Statistic models of price dynamics

Chapter 8 of Mantegna and Stanley's book

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R. N. Mantegna and H. E. Stanley:
An introduction to econophysics
Correlations and complexity in finance

Chapter 8:
Levy stable model
Student's t-distribution
Mixture of Gaussians
Truncated Levy flights

Comments on St. Peterburg paradox

Motivation: News from Z. Lakner, S. Hegyi

Econophysics

28 languages

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Econophysics is a non-orthodox (in economics) interdisciplinary research field, applying theories and methods originally developed by physicists in order to solve problems in economics, usually those including uncertainty or stochastic processes and nonlinear dynamics. Some of its application to the study of financial markets has also been termed statistical finance referring to its roots in statistical physics. Econophysics is closely related to social physics.

Basic problem: develop models adopted to practice.

New input from Zoltán Lakner

New book from S. Hegyi:

Székely J. Gábor
Paradoxonok a véletlen matematikájában

News from S. Hegyi

Székely J. Gábor
Paradoxonok a véletlen matematikájában

Székely J. Gábor

Paradoxonok a véletlen matematikájáb

2. átdolgozott kiadás

Előszó		8
1.	Klasszikus paradoxonok a valószínűségszámításban	11
1.	A kockázás paradoxona	12
2.	De Méré lovag paradoxona	16
3.	Az osztozkodás paradoxona	21
4.	A függetlenség paradoxona	24
5.	A szerencsejátékok két paradoxona: bridzs és lottó	27
6.	Ajándékozási paradoxon	33
7.	Pétervári paradoxon	39
8.	Az emberi halandóság paradoxona	43
9.	A nagy számok Bernoulli-törvényének paradoxona	47
10.	De Moivre paradoxona; energiatakarékosság	51
11.	Bertrand-féle paradoxon	57
12.	A játékelmélet egy paradoxona	62
13.	Villámparadoxonok	70

Stochastic models of price dynamics

Prices of financial assets have a stochastic nature.
Crutial for rational pricing of derivative products issued on it.
The full characterization of a stochastic processs requires the
knowledge of conditional probablities of all orders:

Si ... cs:
E.g.: ln Y(t) ... istributed –
R ... ,

Better/m ... avier tails

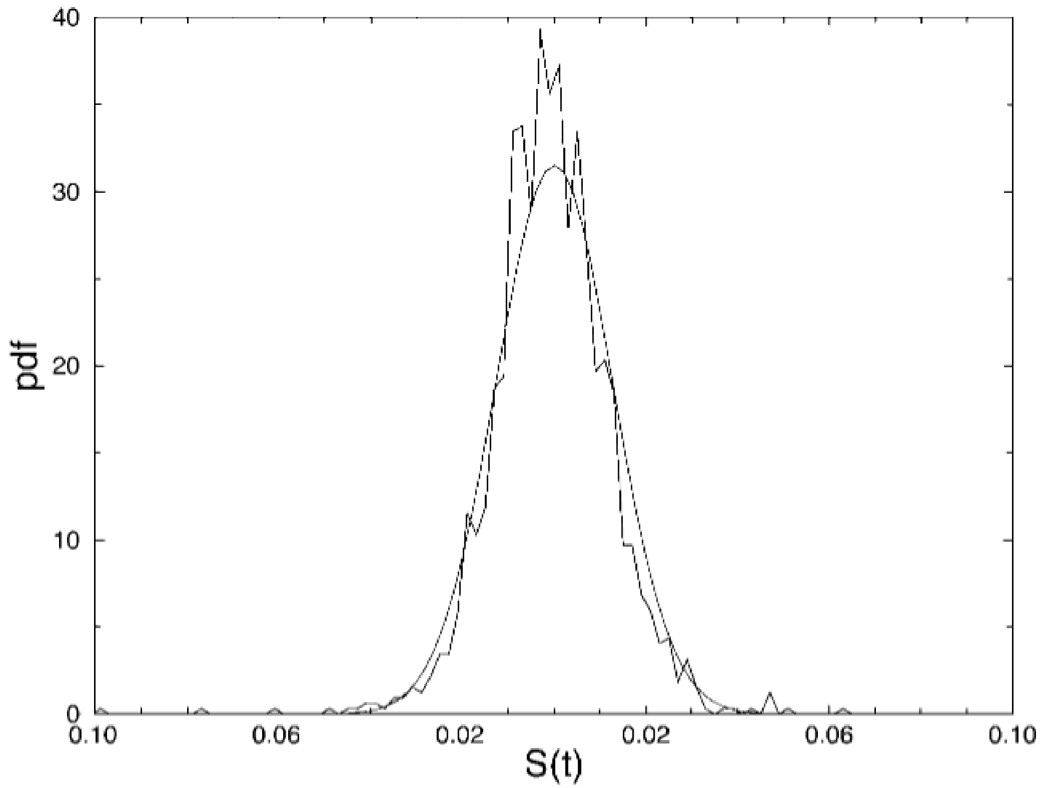


Fig. 8.1. Empirical pdf for the logarithm of daily price differences of Chevron stock traded in the New York Stock Exchange in the period 1989 to 1995. The smooth line is the Gaussian pdf with the same variance calculated from the data.

Chapter 8: examples for GGD

Similarly the famous Black and Scholes formula assumes Gaussian. Several improvements, generalized Gaussians are available by now. Some generalizations are based on the followings:

(i) the finiteness or infiniteness of the second and higher moments of the distribution;

(ii) the nature of stationarity present on a short time scale or asymptotically;

(iii) the continuous or discontinuous character of $Y(t)$ – or $\ln Y(t)$; and

(iv) the scaling behavior of the stochastic process.

Examples for generalized Gaussian distributions (GGD):

8.1 Levy stable (non-Gaussian) model

8.2 Student's *t* distribution

8.3 Mixture of Gaussians

8.4 Truncated Levy flight

Section 8.1: Levy stable model

Proposed by B. Mandelbrot in 1963 for modeling cotton ln Y(t).

The Variation of Certain Speculative Prices

Benoit Mandelbrot

The Journal of Business, Vol. 36, No. 4. (Oct., 1963), pp. 394-419.

Stable URL:
http://links.jstor.org/sici?sici=0021-9398%28196310%2936%3A4%3C394%3ATVOCSP%3E2.0.CO%3B2-L

The Journal of Business is currently published by The University of Chicago Press

Seconded by Fama (1965)
Stable for convolution
Generalized central limit theorems
Infinite second moment for $\alpha < 2$.
Infinite first moment for $\alpha < 1$.
Gaussian is recovered for $\alpha = 2$.
Top cited, paradigm shifting paper.

B. ADDITION OF MORE THAN TWO STABLE RANDOM VARIABLES

Let the independent variables U_n satisfy the condition (PL) with values of α, β, γ, and δ equal for all n. Then, the logarithm of the characteristic function of

$$S_N = U_1 + U_2 + \ldots U_n + \ldots U_N$$

is N times the logarithm of the characteristic function of U_n, and it equals

$$i\,\delta N z - N\gamma|z|^{\alpha}[1 + i\beta(z/|z|)\tan(\alpha\pi/2)]\,,$$

so that S_N is stable with the same α and β as U_n, and with parameters δ and γ multiplied by N. It readily follows that

$$U_n - \delta \text{ and } N^{-1/\alpha}\sum_{n=1}^{N}(U_n - \delta)$$

have identical characteristic functions and thus are identically distributed ran-

Section 8.2: Student's *t* distribution

Proposed by P. K. Clark, Econometrica **41** (1973) 135-256

$$P(z) = \frac{C_n}{(1 + z^2/n)^{(n+1)/2}} \tag{8.1}$$

of a stochastic process

$$z \equiv \frac{x\sqrt{n}}{\sqrt{y_1^2 + \cdots + y_n^2}}, \tag{8.2}$$

obtained from independent stochastic variables $y_1, y_2, \ldots, y_n$ and x, each with normal density, zero mean, and unit variance. Here

$$C_n \equiv \frac{\Gamma[(n+1)/2]}{\sqrt{\pi n}\,\Gamma(n/2)}. \tag{8.3}$$

When $n = 1$, $P(z)$ is the Lorentzian distribution. When $n \to \infty$, $P(z)$ is the Gaussian distribution. In general, $P(z)$ has finite moments for $k < n$. Hence a stochastic process characterized by a Student's t-distribution may have both finite and infinite moments. By varying the control parameter n (which controls the finiteness of moments of order k), one can approximate with good accuracy the log price change distribution determined from market data at a given time horizon [19].

