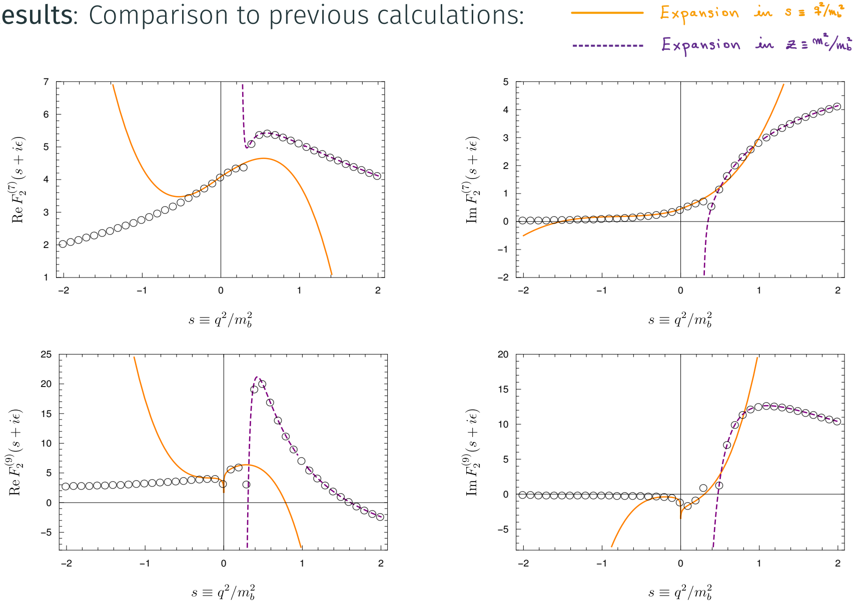
$$G(w_1,\dots,w_n;y) = \int_0^y \frac{dt}{t-w_1} G(w_2,\dots,w_n;t)\,; \quad G(;y)=1\,; \quad G(\vec{0}_n;x) = \frac{\log^n x}{n!}$$

i.e

$$G(1;x) = \log(1-x)\,, \quad G(0,1;x) = -Li_2(x)\,, \quad G(0,0,1;x) = -Li_3(x)\dots$$

Fast numerical evaluation of general GPLs in the complex plane available (C++, python, matlab, ...)

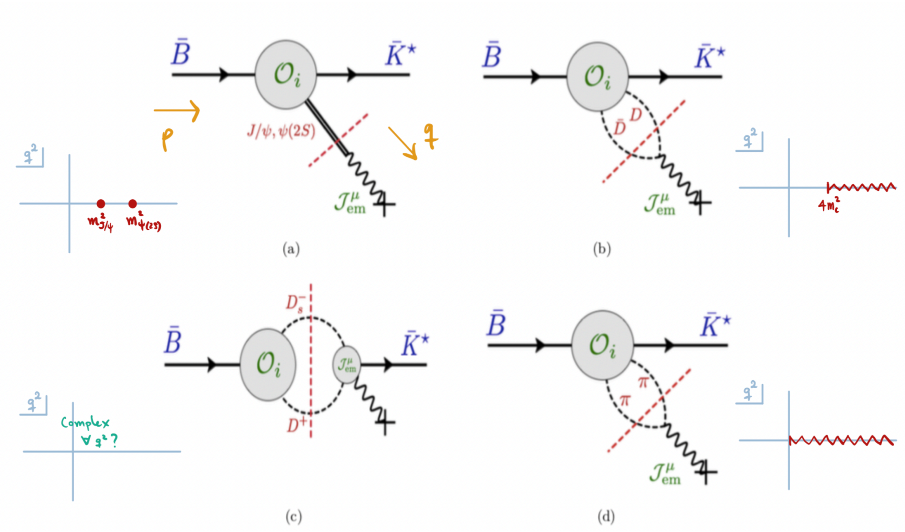
Results: Comparison to previous calculations:



