Response of a storage-ring beam to a gravitational wave

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K. Oide (UNIGE/CERN/KEK)

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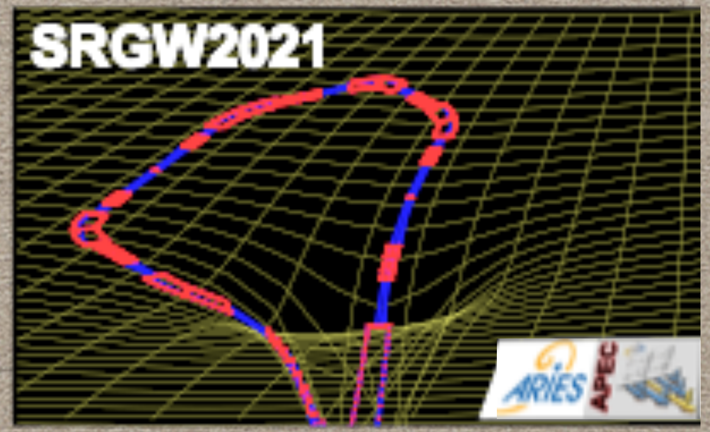
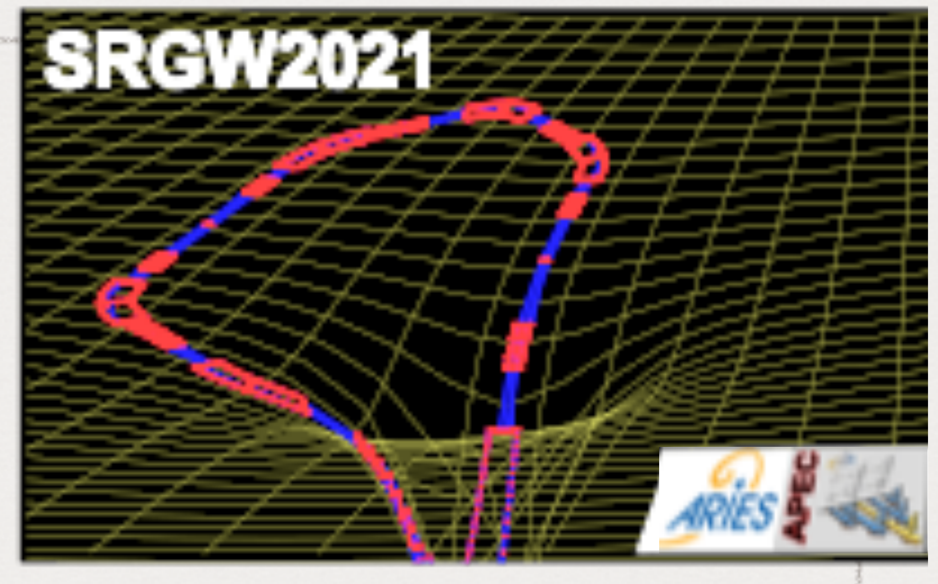
These are basically copying my previous presentation at SRGW2021:
https://indico.cern.ch/event/982987/contributions/4199474/attachments/2182082/3687703/Beam_response_Oide_210203.pdf
https://arxiv.org/abs/2105.00992





1 Orbit deviation by gravitational wave

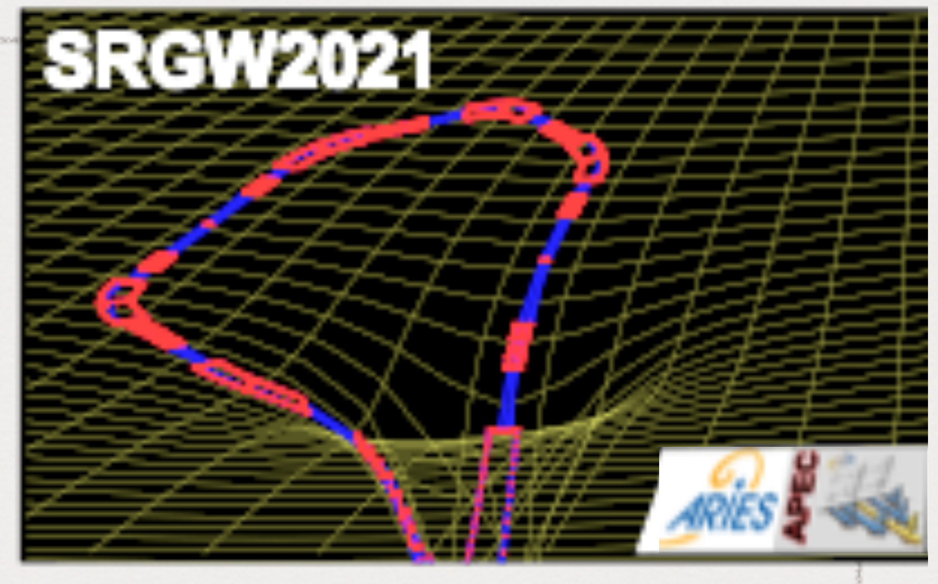
An object at distance X^μ transverse to the wave from an origin receives an acceleration by gravitational radiation (GR) wave:

$$\frac{\mathrm{d}^2 X_\mu}{\mathrm{d}t^2} = \frac{1}{2}\ddot{h}_{\mu\nu} X^\nu \ , \tag{1}$$

where $h_{\mu\nu}$ has the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

with two linear polarization components h_+ and $h_\times$. For the time being, let us consider the case of linear polarization with $h_+ = h$ and $h_\times = 0$.



