SRGWmb2025 – Storage Rings & Gravitational Waves – mini-brainstorm

Cosmological and astrophysical models for GW sources

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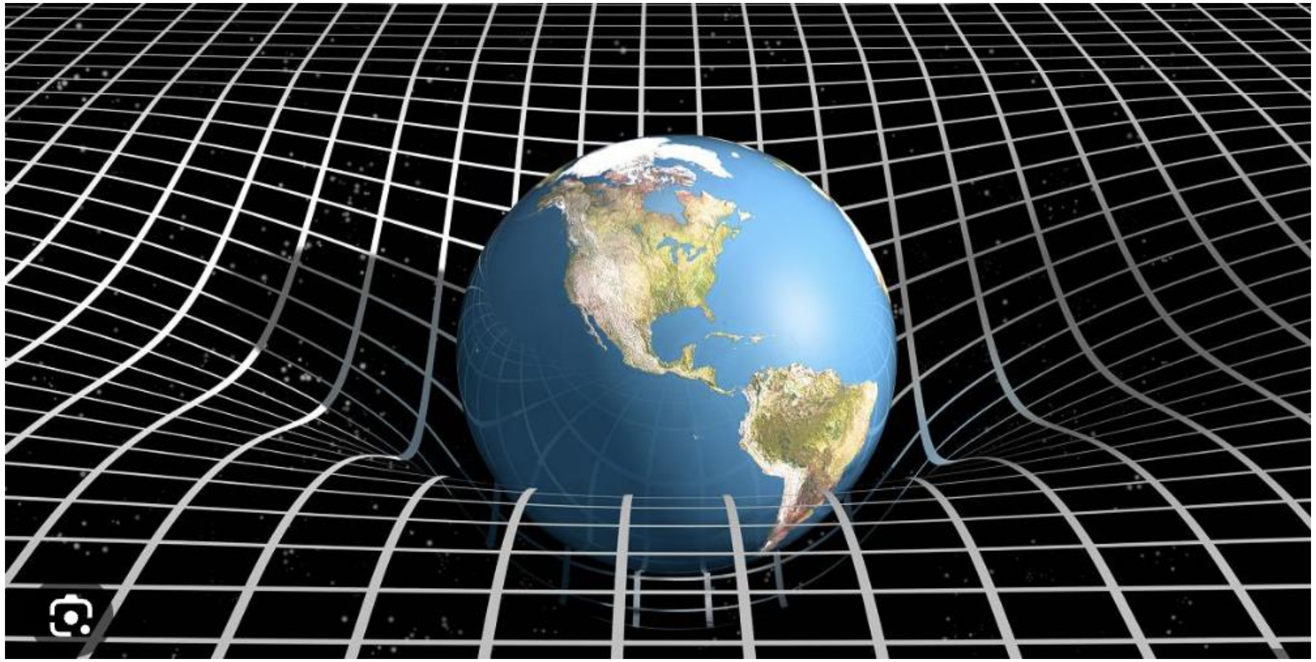
Gravity and Gravitational Waves

Gravity: A Fundamental Property of Space-Time

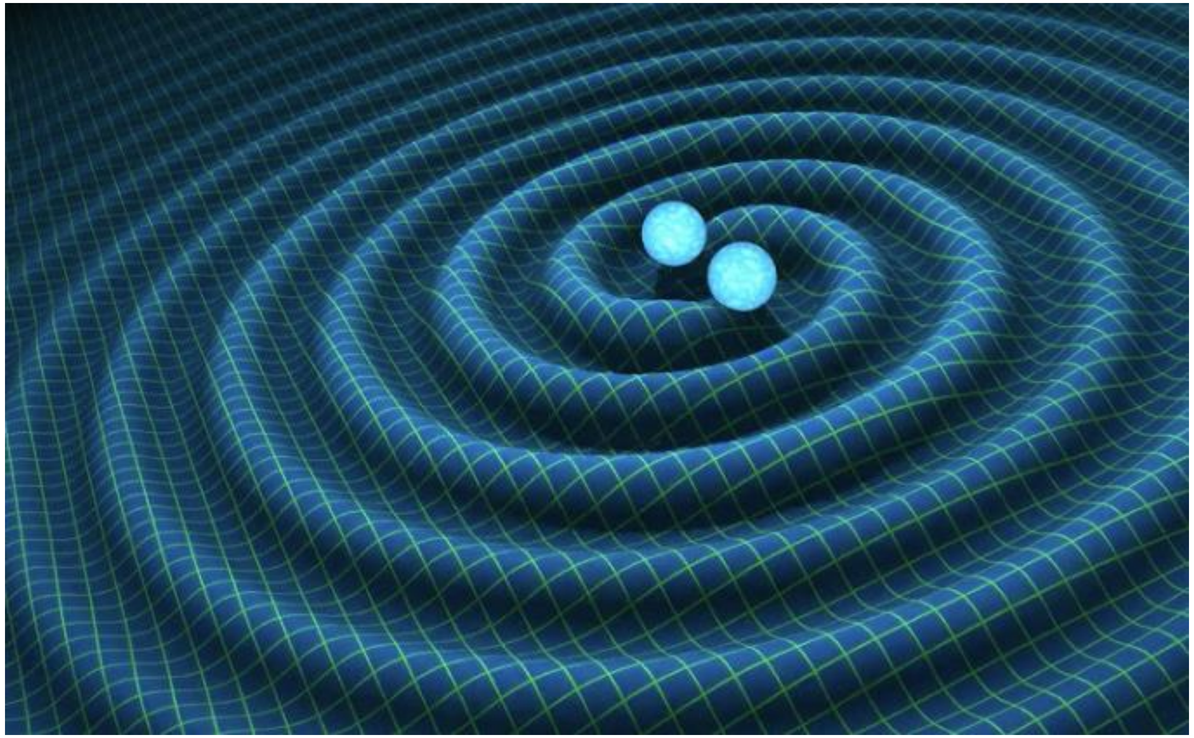
1. Space-time is dynamic and ever-changing.
2. **Universality**: Gravity acts on all objects with mass or energy, regardless of their composition.
3. Matter and energy are influenced by gravity and determine the spacetime they are moving in.
4. Gravity dominates at large scales, shaping the structure of the universe.
5. **Long-ranged and Attractive**



