Handling photons in lattice QCD

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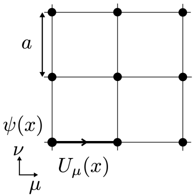
Outline

1. Intro to lattice QCD, and its frontier topics
2. Computational challenges posed by photons
3. Two recent ideas for handling ‘dynamical’ photons, i.e. whose momentum is an integration variable
4. Summary & epilogue

The lattice regularization of QCD K.G. Wilson 1974



Gluon 'link' variables:

$$U_\mu(x) = e^{iag_0 A_\mu(x)} \in SU(3)$$

Quarks: on-site Grassmann variables,

$$\psi_1\psi_2 = -\psi_2\psi_1$$

Action: has exact gauge invariance.

Finite volume: work on $L \times L \times L$ torus – periodic boundary conditions.

Euclidean path integral: finite number of compact degrees of freedom

$$Z = \int \mathcal{D}U_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S_G[U] - \bar{\psi}D[U]\psi} = \int \mathcal{D}U_\mu \, \det D[U] e^{-S_G[U]}$$

QCD $\leftrightarrow$ 4d statistical mechanics system $\Rightarrow$ importance sampling Monte-Carlo

Continuum limit: $g_0^2 \sim 1/\log(1/a)$ (asymptotic freedom)

Correlation functions and parameters of lattice QCD

$$\left\langle j_\mu(x) j_\nu(0) \right\rangle \overset{x_0>0}{=} \left\langle \mathrm{vac} \middle| j_\mu(0) e^{-Hx_0 + i\vec{P}\cdot\vec{x}} j_\nu(0) \middle| \mathrm{vac} \right\rangle$$
$$= \left\langle -\mathrm{Tr}\{S(x,0)\gamma_\nu S(0,x)\gamma_\mu\} + \mathrm{Tr}\{S(x,x)\}\,\mathrm{Tr}\{S(0,0)\} \right\rangle_{\mathrm{SU(3)\ gauge\ field}}$$

Connected   Disconnected diagram

- bare parameters: $m_u = m_d$, m_s and g_0
- fix their values by computing am_π, am_K and (typically) calibrate the lattice spacing via $a = (am_\Omega)/m_\Omega^{\mathrm{PDG}}$.
- electromagnetic effects are usually included as a correction: 1st order expansion around isosymmetric QCD [de Divitiis et al 1303.4896 (PRD)].

Current frontier topics in lattice QCD

(a selection inspired by recent Lattice conferences)

New ideas for handling the Euclidean path integral:

- simulations in very large volumes
- machine-learning techniques

Hadronic contributions to precision observables:

- $(g-2)_\mu$
- running of the Standard Model gauge couplings
- CKM matrix elements, weak decays

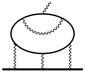
Real-time physics & inverse problems:

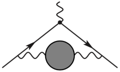
- hadronic scattering amplitudes
- nucleon structure from large-momentum frames (PDFs, GPDs, ...)
- quark-gluon plasma & physics of the Early Universe

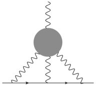
Signal-to-noise and sign problem:

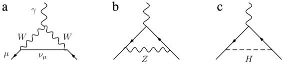
- locality and multi-level algorithms
- the sign problem for non-zero baryon density
- quantum computing

$(g-2)_\mu$: a history of testing the Standard Model









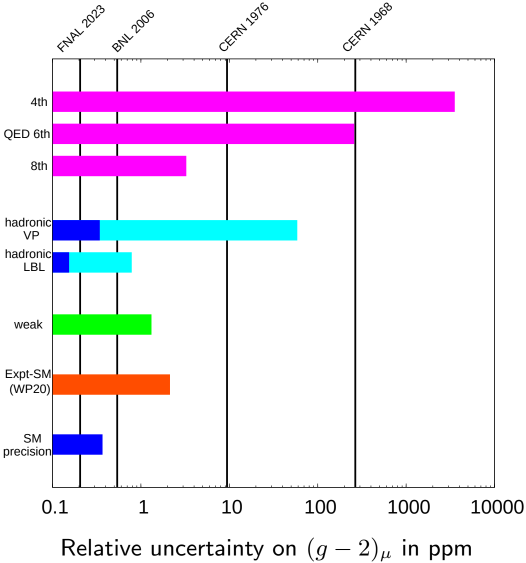
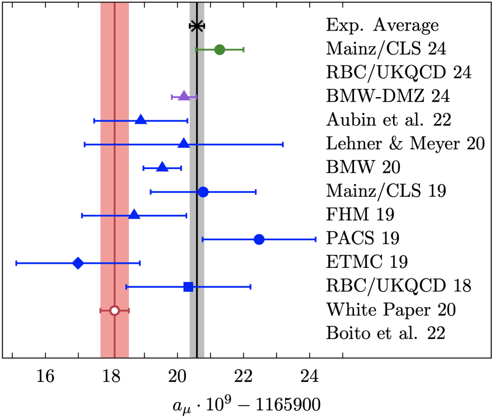


Figure inspired by [Jegerlehner 1705.00263].

