Causality analysis and late time attractors for spin hydrodynamics

Shi Pu (USTC)

West lake workshop on nuclear physics 2024, Hangzhou
Oc. 18, 2024

Based on:

- D.L. Wang, SP, Y. Li, arXiv: 2408.03781
- X. Ren, C. Yang, D.L. Wang, SP, **Phys.Rev.D** 110 (2024) 3, 3
- D.L. Wang, SP, **Phys.Rev.D (Lett),** 109 (2024) 3, L031504
- X.Q. Xie, C. Yang, D.L. Wang, SP, **Phys.Rev.D** 108 (2023) 9, 094031

Outline

- **Introduction to spin hydrodynamics**
- **Causality analysis on spin hydrodynamics**
  - **Causality conditions for spin hydrodynamics in linear mode analysis**
  - **Thermodynamic stability analysis for spin hydrodynamics**
- **New improved causality criterion**
- **Attractors and focusing behavior in spin hydrodynamics**
- **Summary and outlook**

Introduction to spin hydrodynamics

Spin in high energy physics

Striking spin effects have been observed in high energy reactions since 1970s

"Proton spin crisis" 质子自旋危机

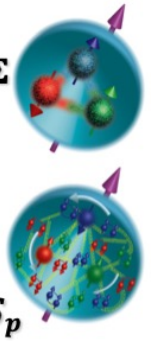
夸克模型：

夸克自旋之和Σ $=$质子自旋S_p

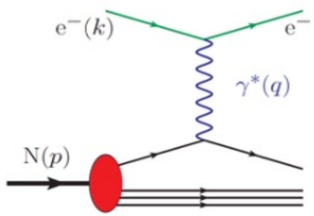
DIS实验：

89年: $\Sigma \sim 0$

目前: $\Sigma \sim 20\% S_p$

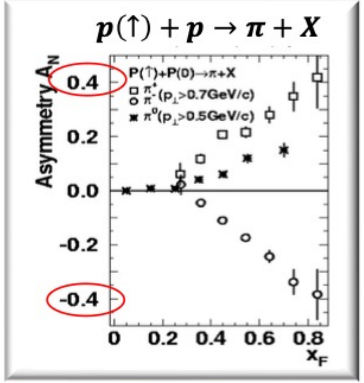


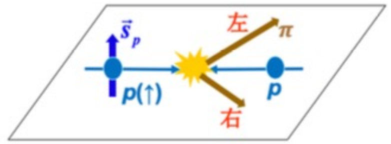
$$e + p \rightarrow e + X$$



EMC, PLB 206.364 (1988)

"Single spin left-right asymmetry (SSA)"



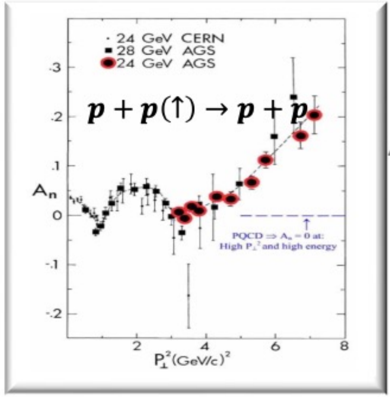


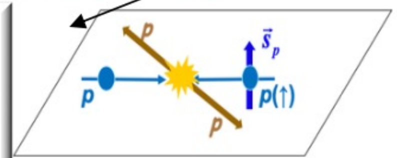
$$A_N \equiv \frac{N(\uparrow) - N(\downarrow)}{N(\uparrow) + N(\downarrow)}$$

e.g. FNAL E704, PLB264, 462 (1991)

Predictions of pQCD ～ 0

"Spin analyzing power in $pp \rightarrow pp$"

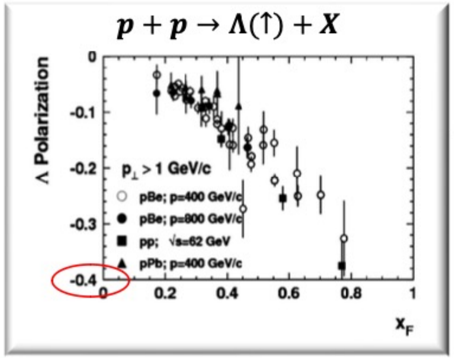


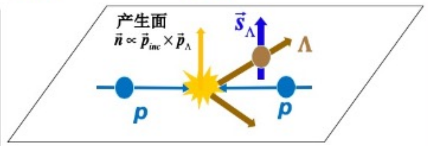


$$A_N \equiv -\frac{N(\uparrow) - N(\downarrow)}{N(\uparrow) + N(\downarrow)}$$

e.g. D. Grab *et al.*, PRL41, 1257 (1978)

"Transverse polarization of hyperon in $pp \rightarrow \Lambda X$"



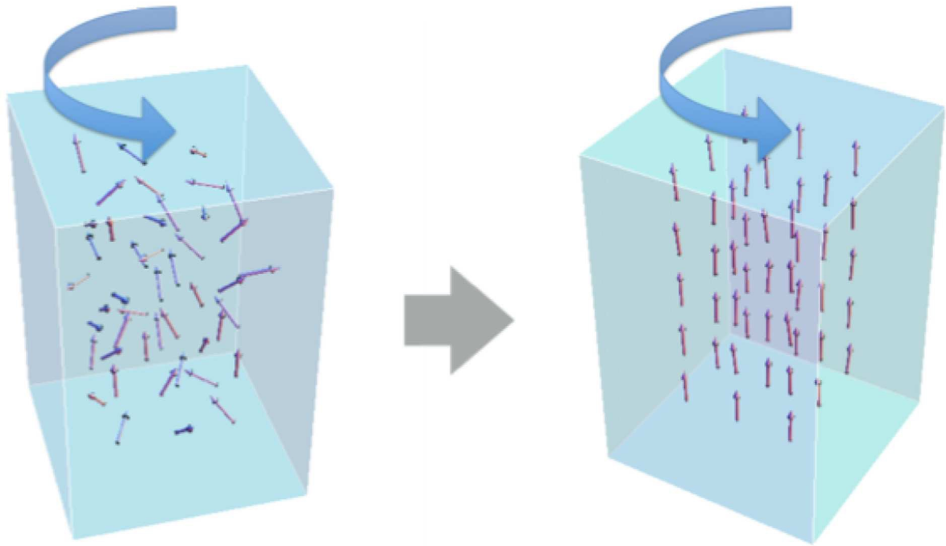


$$P_\Lambda \equiv \frac{\sigma(\uparrow) - \sigma(\downarrow)}{\sigma(\uparrow) + \sigma(\downarrow)}$$

e.g. S.A. Gourlay *et al.*, PRL56, 2244 (1986)

Slides copy from Prof. Zuo-tang Liang's review talk

Barnet and Einstein-de Hass effects



Barnett effect:

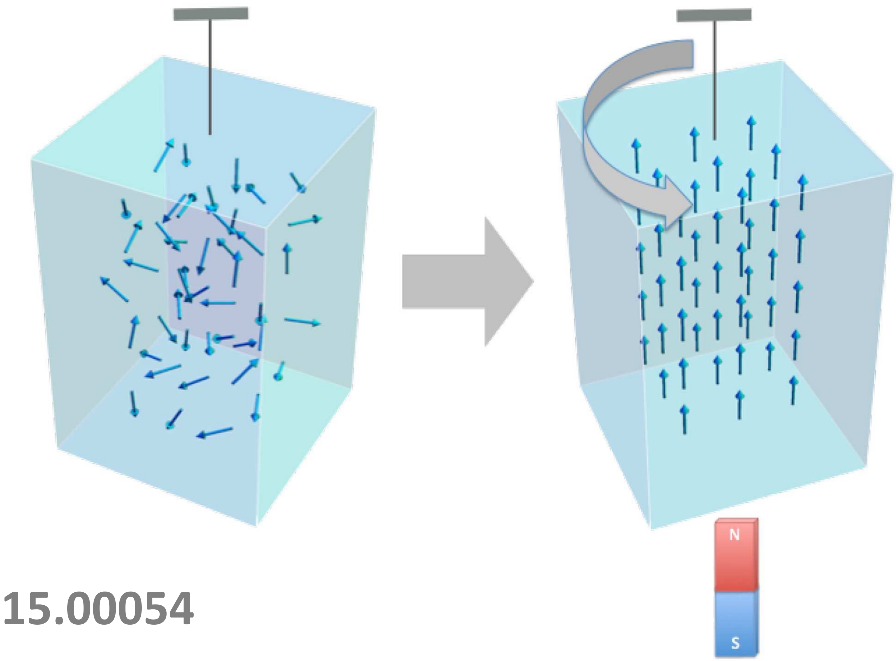
Rotation $\Longrightarrow$ Magnetization

Barnett, Magnetization by rotation, Phys Rev. (1915) 6:239–70.

Einstein-de Haas effect:

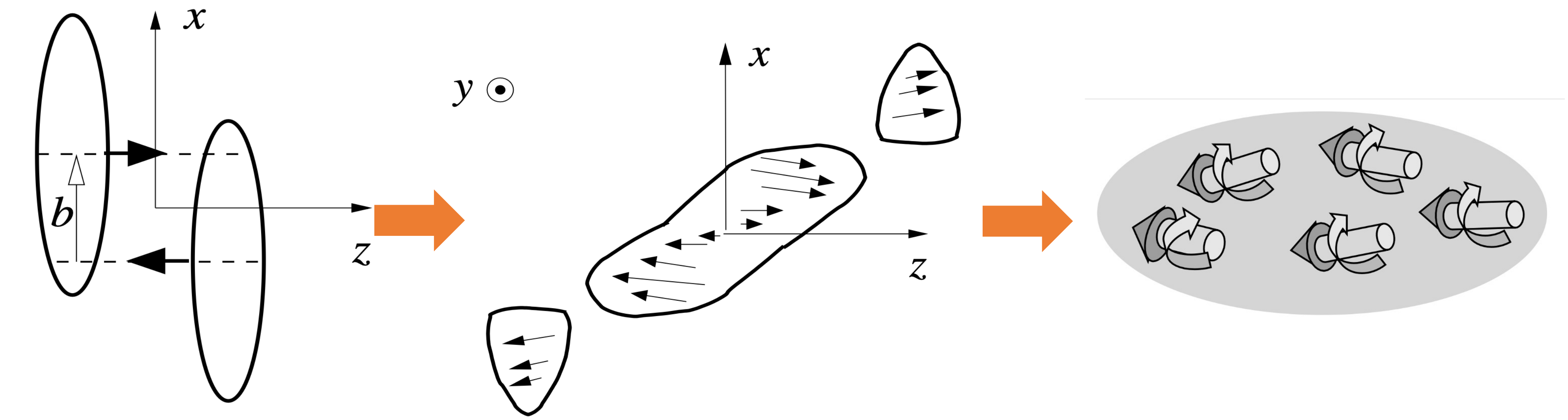
Magnetization $\Longrightarrow$ Rotation

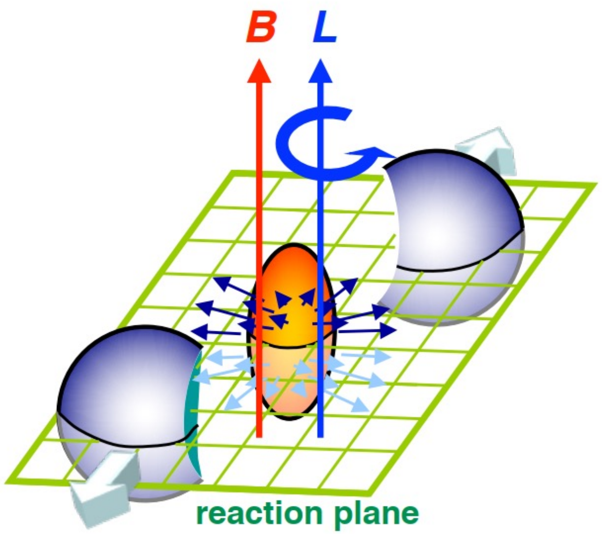
Einstein, de Haas, Experimental proof of the existence of Ampere's molecular currents. Verh Dtsch Phys Ges. (1915) 17:152.