Gravity and Gravitational Waves

Gravitational Waves (GW)

1. Often described as the "sound waves" of space-time.
2. Represent ripples in the curvature of space-time itself.
3. Provide a groundbreaking new window into the universe.
4. Offer complementary insights to those obtained through other cosmic messengers (e.g., light, particles).
5. Encode information about the total mass-energy and momentum distribution of astronomical sources.



Why studying GWs

Fundamental Physics

- **Gravity and Space-Time**
  - Exploration of strong-field gravity and relativistic motion.
  - Understanding black hole properties and the event horizon.
  - Window to the Early Universe and Quantum Gravity.
- **Equation of State of Matter Beyond Nuclear Density**
  - Insights into neutron star physics and ultra-dense matter.

Cosmology

- Investigating dark energy and the universe's accelerated expansion.
- Exploring the nature and distribution of dark matter.
- Inflation

Astrophysics

- Understanding compact objects and their role in stellar evolution.
- Studying the nucleosynthesis of elements in stars.
- Mapping large-scale cosmic structures.
- Examining gamma-ray bursts and their origins.

Einstein Equations

The Einstein equations are written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}, \qquad \kappa^2 = 8\pi G = M_P^{-2}\,, \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum (stress energy) tensor. This is a set of 10 coupled non-linear equations which can be solved (in principle) when the energy-momentum is given. In the vacuum ($T_{\mu\nu} = 0$) the most well known and studied solution is the Schwarzschild solution which describes a static, spherically symmetric spacetime (**Black Hole**) with metric

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad r_s = 2GM. \tag{2}$$

Similar solutions are the **Kerr** (spinning BH) and the **Reissner–Nordström** (charged BH). Analytic solutions of two or more BHs are not known.

Cosmology Review

The Standard Model for Cosmology describes the Universe as made of different forms of energy densities acting as sources of the gravitational field. The latter in turn is described by a Friedmann-Lemaître-Robertson-Walker geometry (FLRW) with metric

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) , \tag{3}$$

where the three-dimensional constant-time slice is a maximally symmetric space, the Riemann tensor of which satisfies (for constant t)

$$R_{ijkl} = \frac{k}{a^2}(g_{ik}g_{jl} - g_{il}g_{jk}) , \quad k = 0, \pm 1 .$$

For $k > 0, k < 0$ and $k = 0$ we refer to closed, open or flat Universe, respectively. $a(t)$ is the "scale factor" which tells us how big is the 3D slice at (comoving) time t.

For a Universe filled by a perfect fluid, described by an energy density $\rho(t)$ and a pressure $p(t)$, we have

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \qquad T^\mu_\nu = \begin{pmatrix} -\rho & \\ & p\ \delta^i_j \end{pmatrix}$$

where U^μ is the relativistic four-velocity vector. We may impose an equation of state

$$p = w\,\rho\,, \qquad |w| \leq 1.$$

The stress tensor conservation implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}\,,$$

which gives that

$$\rho \propto a^{-3(1+w)}\,.$$

Matter Content

The value of w is determined by the nature of the fluid:

$w = -1$	de Sitter (cosmological constant)
$w = -\frac{2}{3}$	domain walls
$w = -\frac{1}{3}$	string gas (also curvature)
$w = 0$	dust (pressuless matter)
$w = \frac{1}{3}$	radiation
$w = 1$	stiff matter

Friedmann Equation

With a multicomponent perfect fluid filling the Universe, the Einstein equations give

$$H^2 = \frac{8\pi G}{3} \sum_i \rho_i \,, \qquad H = \frac{\dot{a}}{a}$$

If we also define the critical energy density

$$\rho_c = \frac{8\pi G}{3H^2} \,, \tag{4}$$

we can write Friedmann equation as

$$\sum_i \Omega_i = 1 \,, \tag{5}$$

where the density parameters Ω_i are defined as

$$\Omega_i = \frac{\rho_i}{\rho_c} \,.$$

Given that the Universe is composed of matter, radiation and dark energy (cosmological constant?), we may express the Friedmann equation at redshift $z = a_0/a(t) - 1$ as