Impact of hadronic vacuum polarisation in $a_\mu = \frac{1}{2}(g-2)_\mu$



Grey band = direct measurement of a_μ

Precise results for $a_\mu^{\rm hvp}$ by two lattice collaborations lead to
a Standard Model prediction in agreement with the experimental world average.

Djukanovic, von Hippel, Kuberski, HM, Miller, Ottnad, Parrino, Risch, Wittig 2411.07969.

Challenges posed by photons in lattice QCD (1)

- Elastic scattering $e^- \,\mathrm{hadron}(\vec{p}) \to e^- \mathrm{hadron}(\vec{p}\,')$: to O($\alpha$), can be handled by computing form factors at spacelike kinematics.

- Annihilation process $e^+e^- \to \gamma^* \to \pi^+\pi^-$: at low energies $\sqrt{s} \lesssim 0.8$ GeV, form factor at timelike kinematics is accessible via finite-volume techniques (Lüscher, Lüscher-Lellouch) [HM 1105.1892 (PRL)].

  At higher energies, one faces an inverse (Laplace) problem.

Challenges posed by photons in lattice QCD (2)

- $\pi^0/\eta/\eta' \to \gamma\gamma^{(*)}$: handling lightlike kinematics is possible [Jung, Ji hep-lat/0101014 (PRL)], but numerically challenging:

$$-\int d^4x\, e^{\omega_1 x_0 - i\vec{q}_1\cdot\vec{x}}\, \langle 0|\mathrm{T}\{V_i(x)V_j(0)\}|\pi(\vec{p})\rangle = \epsilon_{ij\alpha\beta} q_1^\alpha q_2^\beta\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)$$

The $x_0 \to +\infty$ regime must be controlled extremely well.

- Photon emissivity of the quark-gluon plasma: energy moments of the photon spectrum can be computed without involving an inverse problem [HM 1807.00781 (EPJA)]. With $\omega_n = 2\pi T n$, $n \in \mathbb{Z}$:

$$\int_0^\beta dx_0 \int d^3x\, (e^{i\omega_n x_0} - e^{i\omega_n x_2})\, e^{\omega_n x_3}\, \big\langle V_1(x)V_1(0)\big\rangle = \frac{\omega_n^2}{\pi}\int_0^\infty \frac{d\omega}{\omega}\,\frac{\sigma(\omega)}{\omega^2+\omega_n^2}\,,$$

$$\frac{d\Gamma(\omega)}{d\omega} = \frac{\alpha}{\pi}\,\frac{2\omega\,\sigma(\omega)}{e^{\omega/T}-1}.$$

Dynamical photons

Two methodological ideas for handling 'dynamical' photons in lattice QCD:

I. use coordinate-space methods

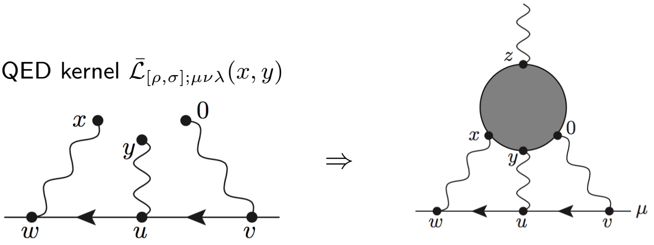
- $\star$ motivation: keep the observable *local*, not spread over the entire volume

II. where needed, use a Pauli-Villars-type UV cutoff $\Lambda \ll a^{-1}$ on the photon virtuality.

- $\star$ motivation: simplifies renormalisation, more continuum-like.

