Review and status of the dynamics in Cyclotron with GW

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Storage Rings & Gravitational Waves – mini-brainstorm

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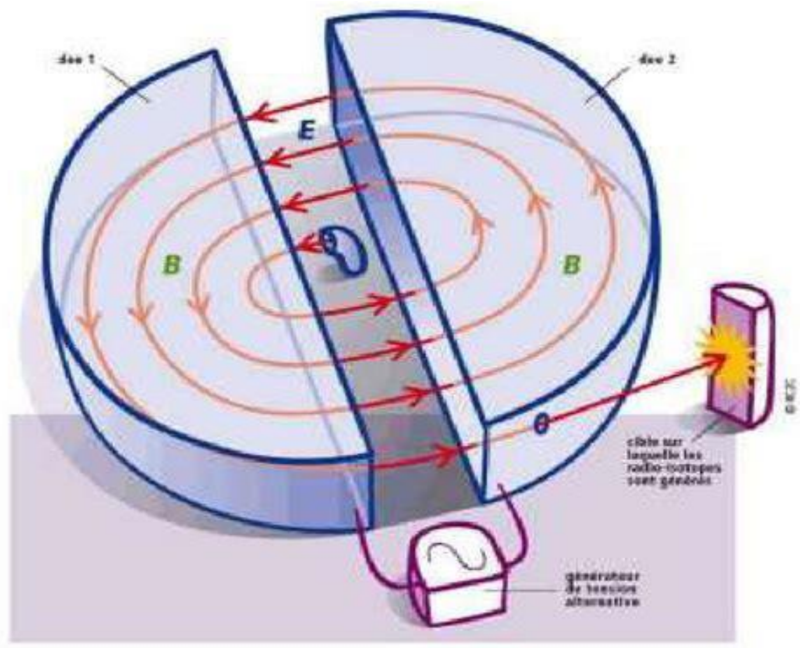
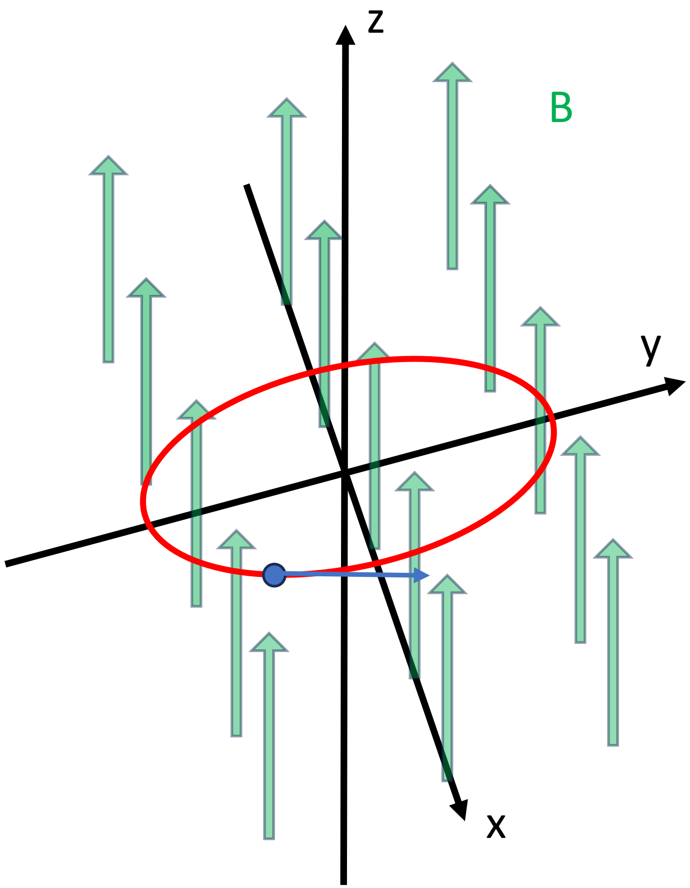
Cyclotron



System studied here



Covariant equation of motion

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma_{\lambda\nu}{}^{\mu}\dot{x}^\lambda \dot{x}^\nu = \frac{q}{m} F^\mu{}_\nu \dot{x}^\nu$$