Finite k-th moment for k < n.
Both finite and infinite moments.
Shape not stable.
No scaling relations.

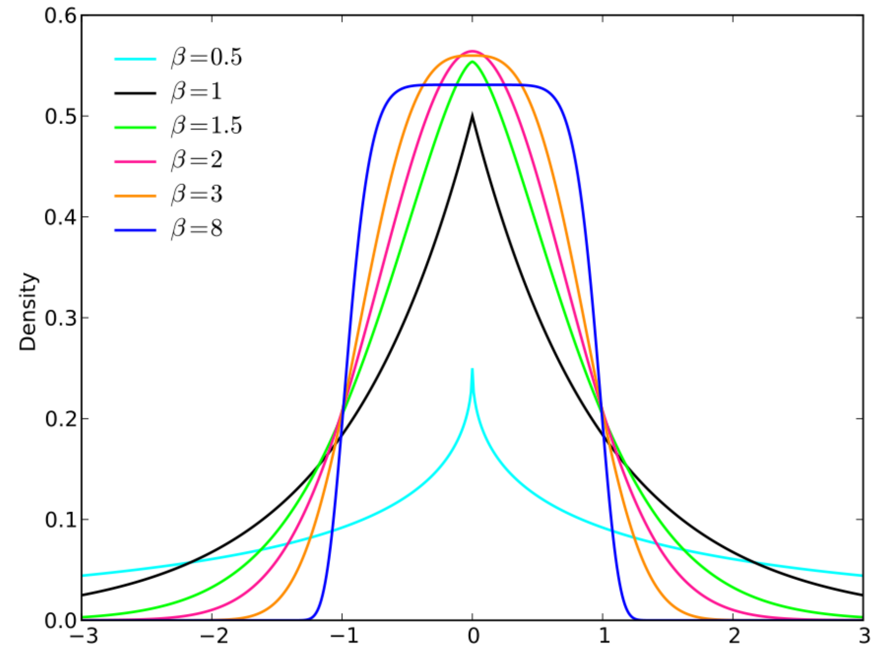
Section 8.3: Mixture of Gaussians

Clark interpreted the leptokurtic behavior observed in empirical analyses as the result of the fact that the trading activity is not uniformly distributed during the trading interval. In his model the second moment of the $P[S[\Omega(t)]]$ distribution is always finite provided $P(\Omega)$ has a finite second moment. The specific form of the distribution depends on the distribution of the directing process $\Omega(t)$. In general, the $P[S[\Omega(t)]]$ distributions do not possess scaling properties.

No scaling relations.
Not very interesting for us.
With sufficiently large number of mixed Gaussians, datasets with large number of peaks can be described.

Added: Generalized Gaussian Distributions

Parameters	μ location (real) α scale (positive, real) β shape (positive, real)
Support	$x \in (-\infty; +\infty)$
PDF	$\frac{\beta}{2\alpha\Gamma(1/\beta)} \; e^{-(\lvert x-\mu \rvert/\alpha)^\beta}$ Γ denotes the gamma function
CDF	$\frac{1}{2} + \text{sign}(x-\mu)\frac{1}{2\Gamma(1/\beta)}\gamma\left(1/\beta, \left\lvert\frac{x-\mu}{\alpha}\right\rvert^\beta\right)$ where β is a shape parameter, α is a scale parameter and γ is the unnormalized incomplete lower gamma function.

