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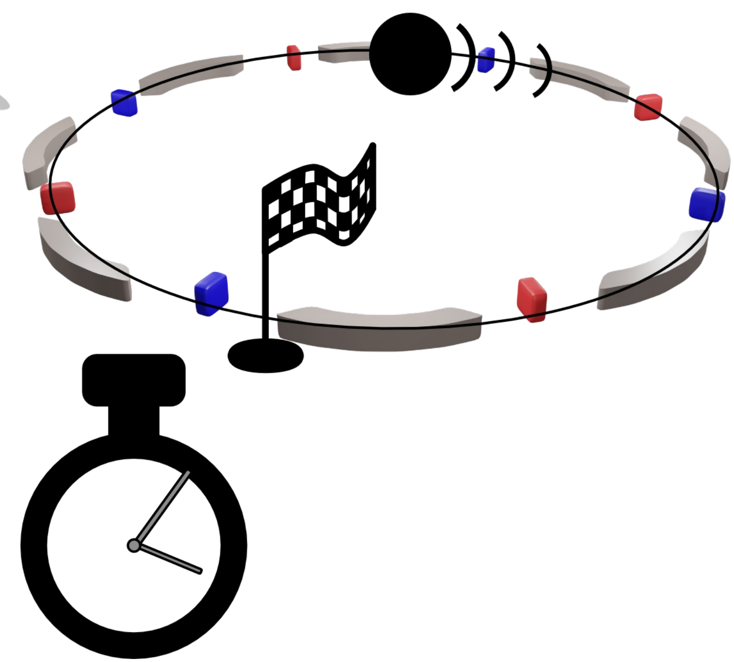
CONCEPT STUDY OF A STORAGE RING-BASED GRAVITATIONAL WAVE OBSERVATORY

Storage Rings & Gravitational Waves – mini-brainstorm

THORBEN SCHMIRANDER
10.02.2025

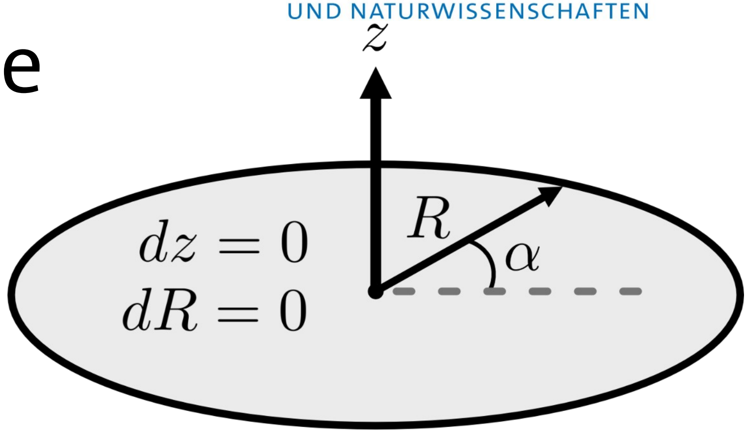
Idea

- Variation of circulation time encodes GW signature
- Need: Particle on fixed circular trajectory in longitudinal free fall
- can circulate for minutes up to hours: mHz GW



a **VERY** precise clock

Frame of reference



metric for ring with fixed radius (cyl. coords):

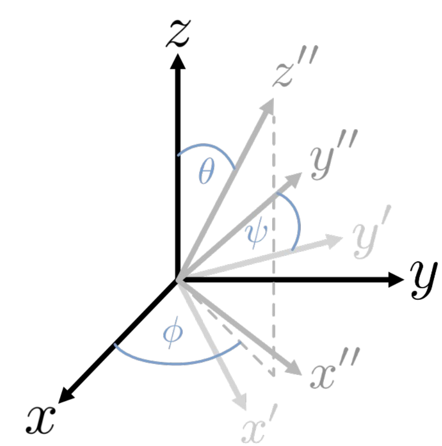
$$ds^2 = -c^2 dt^2 + (1 + h_{\theta\phi\psi}(t,\alpha))\, R^2 d\alpha^2$$

Effective GW strain includes Earth's rotation:

$$h_{\theta\phi\psi}(t,\alpha) = h_+(t)\left(f_s^+ \sin^2\alpha + f_c^+ \cos^2\alpha + \dots\right) + h_\times(t)(\dots)$$

Euler angles enter via e.g.:

$$f_s^+ = \left(\cos^2\theta \cos^2\phi - \sin^2\phi\right)\cos 2\psi - \left(\cos\theta \sin 2\phi\right)\sin 2\psi$$



See: Rao et al., PRD **102,** 122006 (2020)

Particle in a Storage Ring

From geodesic equations: $$\frac{d^2 l}{dt^2} = -\frac{1}{1+h_{\theta\phi\psi}(t)} \frac{dh_{\theta\phi\psi}(t)}{dt} v_0$$

longitudinal acceleration

For fixed radius:

$$a_{||}(t) \approx -\dot{h}_{\theta\phi\psi}(t) v_0$$

$$\Delta T^{\text{fixed}} = \frac{\Delta l^{\text{fixed}}}{v_0}$$

$$= -\int_0^t \left(h_{\theta\phi\psi}(t') - h_{\theta\phi\psi}(0)\right) dt'$$

~ *relativistic result*

Main idea:

- GW strain as classical force
- Find transformation between ring with fixed radius and storage ring simulation
- Find reference to make noise terms comparable

See: Rao et al., PRD **102,** 122006 (2020)

Particle in a Storage Ring

For storage ring:

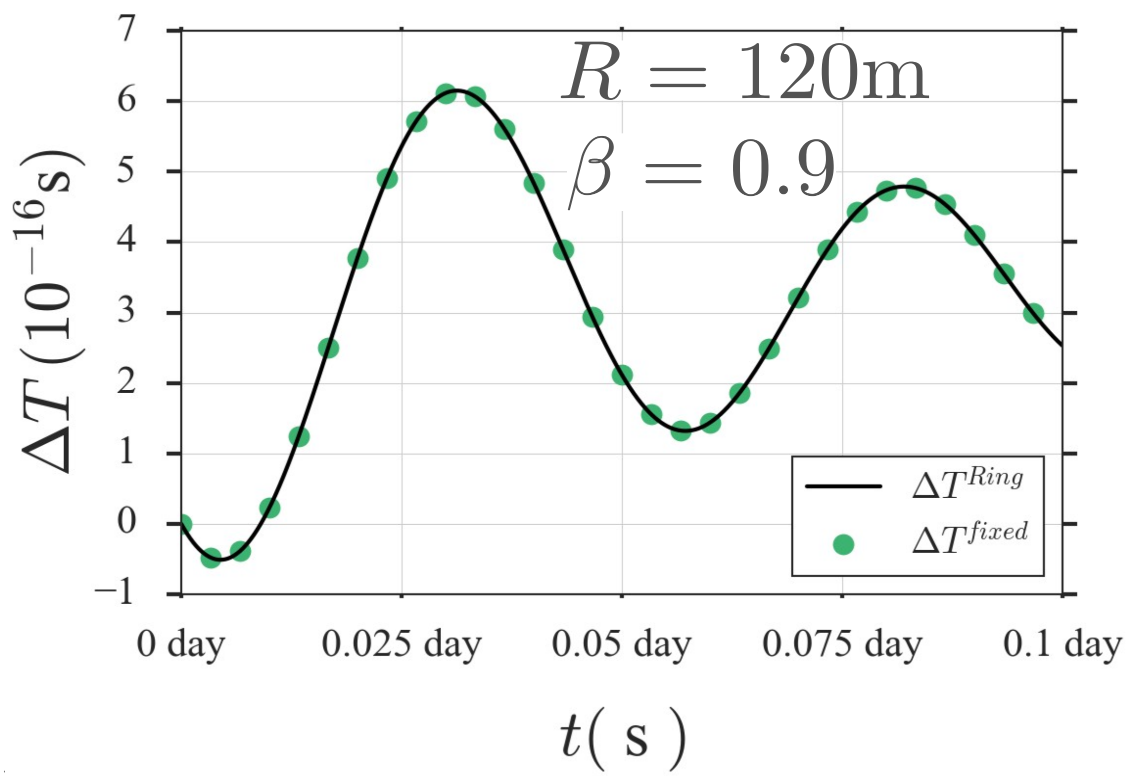
$$\Delta T^{\mathrm{fixed}} = -\frac{1}{\eta}\frac{1}{\gamma^2}\left(1-\frac{v_0^2}{2c^2}\right)\Delta T^{\mathrm{Ring}} - \left(1-\frac{v_0^2}{2c^2}\right)\overline{h_{\theta\phi\psi}(t_0)t}$$