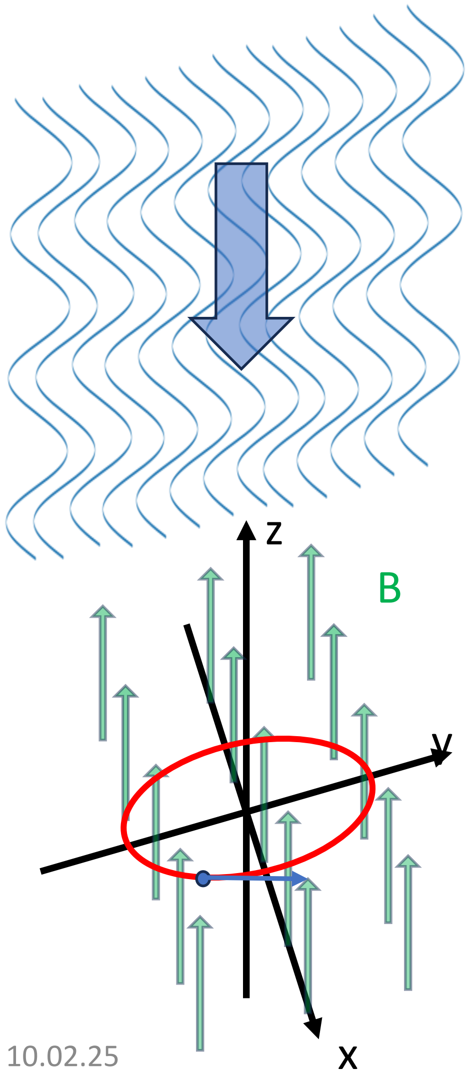
Riemann-Christoffel connection

Electromagnetic tensor

Proper time → $c^2 d\tau^2 = -g_{\mu\nu}\, dx^\mu dx^\nu.$

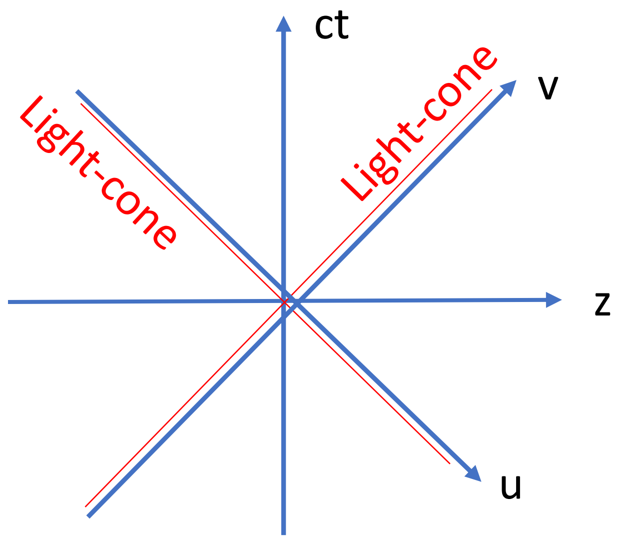
These equations have the same form in any system of coordinates

Gravitational wave



$$u = ct - z,$$
$$v = ct + z$$

Metrics of GW in (u,v)



$$g_{\mu\nu}\, dx^{\mu} dx^{\nu} = -\, du dv - K(u,x,y) du^2 + dx^2 + dy^2.$$

In the system of coordinates (u, x, y, v)

$$\tilde{\Gamma}^{v}_{uu} = K_{,u}$$

$$\tilde{\Gamma}^{x}_{uu} = \frac{1}{2}\tilde{\Gamma}^{v}_{xu} = \frac{1}{2}K_{,x}$$

$$\tilde{\Gamma}^{y}_{uu} = \frac{1}{2}\tilde{\Gamma}^{v}_{yu} = \frac{1}{2}K_{,y}\ .$$

Ricci tensor

$$\tilde{R}_{\mu\kappa} = \tilde{R}^{\lambda}_{\mu\lambda\kappa} = \frac{\partial\tilde{\Gamma}^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial\tilde{\Gamma}^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \tilde{\Gamma}^{\eta}_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\eta\kappa} - \tilde{\Gamma}^{\eta}_{\mu\kappa}\tilde{\Gamma}^{\lambda}_{\eta\lambda}$$

Einstein equation in vacuum

$$\tilde{R}_{\mu\nu} = 0$$

Only equation!!

$$\tilde{R}_{uu} = 0$$

Einstein equation in vacuum

$$\tilde{R}_{uu} = -\frac{1}{2}(\partial_x^2 + \partial_y^2)K = 0$$

Simpler non-trivial solution

$$K = k_+(u)(x^2 - y^2) + 2k_\times(u)xy$$

as $u = ct - z$, it follows that K is a wave propagating along z.