Figures: copy from paper doi: 10.3389/fphy.2015.00054

OAM to polarization in HIC

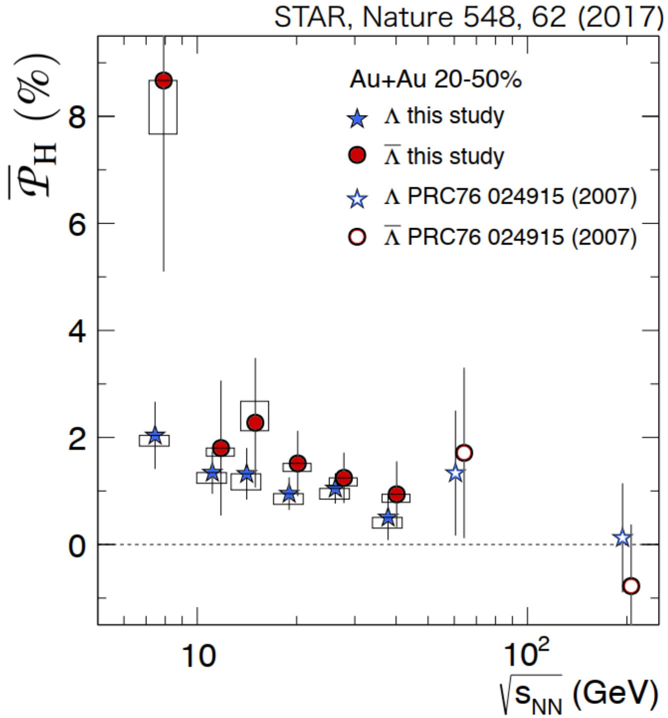




- **Huge global orbital angular momenta ($L\sim10^5\hbar$) are produced in HIC.**
- **Global orbital angular momentum leads to the polarizations of Λ hyperons and vector mesons through spin-orbital coupling.**

Liang, Wang, PRL (2005); PLB (2005);
Gao, Chen, Deng, Liang, Wang, Wang, PRC (2008)

Global polarization for Λ and $\bar{\Lambda}$ hyperons

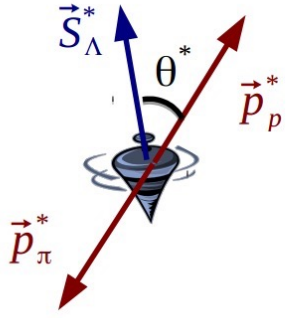


parity-violating decay of hyperons

In case of Λ's decay, daughter proton preferentially decays in the direction of Λ's spin (opposite for anti-Λ)

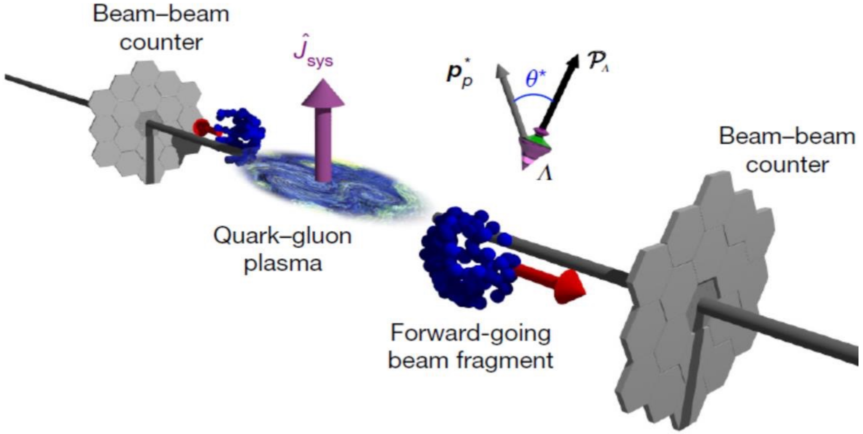
$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha \mathbf{P_\Lambda} \cdot \mathbf{p_p^*})$$

α: Λ decay parameter (=0.642±0.013)
P_Λ: Λ polarization
p_p^*: proton momentum in Λ rest frame



$$\Lambda \to p + \pi^+$$

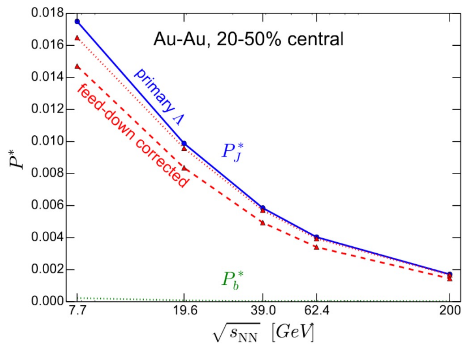
(BR: 63.9%, cτ ~7.9 cm)



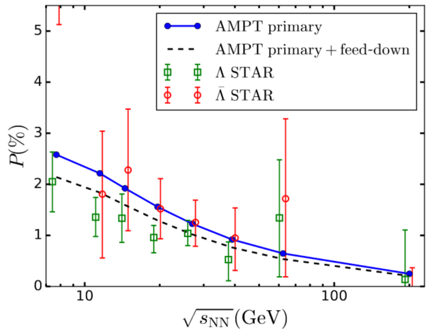
- **The vorticity of QGP can be as large as $(9 \pm 1)\times10^{21}$/s。**
- **It is the most vortical fluid so far.**

Liang, Wang, PRL (2005)
Betz, Gyulassy, Torrieri, PRC (2007)
Becattini, Piccinini, Rizzo, PRC (2008)
Becattini, Karpenko, Lisa, Upsal, Voloshin, PRC (2017)
Fang, Pang, Q. Wang, X. Wang, PRC (2016)
…

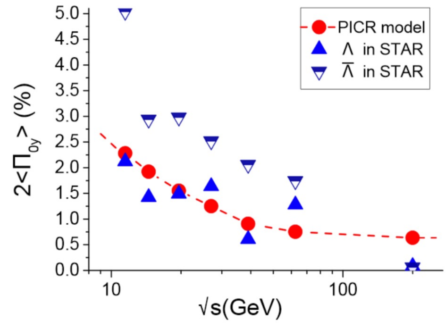
Phenomenological models for global polarization



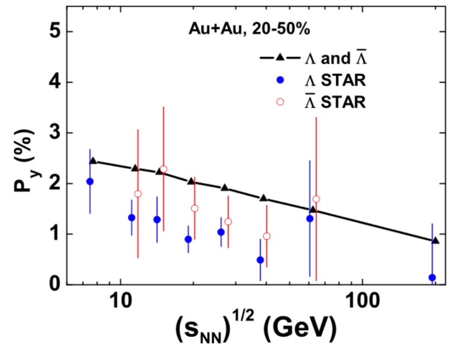
Karpenko, Becattini, EPJC(2017)



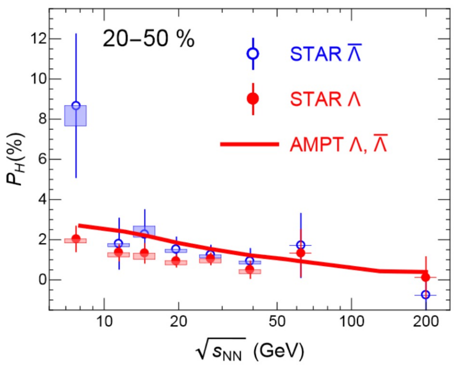
Li, Pang, Wang, Xia PRC(2017)



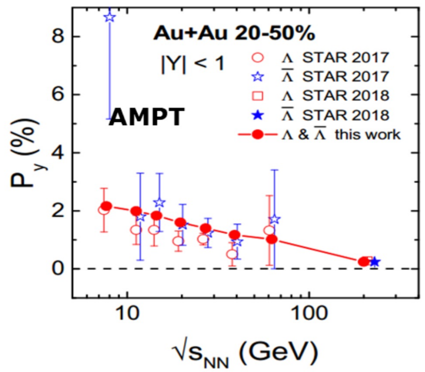
Xie, Wang, Csernai, PRC(2017)



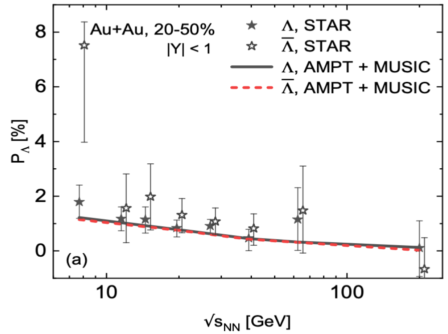
Sun, Ko, PRC(2017)



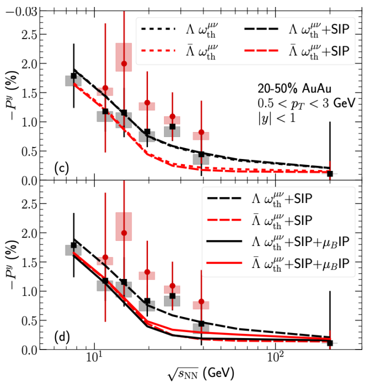
Shi, Li, Liao, PLB(2018)



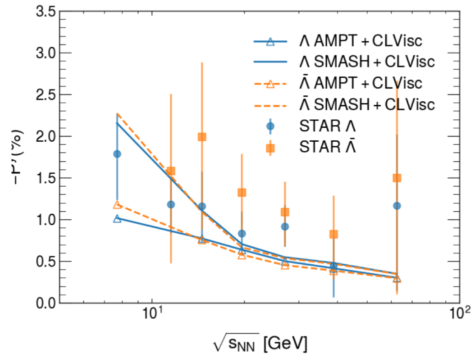
Wei, Deng, Huang, PRC(2019)



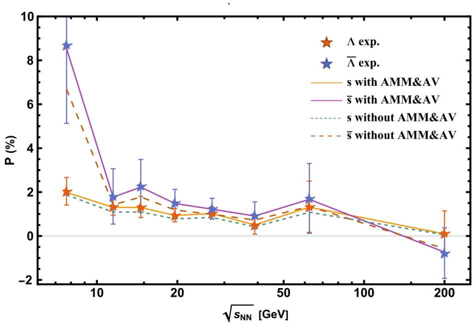
Fu, Xu, Huang, Song, PRC (2021)



S. Ryu, V. Jupic, C. Shen, PRC (2021)



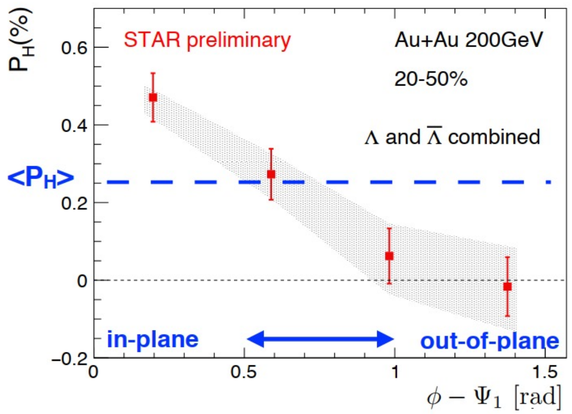
Y.X. Wu, C. Yi, G.Y. Qin, SP, PRC (2022)

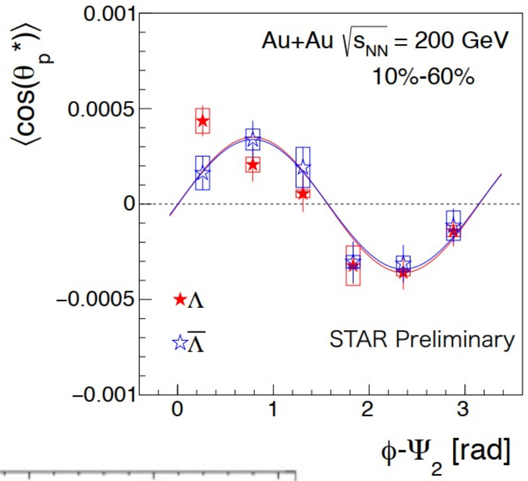


Xu, Lin, Huang, Huang, PRDL (2022)

Local polarization

$$\mathcal{S}^{\mu}_{\text{shear}}(\mathbf{p}) = -\frac{\hbar}{4m_{\Lambda}N}\int d\Sigma\cdot p f_V^{(0)}(1-f_V^{(0)})\frac{\epsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta}{(u\cdot p)T}\frac{1}{2}\{p^\sigma\boxed{(\partial_\sigma u_\nu+\partial_\nu u_\sigma)-Du_\nu}\}$$





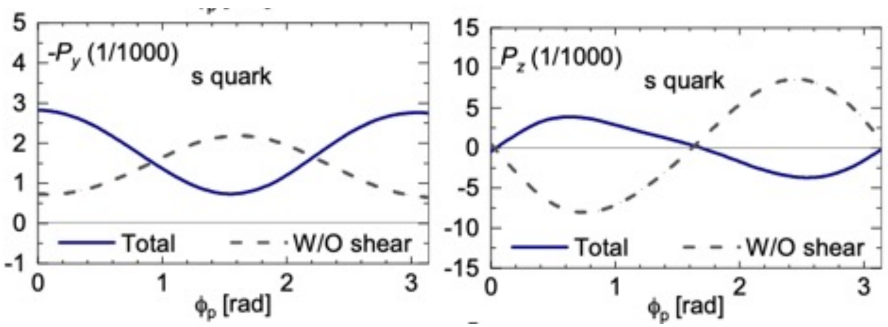
Early works:
(thermal vorticity only)