"slip factor"

Inspiral @ z=0.2

$m_{1/2} = 10^6 M_\odot$

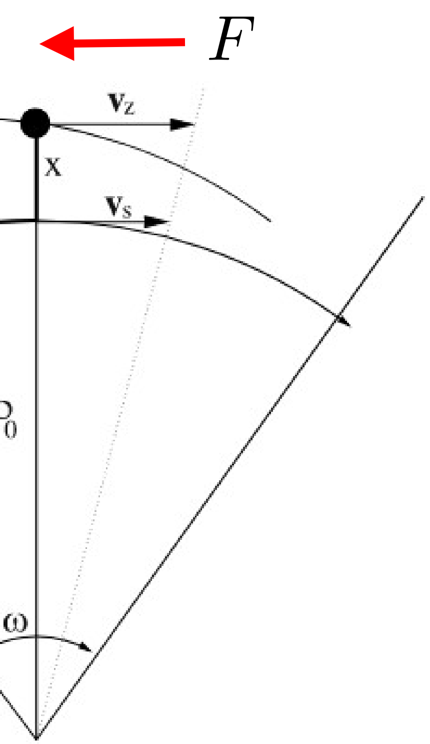
on order of 0.1 fs!!!



See: Schmirander et al. PRD **110**, 082002 (2024

Particle tracking I

Include longitudinal force



radial EOM:

$$\ddot{x} - \omega^2(\rho_0 + x) = -\frac{q}{p} v_s^2 B_y$$

$$v_s = (\rho_0 + x)\omega + \frac{\langle F \rangle}{m\gamma^3\beta c} \cdot s$$

$$x'' + \frac{1-n}{\rho_0} x = \frac{\delta}{\rho_0} - \frac{2z\langle F \rangle}{\rho_0^2 \omega m \gamma^3 \beta c}$$

Green's function approach:

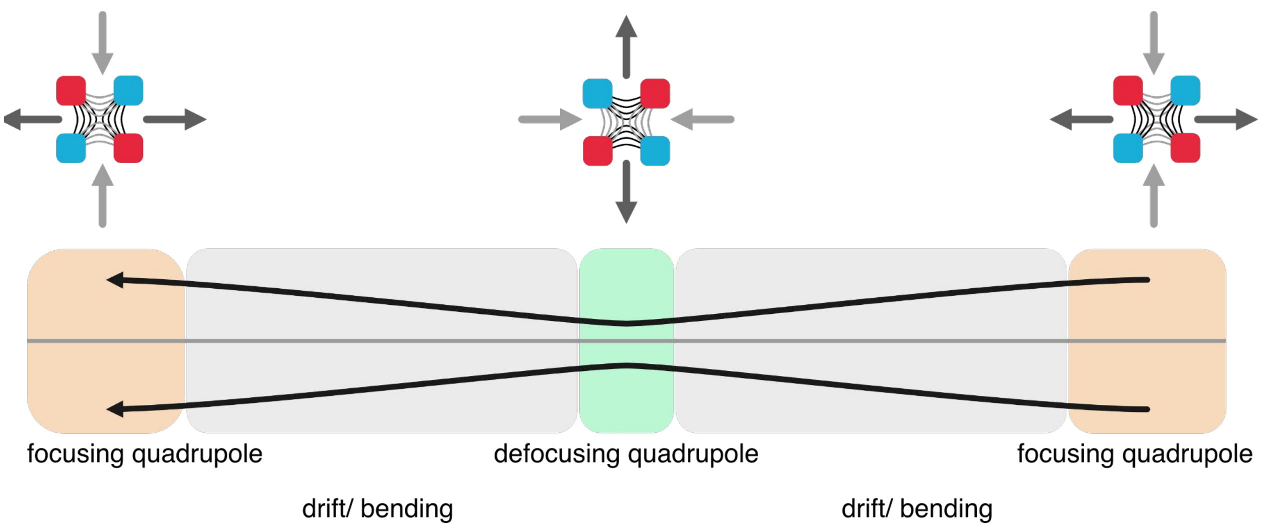
$$x(s) = x_0 c_x(s) + x_0' s_x(s) + \delta d_{x1}(s) + d_{x2}(s)$$

First-order EOM particle tracking: $x_{\text{out}} = M_{\text{drift}} \cdot x_{\text{in}} + \Delta x$

See: e.g. Wiedemann: Part. Acc. Physics, Springer (2015)

Particle tracking II

$$\text{x} = \text{function of} \left\{ \begin{array}{c} \text{radial} \\ \text{longitudinal} \\ \text{azimuthal} \end{array} \right\} \left\{ \begin{array}{c} \text{positions} \\ \text{momenta} \end{array} \right\} \text{relative to a design particle}$$



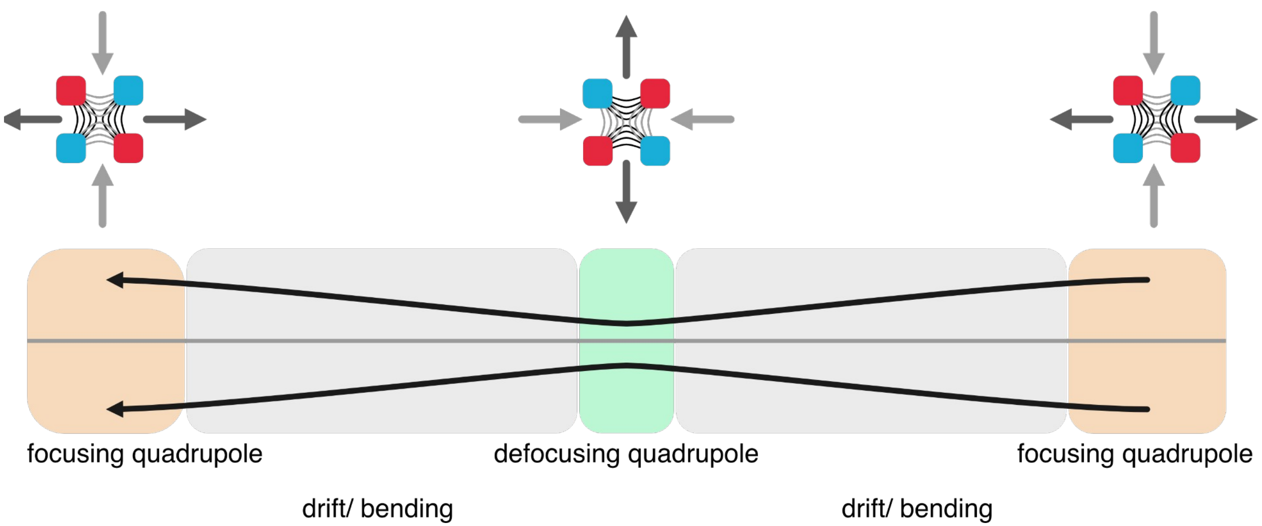
$$\begin{aligned} \text{x}_{\text{out}} &= M_{\text{drift}} \cdot x_{\text{in}} + \Delta x_1 \\ \text{x}_{\text{out}} &= M_{\text{sector}} \cdot x_{\text{in}} + \Delta x_2 \\ &\vdots \end{aligned}$$

↑ GW force

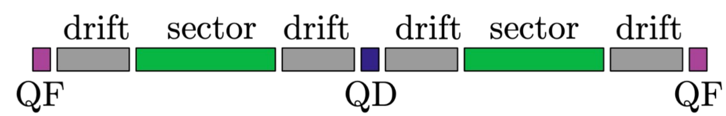
See: e.g. Wiedemann: Part. Acc. Physics, Springer (2015)