I.1 Coordinate-space approach to $a_\mu^{\rm HLbL}$

QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$



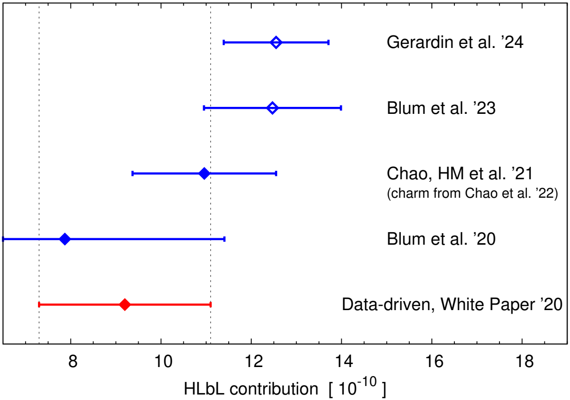
$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \left[\int d^4x \, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED} \; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm =QCD\ blob} \right].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \; z_\rho \left\langle \, j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0) \right\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume

[Asmussen, Gérardin, Green, HM, Nyffeler 1510.08384; 2210.12263 (JHEP).]

I.1 Status of a_μ^{HLbL}



- The three most recent lattice calculations use coordinate-space methods in slightly different variations.
- Given these results, a_μ^{HLbL} is likely to move up by $(2 \div 3) \times 10^{-10}$ in the $(g-2)_\mu$ White Paper '25.

I.2 Pion mass splitting

$$\text{PDG}: \qquad m_{\pi^\pm} - m_{\pi^0} = 4.5936(5)\ \text{MeV}.$$

To order $(\alpha, m_u - m_d)$, the pion mass splitting is a purely electromagnetic effect. In lattice QCD, two quark-level diagrams contribute:

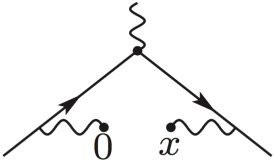
Lattice QCD result obtained with infinite volume, continuum photon propagator:

$$m_{\pi^\pm} - m_{\pi^0} = 4.534(42)_{\rm stat}(43)_{\rm syst}\ \text{MeV}.$$

NB. the intermediate state $\gamma^*\pi$ is given special treatment to avoid enhanced power-law finite-volume effects.

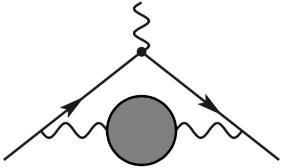
Feng, Jin, Riberdy 2108.05311 (PRL). See also Frezzotti et al 2202.11970 (PRD).

I.3 Hadronic vacuum polarization in $(g-2)_\mu$ HM 1706.01139 (EPJC)



QED kernel $H_{\mu\nu}(x)$

$\Rightarrow$



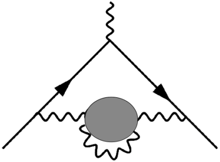
$a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x\; H_{\mu\nu}(x)\, \Big\langle j_\mu(x) j_\nu(0)\Big\rangle_{\rm QCD},$$

$$j_\mu = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d - \frac{1}{3}\bar s\gamma_\mu s + \dots; \qquad H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

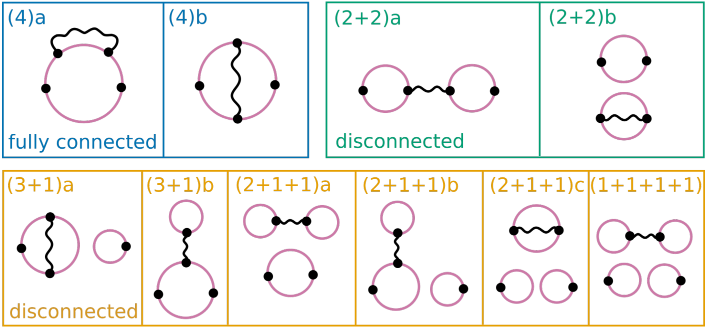
Weight functions $\mathcal{H}_i$ are linear combinations of Meijer's functions.

Kernel $H_{\mu\nu}(x)$ also applicable to the e.m. corrections to HVP



I.3 Electromagnetic corrections to HVP, $a_\mu^{\mathrm{hvp}1\gamma^*}$

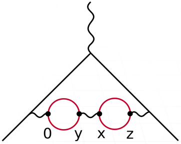
Quark-level Wick-contraction diagrams:



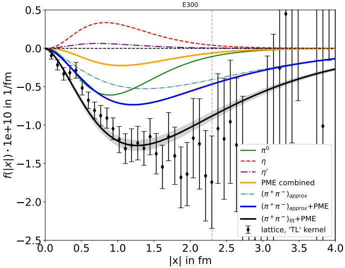
- Current precision target: about 10%.
- Aspect **II.**: Here the internal photon leads to a renormalisation of the QCD parameters $\{m_u, m_d, m_s, m_c, g_0\}$ ... but some diagrams are finite.