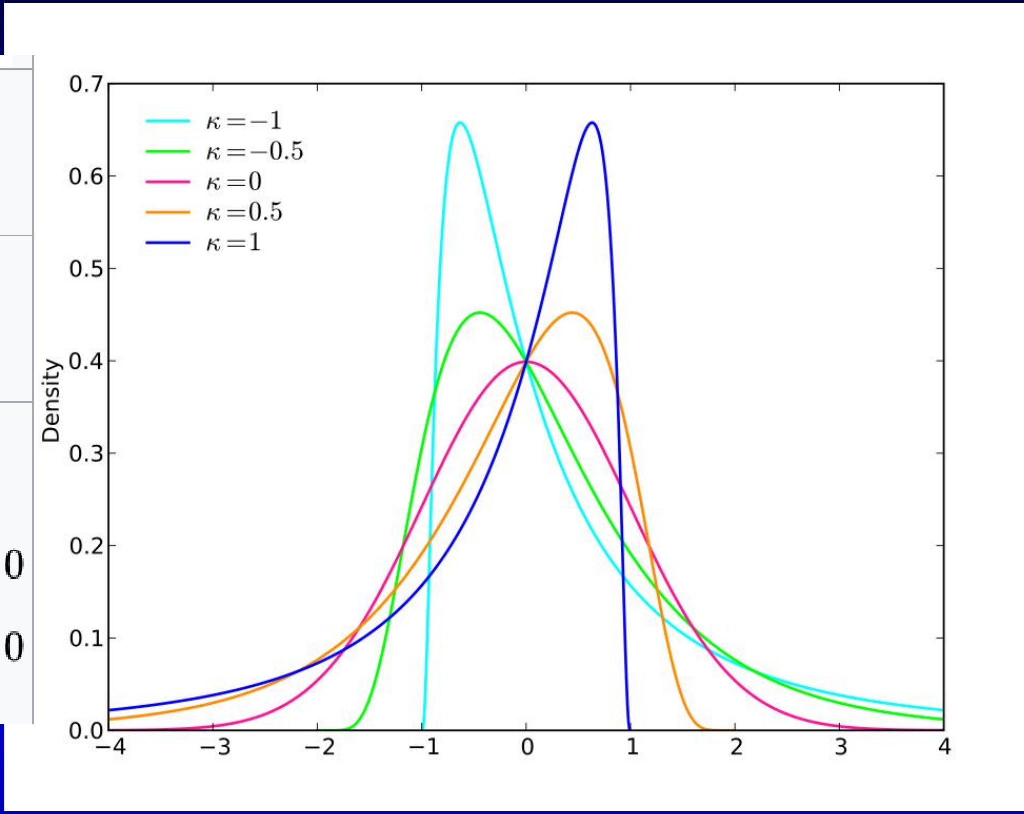


Parameter β similar to Levy index of stability α for $0 < \beta \leq 2$,
but $2 < \beta$ is allowed,
in this case it is not a Fourier-transform of a positive definite probability density.

See: https://en.wikipedia.org/wiki/Generalized_normal_distribution

Asymmetric Generalized Gaussian Ds

Parameters	ξ location (real) α scale (positive, real) κ shape (real)
Support	$x \in (-\infty, \xi + \alpha/\kappa) \text{ if } \kappa > 0$ $x \in (-\infty, \infty) \text{ if } \kappa = 0$ $x \in (\xi + \alpha/\kappa, +\infty) \text{ if } \kappa < 0$
PDF	$\frac{\phi(y)}{\alpha - \kappa(x - \xi)}$, where $y = \begin{cases} -\frac{1}{\kappa}\log\left[1 - \frac{\kappa(x-\xi)}{\alpha}\right] & \text{if } \kappa \neq 0 \\ \frac{x-\xi}{\alpha} & \text{if } \kappa = 0 \end{cases}$ ϕ is the standard normal pdf



Parameter κ similar to Levy asymmetry parameter β but -1 ≤ β ≤ 1,
But in this case support is finite if κ is non-vanishing, while
Levy asymmetric has finite support only if Levy β = ± 1.
Only location, scale and asymmetry but no exponent for the tails.

See: https://en.wikipedia.org/wiki/Generalized_normal_distribution

Section 8.4: Truncated Levy Flights (TLF)

TLF distribution is defined by

$$P(x) \equiv \begin{cases} 0 & x > \ell \\ cP_{\mathrm{L}}(x) & -\ell \le x \le \ell \\ 0 & x < -\ell \end{cases} ,$$

Not stable for convolution.
It has finite means and variances: asymptotically Gaussian.
But how quickly?

How quickly will it converge? To answer this question, we consider the quantity $S_n \equiv \sum_{i=1}^{n} x_i$, where x_i is a truncated Lévy process, and $\langle x_i x_j \rangle =$ const δ_{ij}. The distribution $P(S_n)$ well approximates $P_{\mathrm{L}}(x)$ in the limit $n \to 1$, while $P(S_n) = P_{\mathrm{G}}(S_n)$ in the limit $n \to \infty$. Hence there exists a crossover value of $n, n_\times$, such that (Fig. 8.3)

$$P(S_n) \approx \begin{cases} P_{\mathrm{L}}(S_n) & \text{when } n \ll n_\times \\ P_{\mathrm{G}}(S_n) & \text{when } n \gg n_\times \end{cases} , \qquad (8.5)$$

where $P_{\mathrm{G}}(S_n)$ is a Gaussian distribution. The crossover value $n_\times$ is given by

$$n_\times \simeq A\ell^{\alpha}, \qquad (8.6)$$

Very interesting.