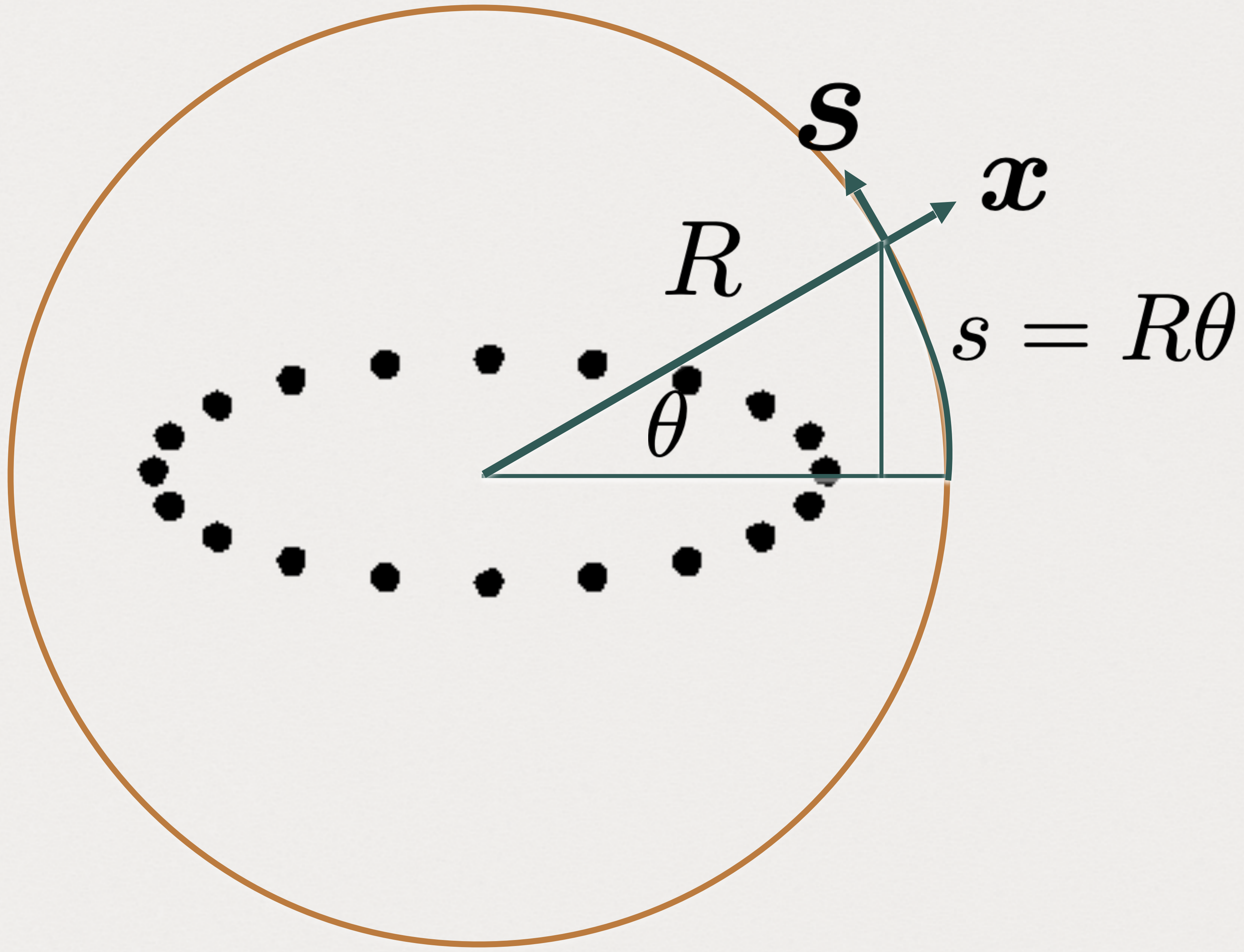
horizontal displacement of the beam particle

The acceleration can be expressed in terms of accelerators for a relativistic particle as:

$$\frac{\mathrm{d}p_x}{\mathrm{d}s} = \frac{1}{2c^2}\ddot{h}R\cos 2\theta \ , \tag{3}$$

where $p_x \approx \mathrm{d}x/\mathrm{d}s$ is the horizontal (radial) momentum divided by the design momentum p_0 of the ring. The coordinate $s \approx ct$ is the path length along the ring. If the GR is a sinusoidal wave $h\cos(\omega_{\mathrm{GR}}t)$ with an angular frequency $\omega_{\mathrm{GR}} = 2\pi c/\lambda = ck$, the above are written as

$$\frac{\mathrm{d}p_x}{\mathrm{d}s} = -\frac{k^2}{2}hR\cos 2\theta\cos(\omega_{\mathrm{GR}}t) \ . \tag{4}$$