Fig. by J. Parrino from 2411.07969.

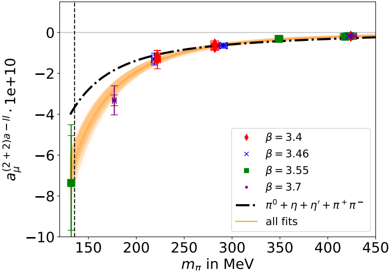
Calculation of one of the largest diagrams in $a_\mu^{\mathrm{hvp1}\gamma^*}$



One-photon irreducible diagram: perturbatively, at least two gluons must be exchanged between the two quark loops



Final 1d integral ($m_\pi = 170\,$MeV)



Chiral extrapolation

- Chiral perturbation theory helps control the noisy tail of the correlator.
- This calculation entered the result of our recent preprint 2411.07969.

J. Parrino, HM et al, Lattice'24 and in preparation.

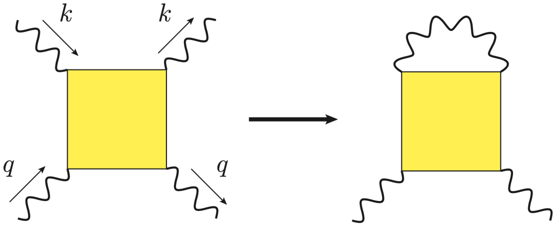
II. Renormalization of QCD parameters due to photons

Focus here on computing to a few permille the (subtracted) vacuum polarisation $\overline{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$.

- Define isoQCD by specific values of (m_π, m_K, m_Ω) that differ at most by O$(\alpha, m_u - m_d)$ from their experimental counterparts.
- Now turn on the coupling of quarks to photons. Work to first order in $(\alpha, m_u - m_d)$.
- The change $\overline{\Pi}_{1\gamma^*}(q^2) = \overline{\Pi}_{\rm full}(q^2) - \overline{\Pi}_{\rm isoQCD}(q^2)$ is an unambiguous prediction of QCD with electrically charged quarks.

Is there an explicit recipe in the continuum to predict $\overline{\Pi}_{1\gamma^*}(q^2)$?

Electromag. correction to hadronic vacuum polarization: general idea



$= \Pi_{\rm 4pt}(q^2, \Lambda)$

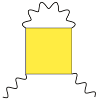
- the leading correction to HVP is expressible in terms of the forward HLbL amplitude, which by itself is a finite, physical amplitude
- regulating the propagator of the internal photon, e.g. $(1/k^2 - 1/(k^2+\Lambda^2))$ is sufficient to make the bare e.m. correction to HVP finite
- after adding the contribution of the quark-mass δm_f and gauge-coupling δg counterterms to the HVP, the $\Lambda \to \infty$ limit can be taken.

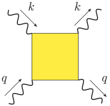
$$\overline{\Pi}^{1\gamma^*}(q^2) = \lim_{\Lambda\to\infty} \Big\{ \overline{\Pi}_{\rm 4pt}(q^2,\Lambda) + \Big(\delta g(\Lambda)\frac{\partial}{\partial g} + \textstyle\sum_f \delta m_f(\Lambda)\,\frac{\partial}{\partial m_f}\Big)\overline{\Pi}_{\rm isoQCD}(q^2)\Big\}.$$

Electromagnetic correction to HVP from forward HLbL amplitude

Master formula:

$$\Pi_{4\text{pt}}(Q^2,\Lambda) = \frac{1}{6Q^4(2\pi)^3}\int_0^\infty dK^2 \underbrace{\left[\frac{1}{K^2}\right]_\Lambda}_{\frac{1}{K^2}-\frac{1}{K^2+\Lambda^2}} \int_0^{K^2Q^2} d\nu^2 \left(\frac{K^2Q^2}{\nu^2}-1\right)^{1/2} \mathcal{M}(\nu,\,K^2,\,Q^2)$$





. . . the relevant forward hadronic light-by-light amplitude being

$$\mathcal{M}(\nu,\,K^2,\,Q^2) = g_{\mu_1\mu_3}g_{\mu_2\mu_4}\mathcal{M}^{\mu_1\mu_2\mu_3\mu_4}(k,q) = 4\mathcal{M}_{TT}-2\mathcal{M}_{LT}-2\mathcal{M}_{TL}+\mathcal{M}_{LL}.$$

NB. $\mathcal{M}$ admits a once-subtracted dispersion relation in the variable $\nu = k\cdot q$, in terms of $\gamma^*\gamma^* \to$ hadrons fusion cross-sections.

