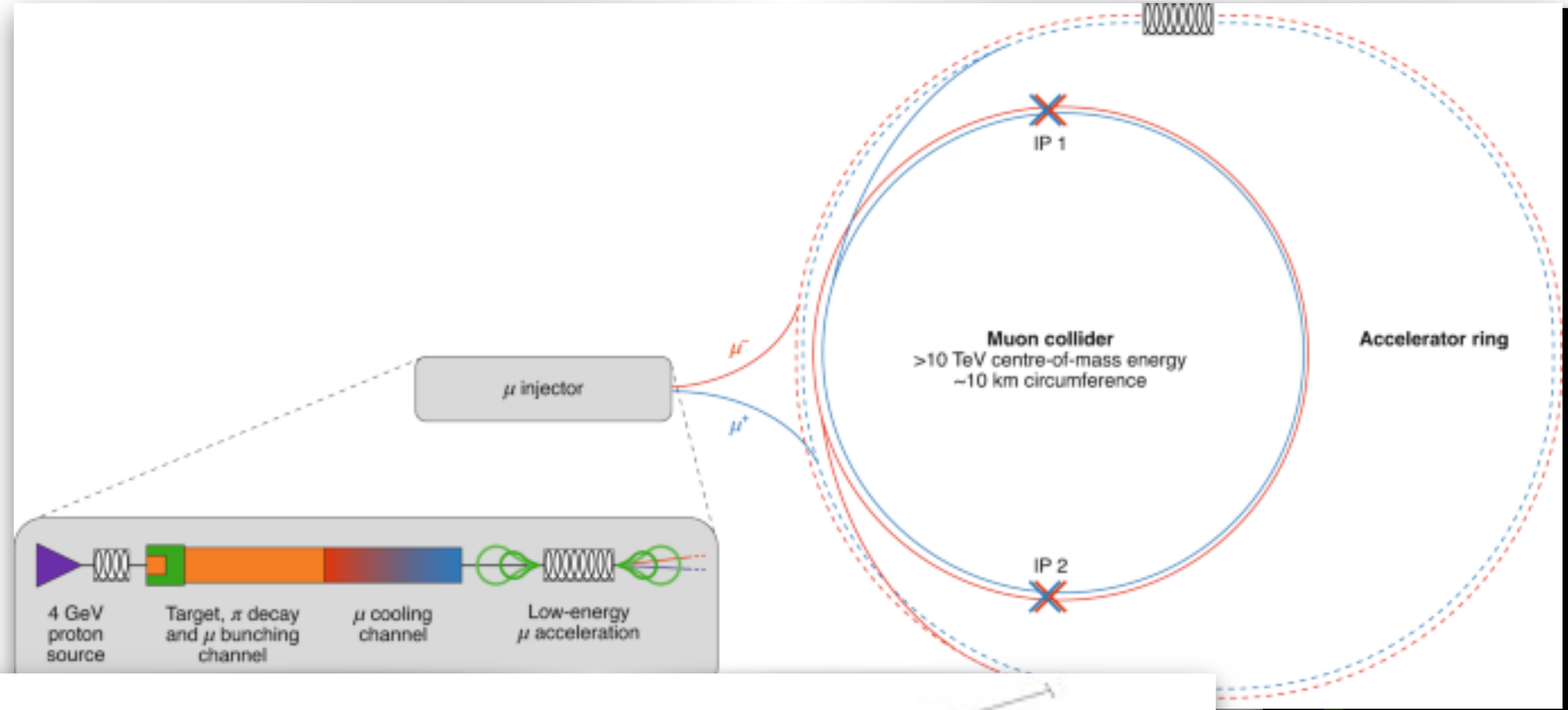
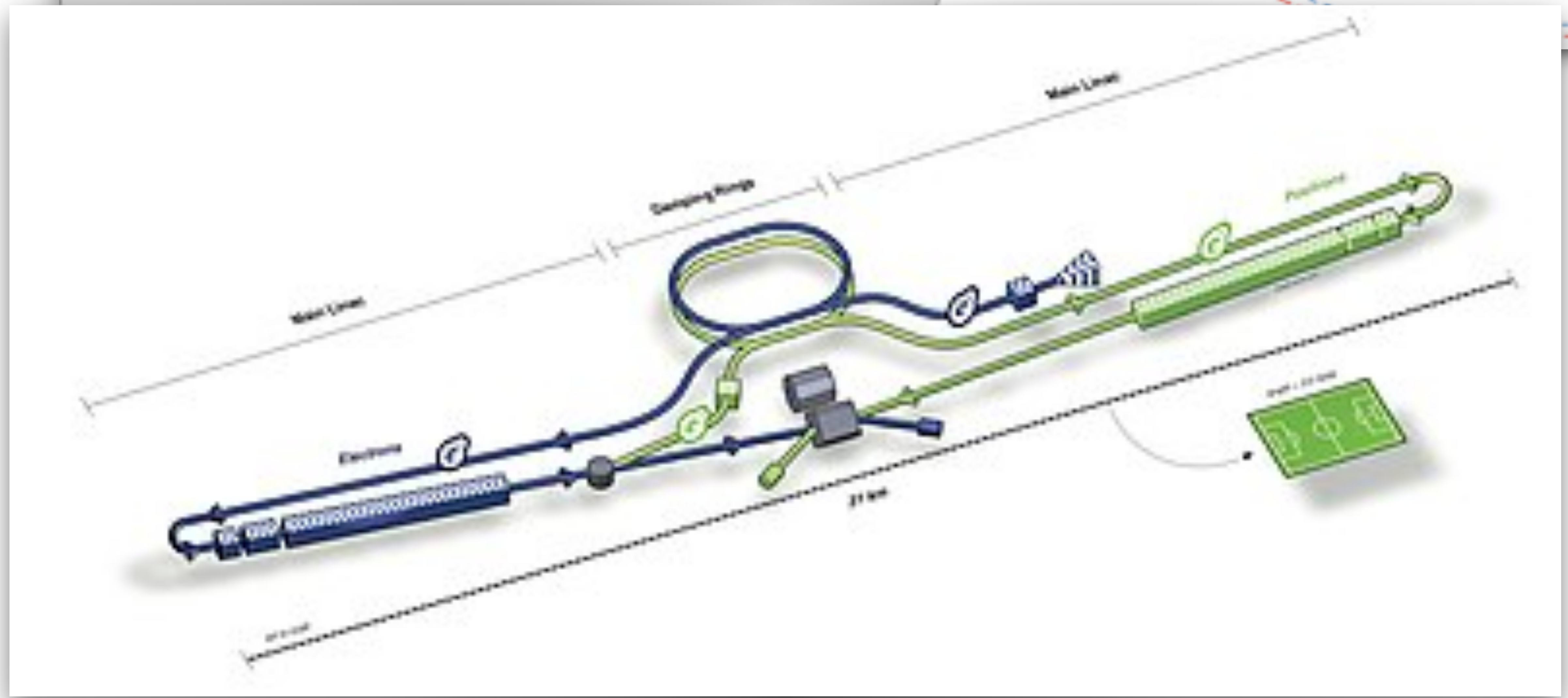
SCATTERING AMPLITUDES: FROM COLLIDER PHYSICS TO GEOMETRY

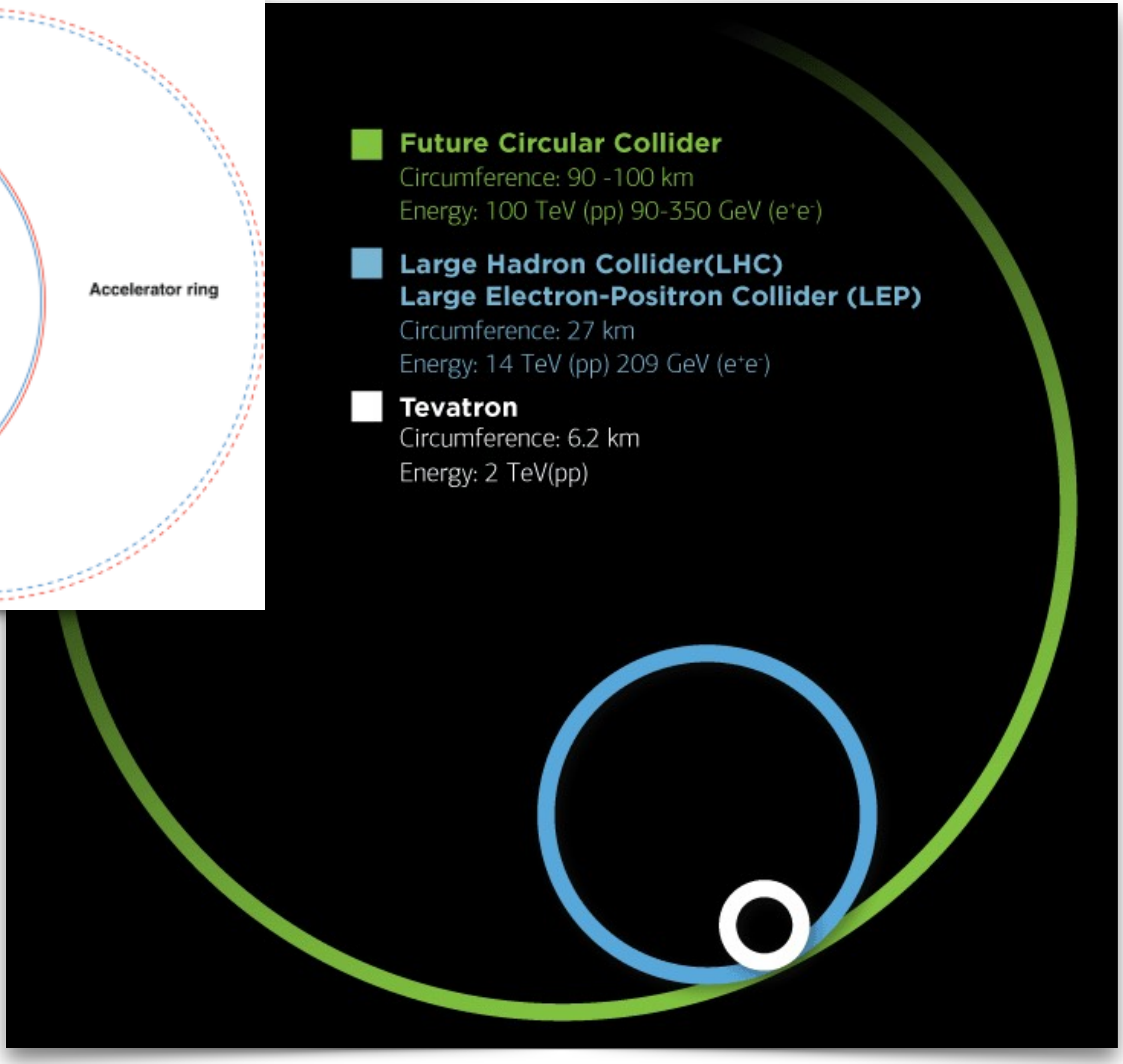
CERN TH Colloquium
August 21st 2024

Lorenzo Tancredi – Technical University Munich

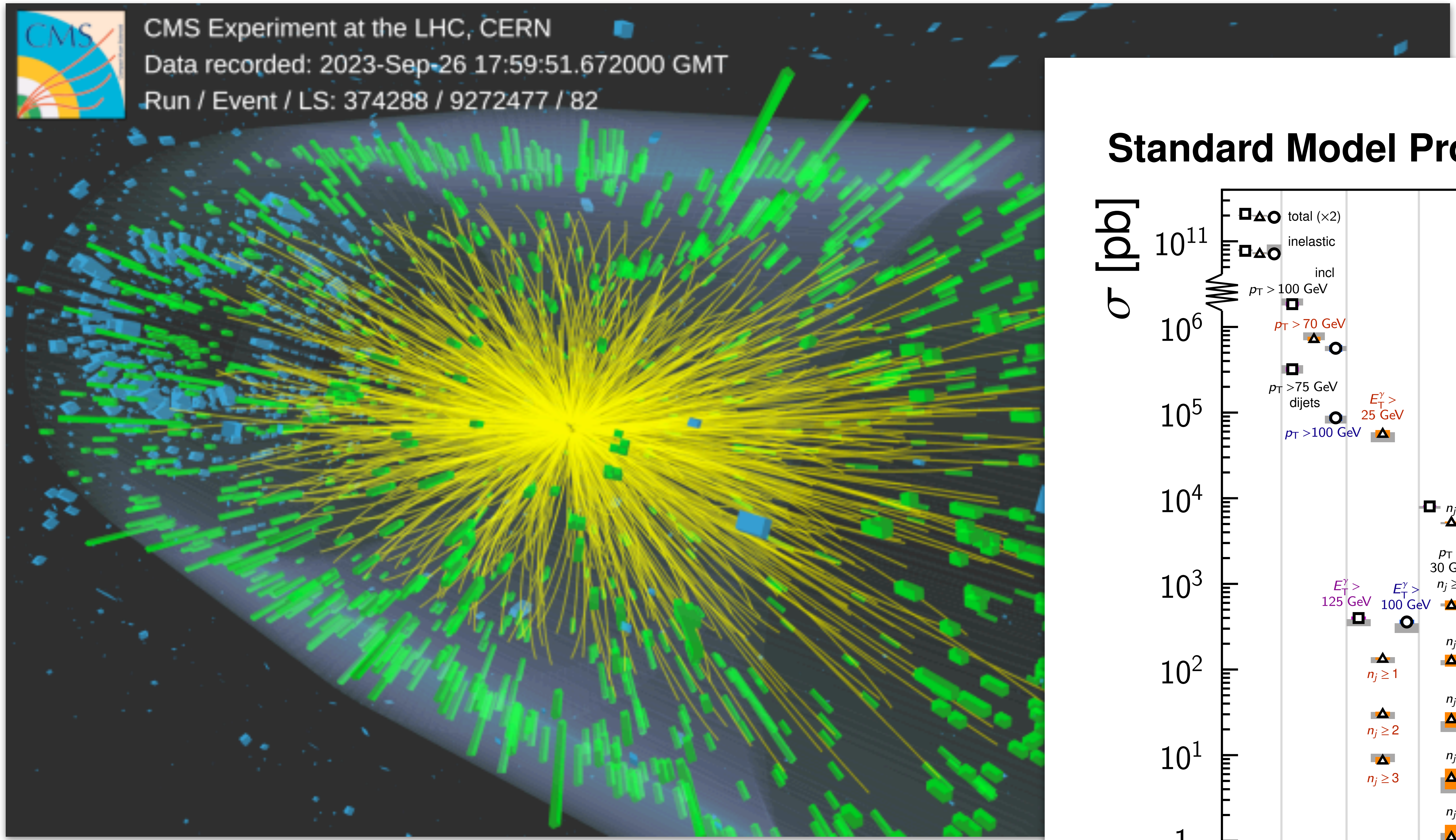
WHY (STILL) COLLIDERS? THE LHC (AND BEYOND)…