Equations of motion

$$
\begin{aligned}
\ddot{u} &= -\frac{q}{m}[B_y\dot{x} - B_x\dot{y}] \\
\ddot{x} + \frac{1}{2}\frac{\partial K}{\partial x}\dot{u}^2 &= \frac{q}{m}[B_z\dot{y} - B_y\dot{z}] - \frac{q}{m}KB_y\dot{u} \\
\ddot{y} + \frac{1}{2}\frac{\partial K}{\partial y}\dot{u}^2 &= \frac{q}{m}[-B_z\dot{x} + B_x\dot{z}] + \frac{q}{m}KB_x\dot{u} \\
\ddot{v} + \frac{\partial K}{\partial u}\dot{u}^2 + 2\frac{\partial K}{\partial x}\dot{x}\dot{u} + 2\frac{\partial K}{\partial y}\dot{y}\dot{u} &= \frac{q}{m}[B_y\dot{x} - B_x\dot{y}]
\end{aligned}
$$

With

$$
\frac{\partial K}{\partial x} = 2k_+(u)x + 2k_\times(u)y
$$

$$
\frac{\partial K}{\partial y} = -2k_+(u)y + 2k_\times(u)x
$$

The simplified case: $B_x = B_y = 0$

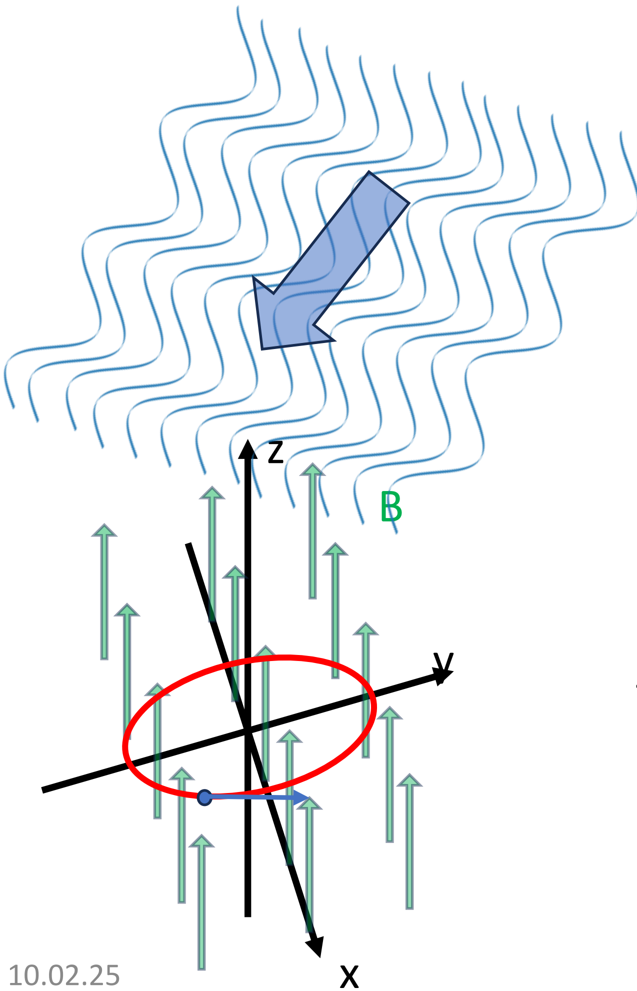
$$
\begin{aligned}
\ddot{u} &= -\frac{q}{m}[\cancel{B_y\dot{x}} - \cancel{B_x\dot{y}}] \\
\ddot{x} + \frac{1}{2}\frac{\partial K}{\partial x}\dot{u}^2 &= \frac{q}{m}[B_z\dot{y} - \cancel{B_y\dot{z}}] - \cancel{\frac{q}{m}KB_y\dot{u}} \\
\ddot{y} + \frac{1}{2}\frac{\partial K}{\partial y}\dot{u}^2 &= \frac{q}{m}[-B_z\dot{x} + \cancel{B_x\dot{z}}] + \cancel{\frac{q}{m}KB_x\dot{u}} \\
\ddot{v} + \frac{\partial K}{\partial u}\dot{u}^2 + 2\frac{\partial K}{\partial x}\dot{x}\dot{u} + 2\frac{\partial K}{\partial y}\dot{y}\dot{u} &= \frac{q}{m}[\cancel{B_y\dot{x}} - \cancel{B_x\dot{y}}]
\end{aligned}
$$