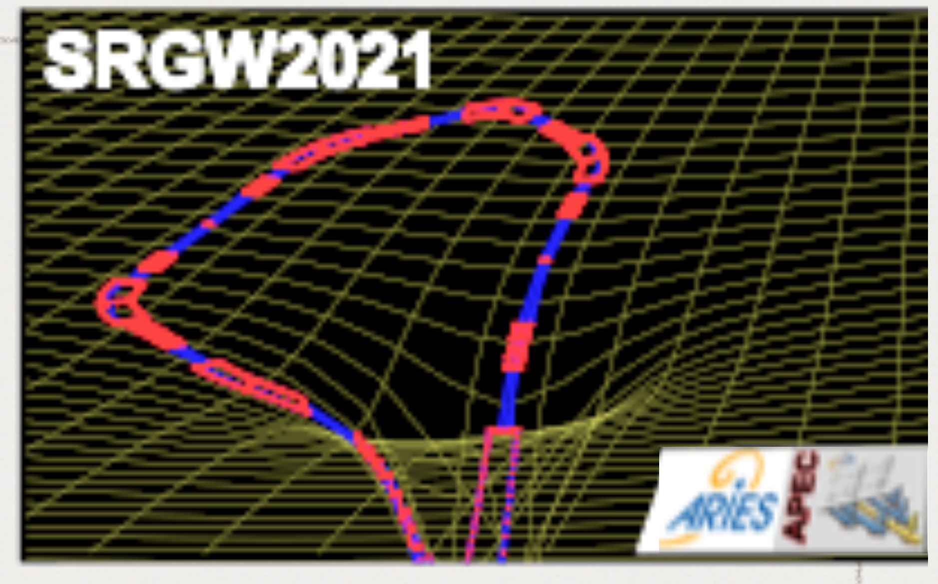
Once we know $\mathrm{d}p_x/\mathrm{d}s$, the displacement x at an arbitrary location s in the beam line is obtained using R_{12}-component of the transfer matrix (Green's function) as:

$$x(s) = \int_0^s ds' R_{12}(s, s')\frac{\mathrm{d}p_x}{\mathrm{d}s}(s') \tag{5}$$

$$= \int_0^s ds' \sqrt{\beta_x(s)\beta_x(s')}\sin\left(\psi_x(s) - \psi_x(s')\right)\frac{\mathrm{d}p_x}{\mathrm{d}s}(s') \ , \tag{6}$$

where $\beta_x(s)$ and $\psi_x(s)$ are the horizontal β-function and the phase advance of the beam line.

https://en.wikipedia.org/wiki/Gravitational_wave



Resonance

Let us assume the ring is a perfect circle with a radius R, and uniformly focused in the horizontal plane, with the β-function β_x. If the GR is incident perpendicular to the ring plane, The horizontal orbit deviation at an angle $\theta = s/R$ generated by the GR force (4) is written as

$$\Delta x = -\frac{k^2 R^2}{2} h \int_0^\theta \beta_x \sin\left(\nu_x(\theta - \theta')\right) \cos(kR\theta') \cos 2\theta' d\theta' \ , \tag{7}$$

where $\nu_x = R/\beta_x$ is the horizontal betatron tune.

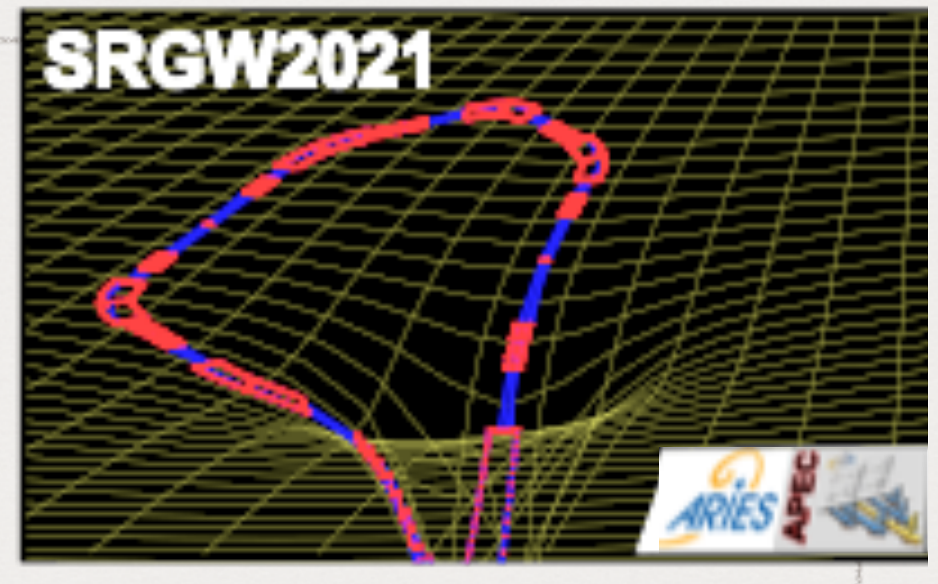
Then the orbit can resonate to the GR if

$$kR \pm 2 = \nu_x \tag{8}$$

resulting an increase the horizontal amplitude by

$$\Delta x \approx \pi k^2 R^2 \beta_x h \approx \pi \nu_x R h \tag{9}$$

per every turn. It is 0.2 fm for $h = 10^{-22}$, $R = 5\,\mathrm{km}$ and $\nu_x = 130$ in the case of the model ring shown later.