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Section 8.4: Truncated Levy Flights (TLF)

Fig. 8.3 ... ssover found t... large n. The ... from [110].

[110] ... 2946

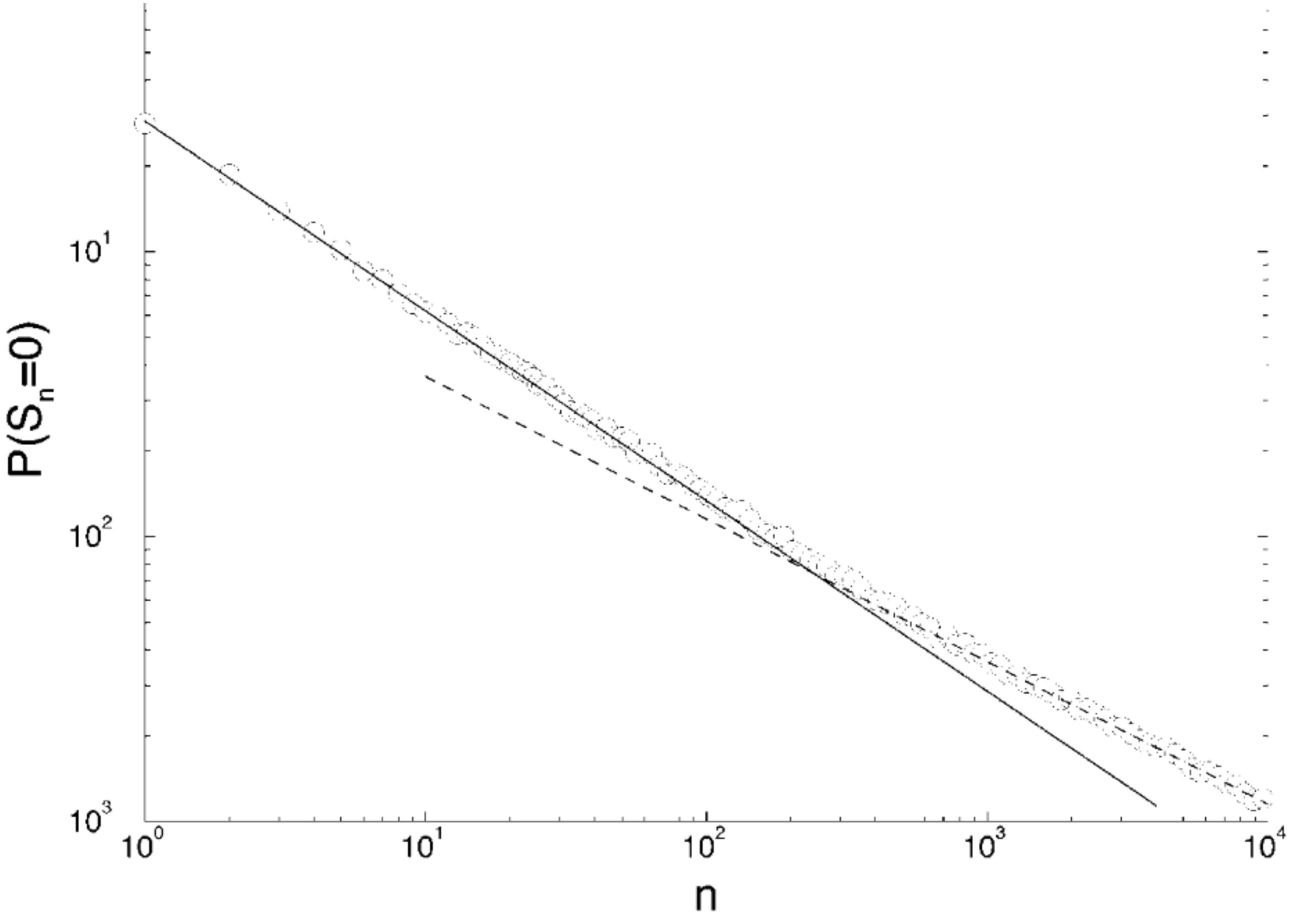

Fig. 8.4. Probability of return to the origin of S_n as a function of n for $\alpha = 1.5$ and $\ell = 100$. The simulations (circles), obtained with 5×10^4 realizations, are compared with the Lévy regime (solid line) and the asymptotic Gaussian regime calculated for $\ell = 100$ (dotted line). Adapted from [114].