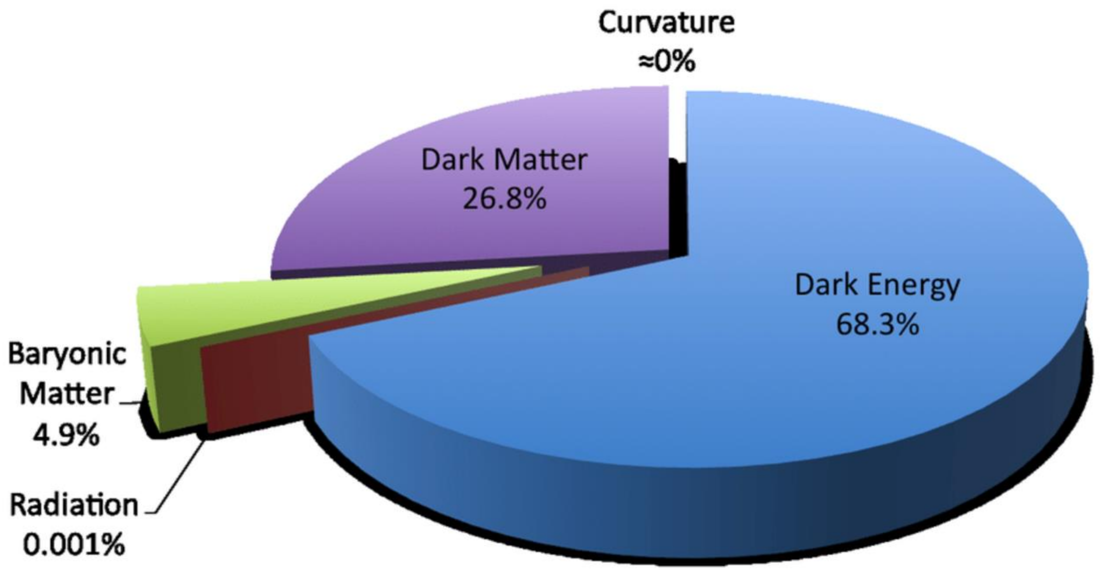
$$H(z)^2 = H_0^2 \left\{ \Omega_{rad}(1+z)^4 + \Omega_m(1+z)^3 + \Omega_\Lambda + \Omega_{GW}(1+z)^\beta \right\} \quad (6)$$

with (in ΛCDM)

$$\Omega_{rad} + \Omega_m + \Omega_\Lambda + \Omega_{GW} = 1.$$

In fact, we know that

$$\Omega_{rad} \approx 10^{-5}$$
$$\Omega_m \approx 0.31$$
$$\Omega_\Lambda \approx 0.69.$$



How much is Ω_{GW}?