Matching calculation at NLO

Asatrian, Greub, Virto 2019

Singularities of the partonic amplitude



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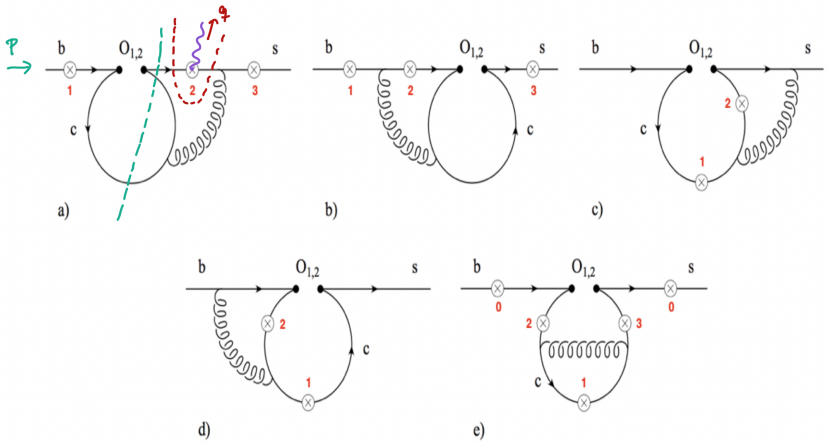
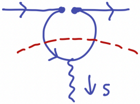
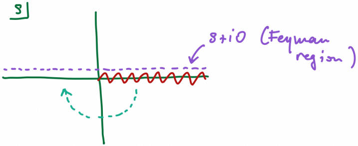


Checking analytic structure of $\mathcal{H}(q^2)$



$\sim \log(-s)$

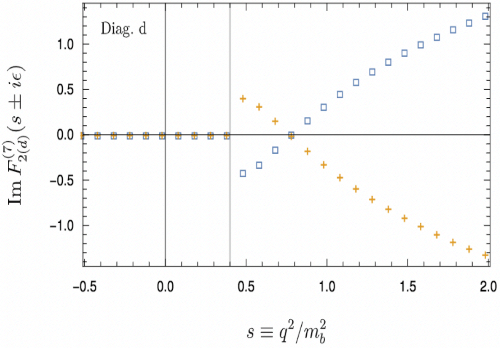


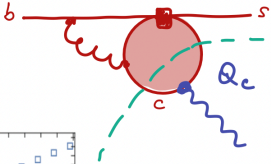
$= \log(s) + i\pi \longrightarrow$ cut is on the other side

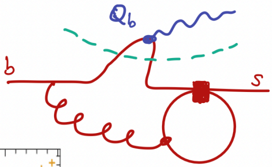
one could write $\log(-s) \to \begin{cases} \log(s) - i\pi & \text{Im}(s) \geq 0 \\ \log(s) + i\pi & \text{Im}(s) < 0 \end{cases}$

The desired analytic continuation is not guaranteed.

Checking analytic structure of $\mathcal{H}(q^2)$







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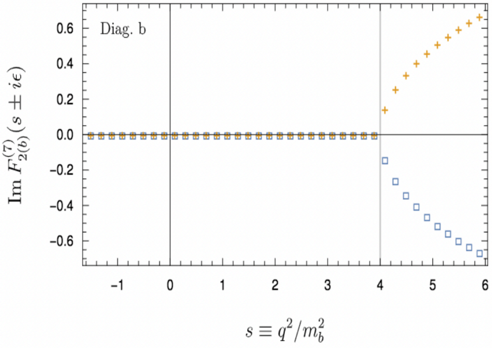
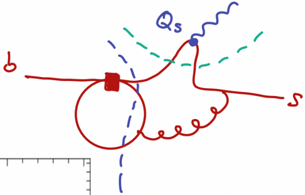
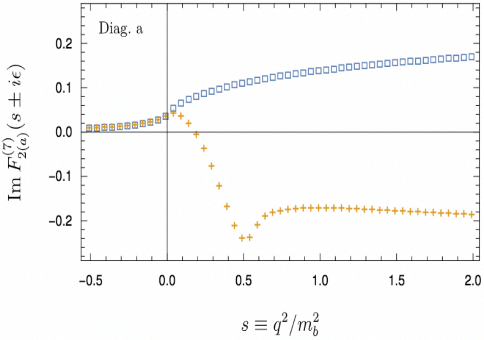