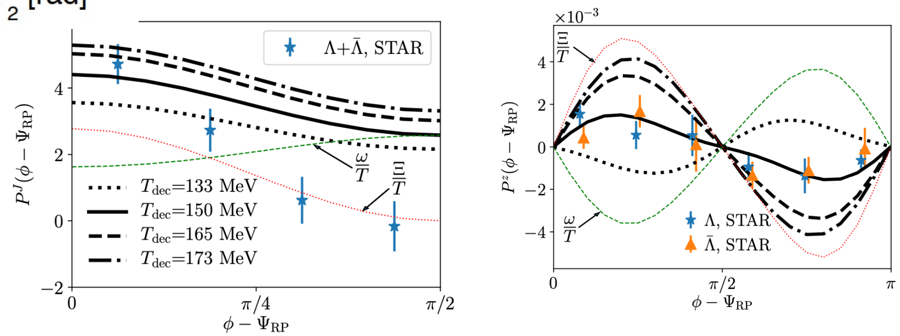
- **UrQMD :**

Becattini, Karpenko, PRL (2018)

- **AMPT:**

Xia, Li, Tang, Wang, PRC (2018)





s quark scenarios (Thermal vorticity + shear)

Fu, Liu, Pang, Song, Yin, PRL 2021

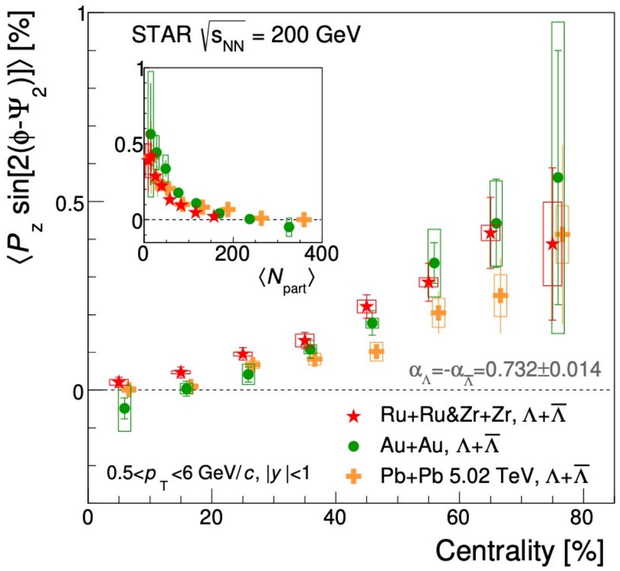
Isothermal equilibrium
(Thermal vorticity + shear)

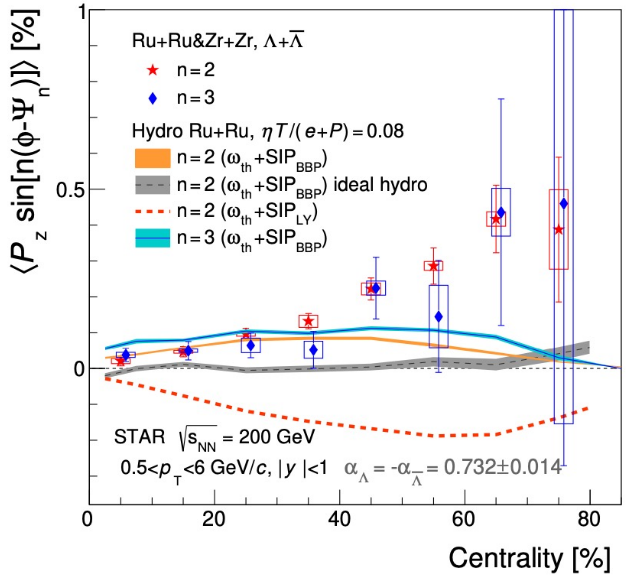
Becattini, Buzzegoli, Palermo, Inghirami, Karpenko, PRL 2021

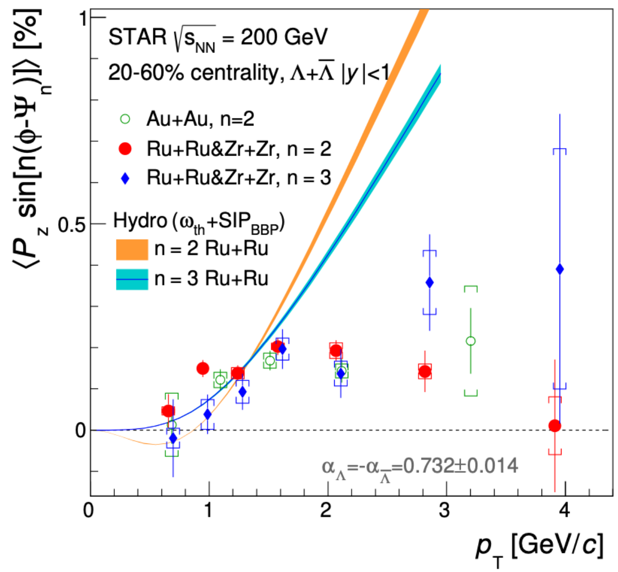
Also see:

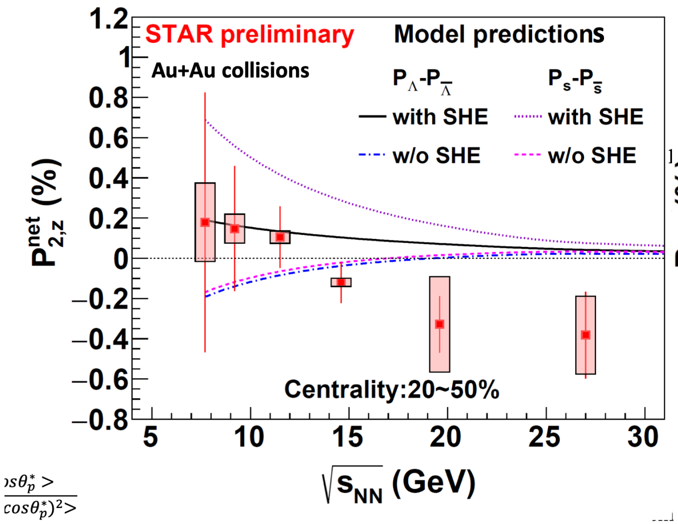
Yi, Pu, Yang, PRC (2021); Yi, Wu, Qin, Pu, PRC (2022)
Ryu, Jupic, Shen, PRC (2021)

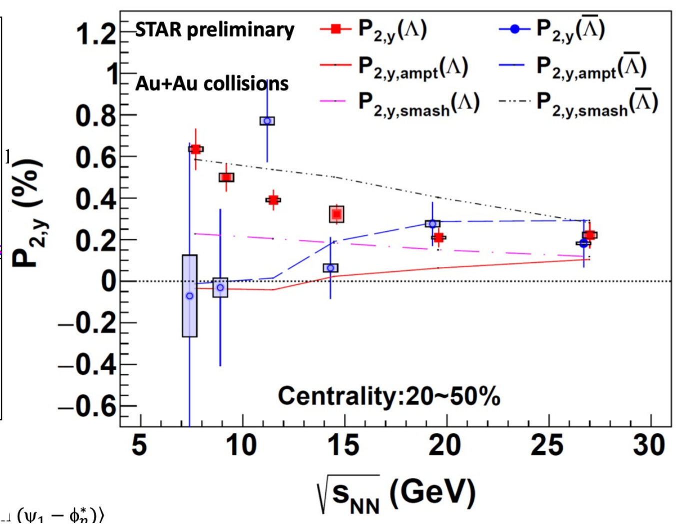
Puzzles in local polarization at AA system

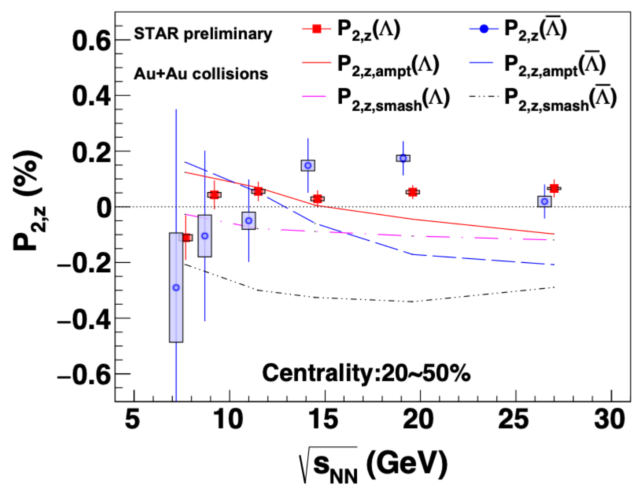




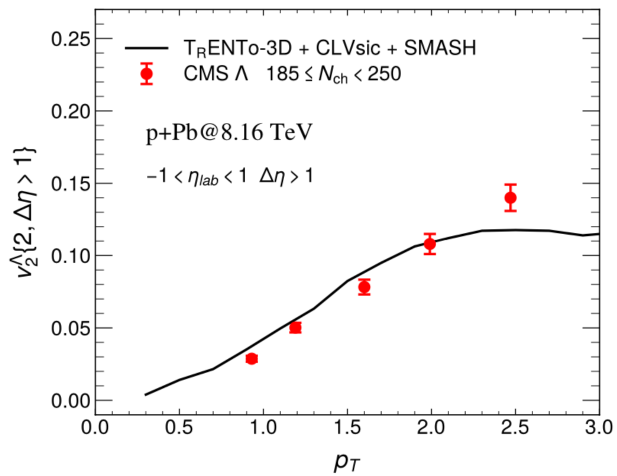


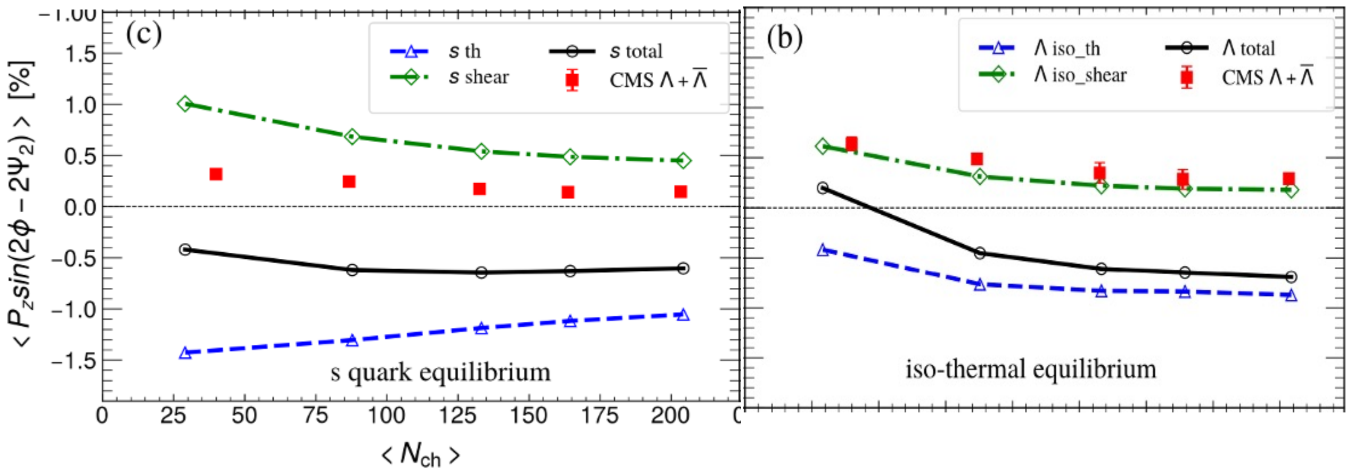






Puzzles in local polarization at pA system



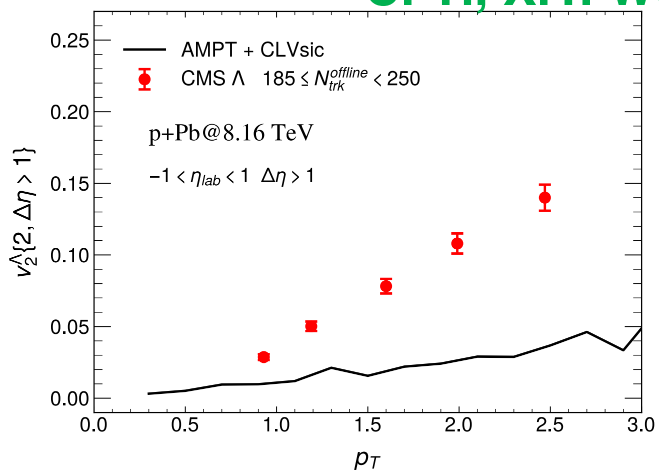


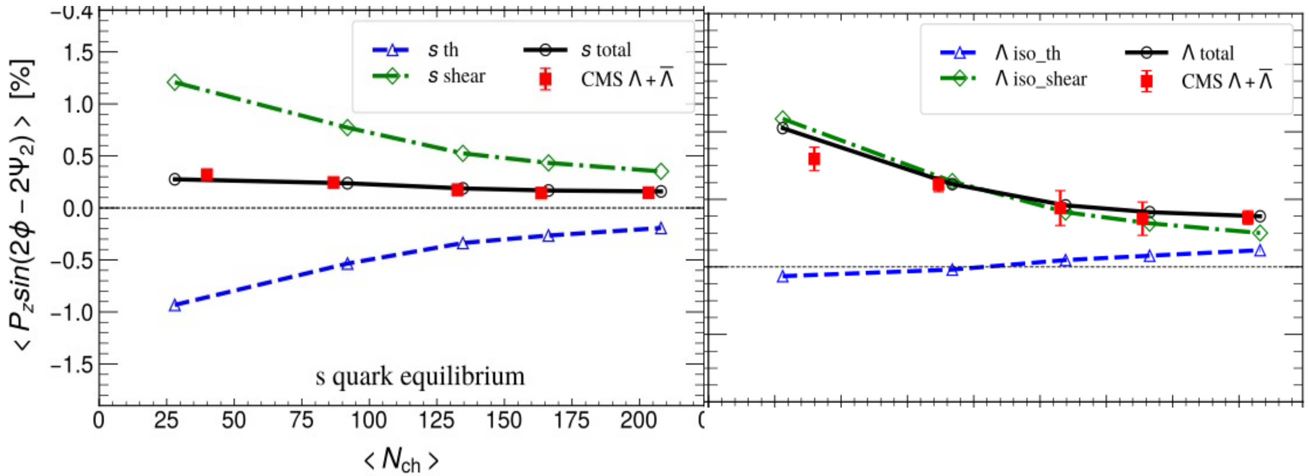
Smaller v2 gives a larger polarization along beam direction ?