Cosmological Sources of Gravitational Waves

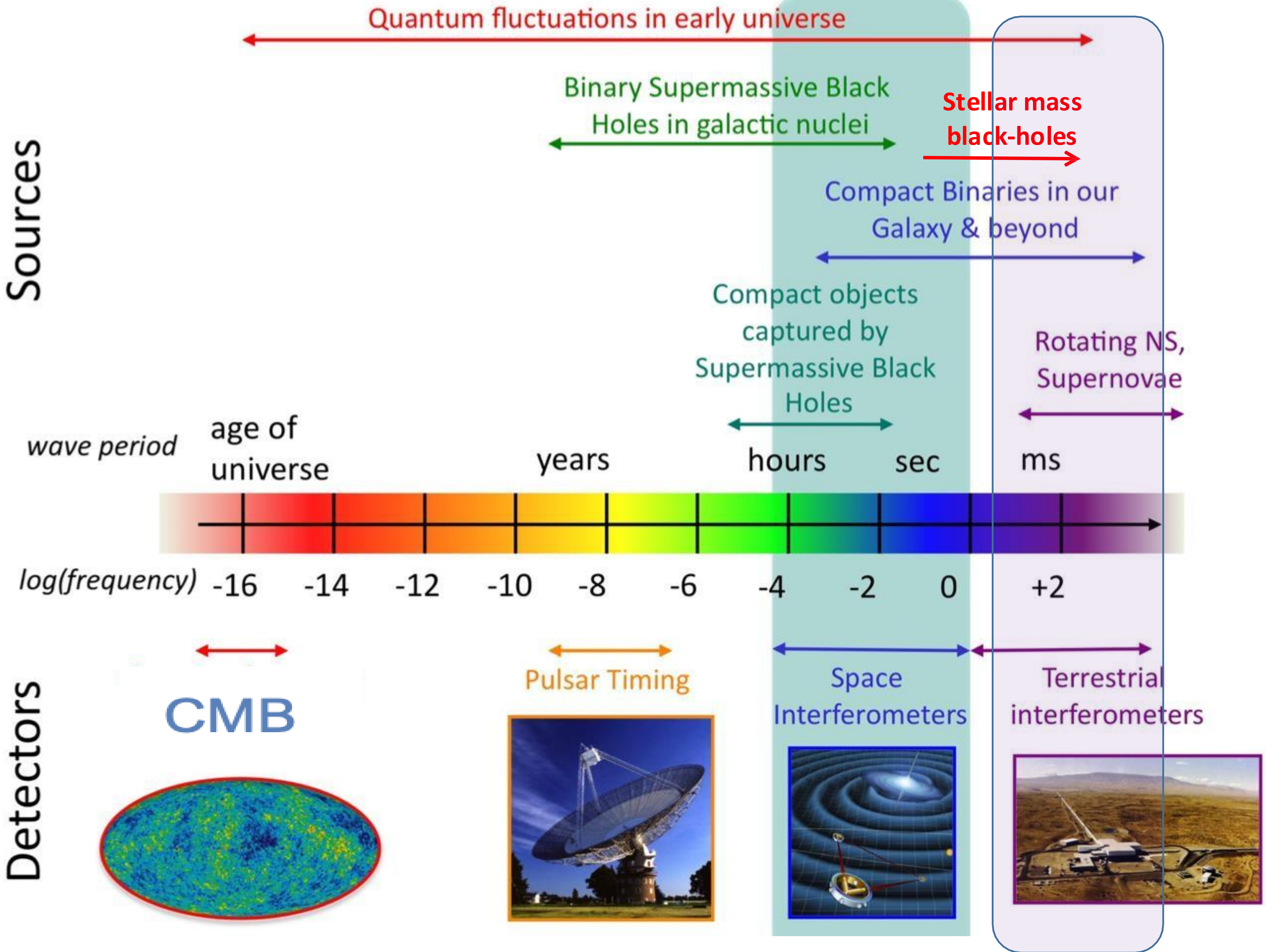
- **Primordial Gravitational Waves (from Inflation)**: Quantum fluctuations during cosmic inflation (a rapid expansion of space in the early universe) could have generated gravitational waves that may be detectable in the Cosmic Microwave Background (CMB) polarization.
- **Cosmic Strings**: Hypothetical topological defects from early universe phase transitions could produce gravitational wave bursts.
- **Phase Transitions in the Early Universe**: Phase transitions (1st order) may have generated stochastic backgrounds of gravitational waves.
- **Collisions of Primordial Black Holes**: If small black holes formed in the early universe, their mergers would create detectable gravitational waves.
- **Extra Dimensions or String Theory Effects**: Some speculative theories predict gravitational waves from high-energy events beyond the Standard Model of physics.

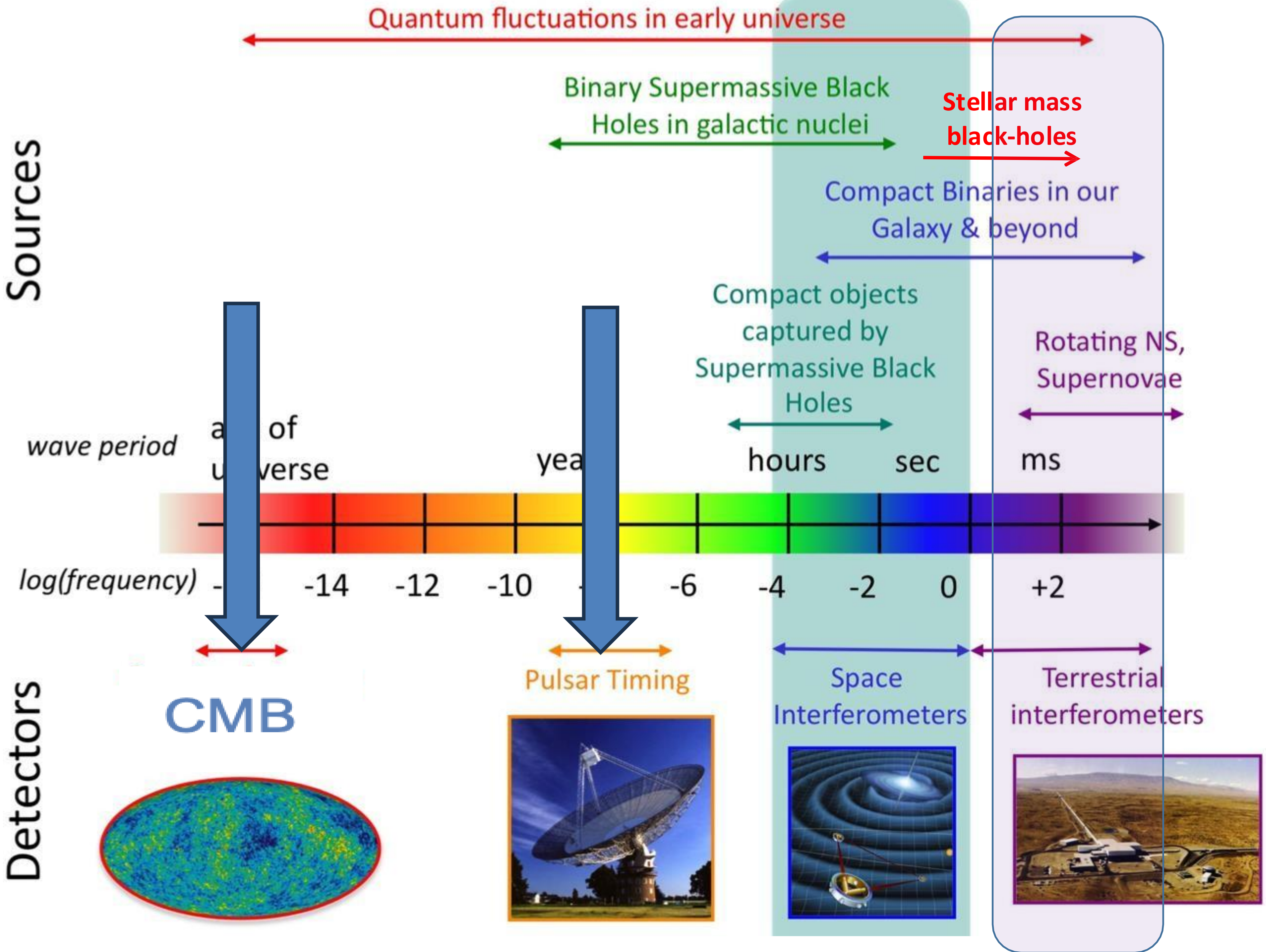
Astrophysical Sources of Gravitational Waves