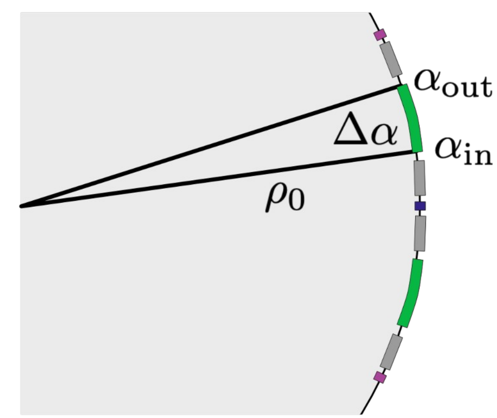
Particle tracking II

$$\mathrm{x} = \text{function of} \left\{ \begin{array}{c} \text{radial} \\ \text{longitudinal} \\ \text{azimuthal} \end{array} \right\} \left\{ \begin{array}{c} \text{positions} \\ \text{momenta} \end{array} \right\} \text{relative to a design particle}$$



"FODO" cell





$$\mathrm{x_{out}} = M_{\mathrm{drift}} \cdot x_{\mathrm{in}} + \Delta x_1$$
$$\mathrm{x_{out}} = M_{\mathrm{sector}} \cdot x_{\mathrm{in}} + \Delta x_2$$
$$\vdots$$

GW force

e.g. for U_2^+:
@ $\beta = 0.32$

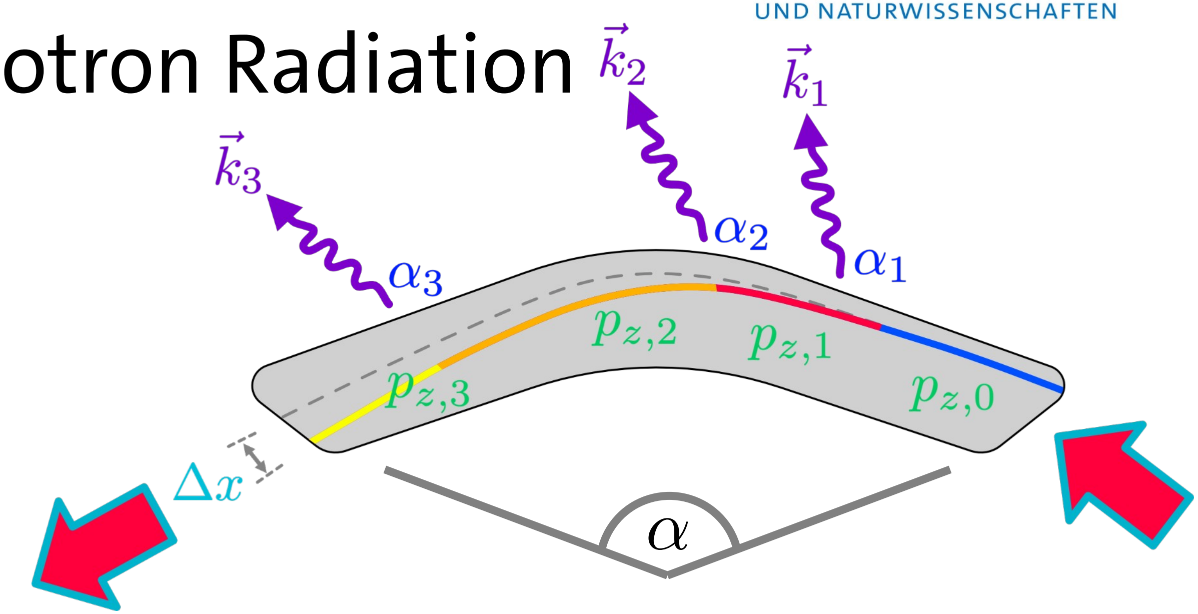
$150\ \mathrm{GeV}$

$L_{\mathrm{FODO}} = 133.5\ \mathrm{m}$

$200 \cdot L_{\mathrm{FODO}}$
$= 26.7\ \mathrm{km}$
$= L_{\mathrm{LHC}}$

See: e.g. Wiedemann: Part. Acc. Physics, Springer (2015)

Synchrotron Radiation

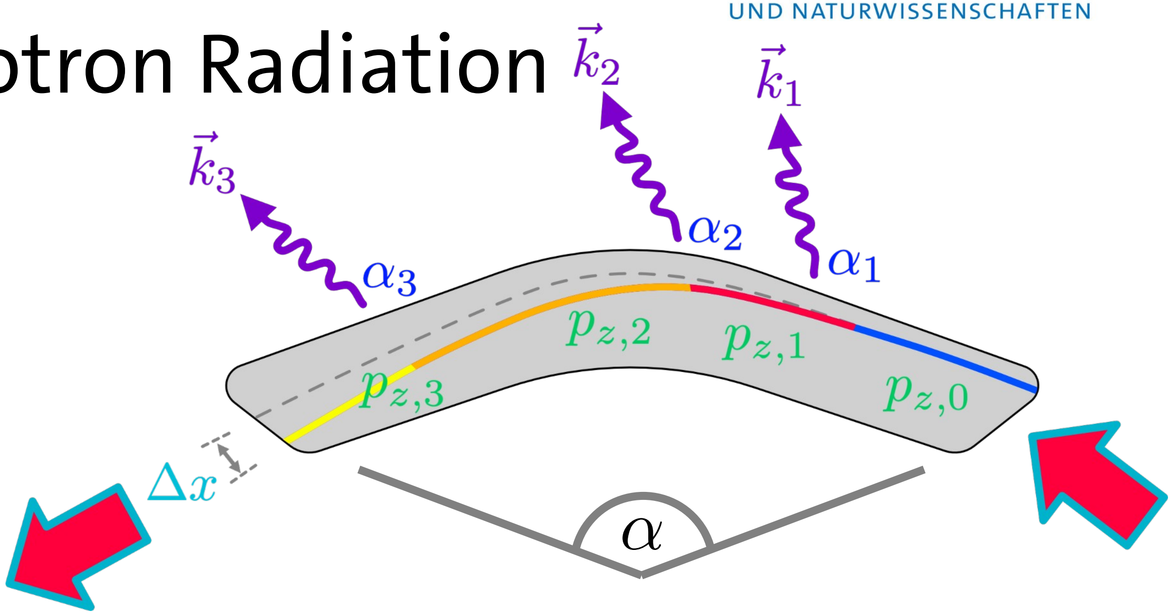


$$x_0(l) = M_{\text{sector}}(\alpha) \cdot x(0)$$

$$x_1(l) = M_{\text{sector}}(\alpha - \alpha_1)\left(M_{\text{sector}}(\alpha_1) \cdot x(0) + \Delta_\gamma(\vec{k}_1)\right)$$

$$\Delta x(\alpha_3, \dots, \vec{k}_3, \dots) = M_{\text{sector}}(\alpha - \alpha_3)\left(\dots + \Delta_\gamma(\vec{k}_3)\right) - x_0(l)$$

Synchrotron Radiation



$$x_0(l) = M_{\text{sector}}(\alpha) \cdot x(0)$$

$$x_1(l) = M_{\text{sector}}(\alpha - \alpha_1) \left(M_{\text{sector}}(\alpha_1) \cdot x(0) + \Delta_\gamma(\vec{k}_1) \right)$$

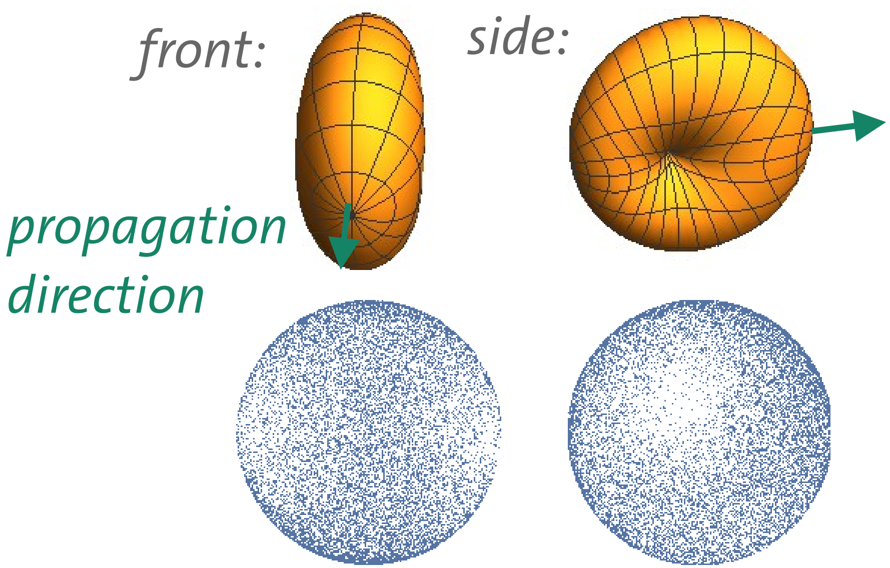
$$\Delta x(\alpha_3, \ldots, \vec{k}_3, \ldots) = M_{\text{sector}}(\alpha - \alpha_3) \left(\cdots + \Delta_\gamma(\vec{k}_3) \right) - x_0(l)$$