Smaller v2, larger shear induced polarization, smaller thermal vorticial induced polarization

Sensitive to initial conditions?

C. Yi, X.Y. Wu, J. Zhu, SP, Q.Y. Qin, arXiv: 2408.04296





Theoretical developments

- **Spin hydrodynamics (macroscopic approach)**
  - Florkowski, Friman, Jaiswal, Ryblewski, Speranza (2017-2018);
  - Montenegro, Tinti, Torrieri (2017-2019);
  - Hattori, Hongo, Huang, Matsuo, Taya PLB(2019) ; arXiv: 2201.12390; arXiv: 2205.08051
  - Fukushima, SP, Lecture Note (2020); PLB(2021); Wang, Fang, SP, PRD(2021); Wang, Xie, Fang, SP, PRD (2022); …
  - S.Y. Li, M.A Stephanov, H.U Yee, arXiv:2011.12318
  - D. She, A. Huang, D.F. Hou, J.F Liao, arXiv: 2105.04060
  - Weickgenannt, Wanger, Speranze, Rischke, PRD 2022; PRD 2022; Weickgennatt, Wanger, Speranza, PRD 2022; arXiv:2306.05936;
  - Peng, Zhang, Sheng, Wang, CPL 2021
- **Quantum kinetic theory with collisions (microscopic approach)**
  - Weickgenannt, Sheng, Speranza, Wang, Rischke, PRD 100, 056018 (2019)
  - Hattori, Hidaka, Yang, PRD100, 096011 (2019); Yang, Hattori, Hidaka, arXiv: 2002.02612.
  - Liu, Mameda, Huang, arXiv:2002.03753.
  - Gao, Liang, PRD 2019
  - Wang, Guo, Shi, Zhuang, PRD100, 014015 (2019) ; Z.Y. Wang, arXiv:2205.09334;
  - Li ,Yee, PRD100, 056022 (2019)
  - Hou, Lin, arXiv: 2008.03862; Lin, arXiv: 2109.00184; Lin, Wang, arXiv:2206.12573
  - Fang, SP, Yang, PRD (2022)
- **Other approaches:**
  - Side-jump effect Liu, Sun, Ko PRL(2020)
  - Mesonic mean-field Csernai, Kapusta, Welle, PRC(2019)
  - Using different vorticity Wu, Pang, Huang, Wang, PRR (2019)
- **Recent reviews:**
  - Gao, Ma, SP, Wang, NST (2020)
  - Gao, Liang, Wang, IJMPA (2021)
  - Hidaka, SP, Yang, Wang, PPNP (2022)
  - Becattini, Buzzegoli, Niida, SP, Tang, Wang, arXiv:2402.04540

Basic conservation equations in canonical form

- **Total angular momentum conservation**

$$\partial_\alpha J^{\alpha\mu\nu}_{\text{can}} = 0 \qquad J^{\lambda\mu\nu} = \underbrace{x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu}}_{\text{Orbital part}} + \underbrace{\Sigma^{\lambda\mu\nu}}_{\text{Spin tensor}},$$

$$\Rightarrow \quad \boxed{\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]},}$$

- **Energy-momentum conservation**

$$\partial_\mu \Theta^{\mu\nu} = 0,$$

- **Currents conservation**

$$\partial_\mu j^\mu = 0,$$

Spin tensor, spin density and chemical potential

Spin density:
has 6 independent components
S^{ij} 3 rotating; S^{0i} 3 boosting

μ, ν is anti-symmetic

$$\Sigma^{\alpha\boxed{\mu\nu}} = u^\alpha S^{\mu\nu} + \Sigma^{\alpha\mu\nu}_{(1)}$$

spin tensor

Parallel to fluid velocity u^μ;
Leading order

Perpendicular to fluid velocity u^μ;
Higher order

Thermodynamic relations

$$e + p = Ts + \mu n + \omega_{\mu\nu} S^{\mu\nu}$$

energy density

pressure

temperature X entropy density

spin chemical potential

spin density

6-d.o.f Spin hydrodynamics

- **By using entropy principle, one can get**

$$\Theta^{\mu\nu} = (e+p)u^{\mu}u^{\nu} - pg^{\mu\nu} + 2h^{(\mu}u^{\nu)} + 2q^{[\mu}u^{\nu]} + \pi^{\mu\nu} + \phi^{\mu\nu},$$

$$q^{\mu} = \lambda[(u\cdot\partial)u^{\mu} + \frac{1}{T}\Delta^{\mu\nu}\partial_{\nu}T - 4\omega^{\mu\nu}u_{\nu}],$$

$$\phi^{\mu\nu} = 2\gamma[T\omega_{th}^{\mu\nu} + 2(g^{\mu\alpha} - u^{\mu}u^{\alpha})(g^{\nu\beta} - u^{\nu}u^{\beta})\omega_{\alpha\beta}]/T.$$

Spin hydrodynamics:

Florkowski, Friman, Jaiswal, Ryblewski, Speranza (2017-2018);
Montenegro, Tinti, Torrieri (2017-2019);
Hattori, Hongo, Huang, Matsuo, Taya PLB(2019) ; arXiv: 2201.12390; arXiv: 2205.08051
Fukushima, SP, Lecture Note (2020); PLB(2021); Wang, Fang, SP, PRD(2021); Wang, Xie, Fang, SP, PRD (2022)
S.Y. Li, M.A Stephanov, H.U Yee, arXiv:2011.12318
D. She, A. Huang, D.F. Hou, J.F Liao, arXiv: 2105.04060Weickgenannt, Wanger, Speranze, Rischke, PRD 2022; PRD 2022; Weickgennatt, Wanger, Speranza, PRD 2022; arXiv:2306.05936

Recent review:

SP, X.G. Huang, "Relativistic spin hydrodynamics", Acta Phys.Sin. 72 (2023) 7, 071202

Causality analysis on spin hydrodynamics

Causality and stability for relativistic systems

- **Causality:**

  The speed of propagating signal cannot be larger than the speed of light.

- **Stability:**

  The small perturbation near the equilibrium (or the solutions of differential equations) must decay with time.

Linear modes analysis (I)

- **Relativistic hydrodynamics:**

  Energy-momentum and currents conservation equations

- **In the linear mode analysis, one considers the perturbations of independent macroscopic variables within the system, e.g. energy density e , number density ρ , etc., near the equilibrium.**

$$\partial_t \varphi(t,\vec{x}) + \mathrm{M}(\partial)\varphi(t,\vec{x}) = 0,$$

$$\varphi(t,\vec{x}) = (\delta e, \delta\rho, ...)^{\mathrm{T}}$$

$$\mathrm{M}(\partial) = \sum_{i=0}^{N} \mathrm{M}^{(i)} \partial_{i_1}\partial_{i_2}...\partial_{i_N}$$

Linear mode analysis (II)

- **We usually consider a plane-wave type perturbation:**

$$\varphi = \varphi_0 e^{-i\omega t + i\vec{k}\cdot\vec{x}}, \quad \varphi_0 = \text{const.}.$$

- **The differential equations in linear mode analysis becomes**

$$0 = \mathcal{P}(\omega, \vec{k}) \equiv \det[\omega + i\mathrm{M}(\vec{k})].$$

- **Stability: perturbation decays with time**

$$\mathrm{Im}\ \omega \leq 0, \qquad \text{for } \vec{k} \in \mathbb{R}^3,$$

- **Causality: group velocity of perturbation is smaller than 1.**

$$\lim_{|\vec{k}|\to+\infty} \left\{ \frac{|\mathrm{Re}\ \omega|}{|\vec{k}|} \leq 1,\ |\omega/\vec{k}| \text{ is bounded} \right\},\ \vec{k} \in \mathbb{R}^3.$$

E. Krotscheck and W. Kundt, Communications in Mathematical Physics 60, 171 (1978)

Why is $|\omega/k|$ bounded?

- **Example: Non-relativistic diffusion equation which is acausal,**

$$\partial_t n - D_n \partial_x^2 n = 0 \qquad\qquad \omega = i D_n k^2$$

- **although**

$$\lim_{k\to+\infty} \frac{|Re\omega|}{|k|} = 0$$

- **Reason:**

$$n \sim n_0 e^{-D_n k^2 t}$$

With any initial value for $n(t_0, x)$, the $n(t_0 + \Delta t, x)$ at $x \to \infty$ can still get the influence. It does not obey the causality.

Applications to relativistic hydro (I)

- The conventional relativistic hydrodynamics up to the first order in gradient expansion is acausal and unstable.

- Relativistic hydrodynamics have been extended to
  - Second order hydro:

    Müller-Israel-Stewart (MIS) theory

    Baier-Romatschke-Son-Starinets-Stephanov (BRSSS) theory

    Denicol-Niemi-Molnar- Rischke (DNMR) theory
  - Generalized first order causal hydro

    Bemfica-Disconzi-Noronha-Kovtun (BDNK) theory

Applications to relativistic hydro (II)

- **Asymptotic causality condition :**

$$\frac{\text{Shear viscosity}}{\boxed{\text{Relaxation time for shear viscous tensor}} \times \left(\text{Energy density} + \text{Pressure}\right)} \leq \frac{3}{4}\left(1 - \text{Speed of sound}^{2}\right)$$

$$\boxed{\frac{\text{剪切粘滞系数}}{\text{弛豫时间} \times (\text{能量密度} + \text{压强})} \leq \frac{3}{4}\left[1 - \text{声速}^{2}\right]}$$

SP, Koide, Rischke, Phys. Rev. D 81, 114039 (2010)

- **Does stability of relativistic dissipative fluid dynamics imply causality?**

Causality conditions for spin hydrodynamics in linear mode analysis

X.Q. Xie, C. Yang, D.L. Wang, SP, **Phys.Rev.D** 108 (2023) 9, 094031



1^{st} order spin hydrodynamics

- **We now consider the small perturbations on top of static equilibrium,**

$$\varphi = \{\delta e, \delta u^i, \delta S^{\mu\nu}\} \qquad \varphi = \varphi_0 e^{-i\omega t + i\vec{k}\cdot\vec{x}}, \qquad \varphi_0 = \text{const.}.$$

- **Main linearized equations for spin hydro becomes**

$$\mathcal{M}_1 \delta\tilde{X}_1 \ = \ 0,$$

$$\delta\tilde{X}_1 \equiv (\delta\tilde{e}, \delta\tilde{\vartheta}^x, \delta\tilde{S}^{0x}, \delta\tilde{\vartheta}^y, \delta\tilde{S}^{0y}, \delta\tilde{S}^{xy}, \delta\tilde{\vartheta}^z, \delta\tilde{S}^{0z}, \delta\tilde{S}^{xz}, \delta\tilde{S}^{yz})^{\mathrm{T}},$$

$$\mathcal{M}_1 \equiv \begin{pmatrix} M_1 & 0 & 0 & 0 \\ A_1 & M_2 & 0 & 0 \\ A_2 & 0 & M_2 & 0 \\ A_3 & 0 & 0 & M_3 \end{pmatrix},$$

Propagating modes in 1st order spin hydro

- **Large k->infinity limit**

$$\omega = -4iD_b\gamma_{\|}^{-1}\lambda'^{-1}k^{-2} + O(k^{-3}),$$
$$\omega = -ic_s^{2/3}\gamma_{\|}^{1/3}\boxed{k^{4/3}} + O(k),$$
$$\omega = (-1)^{1/6}c_s^{2/3}\gamma_{\|}^{1/3}\boxed{k^{4/3}} + O(k),$$
$$\omega = (-1)^{5/6}c_s^{2/3}\gamma_{\|}^{1/3}\boxed{k^{4/3}} + O(k),$$
$$\omega = -2iD_b + O(k^{-1}),$$
$$\omega = 2iD_s\gamma_\perp(\gamma' + \gamma_\perp)^{-1} + O(k^{-1}),$$
$$\omega = \pm ik\sqrt{2\lambda'^{-1}(\gamma' + \gamma_\perp)} + O(k^0),$$

$$\omega = i(\gamma' + \gamma_\perp)\boxed{k^2} \text{ as } k \to \infty,$$

Breaks causality criteria

$$\lim_{k\to\infty}\left|\frac{\omega}{k}\right| \text{ is bounded.}$$

1st order spin hydrodynamics is always unstable and acausal!