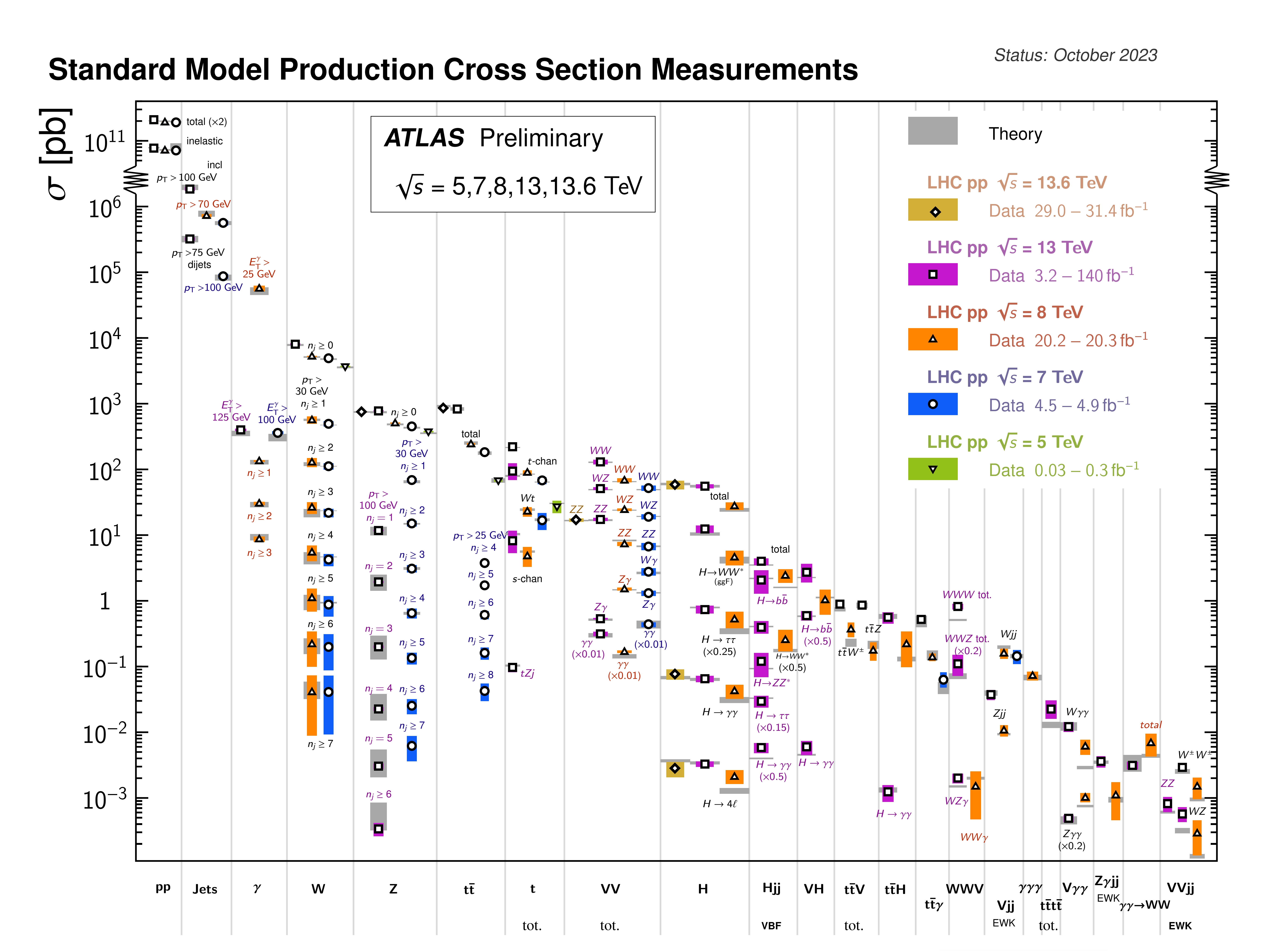






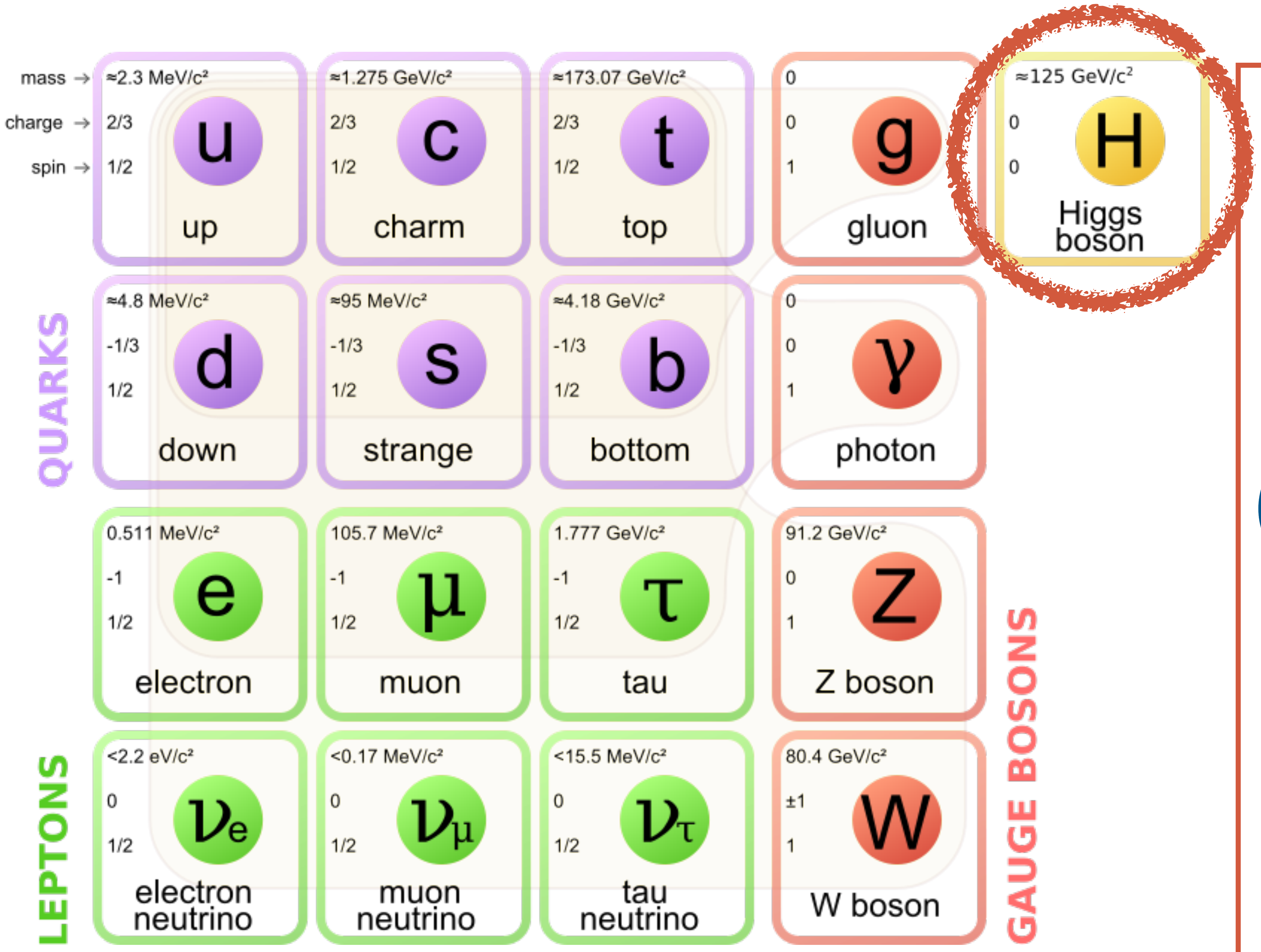
THE LHC HAS BECOME A PRECISION MACHINE

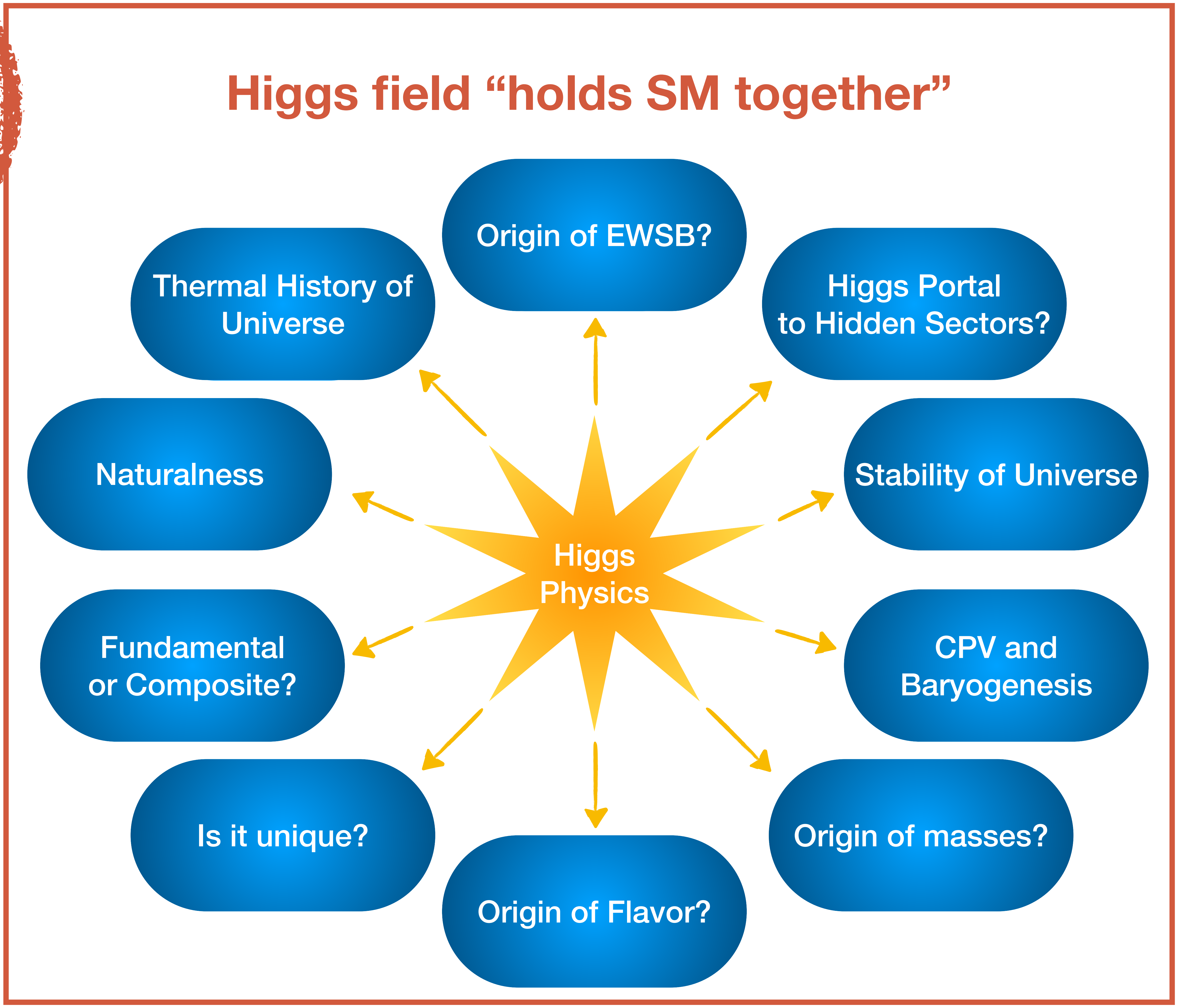




After its discovery in 2012, a lot (but not only) revolving around **Higgs boson's properties**

THE HIGGS BOSON: THE LAST MISSING PIECE

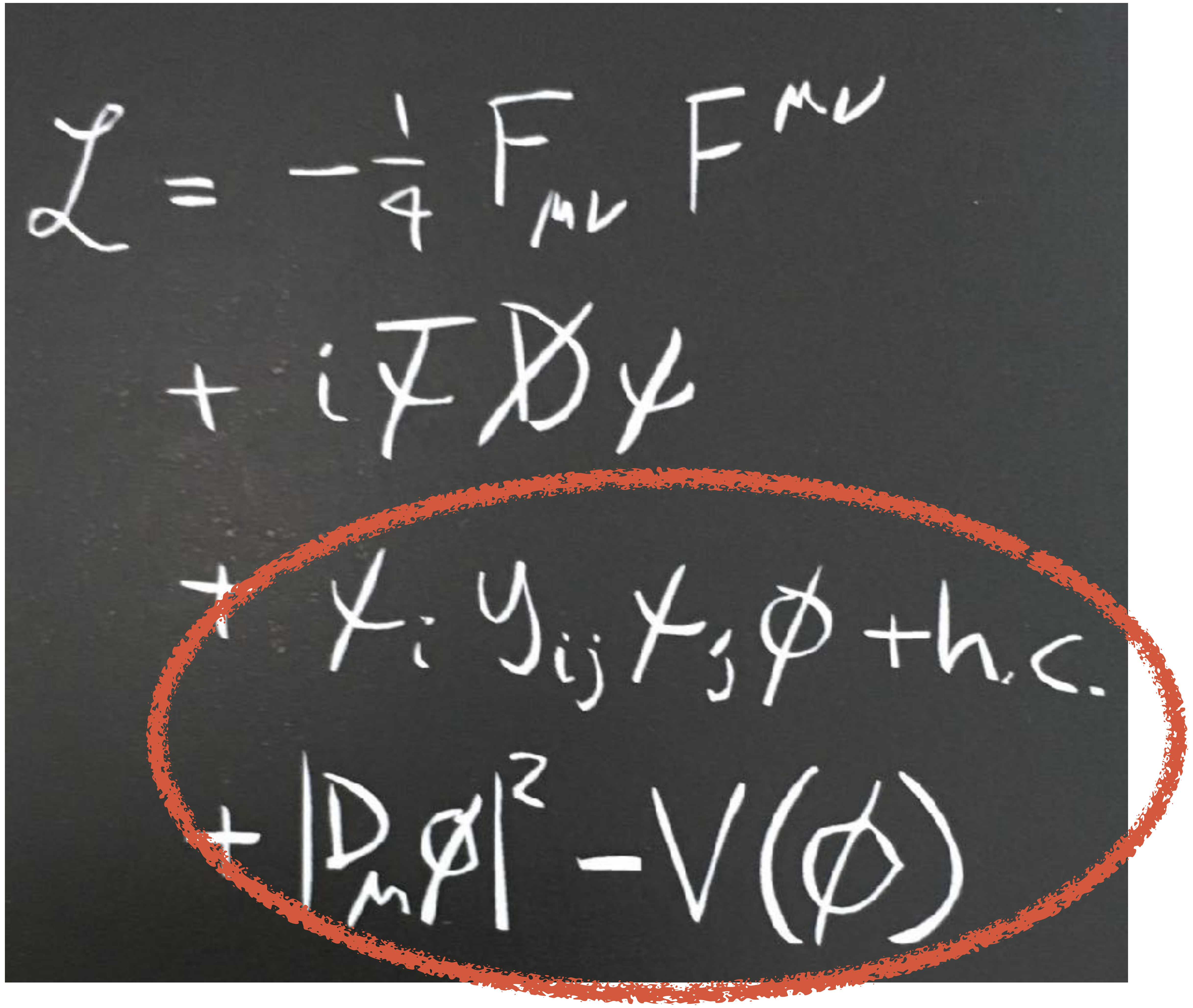




[Snowmass 2022 arXiv:2209.0751]