Checking analytic structure of $\mathcal{H}(q^2)$



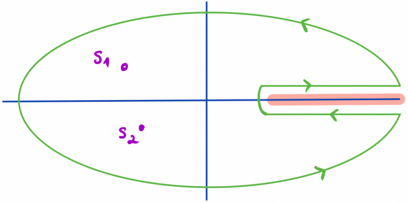
Direct check of analytic structure at two loops: Asatian, Greub, Virto 2019

$$F(s_1) - F(s_2) = \frac{s_1 - s_2}{2\pi i} \int_{s_{\rm th}}^{\infty} dt \frac{F(t+i0) - F(t-i0)}{(t-s_1)(t-s_0)}$$

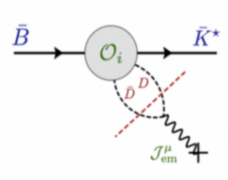
Example:

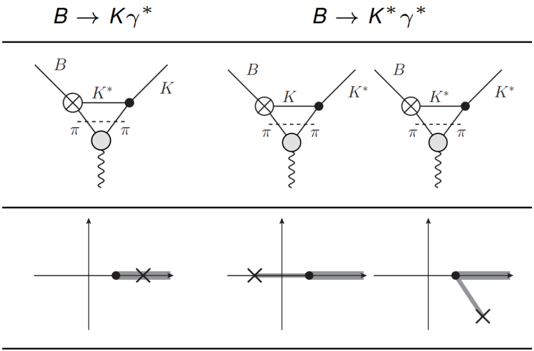
$$F^{(7)}_{2,(b)}(-3+i) - F^{(7)}_{2,(b)}(-1-2i) = 0.0894864 - 0.160827\, i\,,$$

$$\frac{-2+3i}{2\pi i} \int_4^{\infty} dt\, \frac{\mathrm{Disc}\, F^{(7)}_{2,(b)}(t)}{(t+3-i)(t+1+2i)} = 0.0894966 - 0.160839\, i\,.$$



M. Hoferichter, S. Mutke





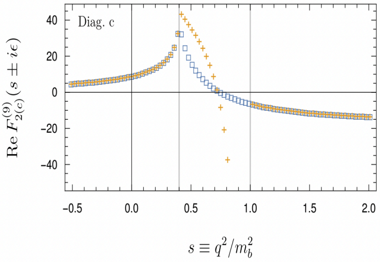
$$\Pi_{P,\lambda}(s) = \underbrace{\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\nu_{P,\lambda}(s')\, g_{P,\lambda}(s')\, F_\pi^{V*}(s')}{s'-s}}_{\equiv \Pi^{\rm norm}_{P,\lambda}(s)} + \underbrace{\frac{1}{\pi}\int_0^1 dx\, \frac{\partial s_x}{\partial x}\, \frac{\nu_{P,\lambda}(s_x)\, \mathrm{disc}\, g_{P,\lambda}(s_x)\, F_\pi^{V*}(s_x)}{s_x - s}}_{\equiv \Pi^{\rm anom}_{P,\lambda}(s)}$$

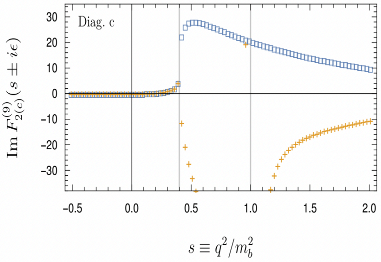
First mentioned by Ciuchini et al 2022 but in the context of the p^2 cut

Are they there? Are they sizable? Can we modify the z-expansion?

Asatrian, Greub, Virto 2019

- The discontinuities in diagrams a and c become purely imaginary for $s > 4z$ and $s > 1$, respectively.
- The contribution from diagrams c features a pole on the real axis when approaching the point $s = 1$ from the negative imaginary plane. This pole is related to an anomalous threshold.





Faster/better implementation of GPLs needed to check the dispersion relation in diagrams (c) [e.g. EOS] – (work by Viktor Kuschke)

Thank you