These are the equation of J.W. van Holten NIKHEF/99-019

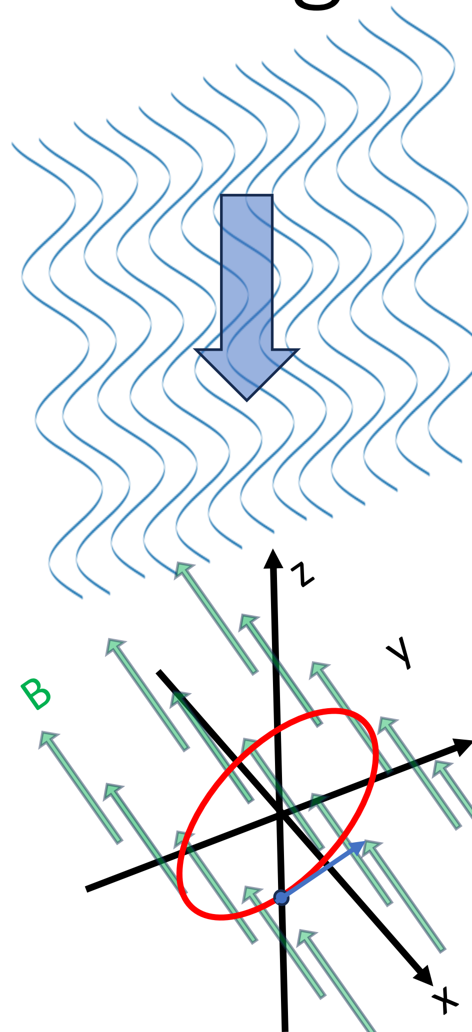
$$
\begin{aligned}
x'' - k_c y' + k_+ x = 0 \\
y'' + k_c x' - k_+ y = 0
\end{aligned}
$$

A transformation is still missing ☹

If GW does not hit not along z?



Equivalent to have B_x, B_y not zero → rotate the axis so that z is parallel to the GW



Equations more complicated …

Case B_x=B_y=0
GW induced resonance

Equations of motion

$$x'' - k_c y' + k_+ x = 0$$
$$y'' + k_c x' - k_+ y = 0$$

$$()' = \frac{d()}{du}$$

Effect of the gravitational wave

$$k_+ = \frac{\omega_+^2}{2c^2} h_+ \cos(\omega_+ t - \alpha)$$

$$u = ct - z,$$
$$v = ct + z$$

For particles at *z=0* we have that *u=ct* and so u=s (the curvilinear "s" used in Accelerators)