HIGGS INTERACTIONS AT THE LHC

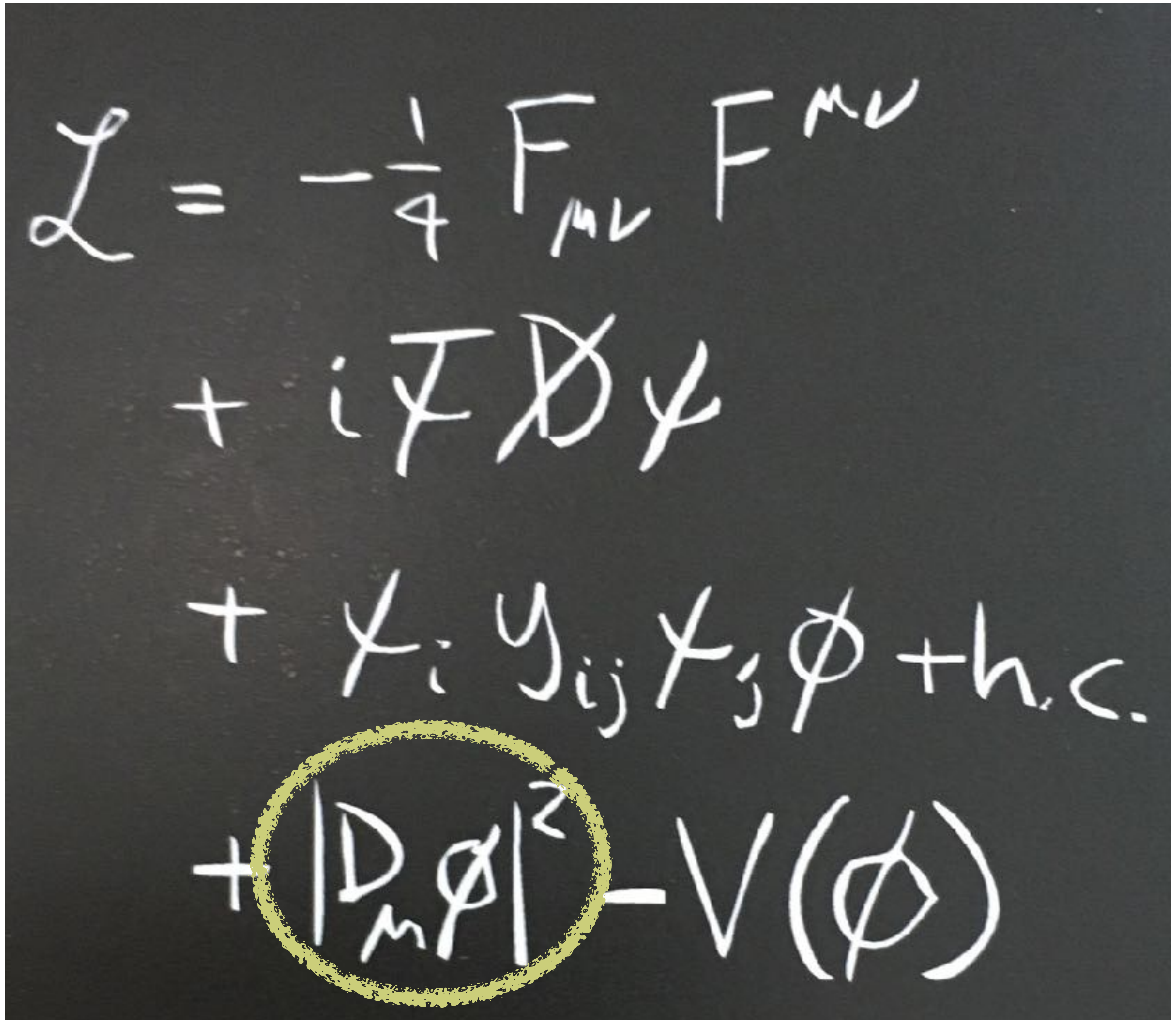
Hints to answer these questions hidden in the **details of Higgs interactions to SM particles**