Biloshytskyi, HM et al 2209.02149 (JHEP)

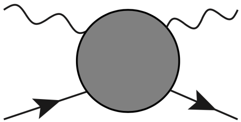
Determining the counterterms induced by the photons

Determine the isoscalar counterterms from three conditions such as

$$M_N^{\rm phys} - M_N^{\rm isoQCD} \stackrel{!}{=} M_N^{\rm self}(\Lambda) + \tfrac{1}{6}\delta(m_u + m_d - 2m_s)(\Lambda)\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle$$
$$+\tfrac{1}{3}\delta(m_u + m_d + m_s)(\Lambda)\langle N|\bar{u}u + \bar{d}d + \bar{s}s|N\rangle + \delta g^{-2}(\Lambda)\ \langle N|\tfrac{1}{2}{\rm Tr}\{G_{\mu\nu}G_{\mu\nu}\}|N\rangle$$

for the average nucleon mass, and $(m_u - m_d)(\Lambda)$ from the mass splitting.

Forward Compton amplitude on hadron H
$T_i(\nu = q\cdot p/m, q^2),\ i = 1, 2$:

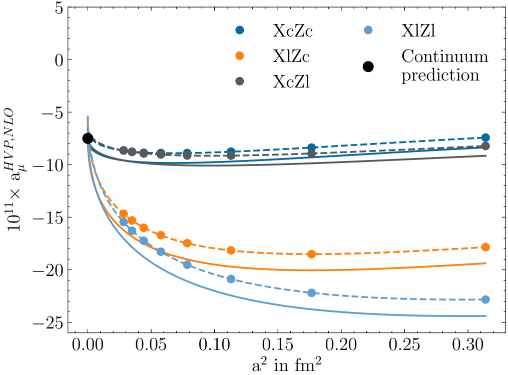


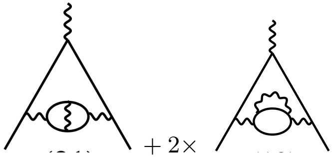
$$M_H^{\rm self}(\Lambda) = \frac{e^2}{2M_H}\int \frac{d^4Q}{(2\pi)^4}\left[\frac{1}{Q^2}\right]_\Lambda \big(3Q^2 T_1(iQ_0, -Q^2) + (2Q_0^2 + Q^2)T_2(iQ_0, -Q^2)\big)$$

...followed by a dispersive representation of the T_i
via the hadron's structure functions $F_i(x = Q^2/(2M_H\nu), Q^2)$.

Cottingham, Ann.Phy. **25**, 424 (1963); [...]; Gasser, Leutwyler, Rusetsky PLB 814 (2021) 136087.

Lattice calculations reproduce two-loop QED vacuum polarization





$$m_{\ell\text{oop}} = m_\mu$$
$$\Lambda = 3m_\mu$$

D. Erb et al LAT'24

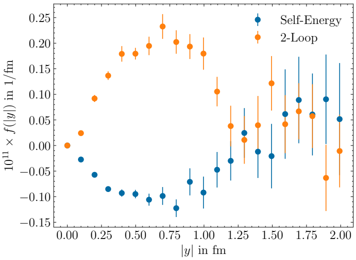
$$a_\mu^{2\text{loop vp}} = -\frac{e^2}{2}\delta_{\mu\nu}\int_{x,y,z} H_{\lambda\sigma}(z)[G_0]_\Lambda(y-x)\,\Big\langle V_\sigma^{\rm em}(z)V_\nu^{\rm em}(y)V_\mu^{\rm em}(x)V_\lambda^{\rm em}(0)\Big\rangle,$$

$$[G_0]_\Lambda(x) = G_0(x) - 2G_{\frac{\Lambda}{\sqrt{2}}}(x) + G_\Lambda(x), \qquad G_m(x) \equiv \int \frac{d^4q}{(2\pi)^4}\,\frac{e^{iq\cdot x}}{q^2+m^2}.$$

The 'continuum prediction' was obtained with the help of dispersive techniques.