- Is given by a “simple” shift: $x(l) = M_{\text{sector}} \cdot x(0) + \Delta x$
- Heavy ions!
- Need distribution for angles and momenta

Sampling of Synchrotron Photons

$$F(\theta, \phi) = \frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}$$

$$\beta = 0.1$$



1. $\{\theta_i, \phi_i\} \in [0,\pi] \times [0, 2\pi]$
2. stratification
3. $f_i \in [0, F(0,0)]$

"rejection sampling"

See: Arvo et al. SIGGRAPH (2004)

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Sampling of Synchrotron Photons

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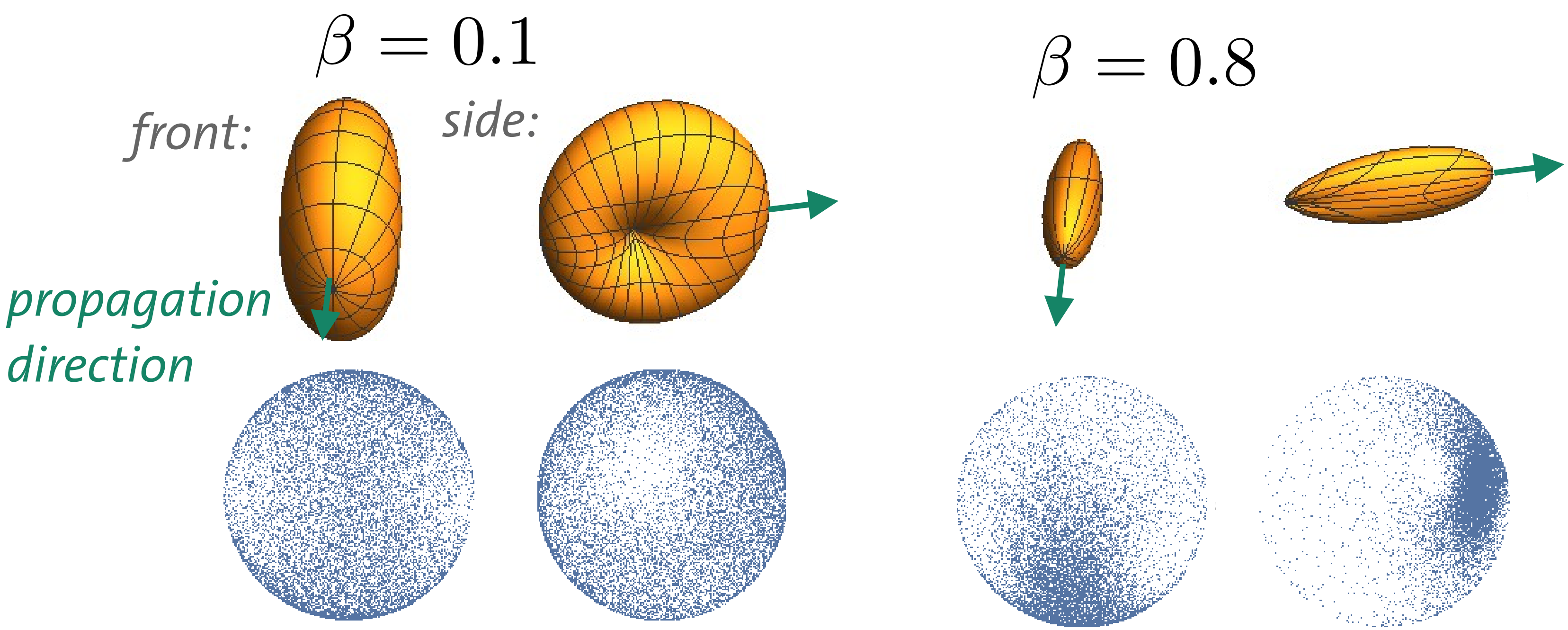


1. $\{\theta_i, \phi_i\} \in [0, \pi] \times [0, 2\pi]$
2. stratification
"rejection sampling"
3. $f_i \in [0, F(0,0)]$

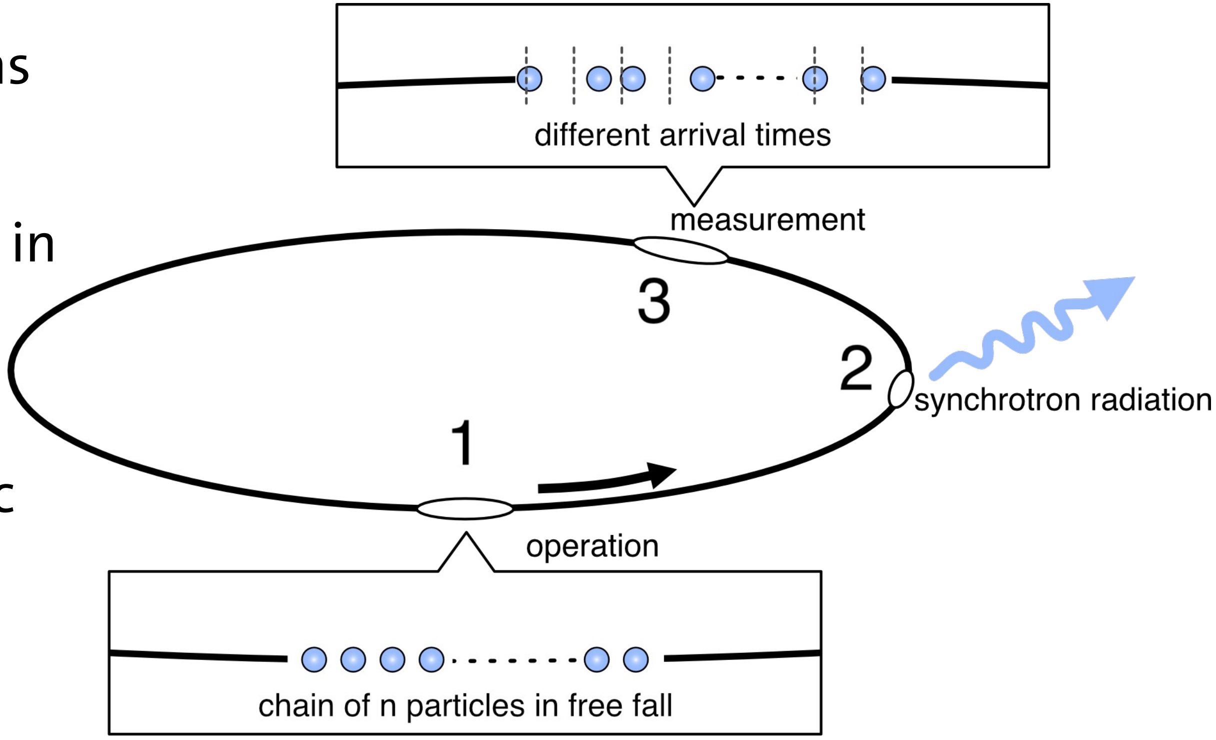
See: Arvo et al. SIGGRAPH (2004)

Concept Experiment Setup

1: operate under stable conditions for many rounds

2: synchrotron radiation emitted in sector magnets

3: measure timing relative to external ticking (from e.g. atomic clock)



- all particle subject to **SAME** GW force + **INDIVIDUAL** SR emission
- expected positions (dashed lines) incl. **AVERAGE** SR emission