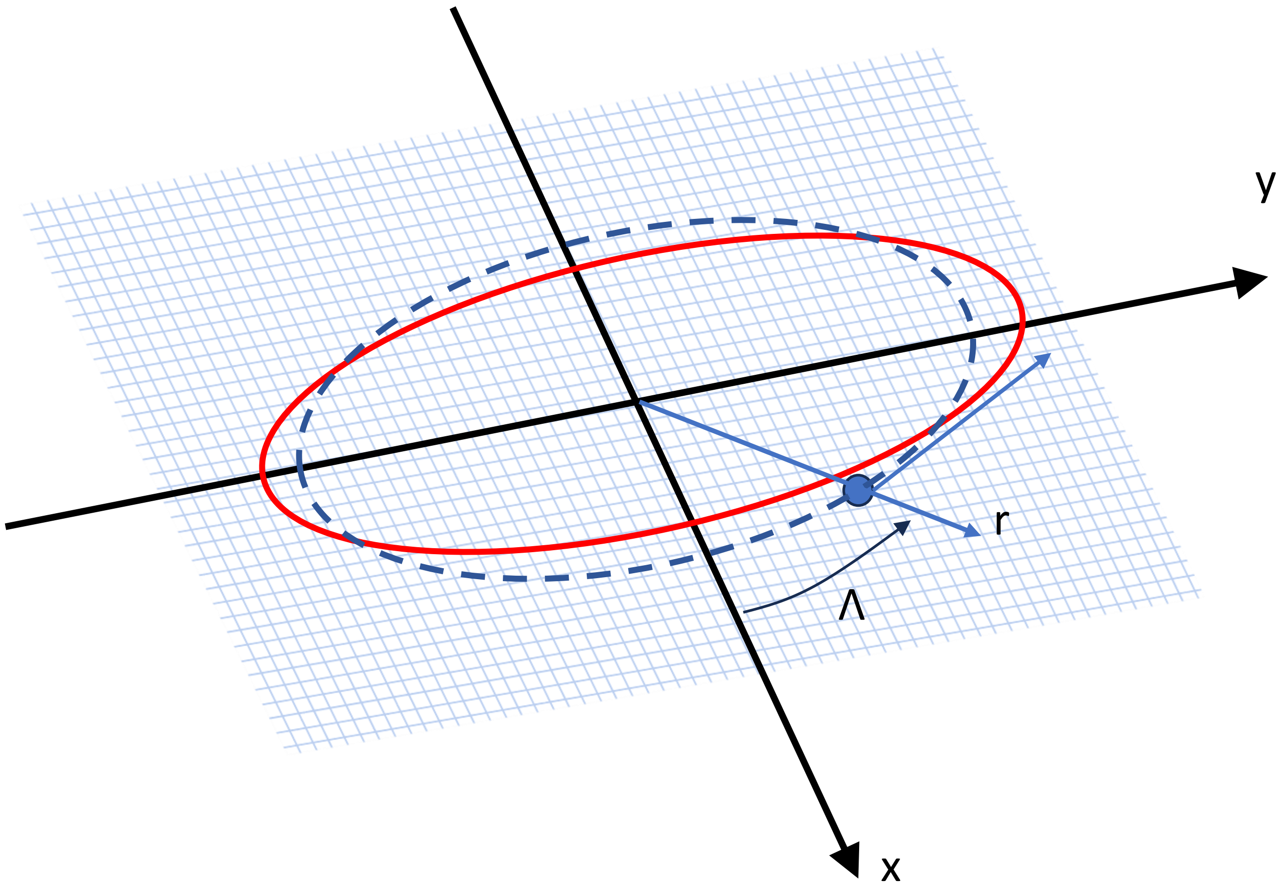
In absence of Gravitational Wave the equations of motion are

$$x_0'' + k_c x_0' = 0$$
$$y_0'' + k_c y_0' = 0$$

$$k_c = \frac{1}{\rho}$$

These equations yield the reference orbit → circular

Dynamics at z=0



$$z = x + iy = re^{i\Lambda} \qquad \Lambda = \frac{s}{\rho}$$

Equation of motion in complex coordinates

$$z'' + i\frac{1}{\rho}z' + k_+ z^* = 0$$

Parameterization of “z”

$$z = (\rho + r - il)e^{-i\frac{s}{\rho}}$$

Radial shift Longitudinal shift

$$\Omega_\pm = \pm\frac{\omega_+}{c} + \frac{2}{\rho} \qquad\qquad C_\pm = \cos(\Omega_\pm s)$$

$$r'' + \frac{r}{\rho^2} = -\frac{\omega_+^2}{4c^2}h_+\left[\rho + \frac{1}{\Omega_+}\right]C_+ - \frac{\omega_+^2}{4c^2}h_+\left[\rho + \frac{1}{\Omega_-}\right]C_-$$

Resonance condition, when $\Omega_- \simeq \frac{1}{\rho}$

Equation of motion of “r” for a resonant dynamics between traveling particle and the “influence” of the gravitational wave is