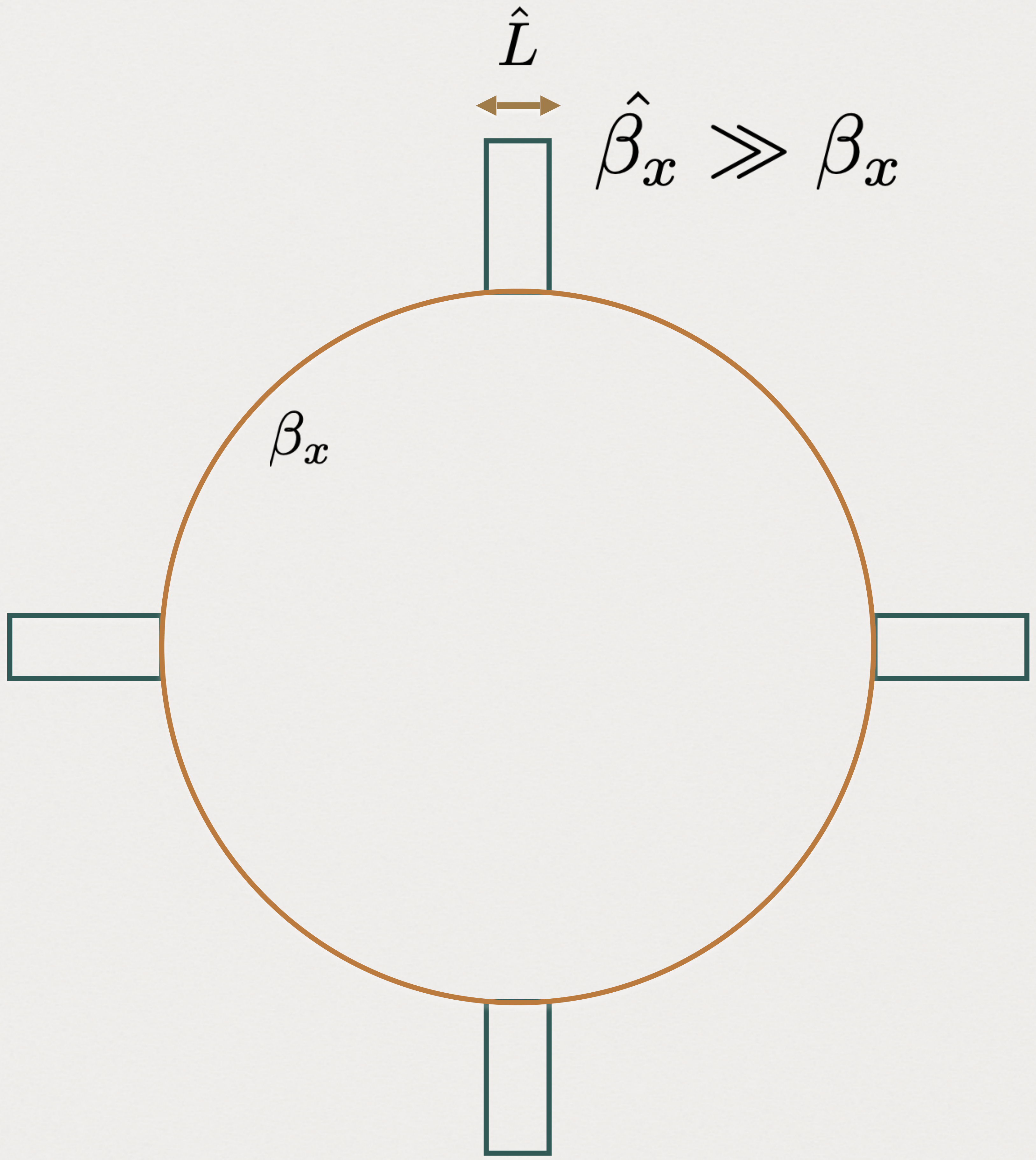


2 Enhancement of the response: Beam Antenna?

As we have seen, even at the exact resonance, the response of the betatron motion to GR seems pretty weak. Is there any way to enhance the response. If we look at Eq. 7, the amplitude of the oscillation is proportional to β_x. So what about having a special section in the ring with a very large β_x? For instance, consider a ring with the circumference $C = 30\,\mathrm{km}$ and the average $\beta_x = 50\,\mathrm{m}$ in the ring. If we have four $\hat{L} = 1\,\mathrm{km}$ sections with $\hat{\beta}_x = 1000\ \mathrm{km}$, the response of $\mathrm{d}p_x/\mathrm{d}s$ to the GR from these special sections is much larger than that from the arc by a factor of

$$\sqrt{\frac{\hat{\beta}_x}{\beta_x}\frac{4\hat{L}}{C}} \approx 19\,. \tag{10}$$

The response to x is enhanced by a factor of 2700, multiplied by another $\sqrt{\hat{\beta}_x/\beta_x}$.