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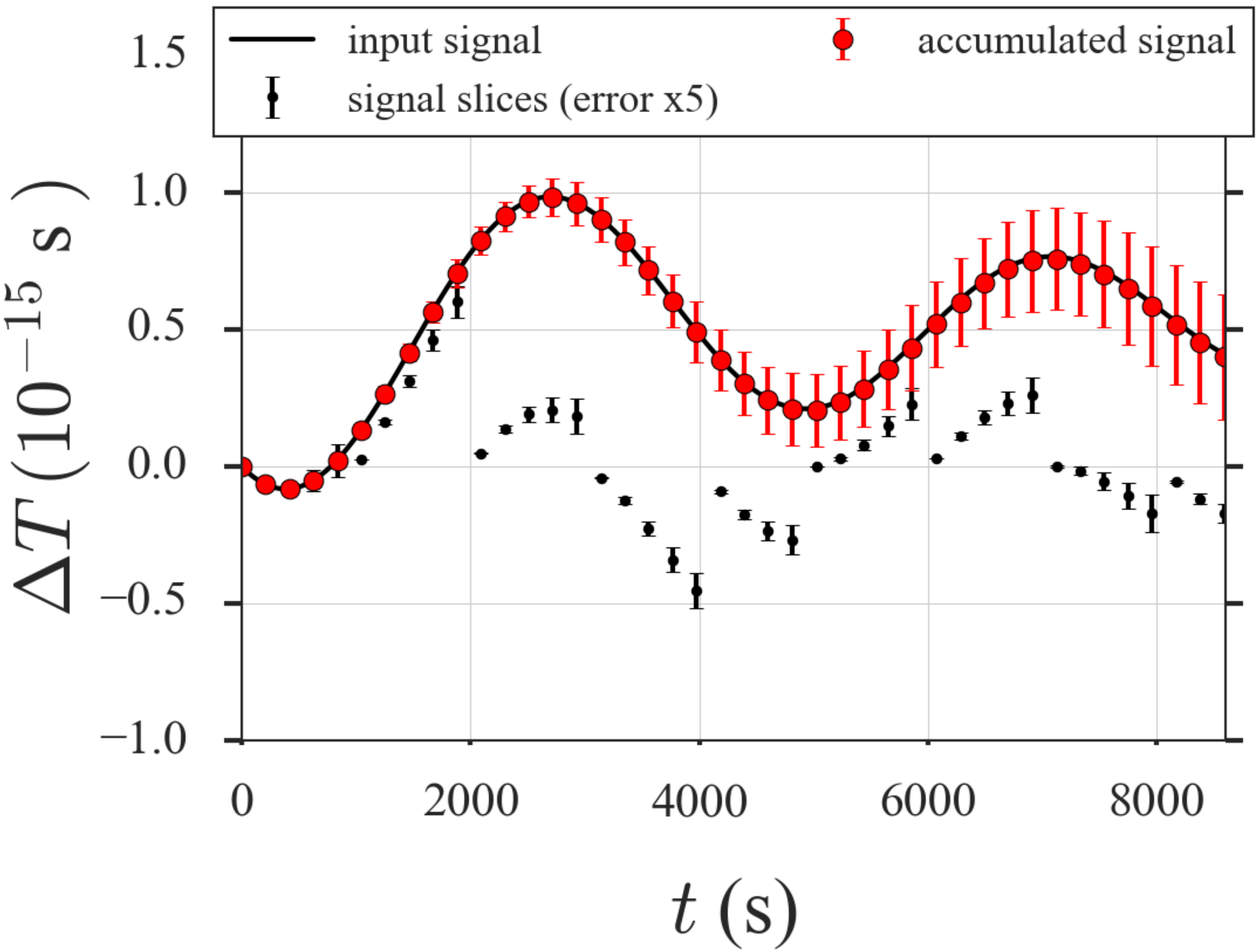
Results

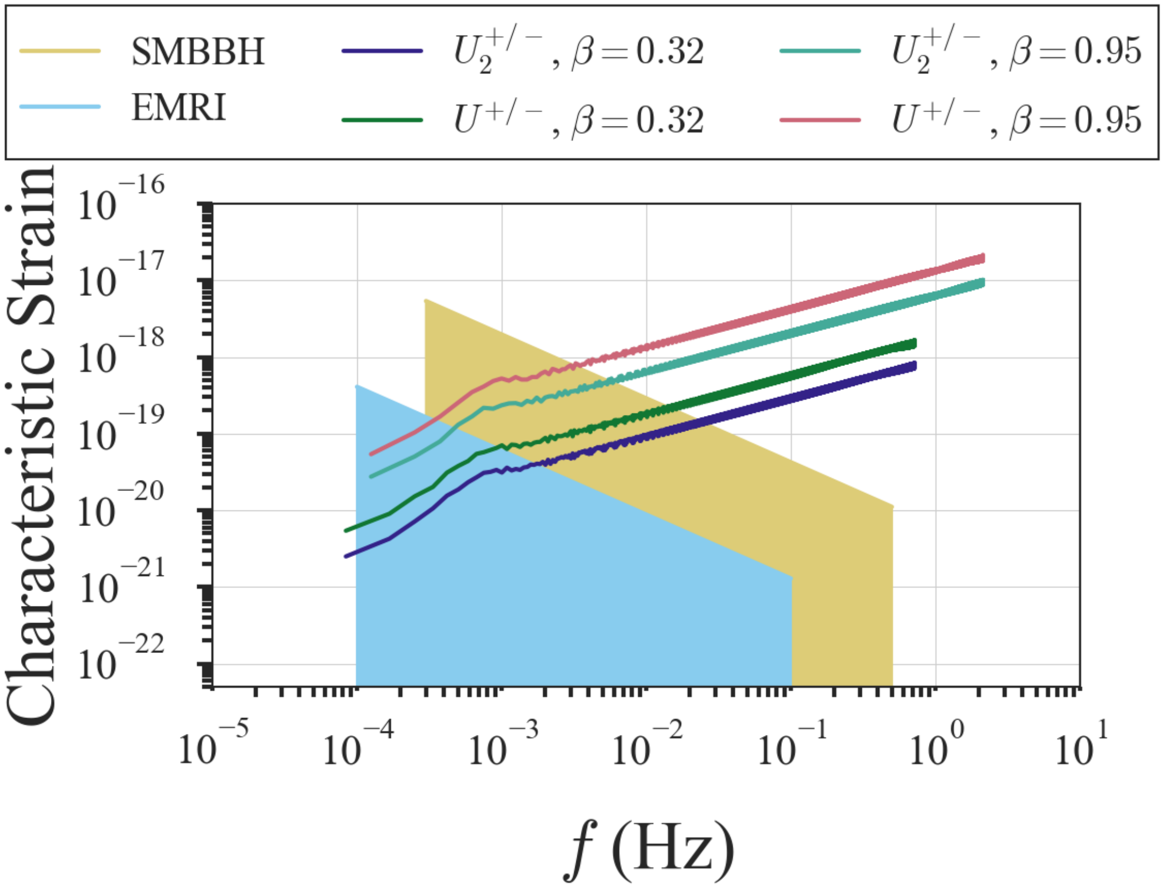


- chain of 100 U_2^+ ions
- dumped after t=10^3 s
- $\beta = 0.32$
- $L = 26.7$ km

error of mean arrival time **ONLY** due to synch. rad. emission power fluctuations

See: Schmirander et al. PRD **110**, 082002 (2024)