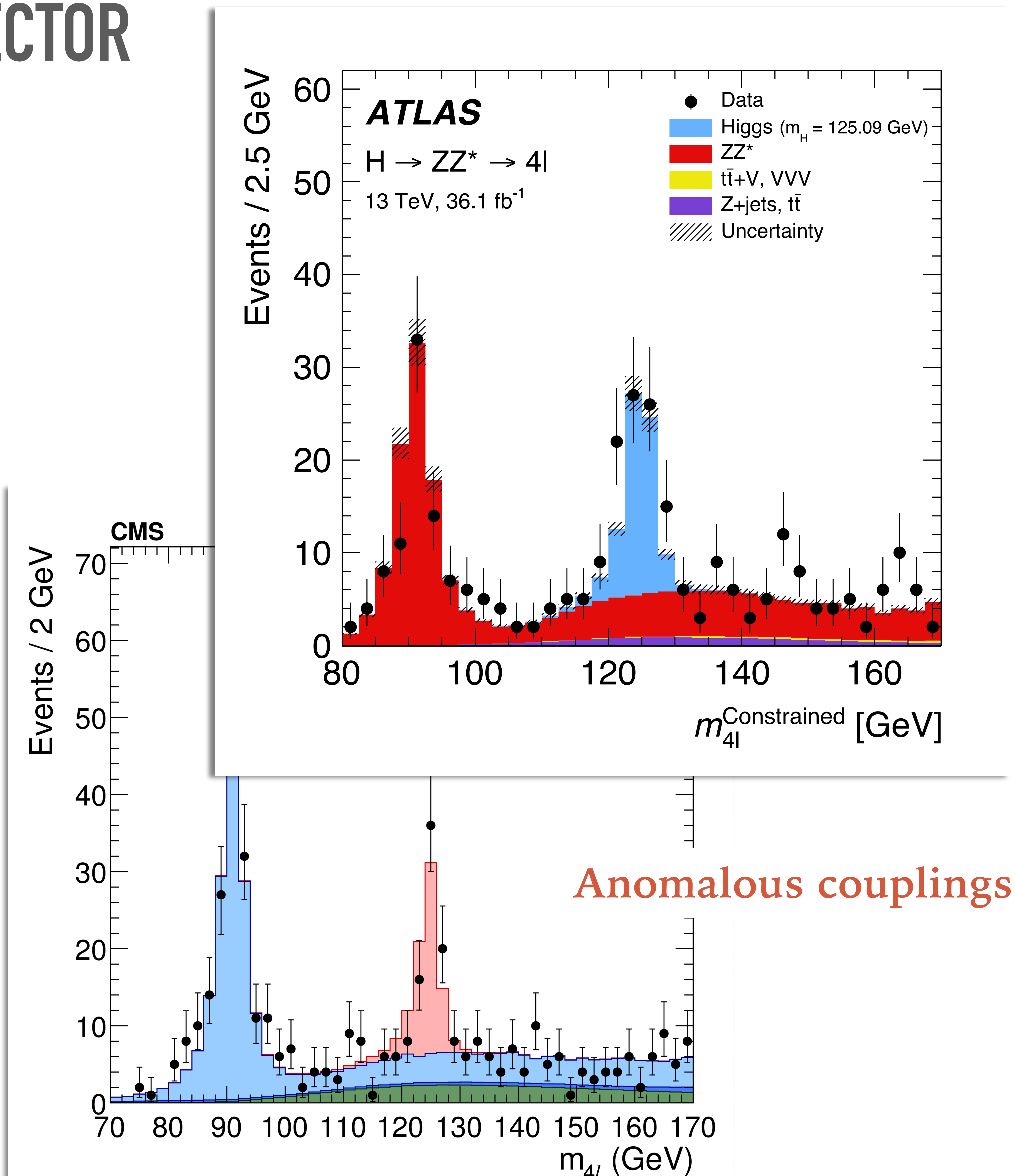


LHC has opened a window for us to peak at Higgs' interactions

HIGGS INTERACTIONS THE GAUGE SECTOR

Higgs discovery through its **couplings to gauge sector**

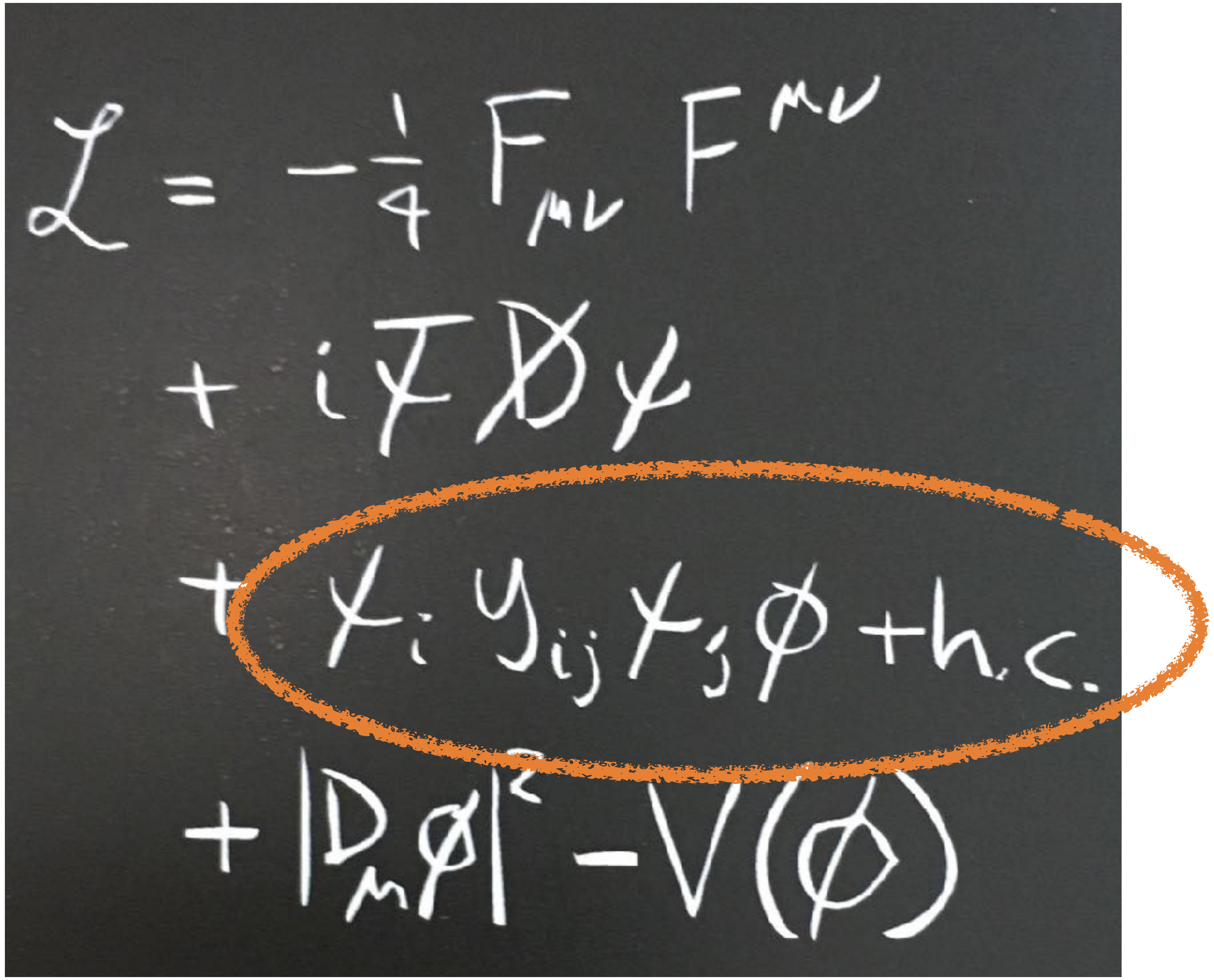


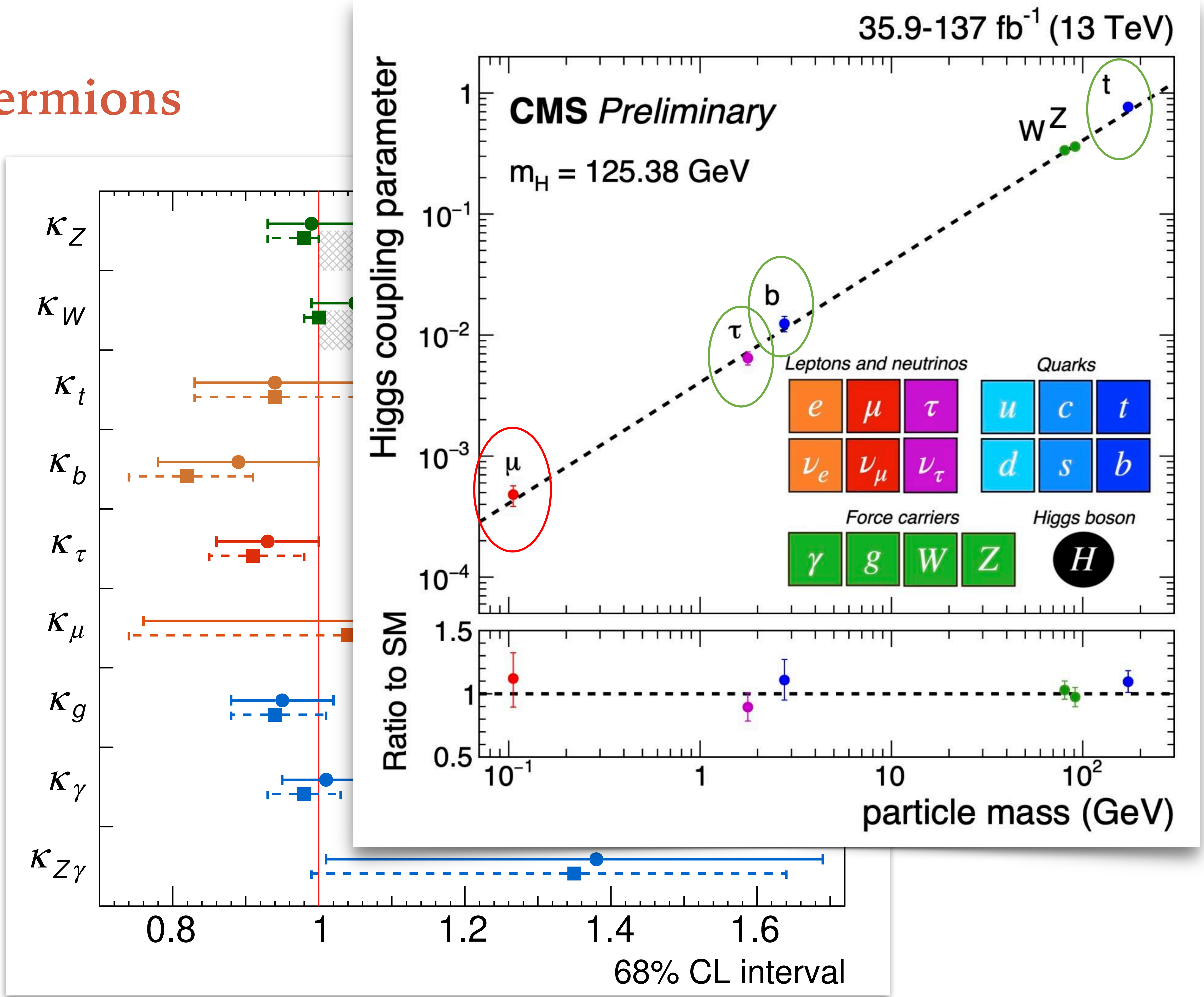


Anomalous couplings?

HIGGS INTERACTIONS THE YUKAWA SECTOR

Run 2 direct observation of H coupling to third family fermions



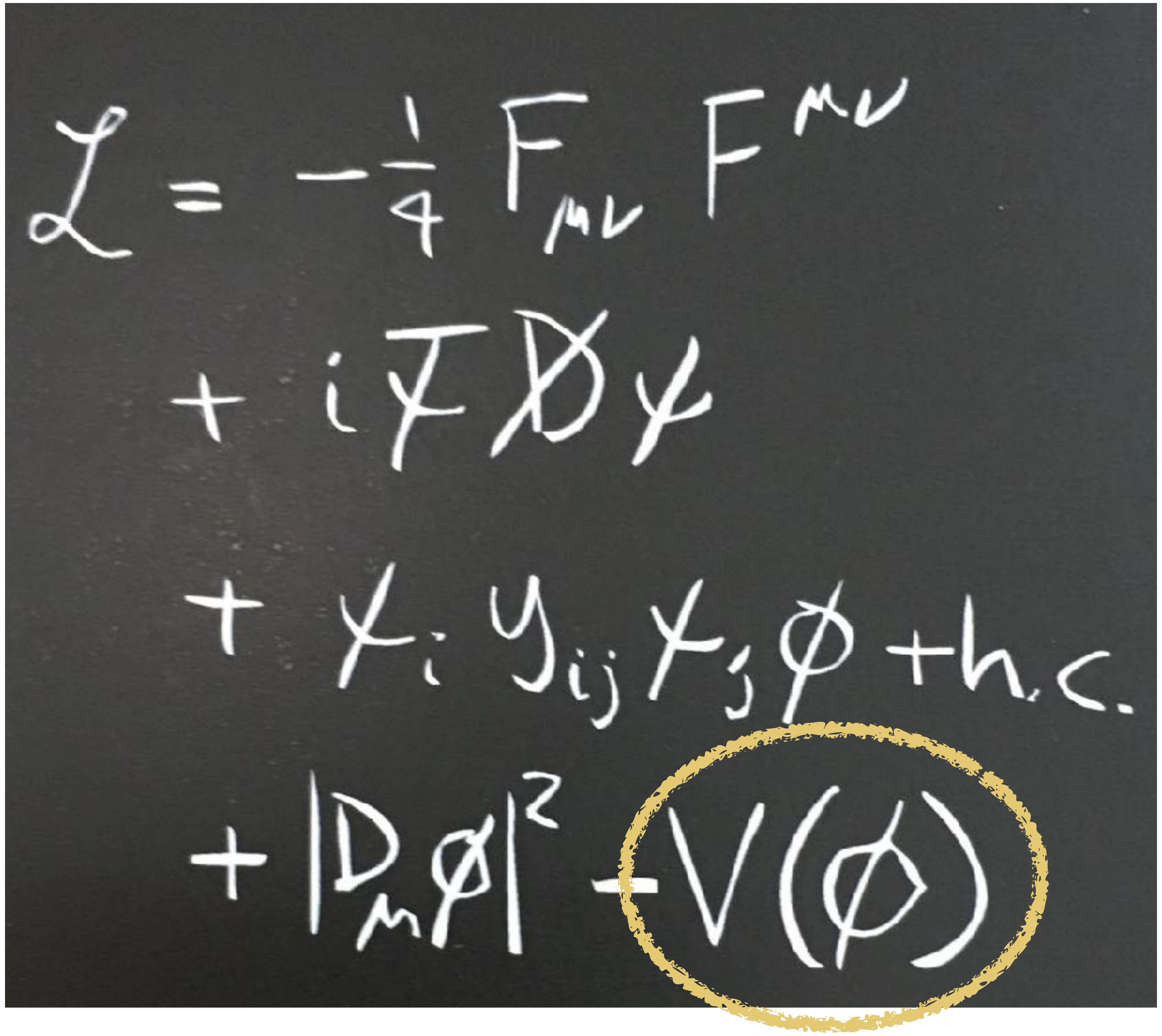


Run 3 and **HL potential:**

1. **Precision** measurements for **third family**
2. Discovery couplings to **second family** (μ & c)

HIGGS SELF INTERACTIONS THE MOST MYSTERIOUS?

HL-LHC first to see the triple-H coupling



We have seen the Higgs but

$$V(\phi) = -\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

is a “toy model”!

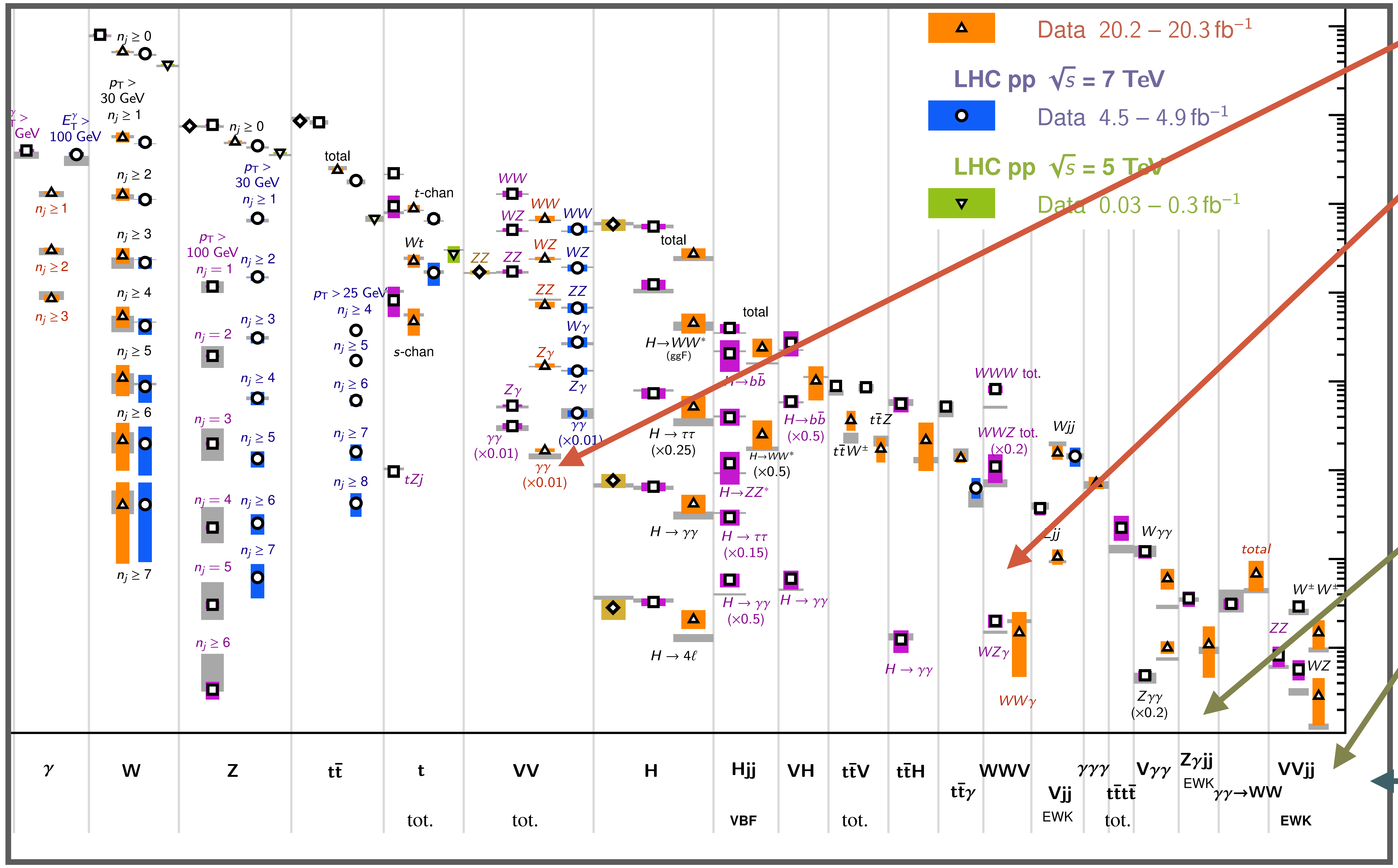
1. more minima?
2. more Higges?
3. microscopic model of SSB?
4. …

Higgs self coupling extremely difficult to measure.