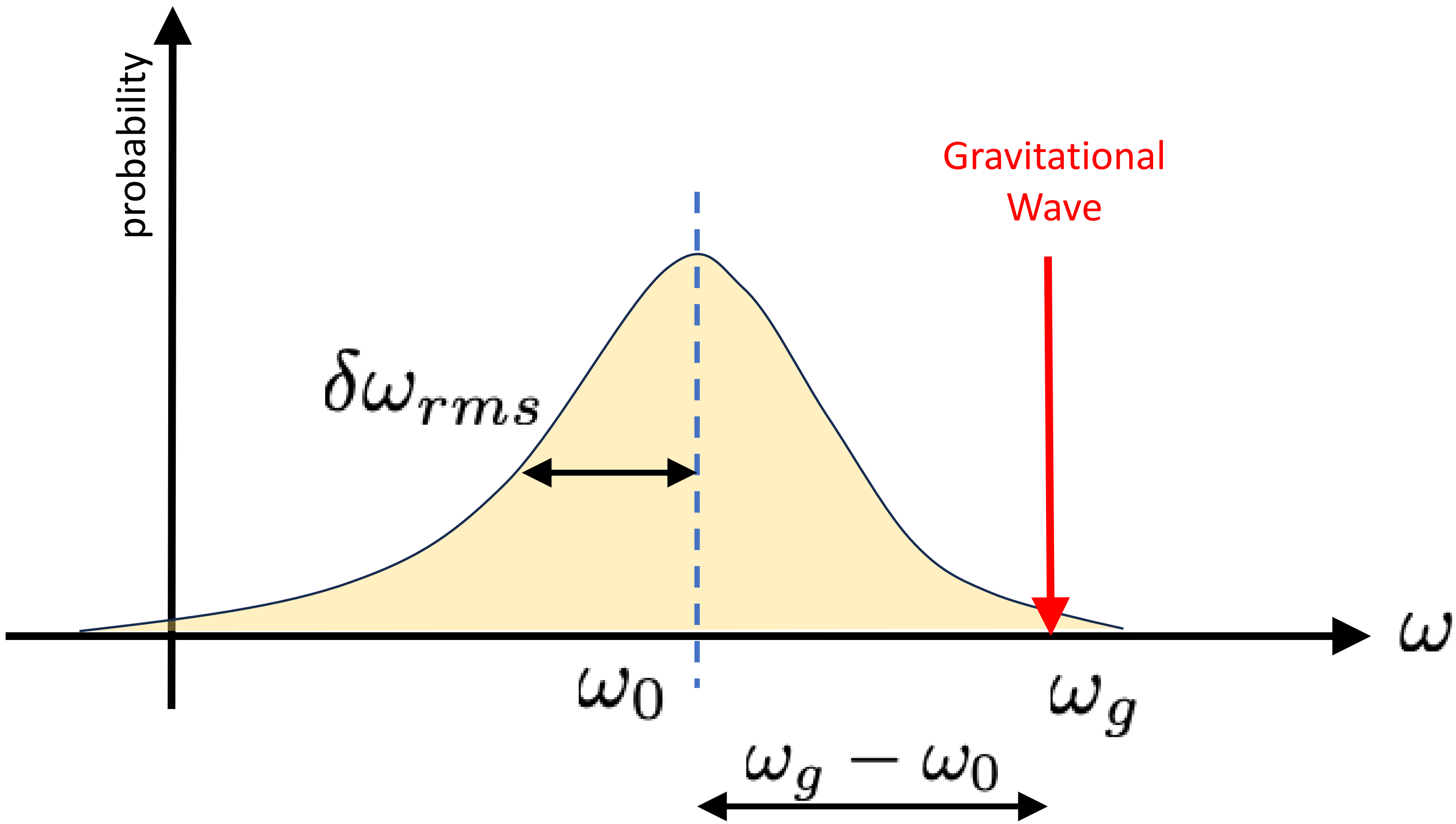
$$r'' + \frac{r}{\rho^2} = -\frac{\omega_+^2}{2c^2}h_+\rho\cos(\Omega_- s)$$