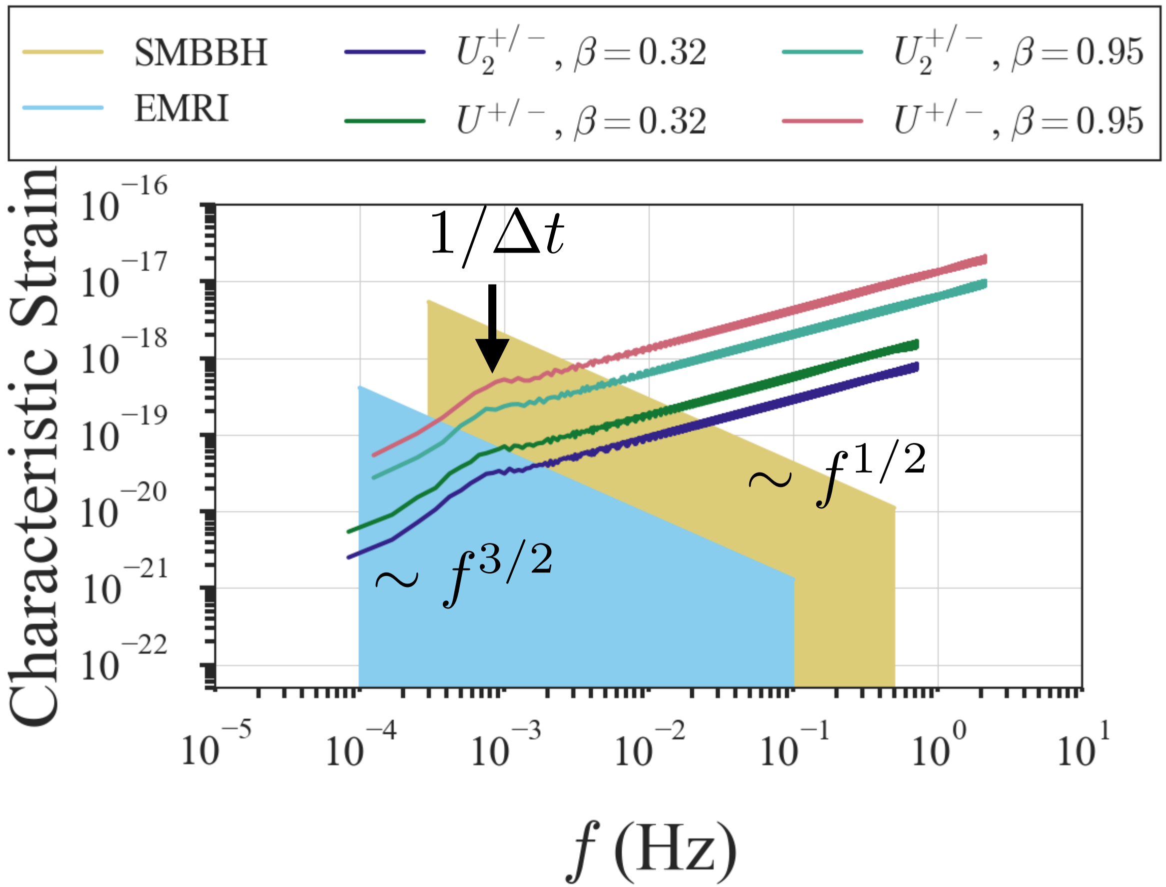
Characteristic Noise Strain



ONLY due to synch. rad. emission power fluctuations , no other noise (read-out, etc.)

See: Schmirander et al. PRD **110**, 082002 (2024)

Characteristic Noise Strain



$$f < \Delta t$$

$$\sqrt{f S_n(f)} \sim f^{3/2}$$

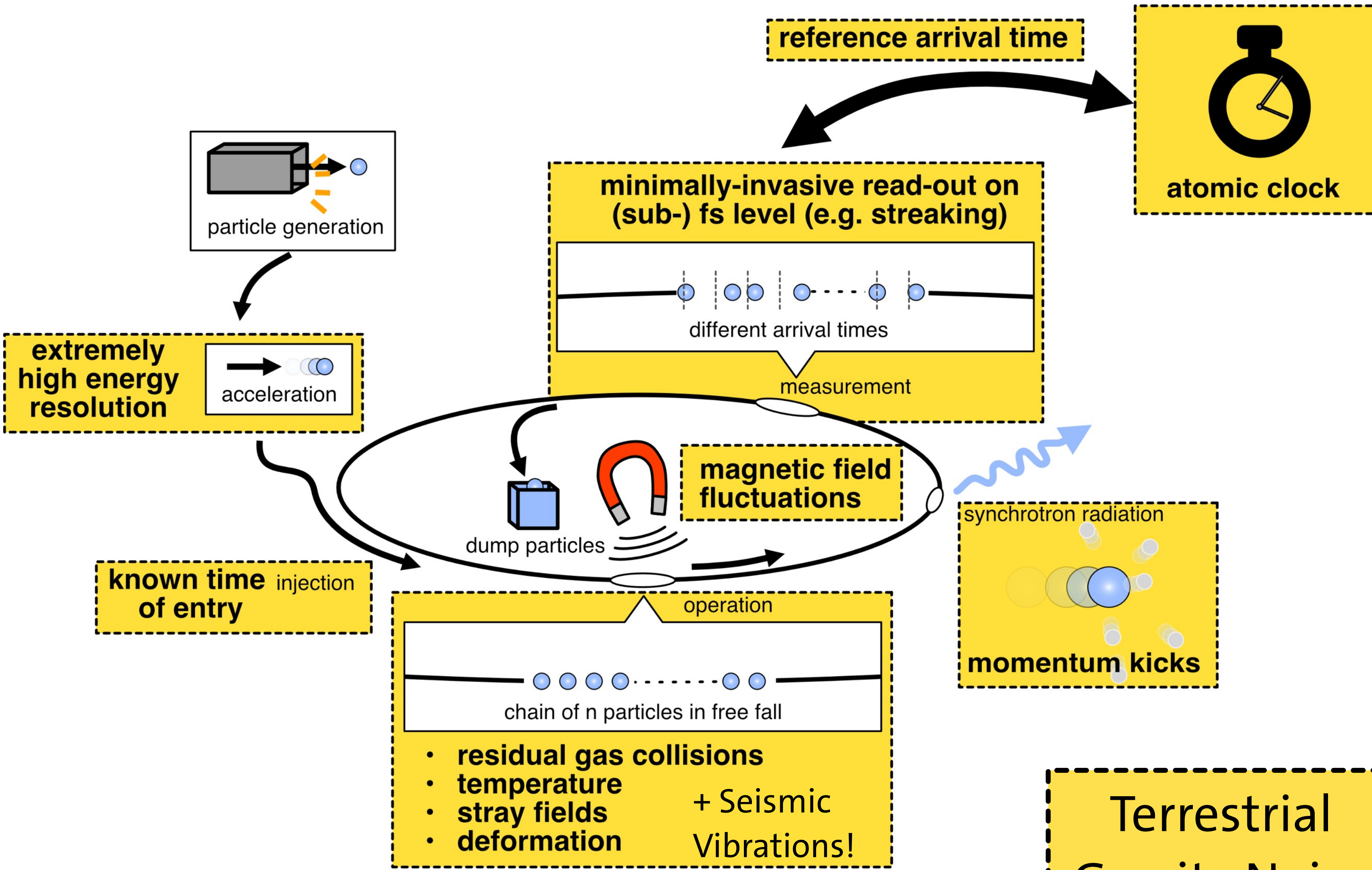
$$f > \Delta t$$

$$\sqrt{f S_n(f)} \sim f^{1/2}$$

ONLY due to synch. rad. emission power fluctuations , no other noise (read-out, etc.)

See: Schmirander et al. PRD **110**, 082002 (2024)

Still open design specifics:



Terrestrial Gravity Noise

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References

PHYSICAL REVIEW D **102,** 122006 (2020)

Detection of gravitational waves in circular particle accelerators

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Storage Rings and Gravitational Waves:
Summary and Outlook

A. Berlin[1], M. Brüggen[2], O. Buchmueller[3], P. Chen[4], R. T. D'Agnolo[5], R. Deng[6], J. R. Ellis[7,×,*], S. Ellis[5], G. Franchetti[8], A. Ivanov[9], J. M. Jowett[8], A. P. Kobushkin[10], S. Y. Lee[11], J. Liske[2], K. Oide[12], S. Rao[2], J. Wenninger[13], M. Wellenzohn[9], M. Zanetti[14], F. Zimmermann[13,×,†]

[×] *Editors*

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PHYSICAL REVIEW D **110,** 022007 (2024)

Detection of gravitational waves in circular particle accelerators II. Response analysis and parameter estimation using synthetic data

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PHYSICAL REVIEW D **110,** 082002 (2024)

Concept study of a storage ring-based gravitational wave observatory: Gravitational wave strain and synchrotron radiation noise

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Conclusion

- Particle in a storage ring: time delay encodes GW signature
- Include GW force & Synchrotron radiation (numerics)
- Concept of experimental setup
- Characteristic noise strain of radiation fluctuation
- open experiment setup questions/ noise sources