- **Binary Black Hole (BBH) Mergers**: Two black holes in a close orbit lose energy via gravitational waves, eventually merging in a highly energetic event.
- **Binary Neutron Star (BNS) Mergers**: Neutron stars spiraling into each other produce GWs and often gamma-ray bursts and kilonovae.
- **Black Hole-Neutron Star (BH-NS) Mergers**: A neutron star merging with a black hole generates GWs and sometimes electromagnetic signals.

These are the most commonly detected GW sources by **LIGO, Virgo, and KAGRA**.

- **Supernovae and Core-Collapse Events.**The violent collapse of a massive star into a neutron star or black hole can generate gravitational waves if the explosion is asymmetric. However, the signals from these events are usually weaker and harder to detect compared to binary mergers.
- **Supermassive Black Hole (SMBH) Mergers** When galaxies collide, their central supermassive black holes (millions to billions of solar masses) eventually merge, emitting gravitational waves at lower frequencies.These are being studied using **Pulsar Timing Arrays (PTAs)** (such as NANOGrav, EPTA) and will be a primary target of **LISA (Laser Interferometer Space Antenna)**.


Sources
Quantum fluctuations in early universe
Binary Supermassive Black Holes in galactic nuclei
Stellar mass black-holes
Compact Binaries in our Galaxy & beyond
Compact objects captured by Supermassive Black Holes
Rotating NS, Supernovae
wave period
age of universe
years
hours
sec
ms
log(frequency)
-16
-14
-12
-10
-8
-6
-4
-2
0
+2
Detectors
CMB
Pulsar Timing
Space Interferometers
Terrestrial interferometers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How we can calculate Ω_{GW}?

As usual, we consider perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad |h| << 1,$$

where $\bar{g}_{\mu\nu}$ is the background metric (here the FLRW metric). Then, the perturbations contribute to the energy-momentum tensor as

$$t_{\mu\nu} = \frac{1}{32\pi G_N} \langle \partial_\mu h_{ij} \partial_\nu h^{ij} \rangle,$$

from where we find that

$$\rho_{GW} = \frac{1}{32\pi G_N} \langle \dot{h}_{ij} \dot{h}^{ij} \rangle.$$

Then we usually define

$$\hat{\Omega}_{GW}(f) = \frac{1}{\rho_c} \frac{d\rho_{GW}}{d \log f},$$

from where

$$\Omega_{GW} = \int_0^\infty \hat{\Omega}_{GW}(f) d \log f.$$

Stochastic GW

A stochastic GW background can be constructed by the superposition of plane GW of all possible frequencies from all possible directions, and therefore will be of the form

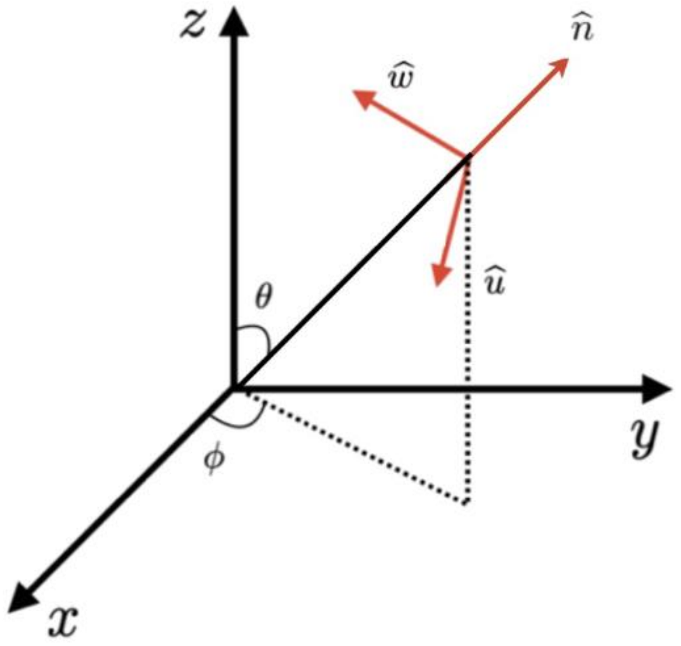
$$h_{ij}^{TT}(t,\vec{x}) = \sum_{A=+,\times} \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\hat{n}\, h_A(f,\hat{n}) e_{ij}^A(\hat{n}) e^{-2\pi i f(t-\hat{n}\cdot\vec{x})}, \tag{16}$$