Minimal extension to 2nd order hydro

- **We add the relaxation time term to the spin hydro**

$$
\begin{aligned}
\boxed{\tau_q \Delta^{\mu\nu} \frac{d}{d\tau} q_\nu} + q^\mu &= \lambda [T^{-1} \Delta^{\mu\alpha} \partial_\alpha T + (u\cdot\partial) u^\mu - 4\omega^{\mu\nu} u_\nu], \\
\boxed{\tau_\phi \Delta^{\mu\alpha} \Delta^{\nu\beta} \frac{d}{d\tau} \phi_{\alpha\beta}} + \phi^{\mu\nu} &= 2\gamma_s \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_{[\alpha} u_{\beta]} + 2\omega_{\alpha\beta}), \\
\tau_\pi \Delta^{\alpha<\mu} \Delta^{\nu>\beta} \frac{d}{d\tau} \pi_{\alpha\beta} + \pi^{\mu\nu} &= 2\eta \partial^{<\mu} u^{\nu>}, \\
\tau_\Pi \frac{d}{d\tau} \Pi + \Pi &= -\zeta \partial_\mu u^\mu,
\end{aligned}
$$

Shear viscous tensor

Bulk viscous pressure

Also see: **Y.C. Liu and X.G. Huang, Nucl. Sci. Tech. 31, 56 (2020), 2003.12482.**

Causality conditions

- **Causality can be satisfied if the following inequalities are fulfilled:**

$$0 \leq \frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi \tau_\Pi} \leq 1 \text{ and } 0 \leq \frac{2\tau_q(\gamma' \tau_\pi + \gamma_\perp \tau_\phi)}{(2\tau_q - \lambda')\tau_\pi \tau_\phi} \leq 1,$$

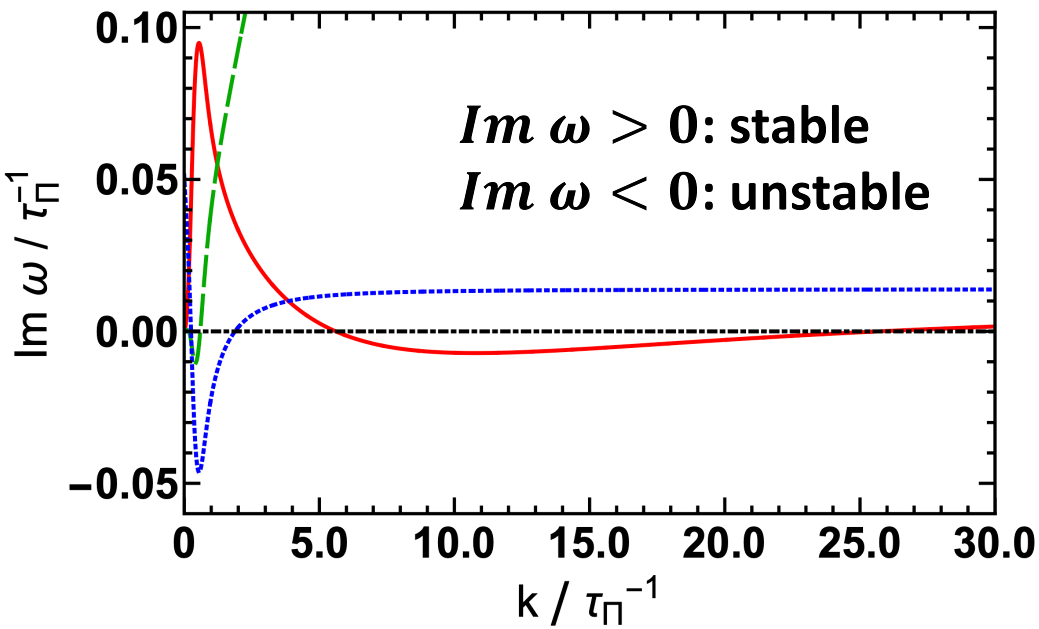
$$\begin{aligned} b_1 &= \{8\gamma_\perp \tau_q \tau_\Pi + \tau_\pi[2\tau_q(3\gamma_{\|} - 4\gamma_\perp) + 3\tau_\Pi c_s^2(3\lambda' + 2\tau_q)]\}^2, \\ b_2 &= 12 c_s^2 \lambda'(2\tau_q - \lambda')\tau_\pi \tau_\Pi[\tau_\pi(3\gamma_{\|} - 4\gamma_\perp) + 4\gamma_\perp \tau_\Pi]. \end{aligned}$$

Non-trivial Stability conditions

- **We implement the conventional stability conditions and derive the following inequality:**

$$
\begin{aligned}
\tau_q &> \lambda'/2, \\
D_s &> 0, \quad D_b < -4c_s\lambda\gamma_{\parallel}^{-1}\left|\chi_e^{0x}\right| \leq 0, \\
b_1 &> b_2 > 0, \quad \frac{c_2}{c_3} > 0.
\end{aligned}
$$

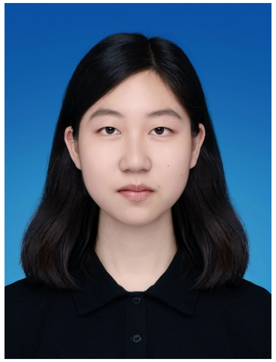
- **The above conditions can make the system be stable at k->0 and k->infinity limits. But, the system is unstable for finite k!**



Conventional stability criteria fails ?!!

Thermodynamic stability in relativistic viscous and spin hydrodynamics

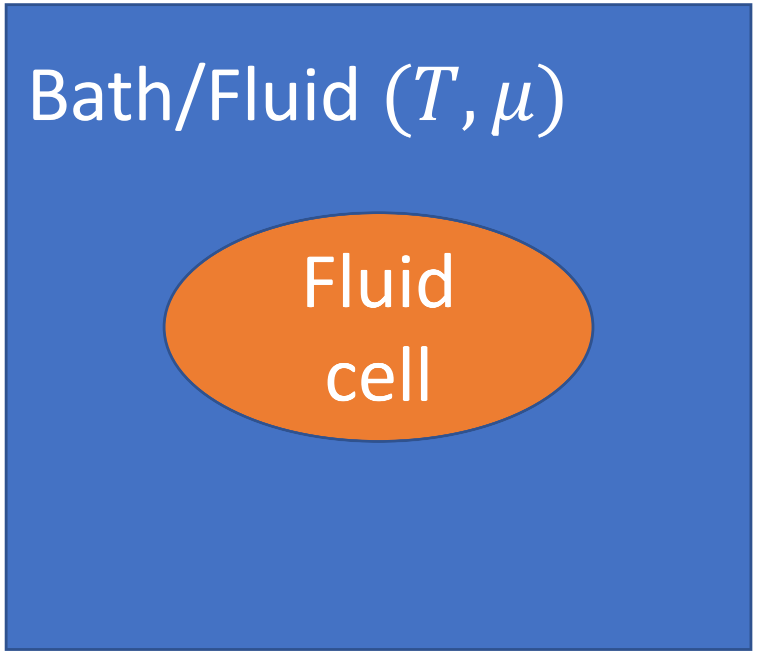
X. Ren, C. Yang, D.L. Wang, SP, **Phys.Rev.D** 110 (2024) 3, 3

Thermodynamic stability

- **Considering a fluid cell with a bath (fluid).**
- **Thermodynamic stability: 2nd law of thermodynamics**

$$\Delta S = \Delta S_F + \Delta S_B \geq 0,$$



L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. Lett. 128, 010606 (2022)

Thermodynamic stability

$$dS = -\sum_a \alpha^a dQ^a.$$

Q^a : **conserved quantities**
α^a : **their thermodynamic conjugates**

e.g. Q: number density, α: chemical potential
Q: energy momentum tensor, α: temperature velocity u^μ/T

$$\Delta S = \Delta S_F + \Delta S_B$$
$$= \Delta S_F - \sum \alpha_B^a dQ_B^a$$
$$= \Delta S_F + \sum \alpha_F^a dQ_F^a \geq 0$$

$$dQ_B^a = -dQ_F^a.$$

Total Q is conserved.

$$\alpha_F^a \simeq \alpha_B^a$$

The bath is huge, fluid cell shares the same α with bath.

L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. Lett. 128, 010606 (2022)

Information current

- **One can also define the information current**

$$E^{\mu} \equiv -\delta s_F^{\mu} - \sum_a \alpha_F^a \delta J_F^{a,\mu},$$

- **The thermodynamic stability (2nd law of thermodynamics) requires,**

$$E \equiv \int d\Sigma\, E^{\mu} n_{\mu} \geq 0,$$

Σ : the space-like three-dimensional surface for fluid cell
n^{μ}: time-like and future- directed normal unit vector for Σ

- **the information current Eμ must satisfy the following conditions, (thermodynamic stability conditions)**

(i). $E^{\mu} n_{\mu} \geq 0$ for any n^{μ} with $n_0 > 0, n^{\mu} n_{\mu} = 1,$

(ii). $E^{\mu} n_{\mu} = 0$ if and only if all perturbations are zero,

(iii). $\partial_{\mu} E^{\mu} \leq 0.$

L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. Lett. 128, 010606 (2022)

Thermodynamic analysis VS linear modes analysis

Thermodynamic analysis	Linear modes analysis
2nd law of thermodynamics	Perturbation should decay with time; Propagating of perturbation cannot be faster than the speed of light.
Information current	Plane-wave type perturbation
(i). $E^\mu n_\mu \geq 0$ for any n^μ with $n_0 > 0, n^\mu n_\mu = 1$, (ii). $E^\mu n_\mu = 0$ if and only if all perturbations are zero, (iii). $\partial_\mu E^\mu \leq 0$.	$\text{Im}\ \omega \leq 0, \qquad \text{for } \vec{k} \in \mathbb{R}^3,$ $\lim_{\lvert\vec{k}\rvert\to+\infty} \left\{ \frac{\lvert\text{Re}\ \omega\rvert}{\lvert\vec{k}\rvert} \leq 1,\ \lvert\omega/\vec{k}\rvert \text{ is bounded} \right\},\ \vec{k} \in \mathbb{R}^3.$

Example: Thermodynamic stability for viscous hydrodynamics

- We consider a relativistic hydro with viscous tensors only.
- We consider the variation of hydrodynamic quantities.

$$
\begin{aligned}
\frac{2n_0 T E^\mu n_\mu}{e+P} = & \frac{n_0^2\tau_\pi}{\eta(e+P)}\sum_{i<j}\left[\delta\pi^{ij}-\frac{1}{n_0\chi_\pi}n_{(i}\delta u_{j)}\right]^2 \\
& +\frac{n_0^2\tau_\pi}{\eta(e+P)}\left[\delta\pi^{11}+\frac{1}{2}\delta\pi^{22}+\frac{1}{2n_0\chi_\pi}(n_3\delta u_3-n_1\delta u_1)\right]^2 \\
& +\frac{3n_0^2\tau_\pi}{4\eta(e+P)}\left[\delta\pi^{22}+\frac{1}{3n_0\chi_\pi}(n_3\delta u_3+n_1\delta u_1-2n_2\delta u_2)\right]^2 \\
& +\sum_{i=1}^{5}a_i(\delta A_i)^2, \quad \geq 0
\end{aligned}
$$

Information current includes the variation of hydro quantities up to the 2nd order.

$$
\boxed{
\begin{aligned}
c_s^2,\tau_\pi,\tau_\Pi &> 0, \\
1-c_s^2-\frac{4\eta}{3\tau_\pi(e+P)}-\frac{\zeta}{\tau_\Pi(e+P)} &> 0,
\end{aligned}
}
$$

They are the same as casual and stable conditions derived in linear modes analysis!