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BACKUP

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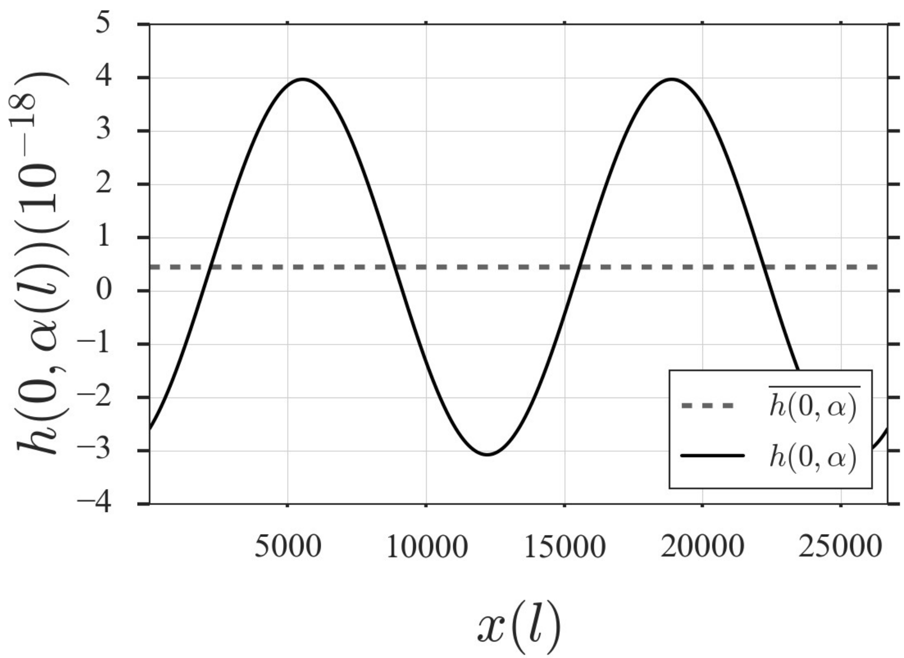
Terrestrial Gravity Noise

Signal Modeling



Chirp mass (red-shift corrected)

$$\mathcal{M} = \frac{(1+z)(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

Frequency $f = \frac{1}{1+z}\frac{\sqrt{G(m_1+m_2)}}{\pi r^{3/2}}$

Luminosity distance

$$d_\mathrm{L}$$

inclination angle

$$h_+ = \frac{4}{d_\mathrm{L}}\left(\frac{\mathcal{G}\mathcal{M}}{c^2}\right)^{5/3}\left(\frac{\pi f}{c}\right)^{2/3}\frac{1+\cos^2(i)}{2}\cos(2\pi f t+\delta_0)$$

$$h_\times = \frac{4}{d_\mathrm{L}}\left(\frac{\mathcal{G}\mathcal{M}}{c^2}\right)^{5/3}\left(\frac{\pi f}{c}\right)^{2/3}\cos(i)\sin(2\pi f t+\delta_0)$$

Sampling of Synchrotron Photons I

By parametrizing the surface of a sphere via $\vec{r}(\theta, \phi)$, the function

$$\sigma(\theta, \phi) = \left| \frac{d\vec{r}}{d\theta} \times \frac{d\vec{r}}{d\phi} \right|$$

is computed, used in the definition of two cumulative distribution functions

$$G(s) := \frac{\int_0^{2\pi} \int_0^{s} \sigma(\theta, \phi) d\theta d\phi}{\int_0^{2\pi} \int_0^{\pi} \sigma(\theta, \phi) d\theta d\phi},$$

$$H(\theta, t) := \frac{\int_0^{t} \sigma(\theta, \phi) d\phi}{\int_0^{2\pi} \sigma(\theta, \phi) d\phi}.$$

These functions are next inverted, such that

$$g(s_1) = G^{-1}(s_1),$$

$$h(s_1, s_2) = H^{-1}(s_1, s_2).$$

While this procedure is more general, in the particular case of a sphere it follows that $H^{-1}(s_1, s_2) = \mathrm{Id}(s_2)$. The inverted functions can be used to map a distributions of angles $\{\theta, \phi\} \in [0, \pi] \times [0, 2\pi]$ onto itself by $(g(s_1), h(s_1, s_2)) : [0, \pi] \times [0, 2\pi] \to [0, \pi] \times [0, 2\pi]$, which turns a stratified sampling of cartesian space into a stratified sampling of the surface of a sphere.

See: Schmirander et al. PRD **110**, 082002 (2024)

Sampling of Synchrotron Photons II

classically emitted power:

$$P_\gamma = \frac{2}{3} r_c m c^3 \frac{\beta^4 \gamma^4}{\rho^2} \qquad u_c = \frac{3}{2} \frac{\hbar c \gamma^3}{\rho}$$

rate of photons:

$$\dot{N} = \frac{15\sqrt{3}}{8} \frac{P_\gamma}{u_c} \qquad \dot{N}_{\text{exp}} = \dot{N} \frac{2\pi\rho}{\beta c}$$

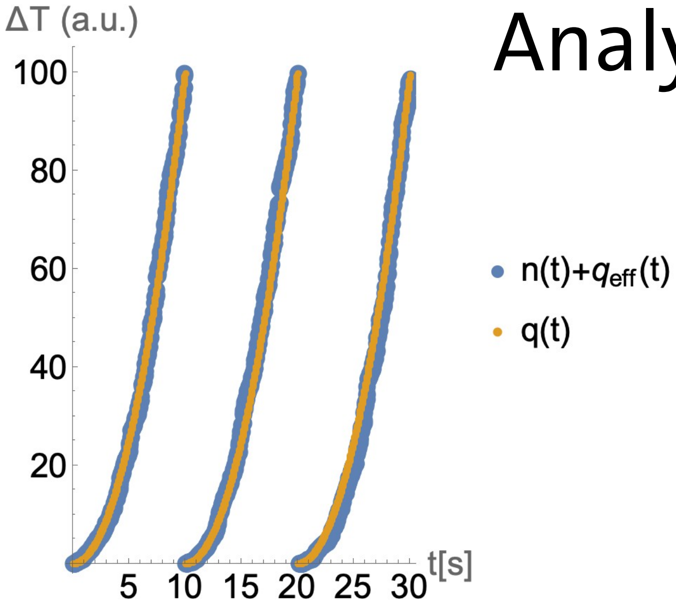
Universal Synchrotron spectrum:

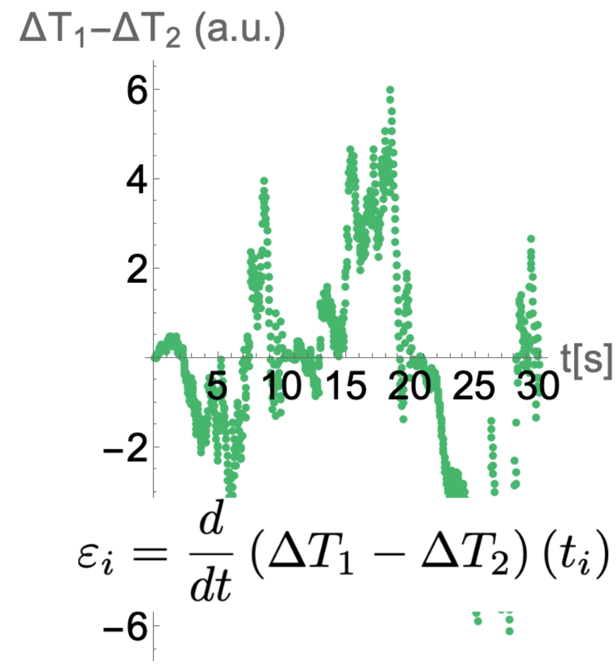
$$S(\xi) = \frac{9\sqrt{3}}{8\pi} \xi \int_\xi^\infty K_{5/3}(x) dx \qquad n(\xi) = \frac{P_\gamma}{u_c^2} \frac{1}{\xi} S(\xi)$$

Probability distribution:

$$\int_0^\infty u_c \frac{n(\xi)}{\dot{N}_{\text{exp}}} d\xi = 1 \qquad \xi \in [10^{-14}, 20]$$