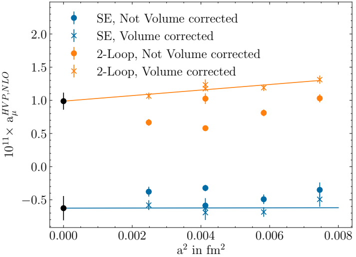
Idem, in QCD

Last integrand:



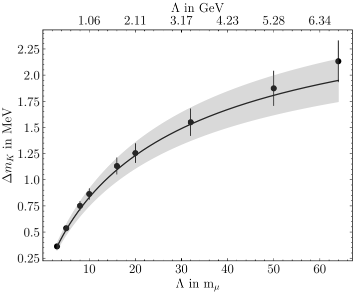
(y is one of the internal vertices)

Continuum extrapolation



$$\Lambda = 16 m_\mu$$

Λ dependence of $(M^{\text{self}}_{K^+} - M^{\text{self}}_{K^0})$ $\longrightarrow$
after continuum extrapolation:
consistent with logarithmic behaviour predicted by OPE



$m_\pi = m_K = 415$ MeV; D. Erb et al LAT'24

Summary

Photon-hadron interactions are essential in many processes, in particular for precision physics.

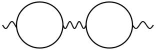
- Photons present particular challenges for lattice QCD:
- Position-space methods help handle the long-distance effects.
- I have argued in favour of taking the continuum limit at fixed cutoff Λ on the photon virtuality.

Epilogue: where will lattice QCD stand in 10 years time?

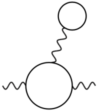
- precision frontier: permille level uncertainties on a range of quantities with impact on phenomenology, including e.m. corrections
- very much larger volumes, together with coordinate-space methods
- method to significantly reduce critical slowing down (via machine-learning techniques?); reach very small lattice spacings; see non-polynomial dependence of observables on the lattice spacing.
- signal-to-noise problems, in particular for nucleon observables: modified importance sampling technique? advanced spectroscopy methods?
- 'real-time' problems: mature calculations based on Lüscher's formalism; quantum computing applications?

The subset of UV-finite diagrams

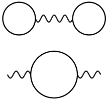
Operator-product expansion and power-counting $\Rightarrow$
about half of the diagrams are UV-finite diagrams.



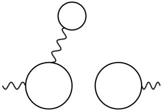
(2+2)a



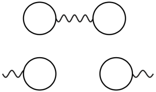
(3+1)b



(2+1+1)a



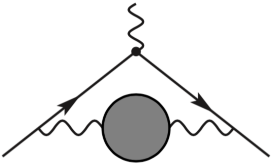
(2+1+1)b



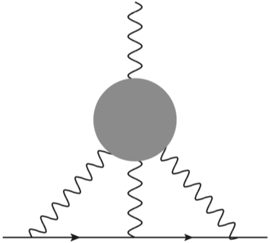
(1+1+1+1)

$\rightsquigarrow$ For these, the internal photon propagator does not need to be regulated.

Dominant uncertainties in SM prediction for $(g-2)_\mu$



Hadronic vacuum polarisation



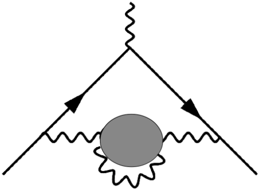
Hadronic light-by-light scattering

HVP: O(α^2), about $700 \cdot 10^{-10}$

$\Rightarrow$ desirable accuracy: 0.2%

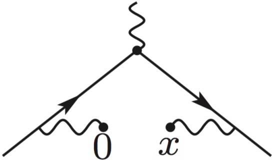
HLbL: O(α^3), about $10 \cdot 10^{-10}$

$\Rightarrow$ desirable accuracy: 10%.



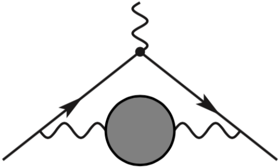
How large are the (overall O(α^3)) QED corrections to the HVP contribution?