G. S. Denicol, T. Kodama, T. Koide, and P. Mota, J. Phys. G 35, 115102 (2008) ;
SP, T. Koide, and D. H. Rischke, Phys. Rev. D 81, 114039 (2010)

X. Ren, C. Yang, D.L. Wang, SP, Phys.Rev.D 110 (2024) 3, 3

Application to spin hydrodynamics

$$
\begin{aligned}
c_s^2, \lambda, \gamma_s, \eta, \zeta, \tau_q, \tau_\phi, \tau_\pi, \tau_\Pi, -\chi_b, \chi_s &> 0, \\
1 - \frac{\lambda'}{2\tau_q} - \frac{4\gamma_\perp}{3\tau_\pi} - \frac{1}{3\tau_\Pi}(3\gamma_\parallel - 4\gamma_\perp) &> 0, \\
1 - \frac{\lambda'}{2\tau_q} - \frac{\gamma_\perp}{\tau_\pi} - \frac{\gamma'}{\tau_\phi} &> 0, \\
1 - c_s^2 - \frac{(1+3c_s^2)\lambda'}{2\tau_q} - \frac{(2\tau_q - c_s^2\lambda')[4\gamma_\perp\tau_\Pi + \tau_\pi(3\gamma_\parallel - 4\gamma_\perp)]}{6\tau_q\tau_\pi\tau_\Pi} &> 0, \\
2 - c_s^2 - \frac{(2+3c_s^2)\lambda'}{2\tau_q} - \frac{4\gamma_\perp\tau_\Pi + \tau_\pi(3\gamma_\parallel - 4\gamma_\perp)}{3\tau_\pi\tau_\Pi} &> 0,
\end{aligned}
$$

- **The above conditions can give the causality condition for spin hydrodynamics.**
- **The thermodynamic stability conditions for spin hydrodynamics are <u>more stringent</u> than those derived from linear mode analysis.**

X. Ren, C. Yang, D.L. Wang, SP, Phys.Rev.D 110 (2024) 3, 3

Improved Causality and stability criteria in linear response theory

D.L. Wang, SP, **Phys.Rev.D (Lett),** 109 (2024) 3, L031504

Motivation

Let me put it in this way, the dumplings were made just for this vinegar.

Problems in conventional analysis

- **A practical challenge arises:**

Commonly, the causality and stability conditions are first derived from the conventional criteria in the rest frame. Then, the verification of these criteria in other reference frames follows. However, this process of examining conditions across different frames is frequently burdensome.

- **A concern arises:**

Conventional causality criterion is incomplete.

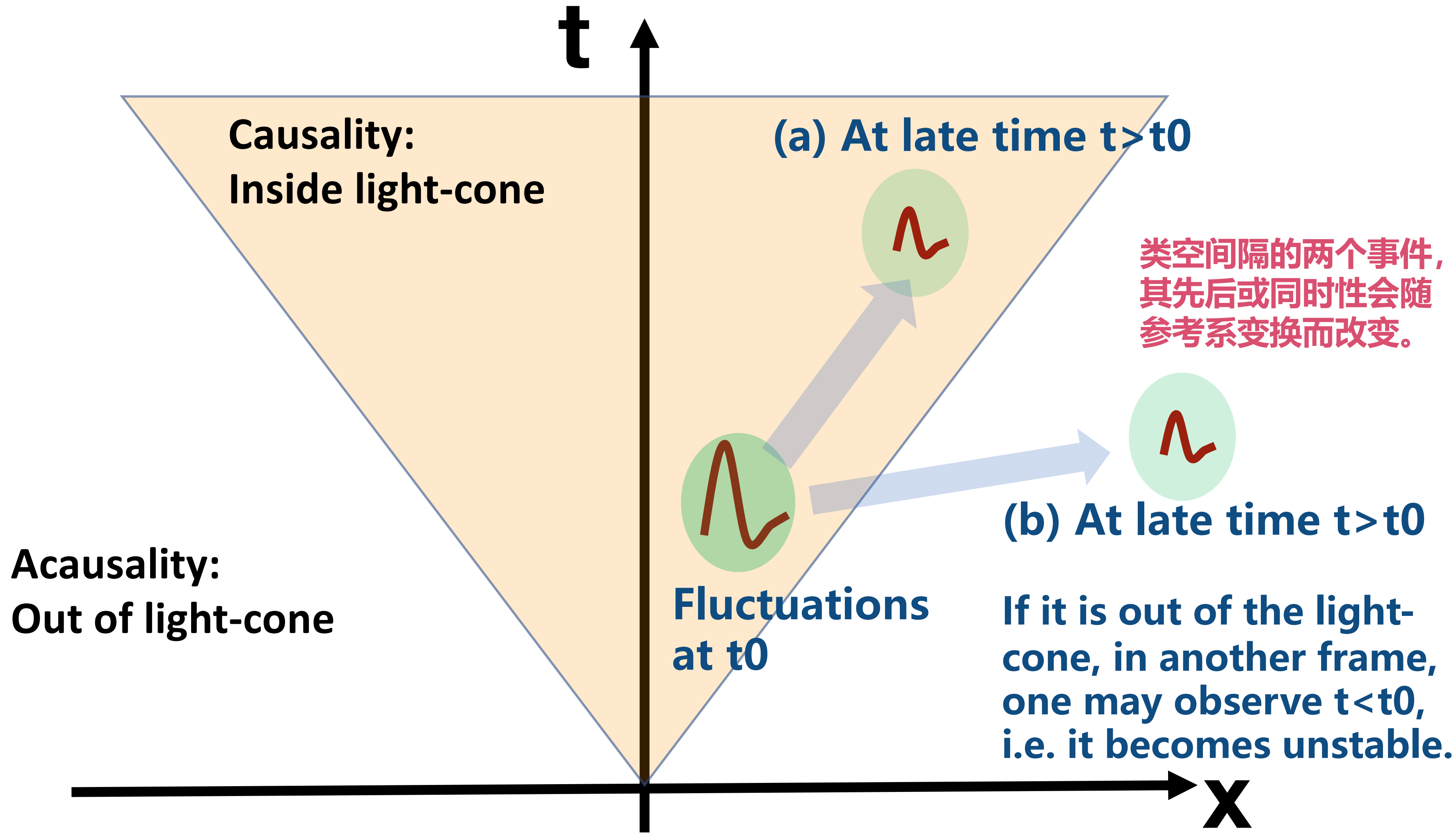
L. Gavassino, M. M. Disconzi, and J. Noronha, 2307.05987

- **An question arises:**

What constitutes the relationship between the stability and the causality criteria?

Connection between causality and stability

- **Acausal propagating can lead to unstable.**



L. Gavassino, Phys. Rev. X 12, 041001 (2022)

Updated stability criterion

- **The improved stability condition for a 3 + 1 dimensional relativistic system is,**

$$\mathrm{Im}\ \omega \leq |\mathrm{Im}\ \vec{k}|, \quad \text{for } \vec{k} \in \mathbb{C}^3.$$

Complex

M. P. Heller, A. Serantes, M. Spaliński, and B. Withers, 2212.07434.

L. Gavassino, Phys. Lett. B 840, 137854 (2023).

- **The imaginary part of k comes from the Lorentz transformation.**

  Assuming the system is stable in one frame, i.e. $\mathbf{Im}\ \boldsymbol{\omega} \leq \mathbf{0}$. Then, if we transform them to another frame, $(\boldsymbol{\omega}, \boldsymbol{k}) \rightarrow (\boldsymbol{\omega}', \boldsymbol{k}')$ by Lorentz transformation, the $\boldsymbol{k}'$ will also have a imaginary part.

Extending stability criterion to all frames

Theorem 1. The stability criterion holds true across all IFR if it is satisfied in a single IFR.

$$\mathrm{Im}\ \omega \leq |\mathrm{Im}\ \vec{k}|, \quad \text{for } \vec{k} \in \mathbb{C}^3.$$

Dong-ling Wang, SP, Phys. Rev. D (Lett), 109 (2024) 3, L031504

Improved causality criterion (I)

Theorem 2. *Suppose that the initial data $\varphi(0,\vec{x})$ for differential equations (1) is smooth with respect to $\vec{x}$, and the volume of the support of $\varphi(0,\vec{x})$ is both finite and non-vanishing. If two constants $R>0$ and $b\in\mathbb{R}$ exist such that*

$$\text{Im }\omega \le |\text{Im }\vec{k}| + b,\ \textit{for } |\vec{k}| > R, \tag{8}$$

then the influence of the initial data propagates with sub-luminal speed.

Simplified version:

$$\text{Im }\omega \le |\text{Im }\vec{k}| + b',\ \text{for } \vec{k} \in \mathbb{C}^3.$$

复数

Theorem 3. *The causality criterion (8) or (9) holds true across all IFR if it is fulfilled in a single IFR.*

Dong-ling Wang, SP, Phys.Rev.D (Lett), 109 (2024) 3, L031504

Improved causality criterion (II)

- Consider the dispersion relation ω = k(1 + i)/2 satisfying the conventional causality condition

$$\lim_{|\vec{k}|\to+\infty}\left\{\frac{|\mathrm{Re}\ \omega|}{|\vec{k}|}\leq 1,\ |\omega/\vec{k}|\ \text{is bounded}\right\},\ \vec{k}\in\mathbb{R}^3.$$

does not obey

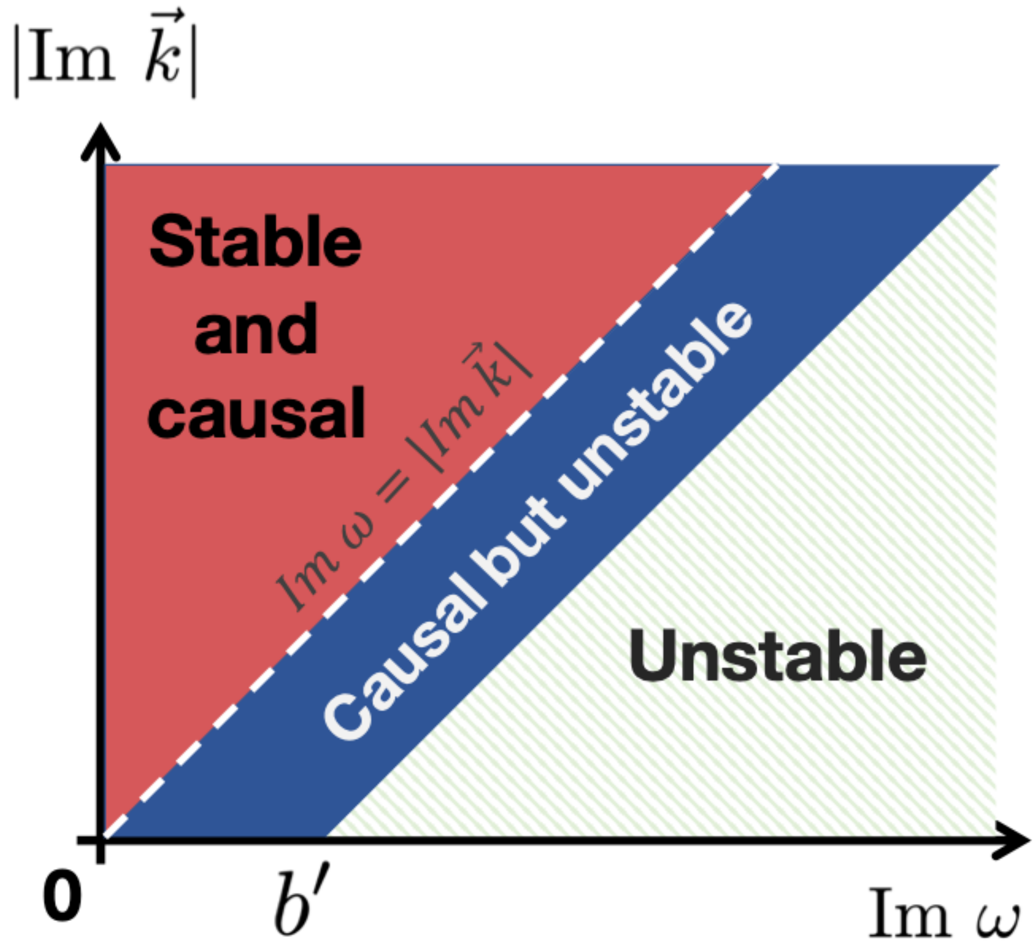
$$\mathrm{Im}\ \omega \leq |\mathrm{Im}\ \vec{k}| + b',\ \text{for}\ \vec{k}\in\mathbb{C}^3.$$

e.g. if we set k be real, then the above inequality becomes, k/2<b'. We cannot find a constant b' to keep this inequality be satisfied for large k limit.

- We know that the above desperation relations are proved to be acausal.

P. D. Lax, Hyperbolic Partial Differential Equations Courant Lecture Notes (American Mathematical Society/Courant Institute of Mathematical Sciences, 2006)

Stability means causality



Conclusion:

Stability in all inertial frame of reference means causality in linear mode analysis.

What is the necessary and sufficient causality criteria?