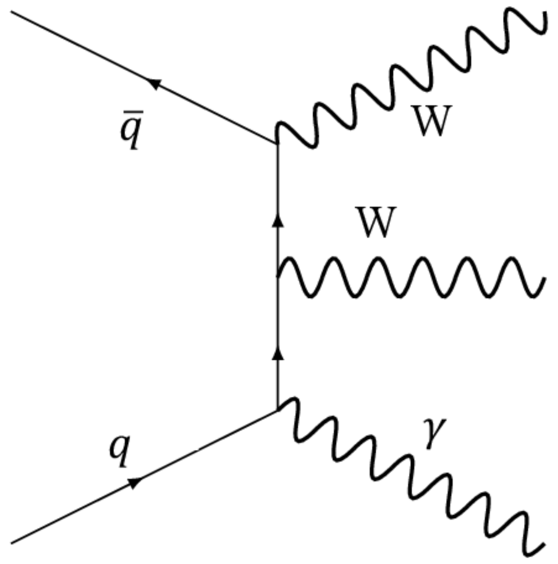
With 2018 estimates 4σ ATLAS+CMS

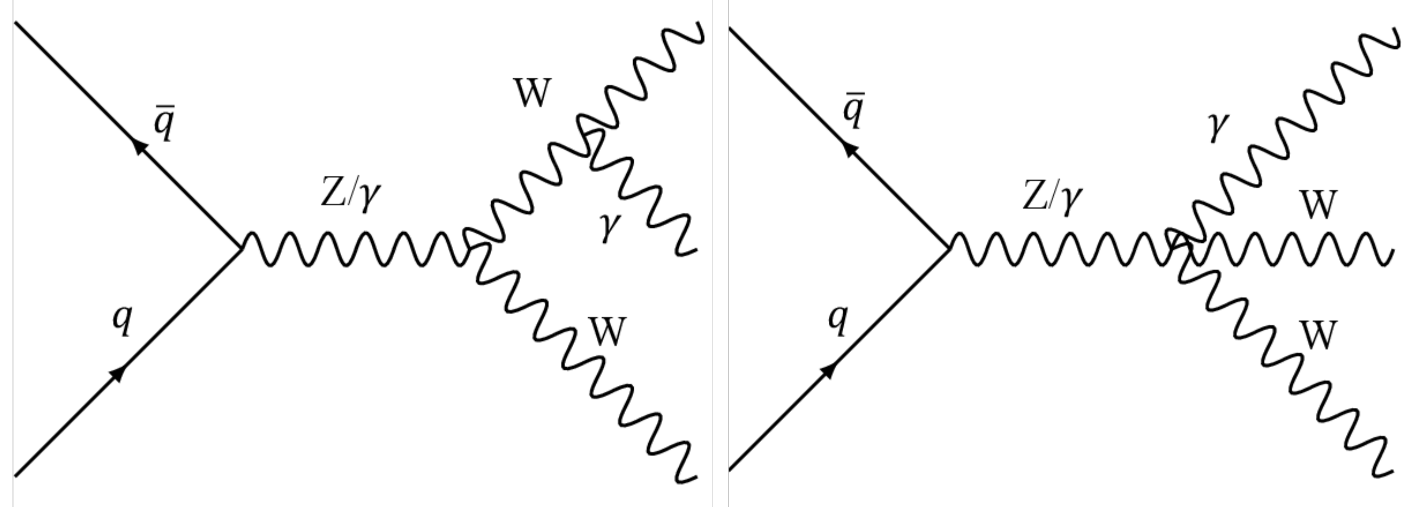
PROBING THE GAUGE SECTOR

Multiboson final states as probe of electroweak sector of SM

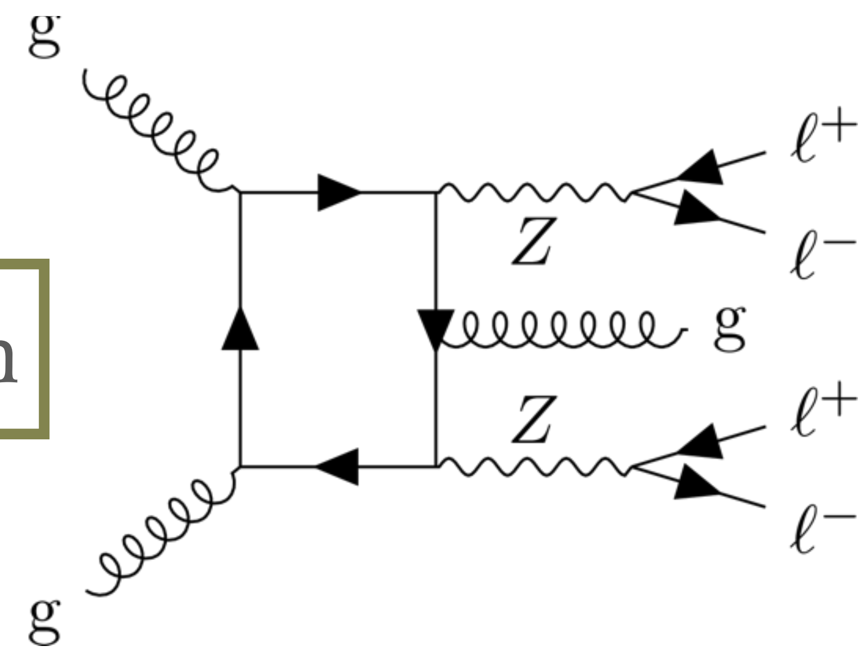


VV & VVV production

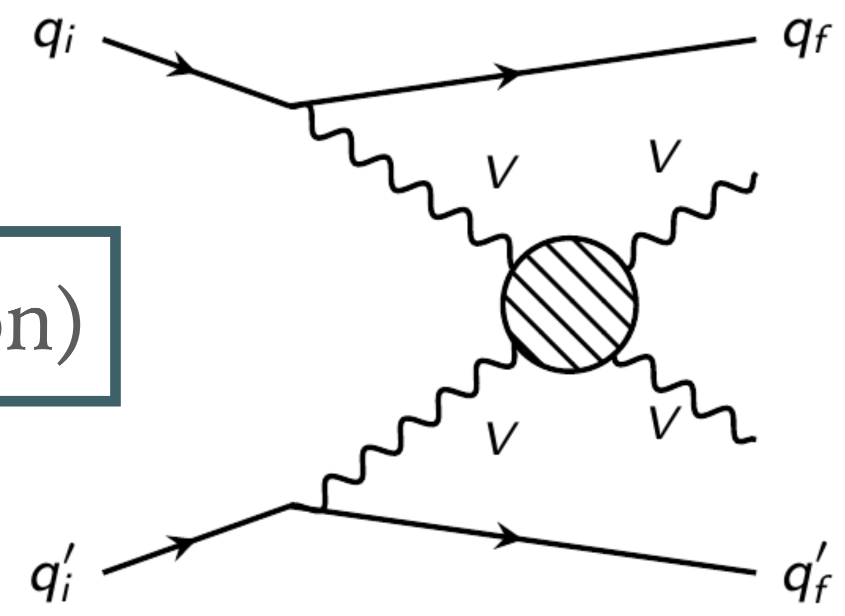




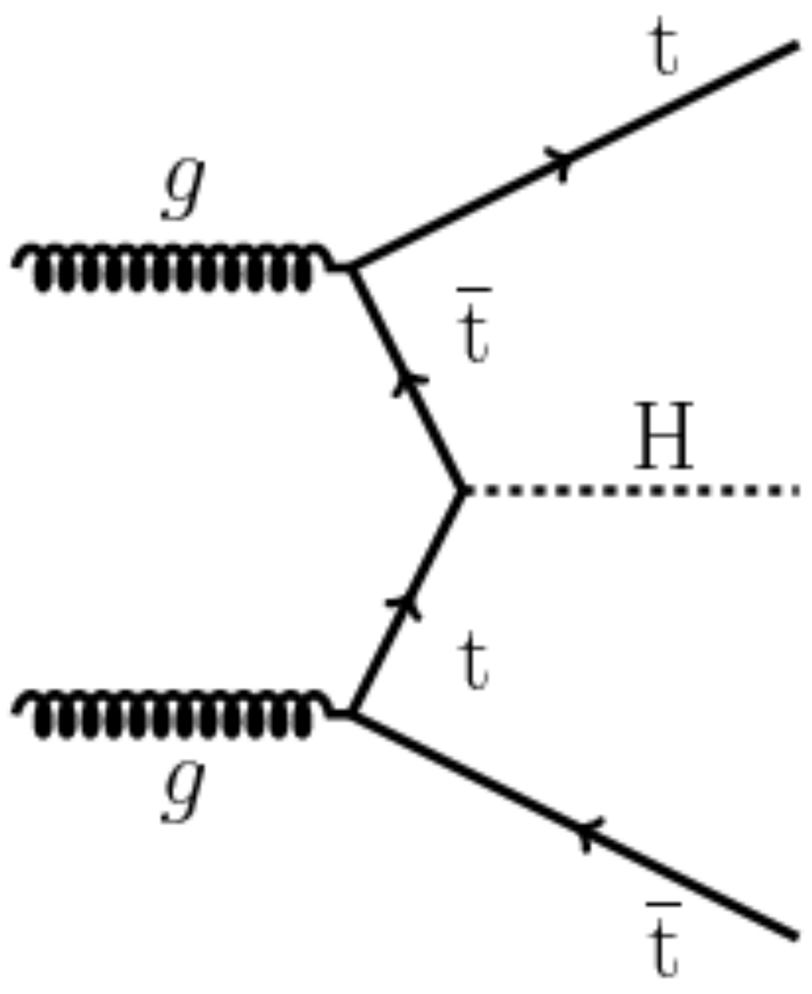
V & VV + jets production



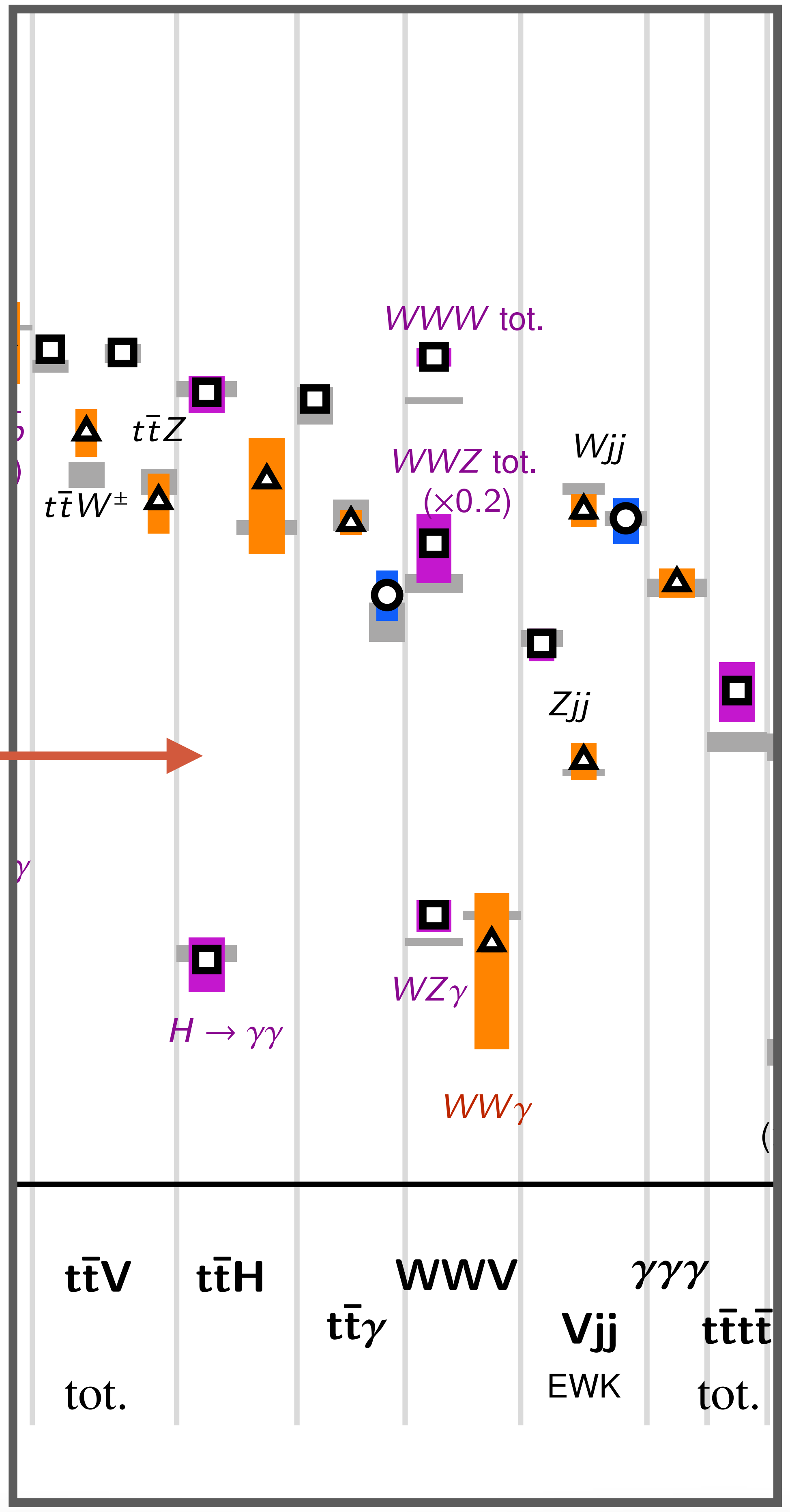
VBF (vector boson fusion)