Similar results to Oide model

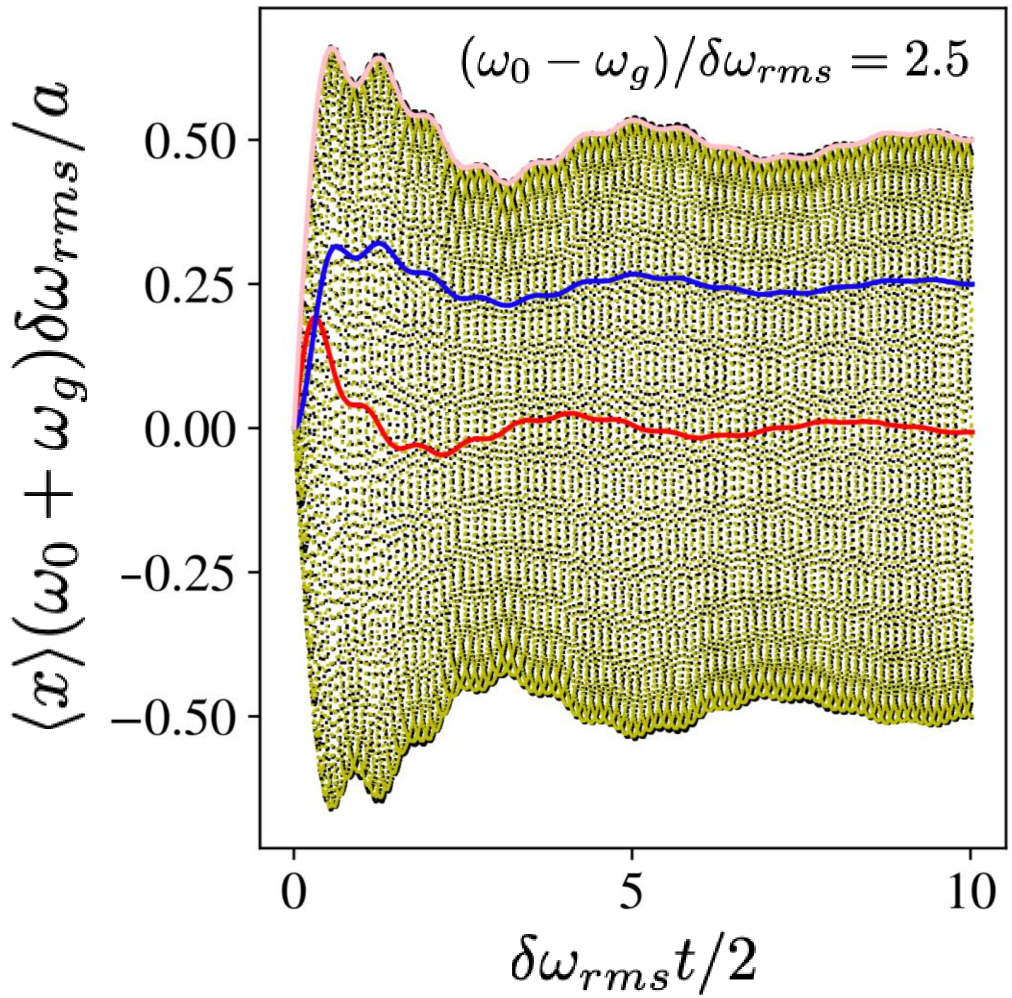
$$r'' + \frac{r}{\rho^2} = -\frac{\omega_+^2}{2c^2} h_+ \rho \cos(\Omega_- s)$$ Cyclotron

$$x'' + \omega^2 x = a \cos(\omega_g t).$$ Oide

Effect of frequency spread



Example for a Uniform distribution of $\delta\omega$



Summary

Distribution of $\delta\omega$ Spread	$\omega_g \sigma t_{max}$	$\langle x \rangle_{max} \omega_g^2 \sigma / a$	$\langle x \rangle_{max,\infty} \omega_g^2 \sigma / a$
Uniform	1.813	0.534	0.453
Parabolic	2.009	0.554	0.526
Gaussian cut at $\rho\sigma_0$	3.141		0.626

with $\rho > 3$, but $\sigma \ll 2/\rho$

$$\sigma = \frac{\sqrt{\langle \delta\omega^2 \rangle}}{\omega_0}$$

Estimates…

From Oide's model → smooth approximation

$$\Delta\hat{x} \simeq -\frac{k^2 R \hat{L}}{2}\hat{\beta}_x h \quad \Rightarrow \quad a = 2\pi\frac{\hat{L}\hat{\beta}_x}{\beta_x} f^2 h$$