1. $\text{Im }\omega \leq |\text{Im }\vec{k}|, \quad \vec{k} \in \mathbb{C}^3.$

Heller, Serantes, Spalinski, Withers, Phys.Rev.Lett. 130, 261601 (2023).
Gavassino, Phys.Lett.B 840, 137854 (2023).
Gavassino, Disconzi, Noronha, arXiv:2307.05987.

2. $\text{Im }\omega \leq |\text{Im }\vec{k}| + b', \text{ for } \vec{k} \in \mathbb{C}^3.$

Dong-ling Wang, SP, Phys.Rev.D (Lett),109 (2024) 3, L031504

3. $0 \leq \lim_{|\vec{k}|\to\infty} \frac{|\text{Re }\omega|}{|\vec{k}|} \leq 1, \quad \lim_{|\vec{k}|\to\infty} \frac{\text{Im }\omega}{|\vec{k}|} = 0, \quad \vec{k} \in \mathbb{R}^3,$

$$\mathcal{O}_\omega \left[F(\omega, \vec{k} \neq 0)\right] = \mathcal{O}_{|\vec{k}|} \left[F(\omega = a|\vec{k}|, \vec{k})\right].$$

Hoult, Kovtun, Phys.Rev.D 109, 046018 (2024).

It is not the end of the story, but merely the beginning of it.

Attractors and focusing behavior in spin hydrodynamics

D.L. Wang, Y. Li, SP, arXiv: 2408.03781

Analytic solutions for spin hydrodynamics

- **Solution for Bjorken type spin hydrodynamics:**

$$\omega^{xy}(\tau) = \omega_0^{xy}\left(\frac{\tau_0}{\tau}\right)^{1/3}\exp\left[-\frac{2\gamma\tau_0}{a_1T_0^3}\left(\frac{\tau^2}{\tau_0^2}-1\right)\right]\left\{1+\left(\frac{2}{3}\frac{\eta_s}{s}+\frac{\zeta}{s}\right)\frac{1}{T_0^4}\right.$$
$$\left.\times\left[\frac{T_0^3}{\tau_0}\left(\left(\frac{\tau_0}{\tau}\right)^{2/3}-1\right)+\frac{\gamma}{a_1}\left(3\left(\frac{\tau}{\tau_0}\right)^2-\frac{9}{2}\left(\frac{\tau}{\tau_0}\right)^{4/3}+\frac{3}{2}\right)\right]\right\}$$
$$+\mathcal{O}\left((\omega_0^{xy}/T_0)^2,(\eta_s/s)^2,(\zeta/s)^2,(\eta_s\zeta/s^2)\right),$$

D.L. Wang, S. Fang, SP, Phys.Rev.D 104 (2021) 11, 114043

- **Solution for Gubser type spin hydrodynamics:**

$$S^{0x}=\frac{4L^2}{\tau}C_+G(L,\tau,x_\perp)^{-1}, \qquad S^{xz}=\frac{4L^2}{\tau}D_+G(L,\tau,x_\perp)^{-1},$$
$$S^{0y}=\frac{4L^2}{\tau}C_-G(L,\tau,x_\perp)^{-1}, \qquad S^{yz}=\frac{4L^2}{\tau}D_-G(L,\tau,x_\perp)^{-1}.$$

D.L. Wang, X.Q. Xie, S.Fang, SP, Phys.Rev.D 105 (2022) 11, 114050

- **Spin density: Power law X exponential decay**

Ordinary hydro variables: power law decay

No spin effects at late time?

Revisited Bjorken type spin hydro (I)

- **For Bjorken type spin hydro, we have**

$$\frac{d^2S^{xy}}{dw^2}+(\Delta_1^{-1}+w^{-1})\frac{dS^{xy}}{dw}+\Delta_1^{-2}(w^{-1}-w^{-2}+8\alpha w^{\Delta_2})S^{xy}=0.$$

$$\mathrm{Kn}^{-1}\approx w\equiv\frac{\tau}{\tau_\phi}. \qquad w\equiv\frac{\tau}{\tau_\phi}=\left(\frac{\tau}{\tau_1}\right)^{\Delta_1},\quad \frac{\tau_\phi\gamma}{\chi}=\alpha w^{\Delta_2},$$

Δ_1,Δ_2 are constant

γ: transport coefficient
τ_ϕ: relaxation time
χ: spin susceptibility

$$f(w)\equiv\Delta_1\frac{w}{S^{xy}}\frac{dS^{xy}}{dw}=\frac{\tau}{S^{xy}}\frac{dS^{xy}}{d\tau}$$

Here, we assume the γ is proper time dependent different with our previous work.

$$\Delta_1 wf'+f^2+wf+w-1+8\alpha w^{2+\Delta_2}=0,$$

Revisited Bjorken type spin hydro (II)

$$\Delta_1 w f' + f^2 + wf + w - 1 + 8\alpha w^{2+\Delta_2} = 0,$$

$$w \equiv \frac{\tau}{\tau_\phi} = \left(\frac{\tau}{\tau_1}\right)^{\Delta_1}, \quad \frac{\tau_\phi \gamma}{\chi} = \alpha w^{\Delta_2}, \qquad f(w) \equiv \Delta_1 \frac{w}{S^{xy}} \frac{dS^{xy}}{dw} = \frac{\tau}{S^{xy}} \frac{dS^{xy}}{d\tau}$$

If $\alpha w^{2+\Delta_2} \to 0$, then the late time beahivor reads

$$\Delta_1 \cancel{w f'} + f^2 + \cancel{wf} + \cancel{w} - 1 + \cancel{8\alpha w^{2+\Delta_2}} = 0,$$

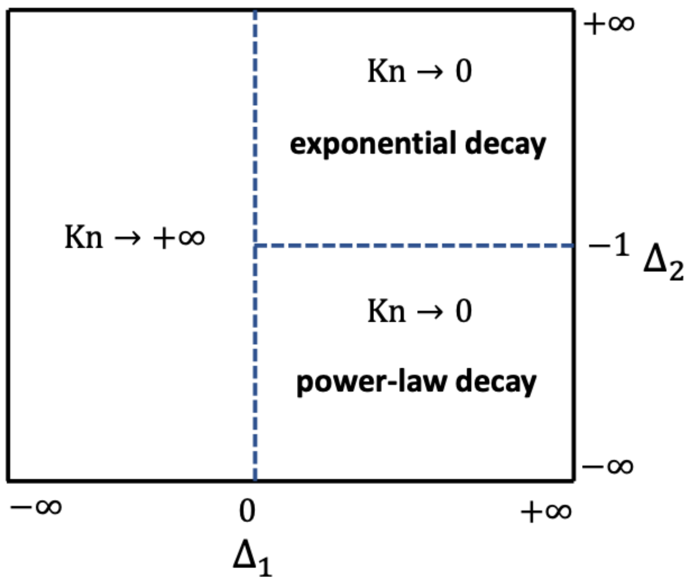
$$\Rightarrow \quad f \sim \pm 1$$

Trivial solution? But one kind of attactors!

Spin density: power law decay

Asymptotic solutions for $S = S^{xy}$

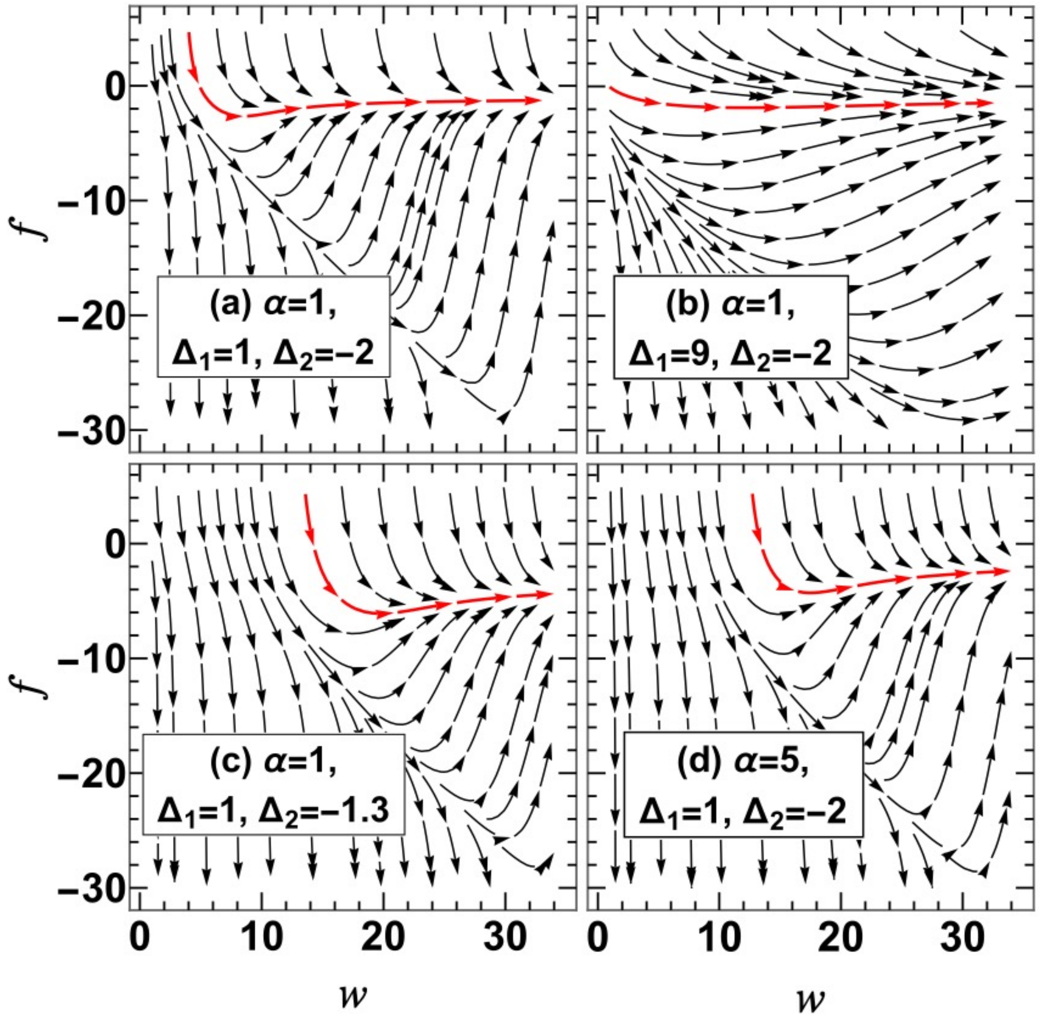
	$w \to +\infty$
$\Delta_2 > 0$	$S_{(1),(2)} \propto e^{-w/(2\Delta_1)}$
$\Delta_2 = 0$	$S_{(1),(2)} \propto e^{-w(1\pm\sqrt{1-32\alpha})/(2\Delta_1)}$
$-1 < \Delta_2 < 0$	$S_{(1)} \propto e^{-w/\Delta_1}$, $S_{(2)} \propto \exp\left[-\frac{8\alpha w^{1+\Delta_2}}{\Delta_1(1+\Delta_2)}\right]$
$\Delta_2 = -1$	$S_{(1)} \propto e^{-w/\Delta_1}$, $\boxed{S_{(2)} \sim w^{-(1+8\alpha)/\Delta_1}}$
$\Delta_2 < -1$	$S_{(1)} \propto e^{-w/\Delta_1}$, $\boxed{S_{(2)} \sim w^{-1/\Delta_1}}$



We focus on the region:

$$\Delta_1 > 0, \quad \Delta_2 \le -1.$$

Late time attactors



$$f_{(1)} \rightarrow -w,$$

$$f_{(2)} \rightarrow \begin{cases} -1-8\alpha, & \Delta_2 = -1, \\ -1, & \Delta_2 < -1, \end{cases}$$

Why late time attactors exist?

- **Assuming spin susceptibility is a constant for simplicity.**

$$\gamma \sim \tau^{1+\Delta_2-1/\Delta_1}$$

When γ is small (or $\Delta_1 > 0, \Delta_2 \leq -1$),

$$\tau_\phi \Delta^{\mu\alpha}\Delta^{\nu\beta}u^\rho\nabla_\rho\phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma\Delta^{\mu\alpha}\cancel{\Delta^{\nu\beta}(\nabla_{[\alpha}u_{\beta]}} + 2\omega_{\alpha\beta}),$$

$$\Longrightarrow \quad \phi^{xy} \approx \phi_0 \exp\left(-\frac{w}{\Delta_1}\right) + \mathcal{O}(\gamma),$$

While ϕ is the source generating spin density

$$\partial_\lambda \Sigma^{\lambda xy} \approx 0,$$

$$\frac{dS^{xy}}{d\tau} + \frac{1}{\tau}S^{xy} \approx 0, \quad \Longrightarrow \quad S^{xy} \approx S_0\frac{\tau_1}{\tau} = S_0 w^{-1/\Delta_1}$$

In this case, spin density decays due to expanding only, just like energy or number density in a Bjorken flow.
Beyond the non-hydro modes?