Analogy: hadronic vacuum polarization in x-space HM 1706.01139



QED kernel $H_{\mu\nu}(x)$

$\Rightarrow$



$a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x \; H_{\mu\nu}(x) \; \Big\langle j_\mu(x) j_\nu(0) \Big\rangle_{\rm QCD},$$

$$j_\mu = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d - \frac{1}{3}\bar s\gamma_\mu s + \dots; \qquad H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

Kernel known in terms of Meijer's functions: $\mathcal{H}_i(|x|) = \frac{8\alpha^2}{3m_\mu^2} f_i(m_\mu |x|)$ with

$$f_2(z) = \frac{G^{2,2}_{2,4}\left(z^2 \Big| \begin{smallmatrix} \frac{7}{2}, 4 \\ 4, 5, 1, 1 \end{smallmatrix}\right) - G^{2,2}_{2,4}\left(z^2 \Big| \begin{smallmatrix} \frac{7}{2}, 4 \\ 4, 5, 0, 2 \end{smallmatrix}\right)}{8\sqrt{\pi} z^4},$$

$$f_1(z) = f_2(z) - \frac{3}{16\sqrt{\pi}} \cdot \left[G^{2,3}_{3,5}\left(z^2 \Big| \begin{smallmatrix} 1, \frac{3}{2}, 2 \\ 2, 3, -2, 0, 0 \end{smallmatrix}\right) - G^{2,3}_{3,5}\left(z^2 \Big| \begin{smallmatrix} 1, \frac{3}{2}, 2 \\ 2, 3, -1, -1, 0 \end{smallmatrix}\right) \right].$$

Lattice implementation aspects

The split-up of the internal photon propagator

$$\frac{1}{k^2} = \frac{1}{k^2+\Lambda^2} + \left(\frac{1}{k^2} - \frac{1}{k^2+\Lambda^2}\right)$$

can be useful to separate the issue of the UV divergence from the IR effects.

The first term can be implemented by placing the photon on the lattice ('standard method', but with a photon mass $\Lambda \sim 400\,\mathrm{MeV}$).

The second term can be implemented with coordinate-space methods, similar to $a_\mu^{\rm HLbL}$,

$$\overline{\Pi}_{4\rm pt}(Q^2,\Lambda) = -\frac{e^4}{2}\delta_{\mu\nu}\int_{x,y,z} H_{\lambda\sigma}(z)(G_0 - G_\Lambda)(y-x)\,\Big\langle V_\sigma^{\rm em}(z)V_\nu^{\rm em}(y)V_\mu^{\rm em}(x)V_\lambda^{\rm em}(0)\Big\rangle,$$

with $H_{\lambda\sigma}(z)$ known analytically [1706.01139] and $G_\Lambda(x) \equiv \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2+\Lambda^2}$.

- Natural formulation on large lattices
- Avoids power-law finite-size effects.

2209.02149.

II. Computing isospin-breaking mass splittings

Determine the $(m_u - m_d)$ value that leads to the correct kaon mass splitting:

$$M_{K^+}^{\rm phys} - M_{K^0}^{\rm phys} \stackrel{!}{=} (M_{K^+}^{\rm self} - M_{K^0}^{\rm self})(\Lambda) + (m_u - m_d)(\Lambda)\ \langle K^+|\bar{u}u - \bar{d}d|K^+\rangle \qquad (\star)$$

Then, predict the proton-neutron mass difference: (similar for D, B, Σ, Ξ mass splittings)

$$M_p^{\rm phys} - M_n^{\rm phys} = \lim_{\Lambda\to\infty} \left\{ (M_p^{\rm self} - M_n^{\rm self})(\Lambda) + (m_u - m_d)(\Lambda)\ \langle p|\bar{u}u - \bar{d}d|p\rangle \right\}$$

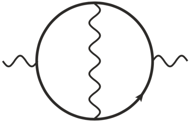
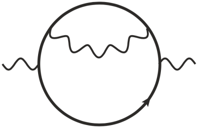
Eliminate $(m_u - m_d)$ using $(\star)$:

$$\begin{aligned} M_p^{\rm phys} - M_n^{\rm phys} \quad &= \quad \lim_{\Lambda\to\infty} \left\{ (M_p^{\rm self} - M_n^{\rm self})(\Lambda) - R_{NK} \cdot (M_{K^+}^{\rm self} - M_{K^0}^{\rm self})(\Lambda) \right\} \\ &\quad + R_{NK} \cdot \left(M_{K^+}^{\rm phys} - M_{K^0}^{\rm phys} \right), \\ R_{NK} \quad &\equiv \quad \frac{\langle p|\bar{u}u - \bar{d}d|p\rangle}{\langle K^+|\bar{u}u - \bar{d}d|K^+\rangle} = 0.45(4). \end{aligned}$$

Possible use: compute the quantities in blue in lattice QCD $\rightsquigarrow$
constraint on the nucleon structure functions (use same regulator Λ !).