Analytical Noise Models





$$\varepsilon_i = \frac{d}{dt}\left(\Delta T_1 - \Delta T_2\right)(t_i)$$

$$\Delta T_1 = \frac{2}{\Delta t}\sum_n^N \frac{1}{k_n}\sum_j^{k_n}(t-(n-1)\Delta t - t_j)\times$$

$$\Theta(t-(n-1)\Delta t - t_j)\Theta(n\Delta t - t)$$

$\sim q_{\mathrm{eff}}(t) + n_T(t)$ effectively "quadratic"

$$\Delta T_2 = \frac{1}{\Delta t^2}\sum_n^N (t-(n-1)\Delta t)^2 \times$$

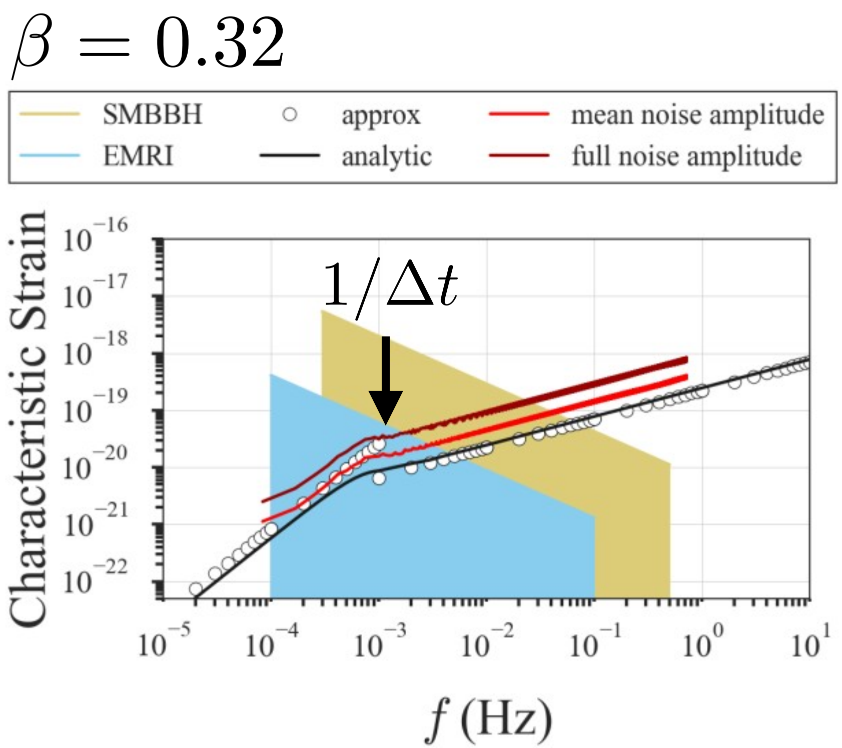
$$\Theta(t-(n-1)\Delta t)\Theta(n\Delta t - t)$$

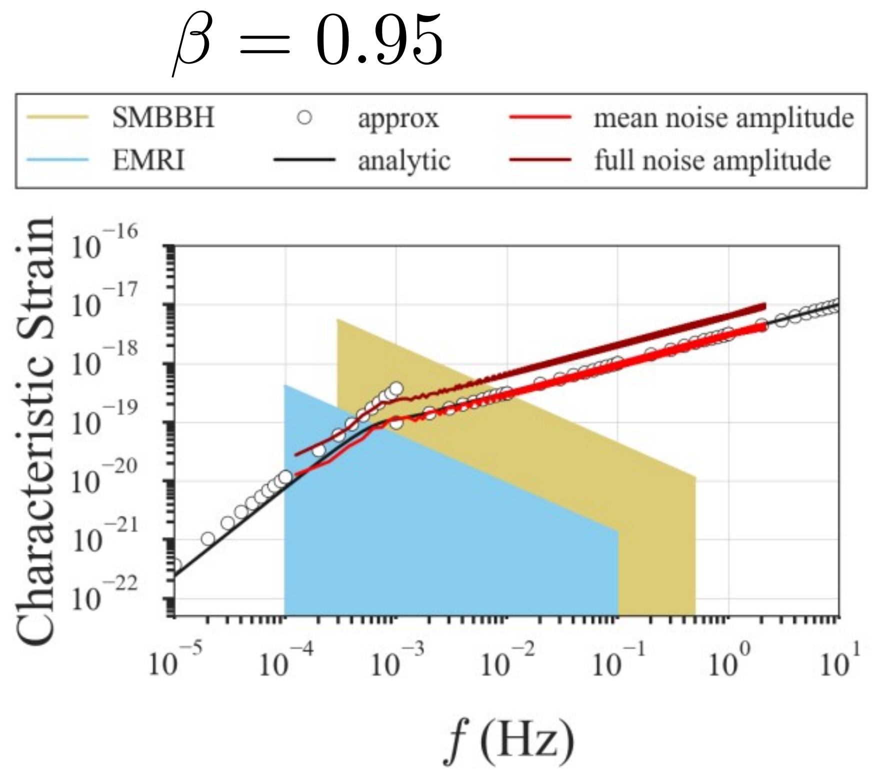
$\sim q(t)$ "quadratic"

For many photons (k) the difference becomes zero, if emitted at average t_j!

$$\lim_{k\to\infty}\left(\tilde{q}_{\mathrm{eff}}(f) + \tilde{n}_T(f)\right) \sim \tilde{q}(f)$$

Analytics vs. Numerics





$$S_n(f) = 4\Delta t S_0^2 \left(\frac{\langle P_{\text{eff}} \rangle}{2p_0 c\gamma^2} \right)^2 \langle \varepsilon \rangle^2 \begin{cases} (2\pi f \Delta t)^2/4 & f < 1/\Delta t \\ 1 & f > 1/\Delta t \end{cases}$$

See: Schmirander et al. PRD **110**, 082002 (2024)

Geodesic Equations

$$\frac{d^2 t}{d\tau^2} + \frac{1}{2}\frac{dh_{\theta\phi\psi}(t)}{dt}\left(\frac{dl}{d\tau}\right)^2 = 0,$$

$$\frac{d^2 l}{d\tau^2} + \frac{1}{1+h_{\theta\phi\psi}(t)}\frac{dh_{\theta\phi\psi}(t)}{dt}\left(\frac{dl}{d\tau}\right)\left(\frac{dt}{d\tau}\right) = 0,$$

$$-\left(\frac{dt}{d\tau}\right)^2 + (1+h_{\theta\phi\psi}(t))\left(\frac{dl}{d\tau}\right)^2 = g_{\mu\nu}\dot{dx^\mu}\,\dot{dx^\nu} = -1.$$

$$h_{\theta\phi\psi}(t,\alpha) = h_+(t)\left(f_s^+ \sin^2\alpha + f_c^+ \cos^2\alpha + f_{sc}^+ \sin 2\alpha\right) + h_\times(t)\left(f_s^\times \sin^2\alpha + f_c^\times \cos^2\alpha + f_{sc}^\times \sin 2\alpha\right)$$

$$f_s^+ = (\cos^2\theta\cos^2\phi - \sin^2\phi)\cos 2\psi - (\cos\theta\sin 2\phi)\sin 2\psi,$$
$$f_c^+ = (\cos^2\theta\sin^2\phi - \cos^2\phi)\cos 2\psi + (\cos\theta\sin 2\phi)\sin 2\psi,$$
$$f_{sc}^+ = \left(\frac{1}{2}(1+\cos^2\theta)\sin 2\phi\right)\cos 2\psi + (\cos\theta\cos 2\phi)\sin 2\psi,$$
$$f_s^\times = (\cos^2\theta\cos^2\phi - \sin^2\phi)\sin 2\psi + (\cos\theta\sin 2\phi)\cos 2\psi,$$
$$f_c^\times = (\cos^2\theta\sin^2\phi - \cos^2\phi)\sin 2\psi - (\cos\theta\sin 2\phi)\cos 2\psi,$$
$$f_{sc}^\times = \left(\frac{1}{2}(1+\cos^2\theta)\sin 2\phi\right)\sin 2\psi - (\cos\theta\cos 2\phi)\cos 2\psi.$$

See: Rao et al., PRD **102,** 122006 (2020)