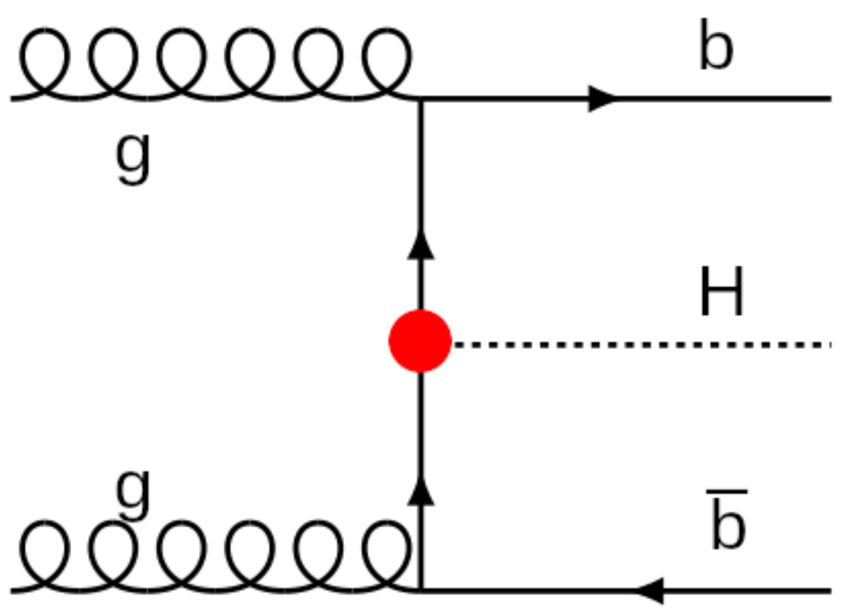


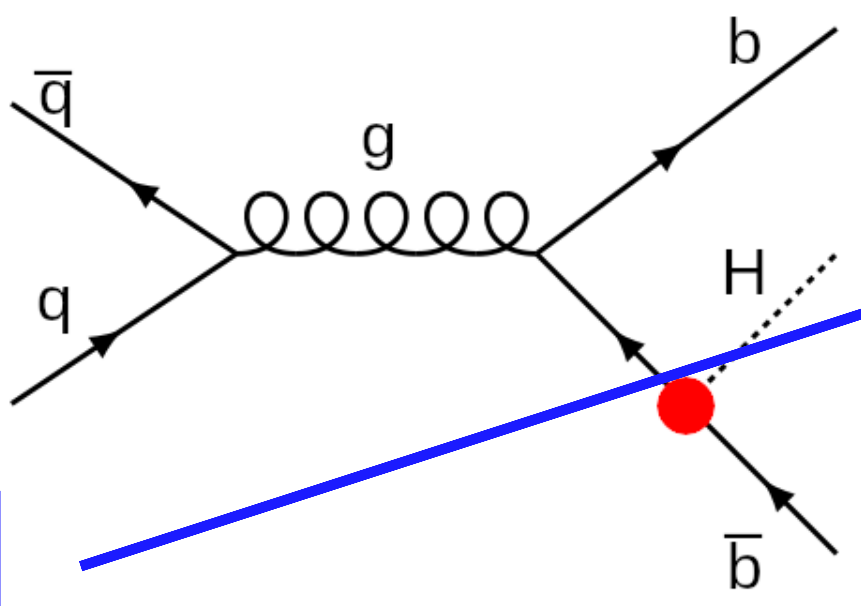
PROBING THE YUKAWA SECTOR

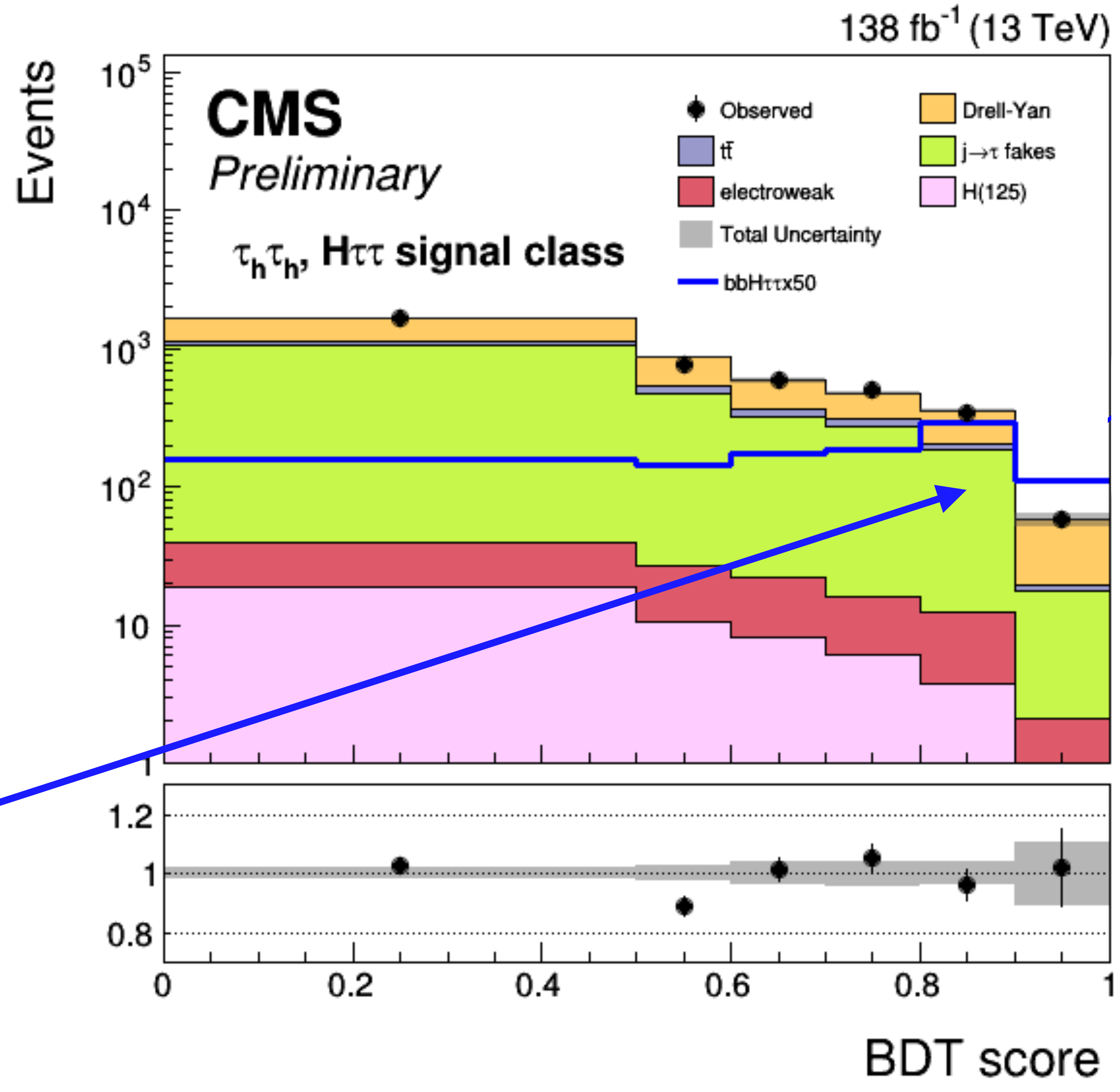


$t\bar{t}H$ production





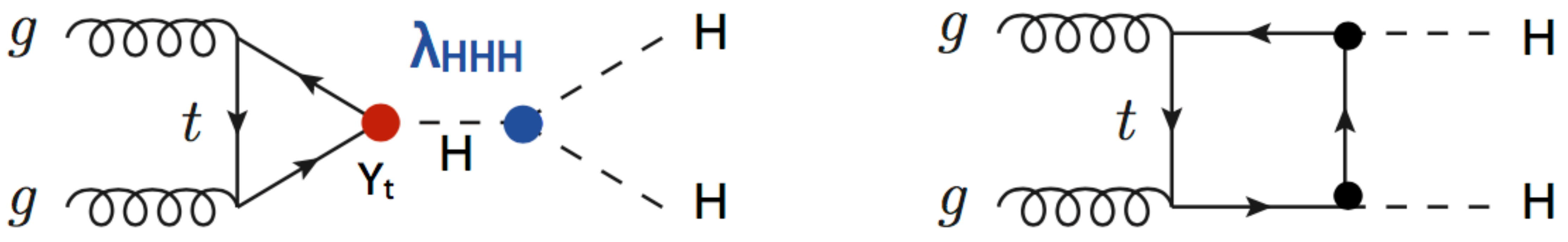




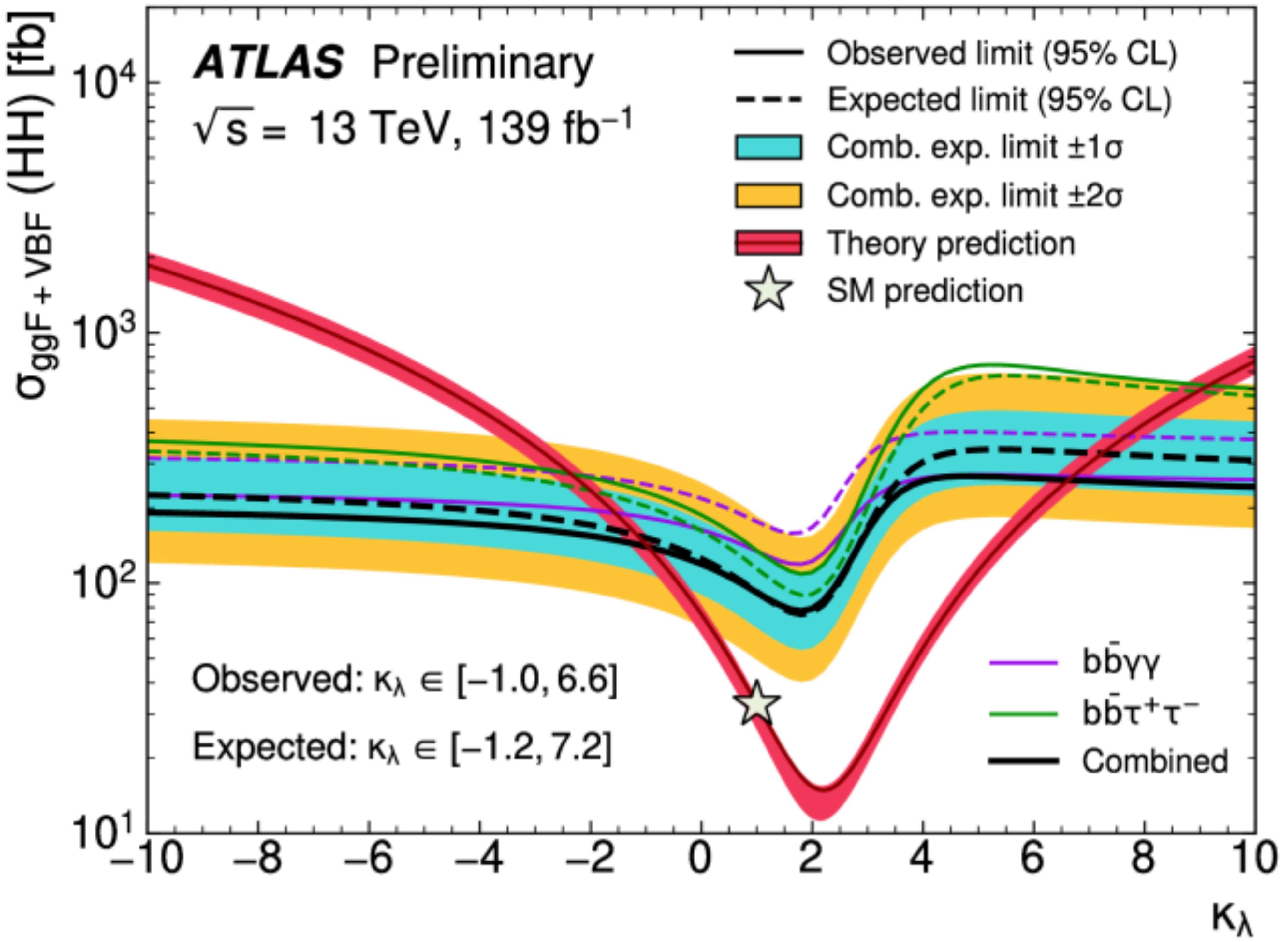
$b\bar{b}H$ production

PROBING H SELF INTERACTION THE MOST CHALLENGING?

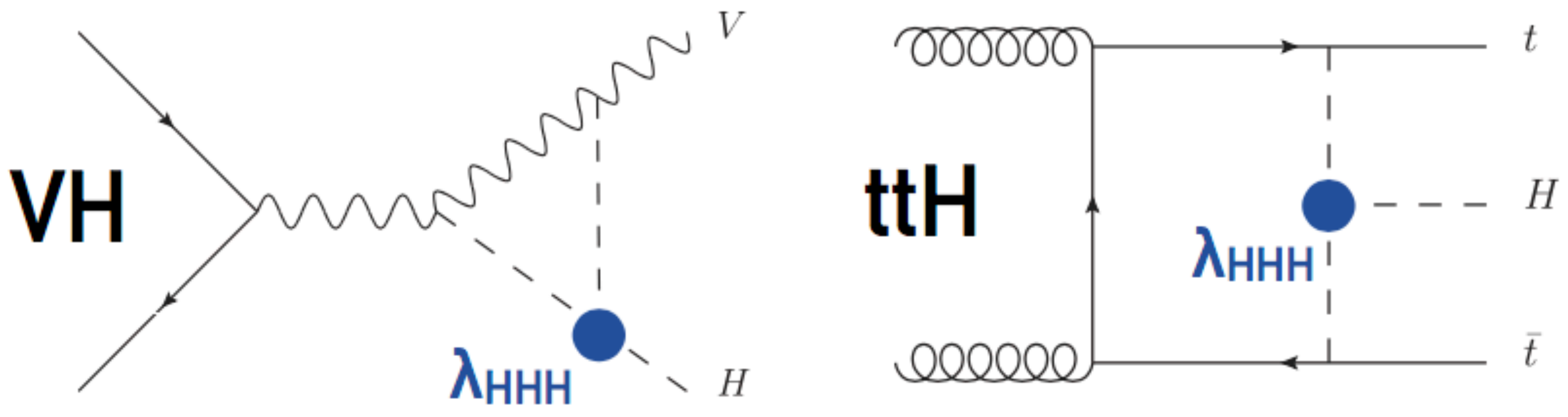
Direct sensitivity in HH production: Progress, but extremely hard to measure even at (HL-)LHC

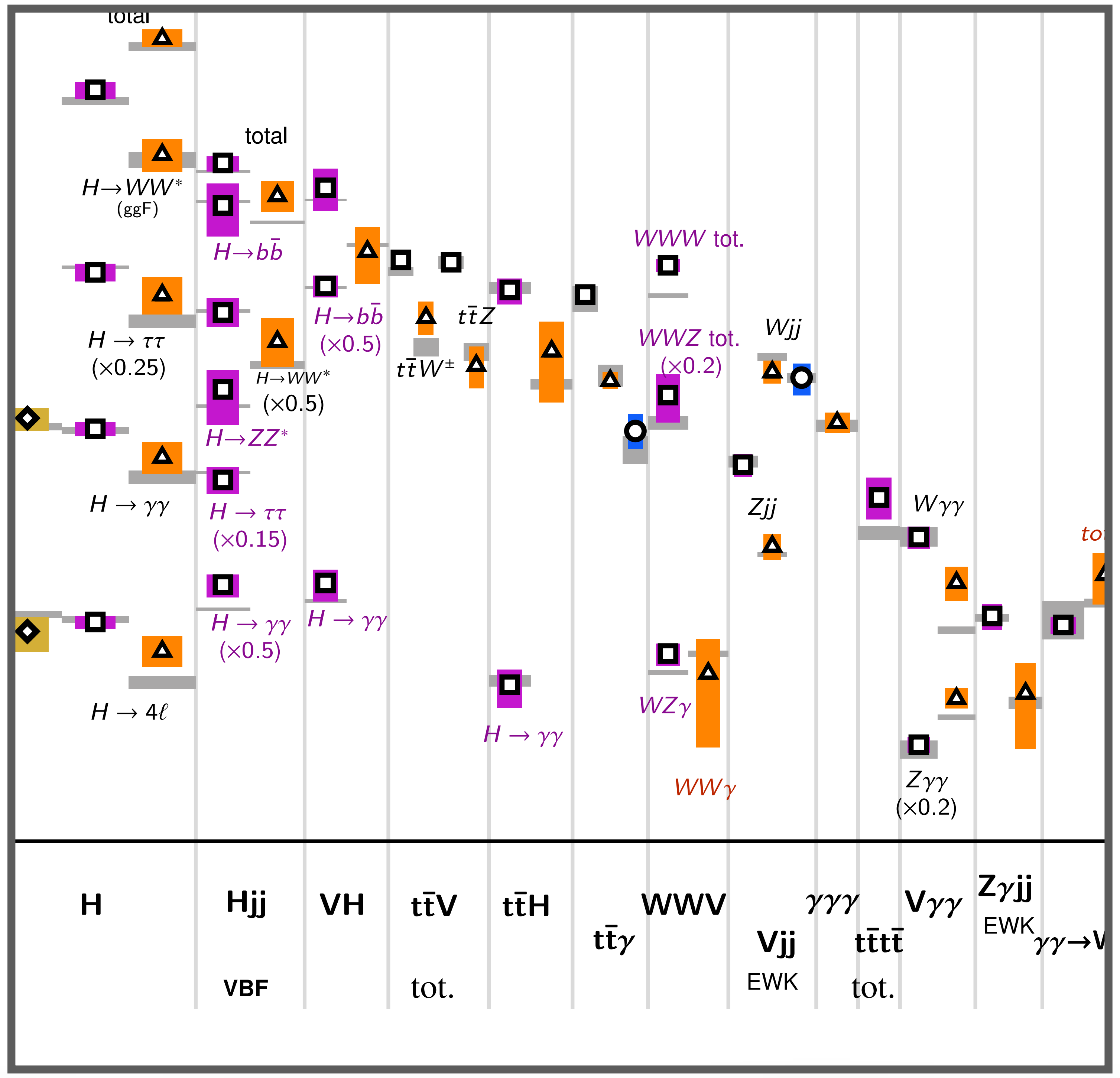


$b\bar{b}\tau\tau + b\bar{b}\gamma\gamma + b\bar{b}b\bar{b}$



Indirect sensitivity through precision studies!