2-loop vacuum polarisation in QED $\overline{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$

The ‘unperturbed’ theory consists of free Dirac fermions of mass $m_{\rm u}$.
It produces VP $\overline{\Pi}^{(1)}(q^2, m_{\rm u})$. At the next order:

$\overline{\Pi}_{\rm 4pt}(q^2, \Lambda) =$  $+\, 2\times$ 

$$\overline{\Pi}^{(2)}(q^2, m) = \lim_{\Lambda\to\infty} \left\{ \overline{\Pi}_{\rm 4pt}(q^2, \Lambda, m_{\rm u}) + \delta m_0(\Lambda)\ \frac{\partial}{\partial m_u} \overline{\Pi}^{(1)}(q^2, m_{\rm u}) \right\},$$

On shell mass $m = m_{\rm u} + m_{\rm self}(\Lambda) + \delta m_0(\Lambda)$.

On-shell renormalization scheme: keep m constant, $m_{\rm u} = m$:

$$\delta m_0^{\rm on\,shell}(\Lambda) = -m_{\rm self}(\Lambda) = -3m\frac{\alpha}{2\pi}\left[\frac{1}{4} + \log\frac{\Lambda}{m} + {\rm O}(\Lambda^{-2})\right].$$

Different scheme X: allow the on-shell mass to change by a finite amount Δm upon switching on the radiative correction, $m_{\rm u} = m - \Delta m$:

$$\delta m_0^{\rm X}(\Lambda) = \delta m_0^{\rm on\,shell}(\Lambda) + \Delta m.$$

A test in QED: two-loop VP from one-loop forward LbL amplitude

$$
\begin{aligned}
&\mathcal{M}(\nu, K^2, Q^2) = 16\alpha^2\Bigg(6 - \Bigg\{\frac{2\log\left[\frac{1}{2}Q\left(\sqrt{Q^2+4}+Q\right)+1\right]}{\sqrt{Q^2+4}}\\
&\times\Big(-4\nu^2Q^2\left[\left(K^2-2\right)\left(K^2+1\right)Q^4+\left(K^2+2\right)\left(7K^2-2\right)Q^2+6K^4+52K^2+16\right]\\
&+K^2Q^4\left(K^2+Q^2+4\right)^2\left[K^2\left(Q^2+4\right)-2Q^2+4\right]+96\nu^4\Big)\Big/\Big(K^4Q^5\left(K^2+Q^2+4\right)^2\\
&+16\nu^4Q-4K^2\nu^2Q^3\left[K^2\left(Q^2+2\right)+2\left(Q^2+4\right)\right]\Big)+\{K\leftrightarrow Q\}\Bigg\}\\
&+\Bigg\{\frac{2\sqrt{1+\frac{4}{K^2+2\nu+Q^2}}\log\left[\frac{1}{2}\left(\sqrt{\left(K^2+2\nu+Q^2\right)\left(K^2+2\nu+Q^2+4\right)}+K^2+2\nu+Q^2+2\right)\right]}{K^2Q^2\left(K^2+Q^2+2\nu+4\right)-4\nu^2}\\
&\times\Big(K^2Q^2(K^2+Q^2+2\nu)-2(K^2+Q^2)(\nu-1)-(K^4+Q^4)-2\nu(\nu+2)\Big)\\
&+\frac{(K^2+Q^2)^2+2\nu(K^2+Q^2)+2\nu(\nu-2)-4}{\nu}\,C_0\left(-K^2,-Q^2,-K^2-2\nu-Q^2;1,1,1\right)\\
&+\{\nu\to-\nu\}\Bigg\}\Bigg), \qquad \text{(lepton mass set to unity)}
\end{aligned}
$$

where $C_0(p_1^2, p_2^2, (p_1+p_2)^2; m_1^2, m_2^2, m_3^2)$ is the scalar one-loop integral [hep-ph/9807565].
Inserting this expression into the master formula gives the same result for $\overline{\Pi}^{(2)}(Q^2)$ as

$$\overline{\Pi}(Q^2) = -\frac{Q^2}{\pi}\int_{4m_\ell^2}^{\infty}\frac{dt}{t(t+Q^2)}\,\mathrm{Im}\Pi(t)$$

using the 1955 Källen-Sabry next-to-leading-order spectral function $\frac{1}{\pi}\mathrm{Im}\Pi(t)$.