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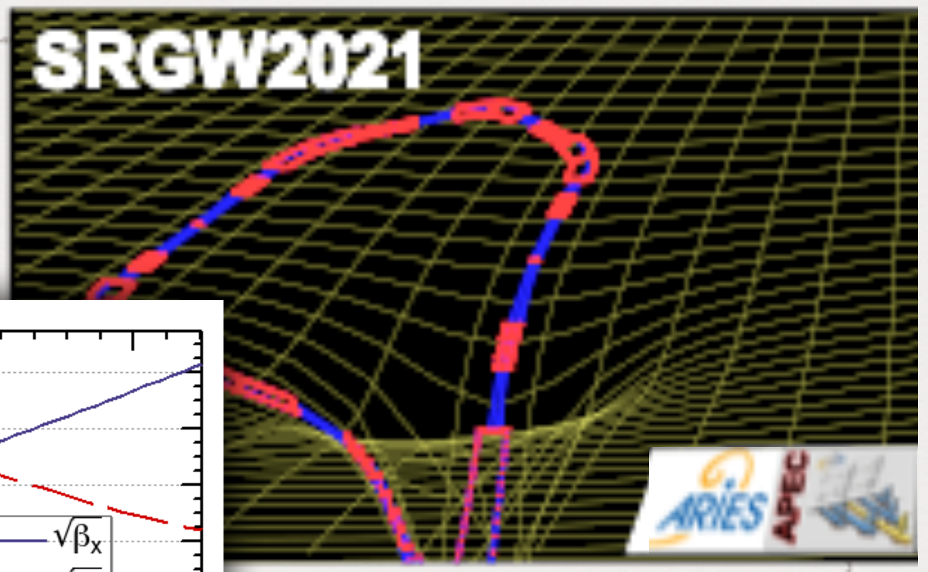
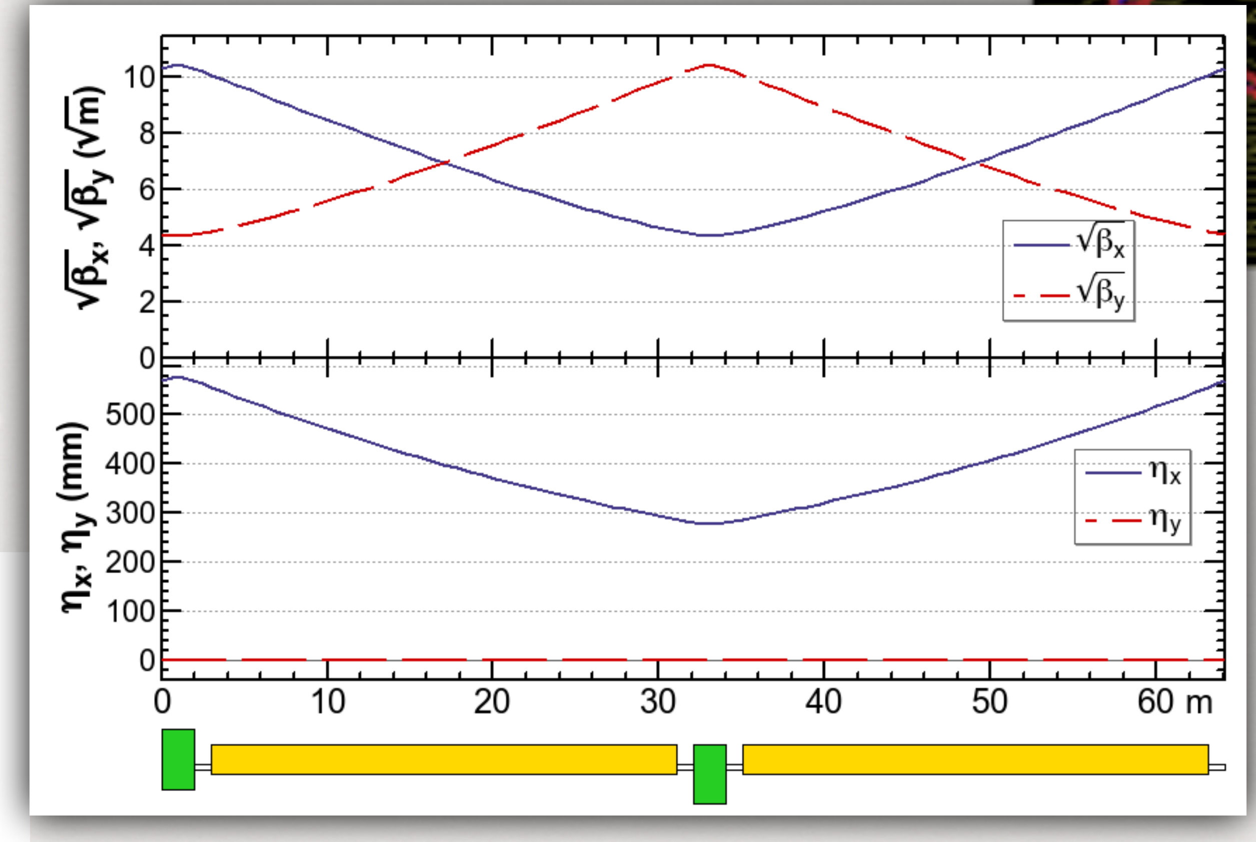
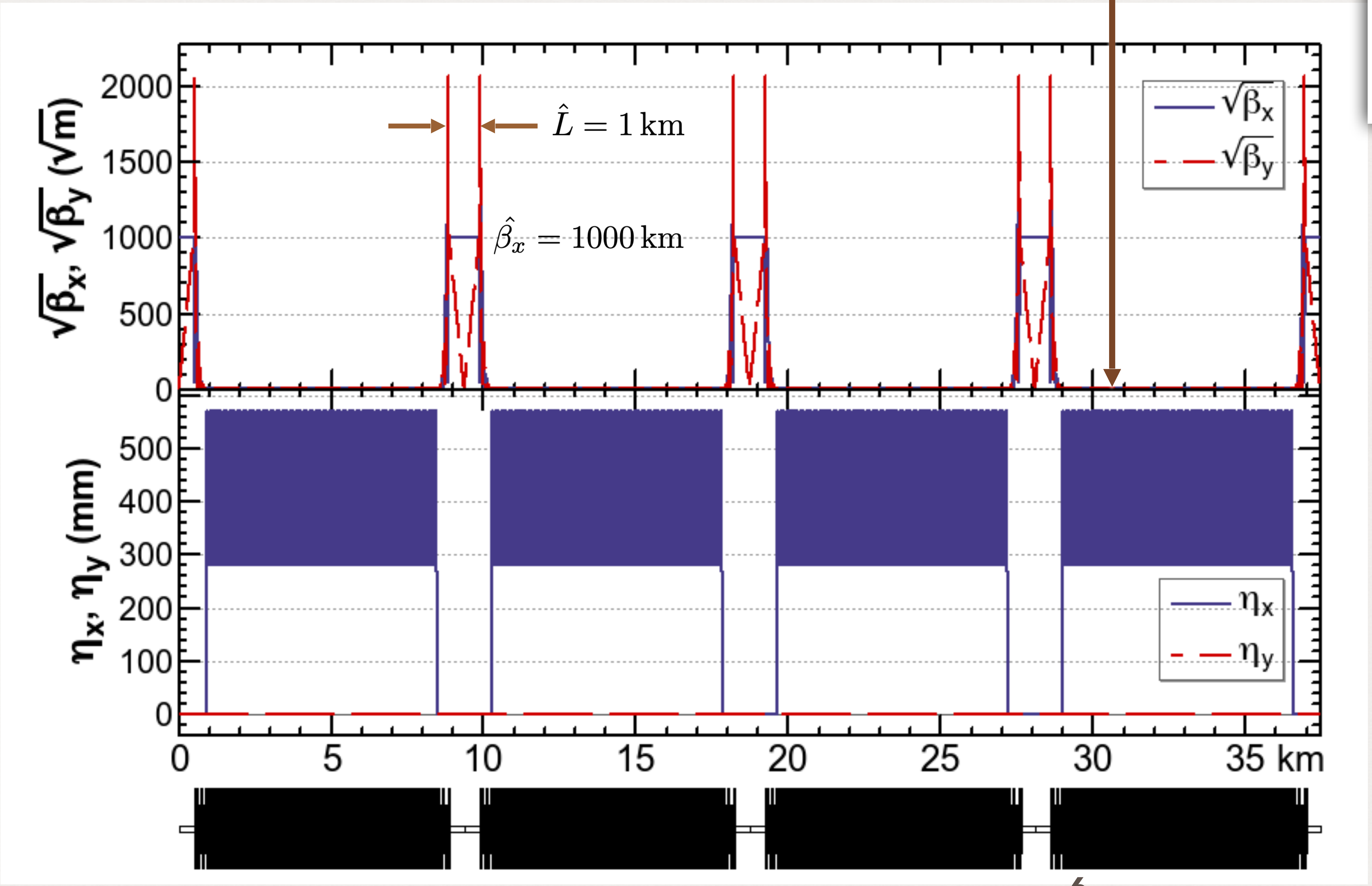
An example ring optics for a GR antenna