where $A = +, \times$ are the polarizations of the GWs, $\hat{n}$ their direction of propagation and f their frequency. The polarization tensors $e_{ij}^A(\hat{n})$ are given by

$$e_{ij}^{+} = \hat{u}_i\hat{u}_j - \hat{w}_i\hat{w}_j, \qquad e_{ij}^{\times} = \hat{u}_i\hat{w}_j + \hat{w}_i\hat{u}_j,$$

where the vectors $\hat{n}$, $\hat{u}$ and $\hat{w}$ are explicitly given by

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$
$$\hat{u} = (\sin\phi, -\cos\phi, 0),$$
$$\hat{w} = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta),$$



The GW energy density is thus

$$\langle \dot{h}_{ab}\dot{h}^{ab}\rangle = \sum_{A}\sum_{A'}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{S^2} d^2\Omega_{\hat{n}} \int_{S^2} d^2\Omega_{\hat{n}'} \, \langle h_A(f,\hat{n}) h^*_{A'}(f',\hat{n}')\rangle$$
$$\times\, e^{A}_{ab}(\hat{n}) e^{ab}_{A'}(\hat{n}') \times 4\pi^2 f f' \tag{17}$$
$$\times \exp\left[2\pi i (f-f')t + 2\pi i(\hat{n}-\hat{n}')\cdot\vec{x}\right].$$

For a *Gaussian-stationary, unpolarized, spatially homogeneous and isotropic* stochastic background the quadratic expectation value of the Fourier modes are

$$\langle h_A(f,\hat{n}) h^*_{A'}(f',\hat{n}')\rangle = \frac{\delta_{AA'}}{2}\,\frac{\delta^{(2)}(\hat{n},\hat{n}')}{4\pi}\,\frac{\delta(f-f')}{2} S_h(f), \qquad 3)$$

where $S_h(f)$ is the one-sided power spectral density (PSD) of the *Fourier modes* of the SGWB. With identities $\int d\hat{\Omega} = 4\pi$ and $\sum_A e^A_{ab} e^{ab}_A = 4$, converting the frequency integration bounds in Eq. (17) to $[0,\infty)$ gives

$$\langle \dot{h}_{ij}\dot{h}^{ij}\rangle = 8\pi^2 \int_0^\infty df f^2 S_h(f). \tag{20}$$

Hence,

$$\boxed{\Omega_{SGWB} = \frac{1}{\rho_c}\frac{\rho_{SGWB}}{d\log f} = \frac{2\pi^2}{3H_0^2} f^3 S_h(f).}$$

Characteristic Strain

The fractional energy density Ω_{SGWB} is the usual quantity that particle physicists and cosmologists refer to. But GW scientists typically talk about the characteristic strain

$$h_c(f) = \sqrt{f\, S_h(f)} \tag{22}$$

The characteristic strain accounts for the number of wave cycles the signal spends in-band through the $\sqrt{f}$ dependence Hence we can write

$$\Omega_{SGWB} = \frac{2\pi^2}{3H_0^2} f^2 h_c^2(f). \tag{23}$$

As is often the case in astronomy and astrophysics, several SGWB sources predict a power-law form for $h_c(f)$, defined as

$$h_c(f) = A_{\alpha,ref} \left(\frac{f}{f_{ref}}\right)^{\alpha} \tag{24}$$

Here α is a spectral index, f_{ref} is a reference frequency that is typical of the detector's band, and $A_{\alpha,ref}$, is the characteristic strain amplitude at the reference frequency. The fractional energy density then scales as

$$\Omega_{SGWB} \propto f^{2\alpha+2}$$

Primordial SGWBs resulting from quantum tensor fluctuations that are inflated to macroscopic scales usually assume a scale-invariant spectrum for which

$$\text{Inflation} \qquad \Omega_{SGWB} \propto \text{const.} \qquad \alpha = -1 \tag{25}$$

On the other hand, a population of circular inspiraling compact-binary systems creates a SGWB with

$$\text{binary cluster} \qquad \Omega_{SGWB} \propto f^{2/3} \qquad \alpha = -2/3. \tag{26}$$

Current Estimates for Ω_{GW}

Observational data constrain Ω_{GW} at different frequency bands:

1. Primordial Gravitational Waves (Inflation)
   - Ω_{GW} is constrained by measurements of the Cosmic Microwave Background (CMB) polarization, particularly the tensor-to-scalar ratio (r).
   - Upper limits on the tensor-to-scalar ratio r from Planck and BICEP/Keck experiments yield:

$$\Omega_{\text{GW}} \lesssim 10^{-15} \quad (\text{at } f \sim 10^{-18} \text{ Hz})$$

2. Astrophysical Stochastic Background (LIGO/Virgo Band)
   - These are produced by sources like compact binary mergers (black holes, neutron stars) and contribute to the stochastic gravitational wave background (SGWB).
   - LIGO-Virgo constraints in the frequency range $10 \text{ Hz} \lesssim f \lesssim 10^3 \text{ Hz}$:

$$\Omega_{\text{GW}} \lesssim 6 \times 10^{-8}$$

3. Cosmic Strings or Phase Transitions

- Depending on model parameters:

$$10^{-10} \lesssim \Omega_{\mathrm{GW}} \lesssim 10^{-6}$$

4. Pulsar Timing Arrays (Low Frequencies)

- Pulsar Timing Array (PTA) experiments constrain Ω_{GW} at frequencies 10^{-9} Hz $\lesssim f \lesssim 10^{-7}$ Hz:

$$\Omega_{\mathrm{GW}} \lesssim 10^{-9}$$

Summary of Ω_{GW} Constraints

- **Inflationary GWs:** $\Omega_{\mathrm{GW}} \lesssim 10^{-15}$
- **Astrophysical SGWB:** $\Omega_{\mathrm{GW}} \lesssim 10^{-8}$
- **Cosmic Strings/Phase Transitions:** $10^{-10} \lesssim \Omega_{\mathrm{GW}} \lesssim 10^{-6}$

These limits indicate that gravitational waves contribute only a tiny fraction of the universe's total energy density.

Inflation

We cannot talk about Cosmological GWs without INFLATION

Cosmological inflation is one of the basic ideas of modern cosmology, which postulates an epoch early in the history of the universe when potential, or vacuum, energy associated to a scalar field, the inflaton, dominated other forms of energy density such as matter or radiation. During such a vacuum-dominated era the scale factor grew exponentially (or nearly exponentially) in time:

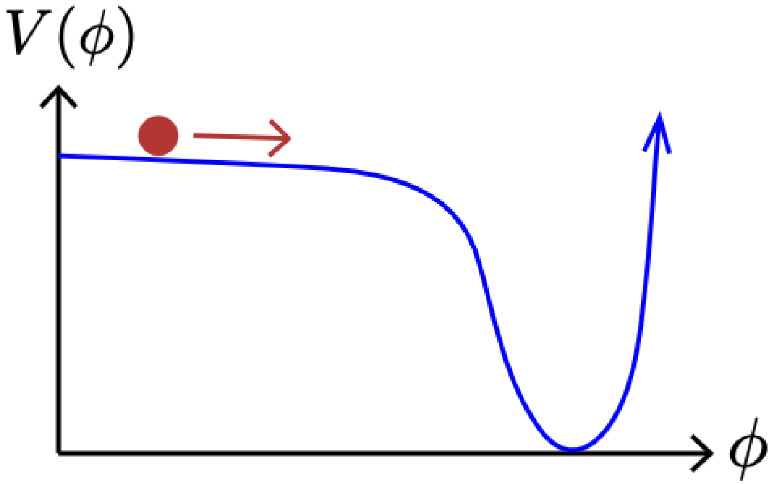
$$a(t) \sim e^{Ht}.$$

The condition for (slow-roll) inflation: $-\dot{H} \ll H^2.$

Eq. of motion:

$$H^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0$$



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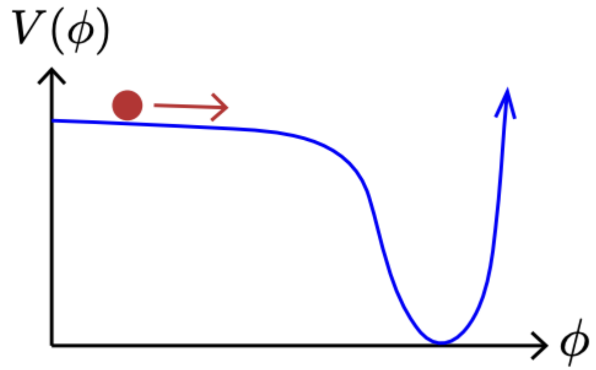

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$$\textbf{Background}: \quad ds^2 = -dt^2 + e^{2Ht}\delta_{ij}dx^i dx^j.$$

$$\textbf{Perturbations}: \quad ds^2 = -dt^2 + e^{2Ht+2\zeta(t,\vec{x})}\Big(\delta_{ij} + h_{ij}(t,\vec{x})\Big)dx^i dx^j.$$