$$fh > 2\frac{\delta Q_{rms}}{Q}\frac{c}{\hat{L}\hat{\beta}_x}\Delta x_{res}$$

Example

Therefore for: $\hat{L} = 1000\,\mathrm{m}$, $\hat{\beta}_x = 10^6$ m, and assuming for any spread $\delta Q_{rms}/Q \simeq 10^{-4}$ then we find

$$fh > 6 \times 10^{-5}\Delta x_{res} \; .$$

From here we find that for a BPM with resolution of $\Delta x_{res} = 1$ nm, then

$$fh \gtrsim 10^{-14} \; ;$$

instead for BPM resolution of $\Delta x_{res} = 1$ pm, then

$$fh \gtrsim 10^{-17} \; .$$

Instead for Localized dipolar kick by optics + GW

$$x'' + \omega^2 x = a\cos(\omega_g t)$$

Localized. → expand in harmonics…

$$x'' + \omega^2 x = a\frac{\Delta t}{\tau}\cos(\omega_{res} t) \qquad \omega_{res} = \omega_g + \frac{2\pi}{\tau} j$$

Many harmonics

$$\frac{1}{\pi}\beta^2 \delta Q_{rms}\frac{c^2}{R\hat{L}}\frac{\langle x\rangle_{max}}{\hat{\beta}_x} = f^2 h$$

Example

Then for $\hat{\beta}_x = 10^6$, $R = 5000, \hat{L} = 1000$, $\delta Q_{rms} = 10^{-4}$ we find

$$\frac{18}{5}\beta^2 \langle x \rangle_{max} = f^2 h$$

at $\beta = 1$

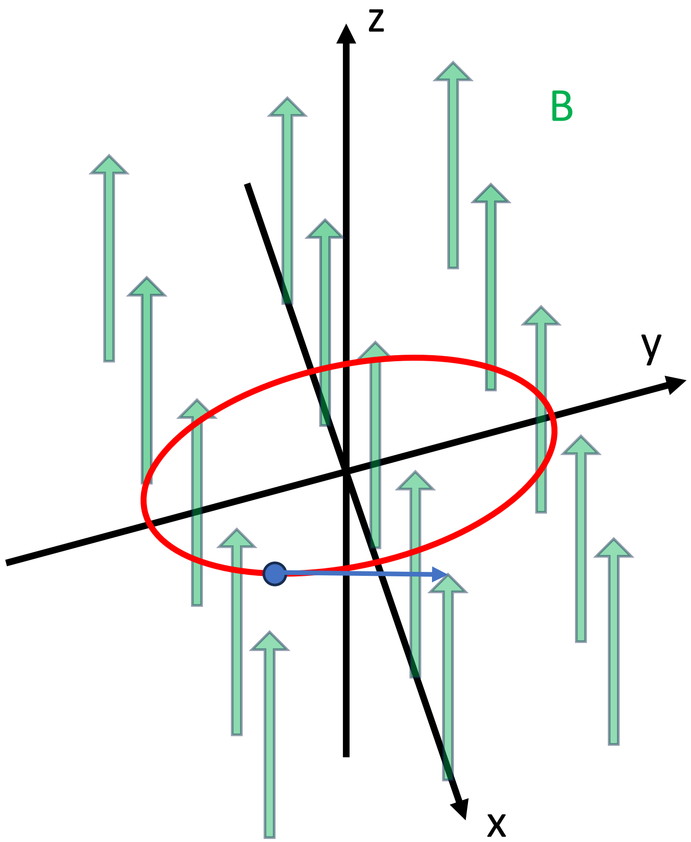
$$0.57 x_{resol.} = f^2 h$$

so for 10^{-9}

$$5.7 \times 10^{-10} = f^2 h \ .$$

Summary/Outlook

- Equations of motion in a constant magnetic field “almost” fully applicable
- Resonant dynamics as for the Oide model
- Assessment of lower limit for $f\,h$, or $f^2\,h$ *(??)*
- Full dynamics for arbitrary Bx, By, Bz difficult, but interesting as maybe it is not very sensitive to tilts and/or ground vibration (?)


z
B
y
x