New discovery: focusing behavior

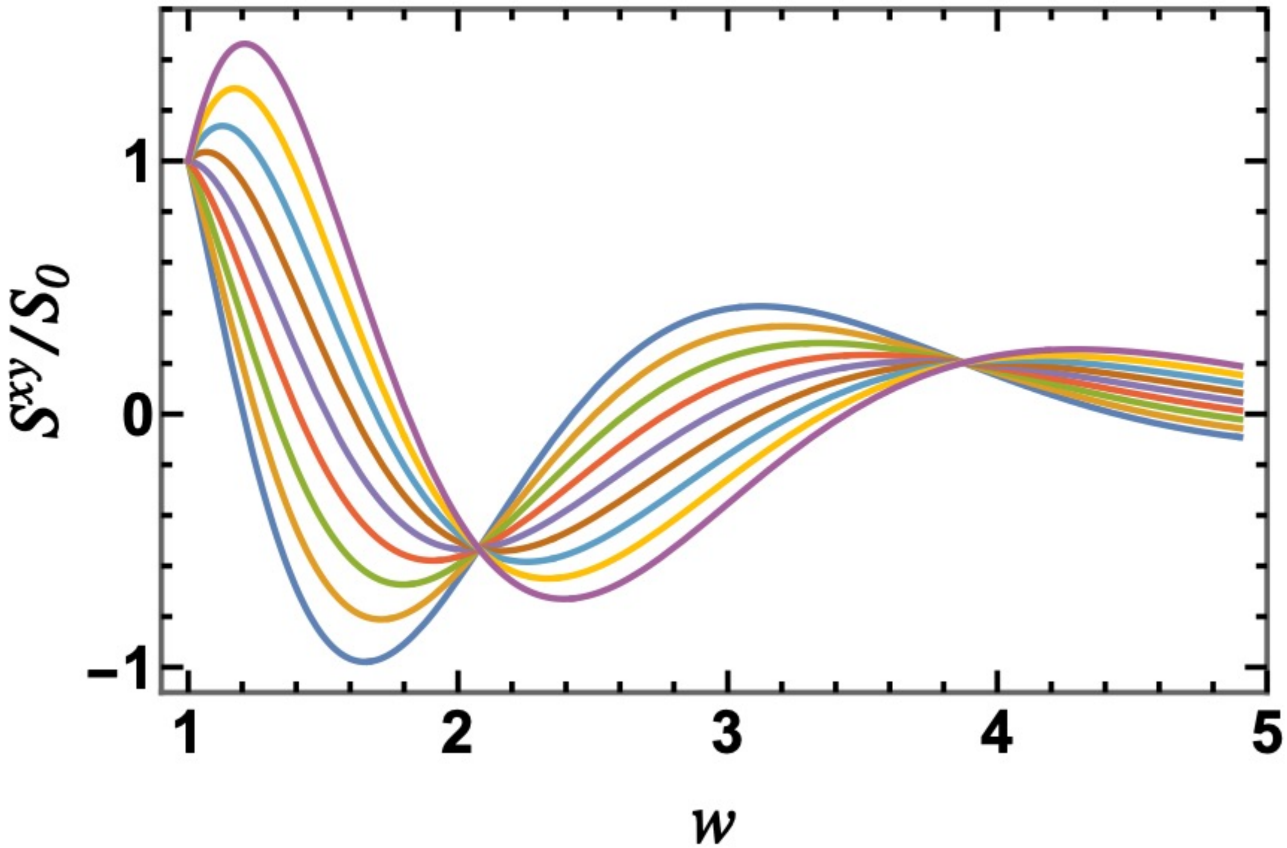


FIG. 5. The focusing behavior for $S^{xy}(w)/S_0$ with different S_0'. The parameters are set to be $\Delta_1 = 1$, $\Delta_2 = -1.5$, and $\alpha = 2$. The initial conditions are chosen as $w_0 = 1$ and $S_0' = -4.9$, -3.7, -2.5, -1.3, -0.1, 1.1, 2.3, 3.5, and 4.7. All solutions $S^{xy}(w)/S_0$ pass through the same point at $w = 2.077$, 3.876, 6.804, and 11.974 (the last two are not shown in this figure).

Summary and outlook

Summary

- We study the causality and stability for spin hydrodynamics. We derived the causality conditions and find the conventional stability criterion (fails) cannot make the system be stable for finite wave length limit.

- The thermodynamic stability provides the constrains, which is more stringent than those derived from linear mode analysis.

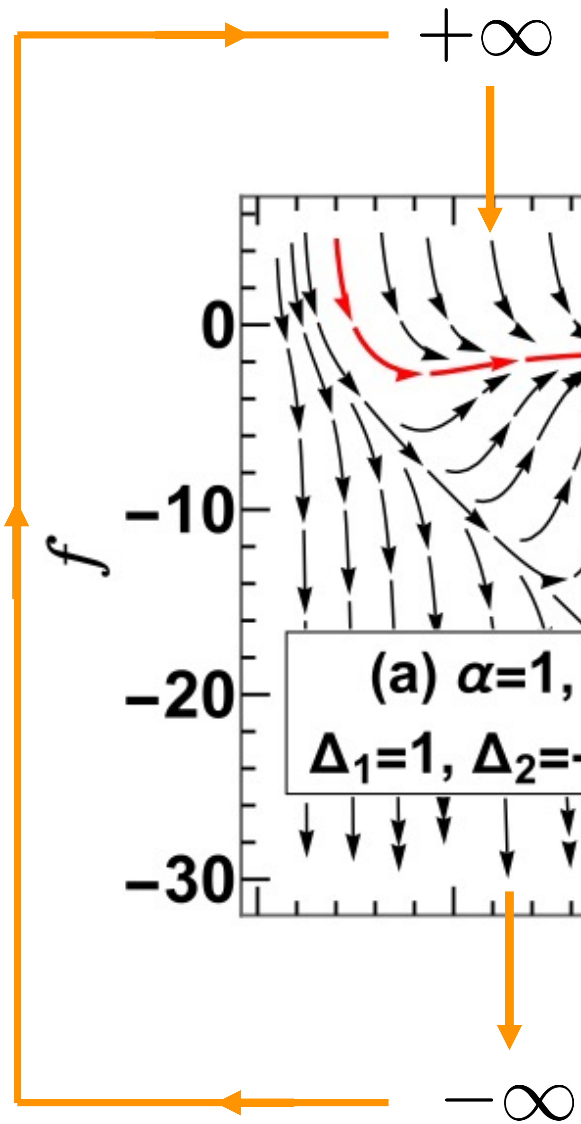
- We introduced and proved an improved causality criteria. By the new criteria, we find that stability in all inertial frame of reference means causality in linear mode analysis.

- We derive the late time attractors and focusing behavior for spin hydrodynamics. It implies that spin density can be treated as other thermodynamic variables in certain region.

Thank you!

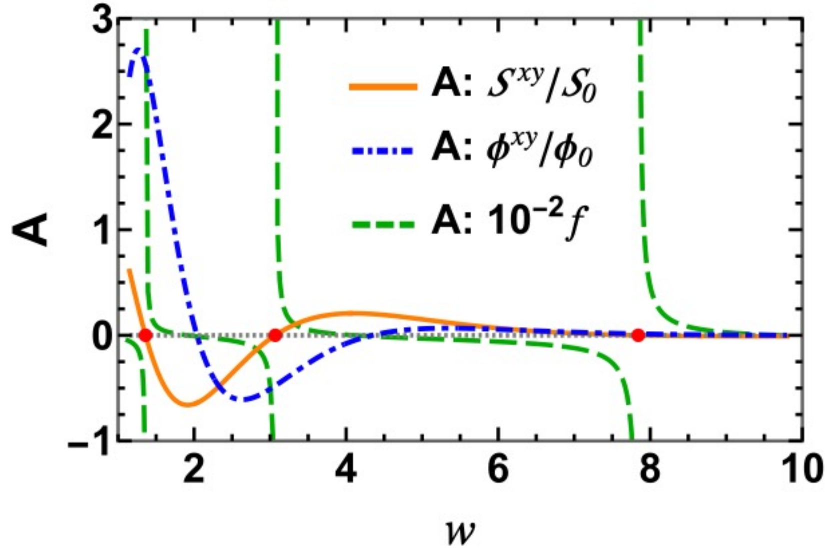
Backup

No singularity for spin density



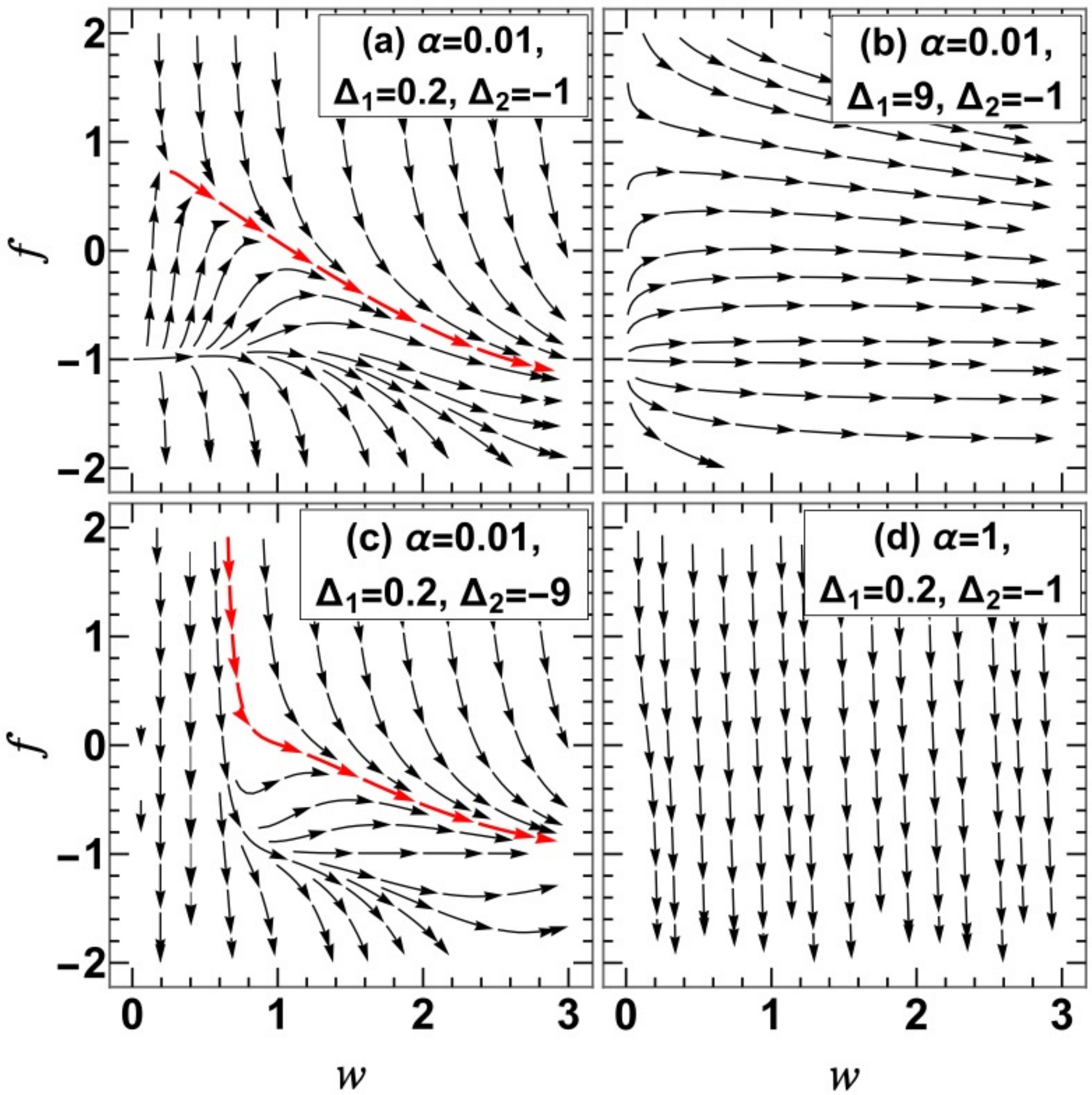
$$f(w) \equiv \Delta_1 \frac{w}{S^{xy}} \frac{dS^{xy}}{dw} = \frac{\tau}{S^{xy}} \frac{dS^{xy}}{d\tau}$$

$$S^{xy} \to 0, \;\; f \to \pm\infty$$



$\Delta_1 = 1, \; \Delta_2 = -2, \text{ and } \alpha = 2$

Early time attractors



A practical challenge arises

- Commonly, the causality and stability conditions are first derived from the conventional criteria in the rest frame.
- Then, the verification of these criteria in other reference frames follows.
- However, this process of examining conditions across different frames is frequently burdensome.