[114] R. N. Mantegna and H. N. Stanley, *Physics Investigation of Financial Markets* *In: The physics of complex systems*. Book issued by IOS Press, (1997) pp. 473-489.

Section 8.4: Truncated Levy Flights (TLF)

Transition from Levy to Gaussian regime. From [114].

Very interesting: effect of truncation dominant for long term.
Similar effects for St. Peterburg Paradox (for next time).

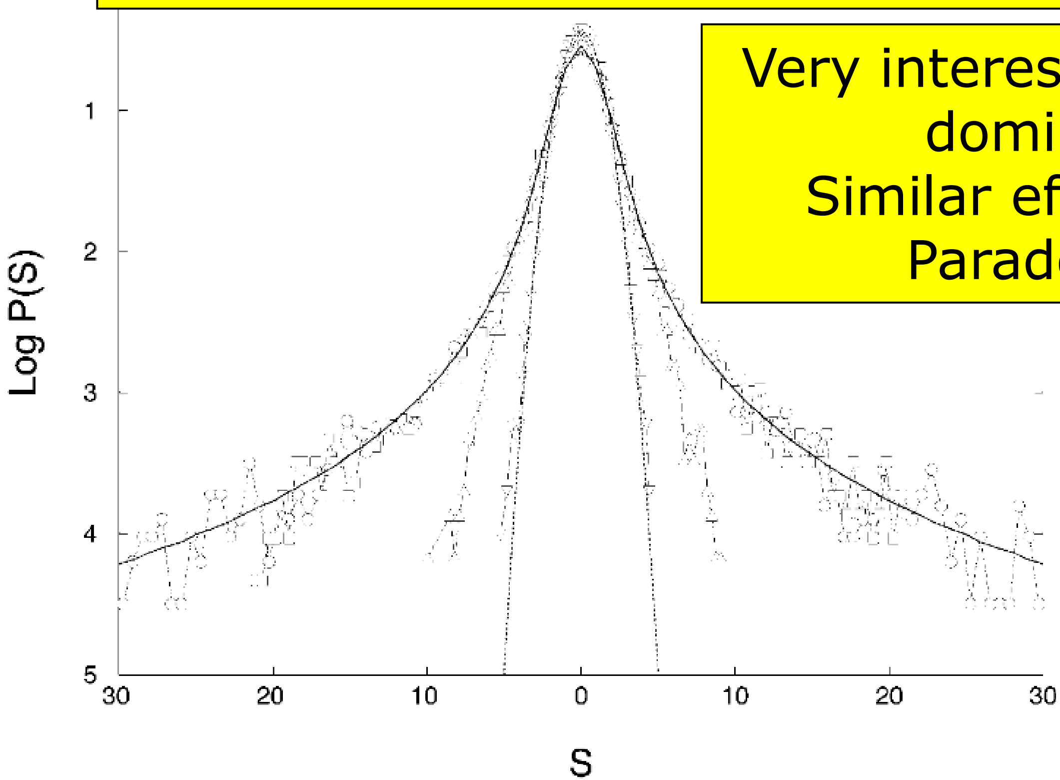

Fig. 8.5. Semi-logarithmic scaled plot of the probability distributions of the TLF process characterized by $\alpha = 1.5$ and $\ell = 100$ for $n = 1$, 10, 100, and 1,000. For low values of n ($n = 1$ (circles) and 10 (squares)) the central part of the distributions is well described by the Lévy stable symmetrical profile associated with $\alpha = 1.5$ and $\gamma = 1$ (solid line). For large values of n ($n = 1{,}000$ (inverted triangles)), the TLF process has already reached the Gaussian regime and the distribution is essentially Gaussian (dotted line). Adapted from [114].

[114] R. N. Mantegna and H. N. Stanley, Physics Investigation of Financial Markets *In: The physics of complex systems*. Book issued by IOS Press, (1997) pp. 473-489.