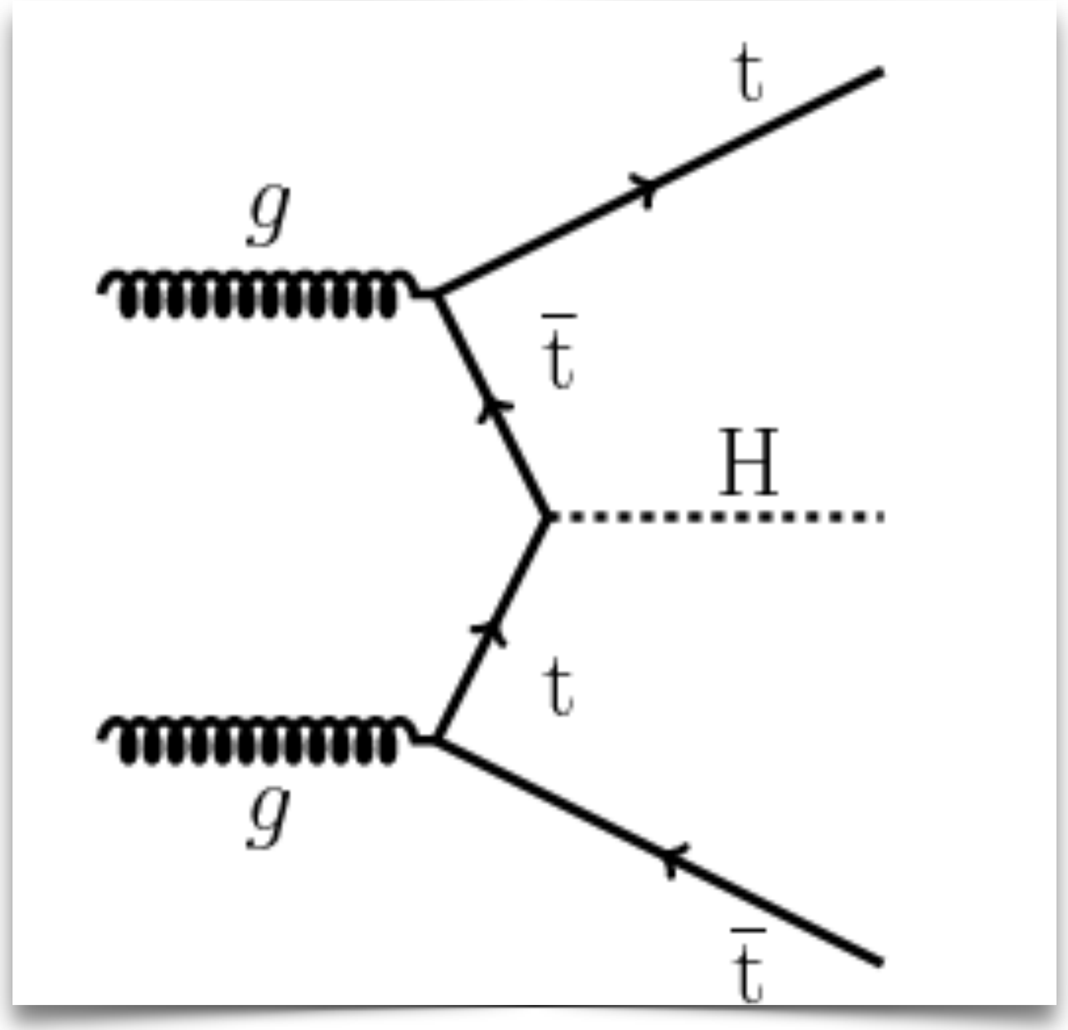


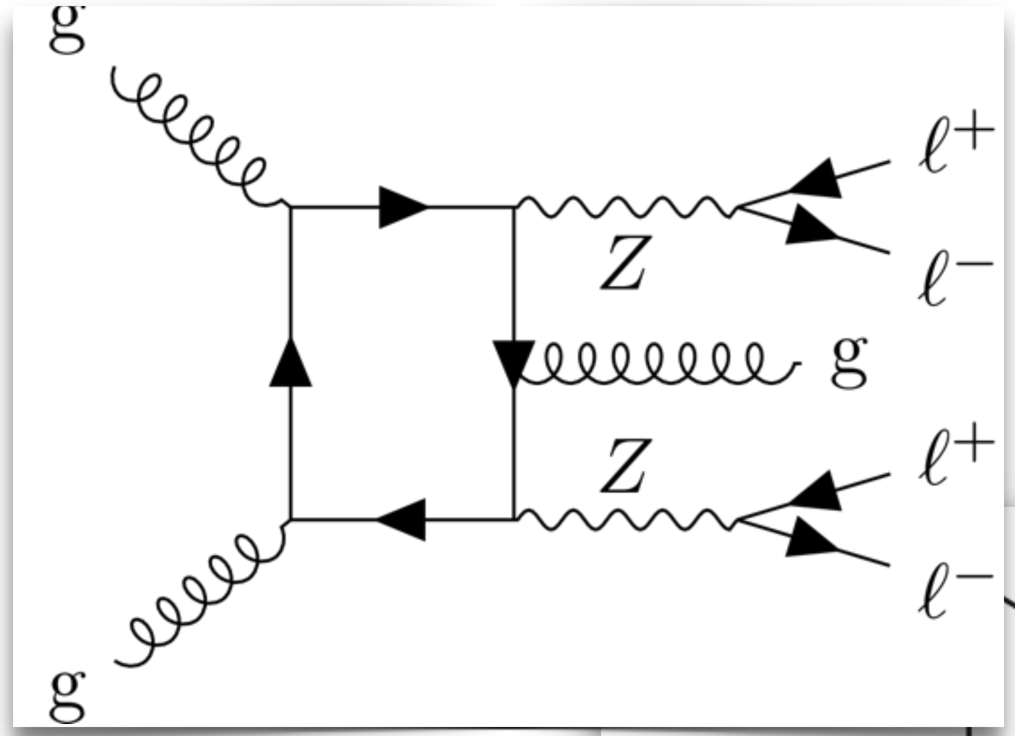
PRECISION STUDIES “OPPORTUNITIES” ALL OVER

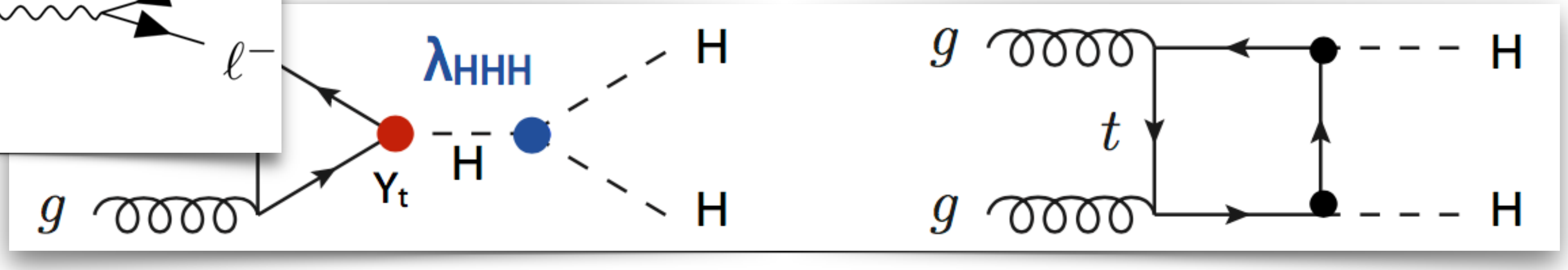
What do they have in common?

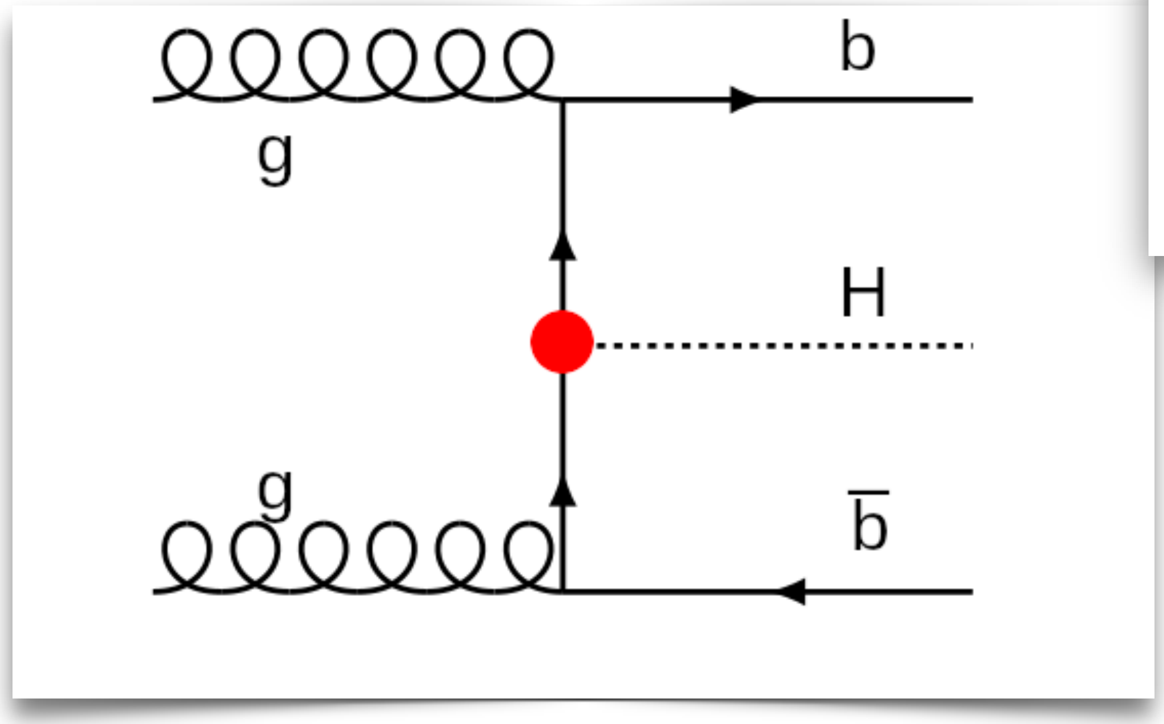
complex final states, QCD & EW corrections