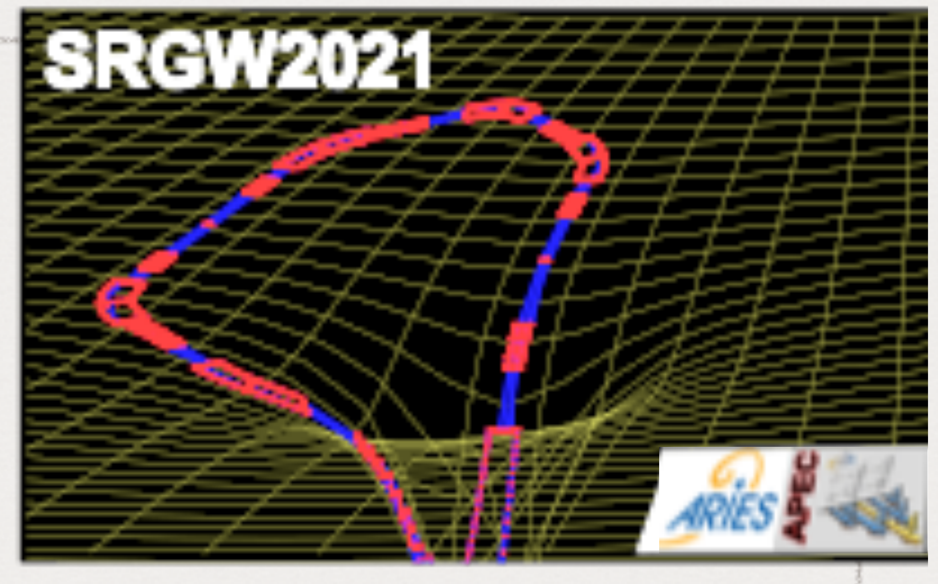


$\overline{\beta_x} \approx 50\,\mathrm{m}$





Parameters		
Particles		p
Beam energy	TeV	1.0
Circumference	km	37.4
Length of an IR, $\hat{L}$	km	1.0
Number if IRs		4
β_x at the IR, $\hat{\beta}_x$	km	1000
β_x ave. in the arc	m	50
Betatron tunes, ν_x/ν_y		130.8/131.3
SR damping time in x	s	73600
SR equiv. emittance	fm	0.198



Response of "antenna"

In the case of such an "antenna", the response to GR in the arc is small compared to the $\hat{\beta}_x$ sections, to we ignore below. Thus Eq. 7 is reduced to

$$\mathit{\Delta}\hat{x} = -\frac{k^2 R\hat{L}}{2} h \sum_{m=0}^{n} \hat{\beta}_x \sin\left(2(n-m)\pi\mathit{\Delta}\nu_x\right) \cos\left(\frac{m\pi}{2} kR\right) \cos m\pi\,, \tag{11}$$

where $\mathit{\Delta}\mu_x$ is the phase advance between two $\hat{L}$ sections, and we have used $\theta = m\pi/2$ at the m-th $\hat{L}$ section.