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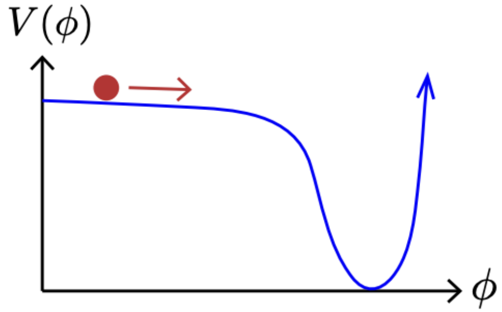

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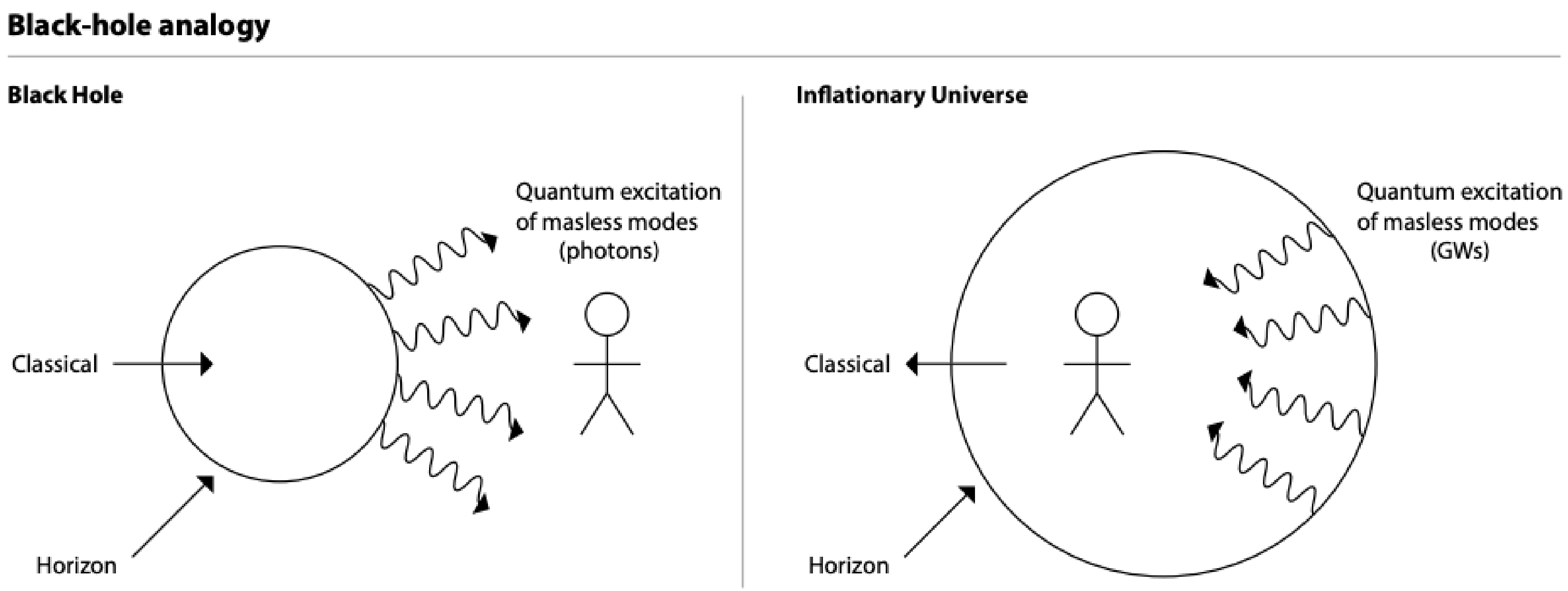
Scalar perturbation Tensor perturbation (GW)

Qunatum fluctuations are stretched so much that become classical

Understanding of inflationary gravitational waves: BH analog.



BH has an event horizon. Classically nothing can escape from horizon. QM demands BH to emit radiation (Hawking) including GWs. Inflationary universe, has a horizon and observer is now inside the horizon. This horizon also radiates GW which remain, after inflation, as a primordial-gravitational-wave background with a nearly scale-invariant spectrum.

A detection of this background will provide extremely useful information about the early Universe. It will help to :

- differentiate inflationary models,
- rule out entire model families,
- probe aspects of the quantum nature of gravity,
- provide information about the equation of state of the Early Universe.

This background leaves a precise imprint on the CMB, resulting in a specific polarization pattern of B-modes, which is the primary probe for its detection, although even in a best case scenario, assuming an almost scale invariant spectrum, the amplitude $\Omega_{GW} \sim 10^{-15}$ is simply too small.

Conclusions

GWs is a new window into Cosmos. GWs provide insights into both astrophysical and cosmological phenomena, and they allow us to study spacetime dynamics under extreme conditions. In addition, GWs offer clues about the origins of the universe and fundamental physics beyond the Standard Model. In general, they are generated by:

- Binary mergers of black holes and neutron stars (directly observed).
- Supernovae and rapidly rotating neutron stars.
- Inflationary gravitational waves, cosmic strings, and early universe phase transitions.

Gravitational waves are more than ripples in spacetime; they are messengers of the universe's most profound mysteries!

THANK YOU !