Equation 11 has resonances at

$$2\mathit{\Delta}\nu_x = \pm kR + N\,, \tag{12}$$

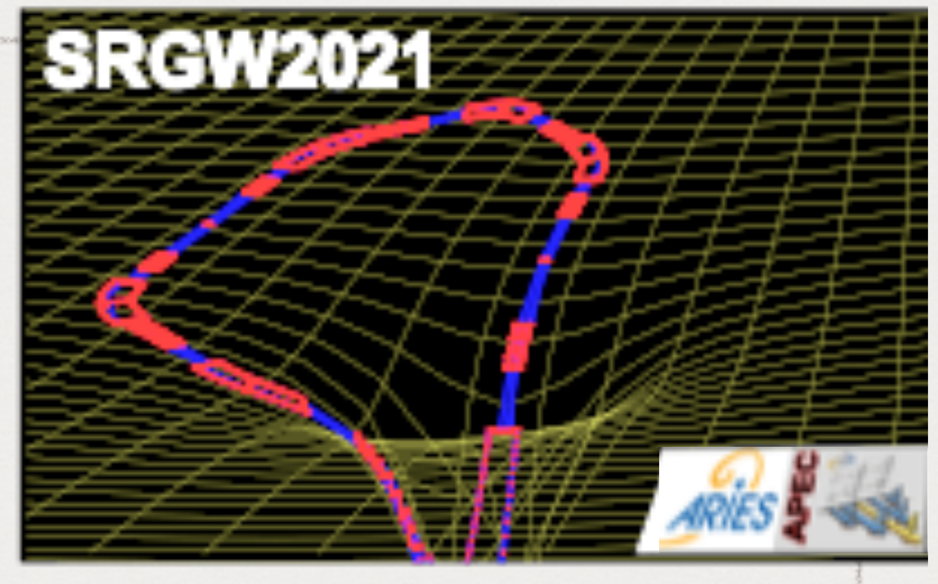
where N is an integer. At the resonance, the horizontal amplitude accumulates by

$$\mathit{\Delta}\hat{x} \approx -\frac{k^2 R\hat{L}}{2} \hat{\beta}_x h \tag{13}$$

per turn.

If we set $\omega_{\mathrm{GR}} = 2\pi \times 1\,\mathrm{kHz}$ and with parameters above, Eq. 13 gives $\mathit{\Delta}\hat{x} \approx 1000h\,\mathrm{m}$, which seems still too small to detect. As the amplitude is proportional to ω_{GR}^2, there may exist a possibility to detect a GR in $\omega_{\mathrm{GR}} = 2\pi \times 1\,\mathrm{MHz}$ range, giving $\mathit{\Delta}\hat{x} \approx 10^9 h$ m ($0.1\,\mathrm{pm}$ for $h = 10^{-22}$), if there are sources.

3 Noise sources



Even if the signal is enhanced by the “antenna”, the problems exist on the noises in the betatron oscillation, which include:

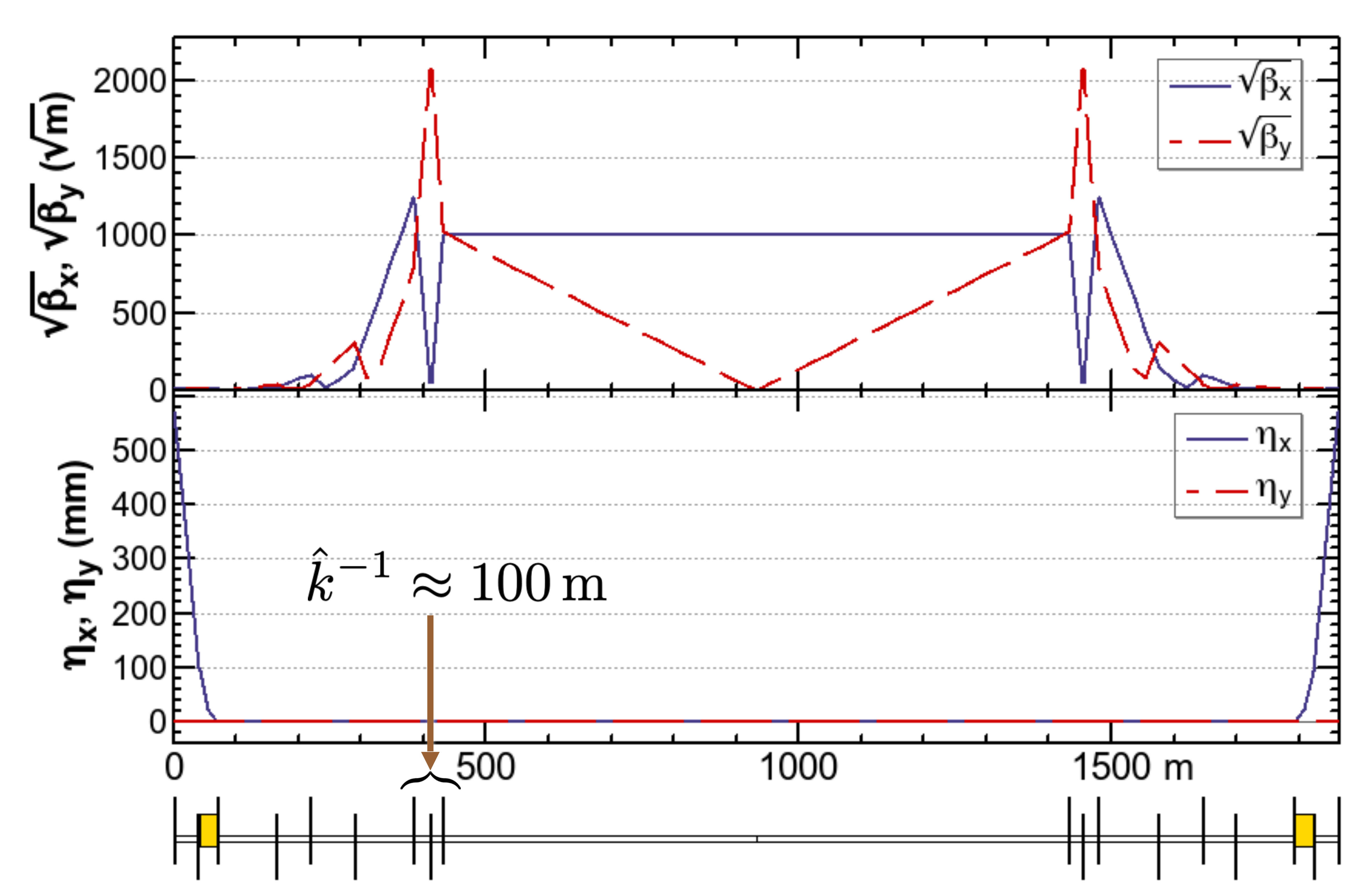
- The excitation by the vibration of the quads, esp. on the both sides of the $\hat{L}$ sections. The triplet-like quad at each side has the focal length $\hat{k}^{-1} \approx 100\,\mathrm{m}$, then its horizontal vibration by Δx_{Q} excites the betatron oscillation of

$$\Delta x_n \approx \sqrt{8}\hat{\beta}\hat{k}\Delta x_{\mathrm{Q}} \approx 3 \times 10^4 \Delta x_{\mathrm{Q}}\,, \qquad (14)$$

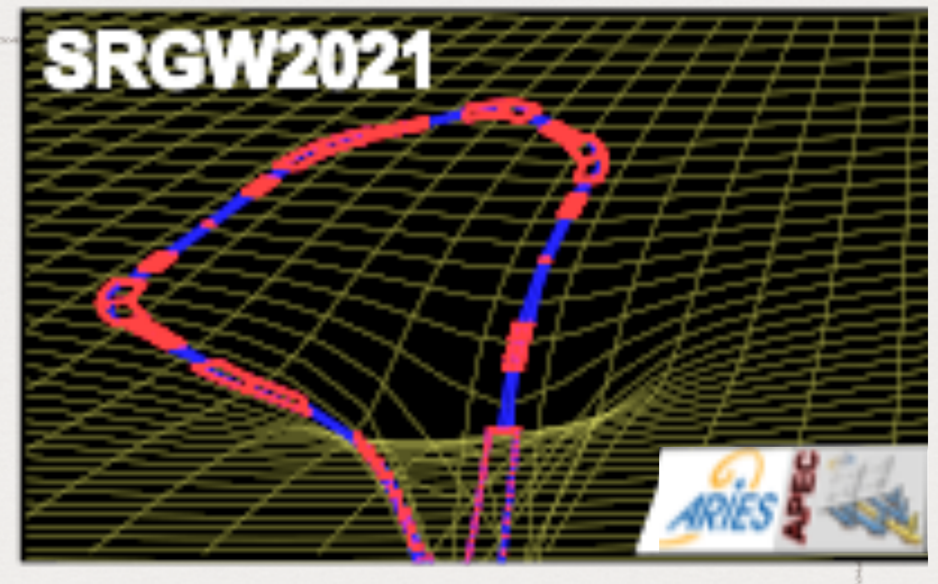
as there are 8 units of such triplets in the ring. The minimum magnitude of vibration Δx_{Q} is given by the *thermal vibration* of the triplet:

$$\Delta x_{\mathrm{Q}}^2 \approx \frac{4k_{\mathrm{B}}T\omega_{\mathrm{Q}}\Delta f}{M_{\mathrm{Q}}\omega_{\mathrm{GR}}^2 Q_{\mathrm{M}}} \qquad (15)$$

where M_{Q}, ω_{Q}, Q_{M}, T, Δf, and k_{B} are the mass, eigen angular frequency, and the Q_{M}-value of the triplet, the temperature, the detection bandwidth, and the Boltzmann constant, respectively. If we plugin $M_{\mathrm{Q}}{=}1000\,\mathrm{kg}$, $\omega_{\mathrm{Q}}/2\pi = 10\,\mathrm{Hz}$, $Q_{\mathrm{M}} = 10$, $T = 4.2\,\mathrm{K}$, $\Delta f = 100\,\mathrm{kHz}$, obtain $\Delta x_{\mathrm{Q}} = 6\,\mathrm{pm}$, which generates the betatron amplitude $\Delta x_n = 0.17\,\mu\mathrm{m}$ through Eq. 14.



More noise sources



- The vibration may be reduced by making Q_{M} higher and lowering ω_{Q}. This is somewhat similar to the mirror suspension system for the GR laser interferometers. Also the detection bandwidth is a subject to be refined.

- Similar effects arise from all quadrupoles in the entire ring.

- The emittance of the beam may accompany an intrinsic fluctuation of the orbit.:

$$\Delta\hat{x}_s = \sqrt{\hat{\beta}_x \varepsilon_x / N_p}\,, \tag{16}$$

  where N_p is the number of particles in the bunch. If this beam has the same invariant emittance as the LHC, $\gamma\varepsilon_x = 4\,\mu\mathrm{m}$ and the beam energy and N_p are 1 TeV and 10^{11}, respectively, the orbit fluctuation becomes $\Delta\hat{x}_s \approx 0.19\,\mu\mathrm{m}$.

- As the LHC's emittance above seems too large for our purpose, we need much smaller one. One and the easiest way to use the synchrotron radiation. This ring has the damping time of about 20 hours and the equilibrium emittance 0.2 fm, which is 5×10^{-8} smaller than the LHC's. The corresponding orbit fluctuation is $\Delta\hat{x}_s \approx 40\,\mathrm{pm}$.

- if we need the SR damping, an RF acceleration is inevitable. It will be another noise source for the beam, esp. if there are residual dispersions at the RF cavities.

4 Summary

- The beam in a storage ring can respond to a wave of gravitational radiation, somewhat similar to the Weber bar.
- The betatron motion of the beam in a ring can resonate to the GR.
- A special beam optics "beam antenna" may enhance the sensitivity,
- There will be many noise sources to overcome: thermal motion of quadrupoles, beam fluctuation due to emittance, synchrotron radiation and the acceleration, etc.