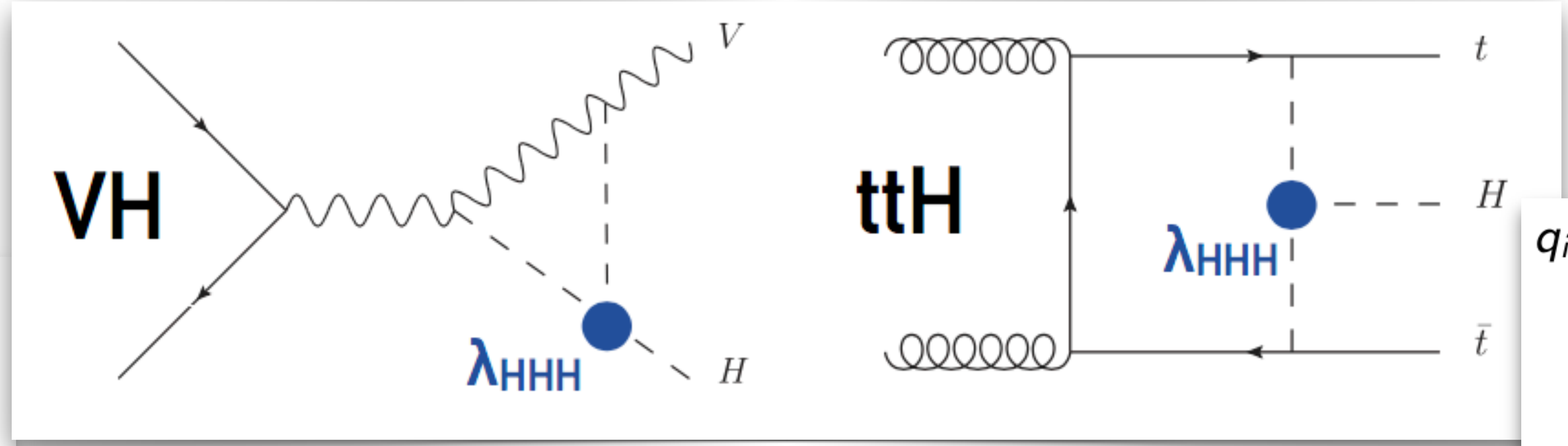
massive external and virtual states

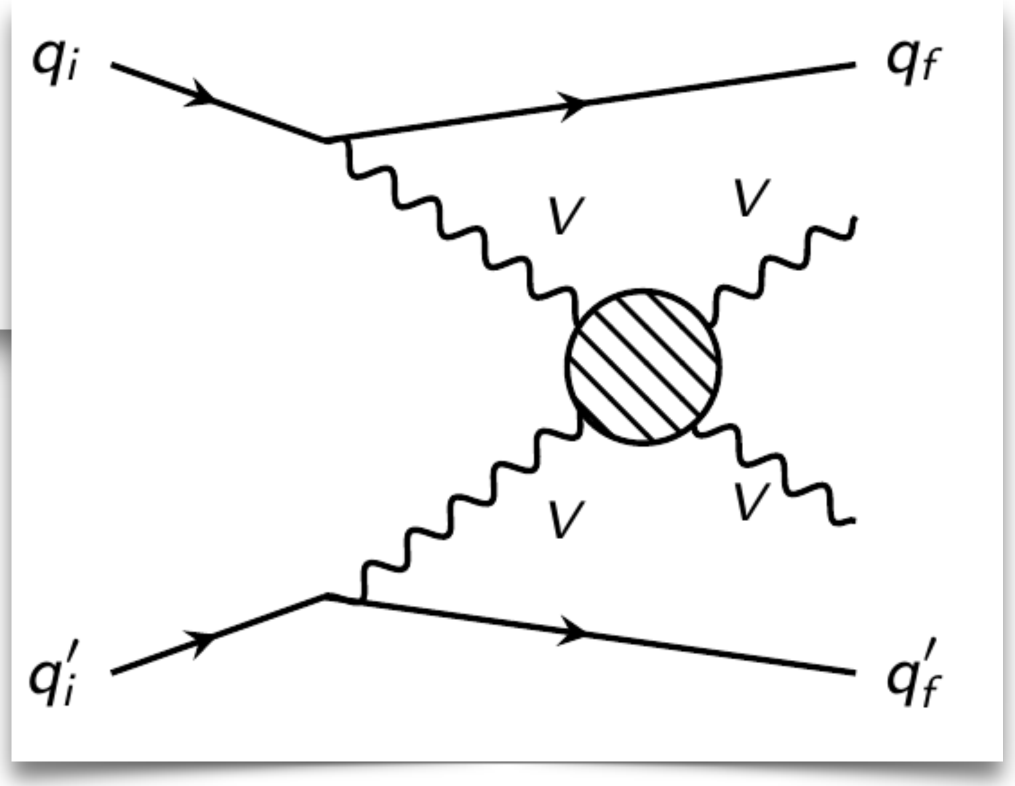






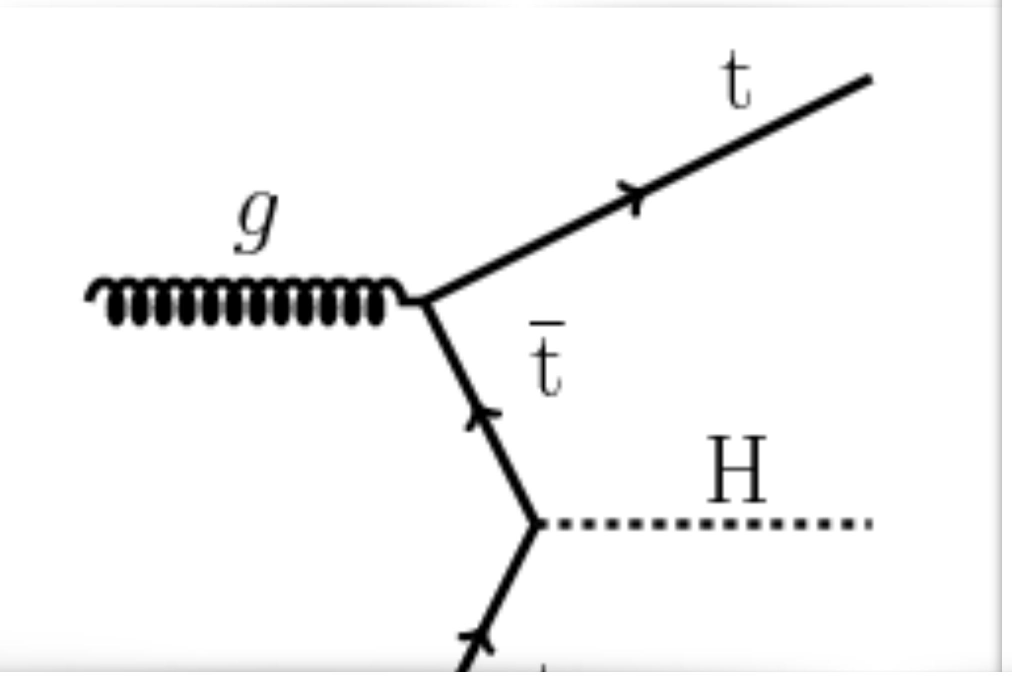


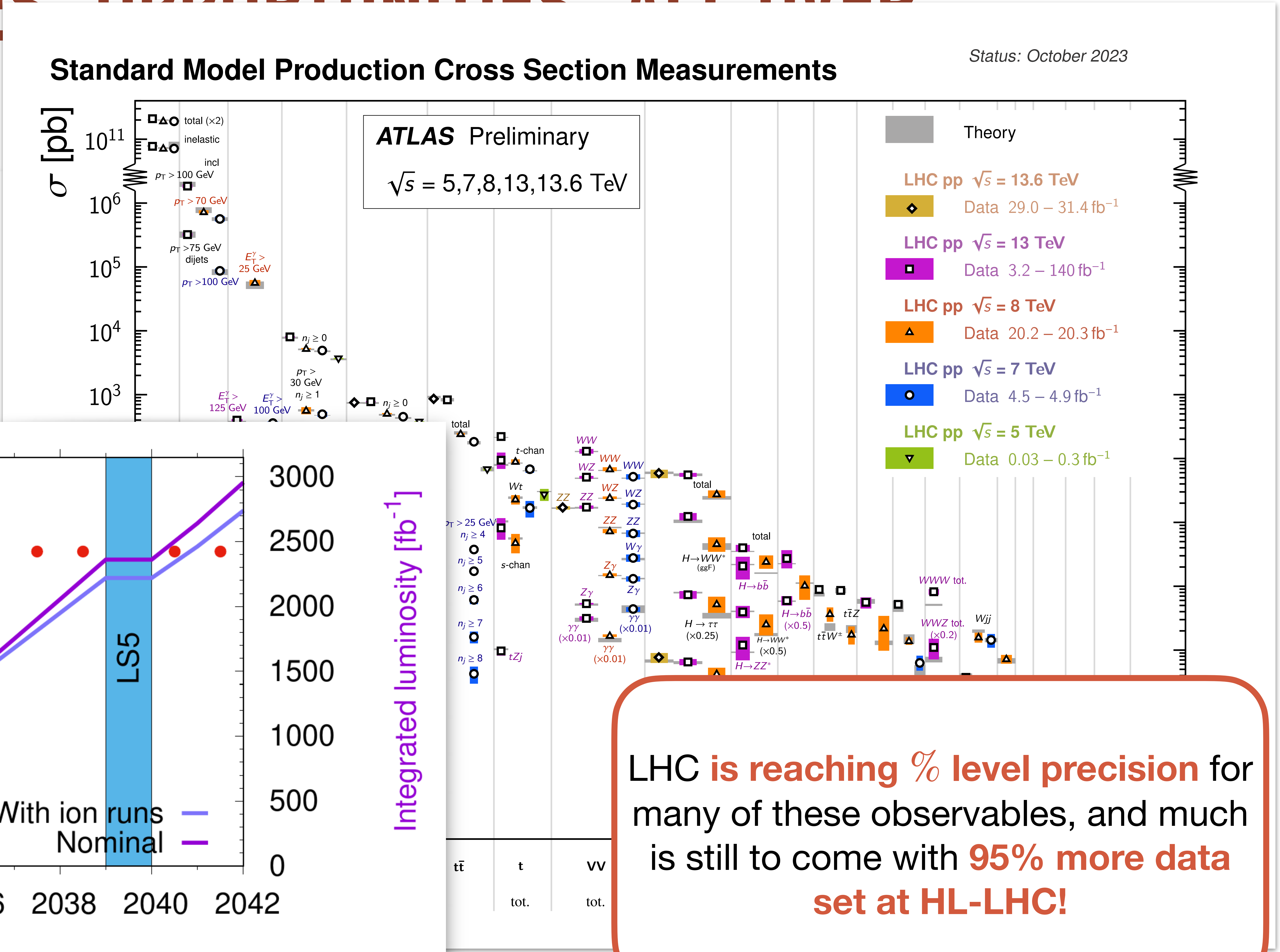


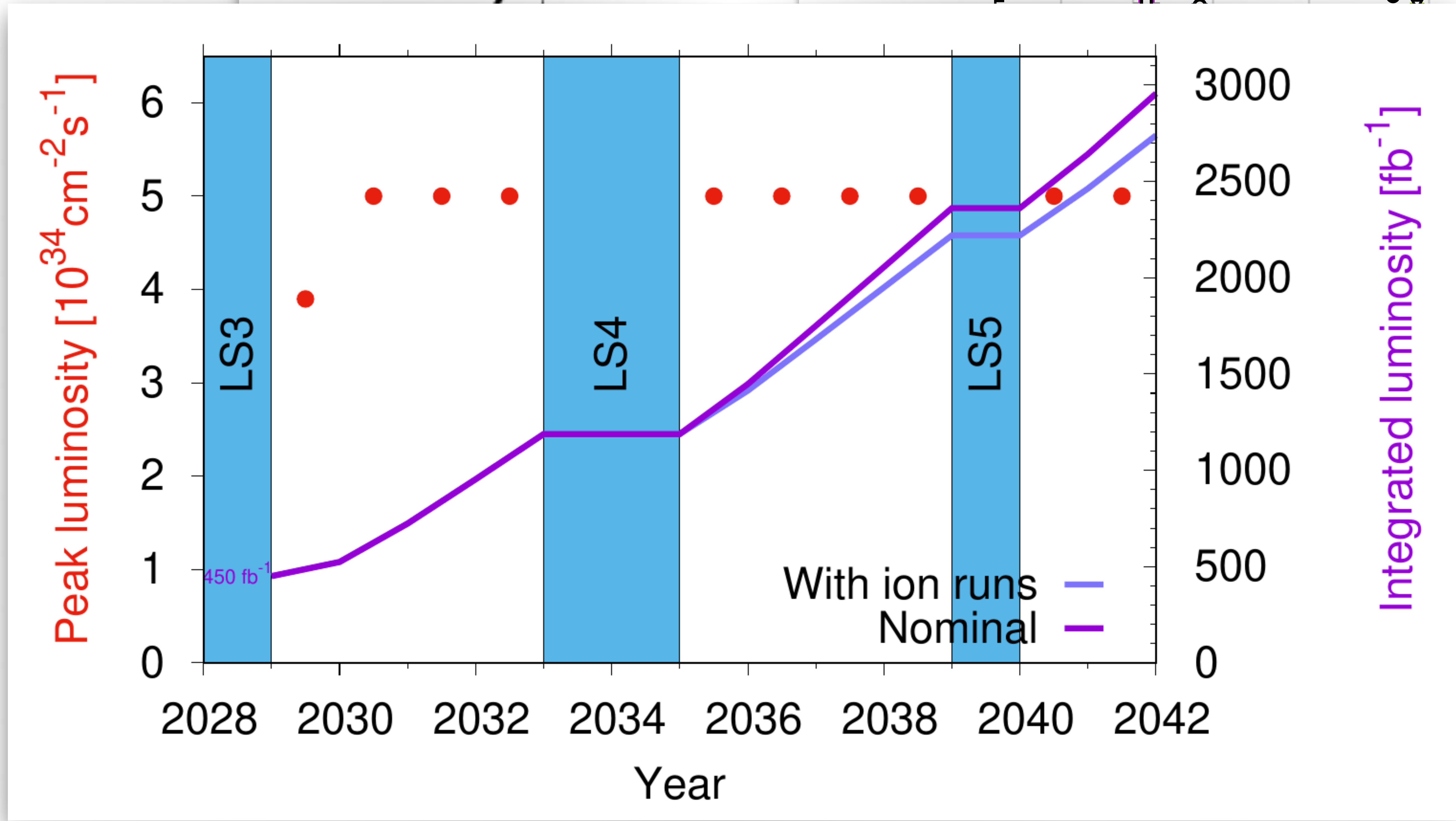


… and many others …

PRECISION STUDIES “OPPORTUNITIES” ALL OVER



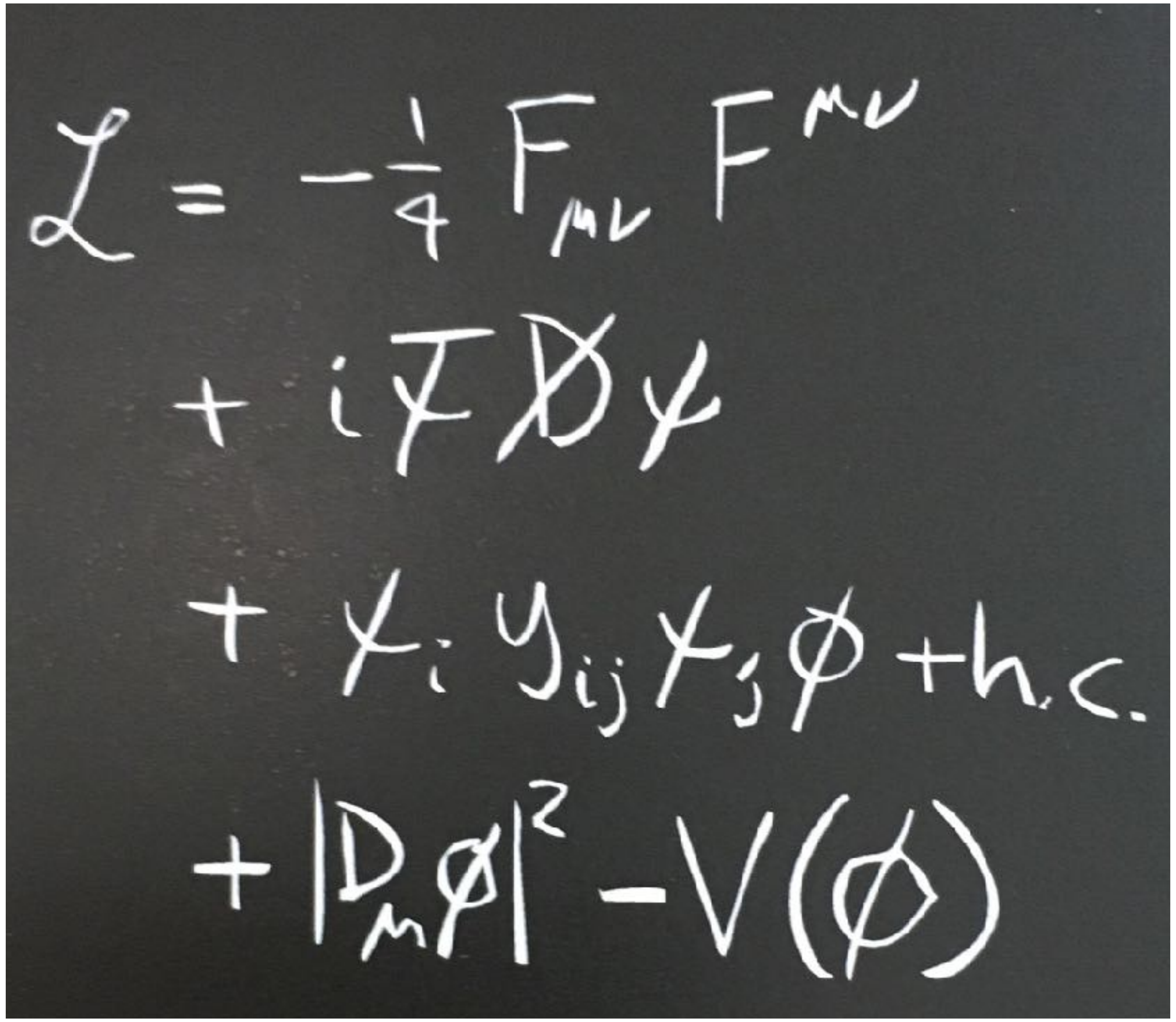




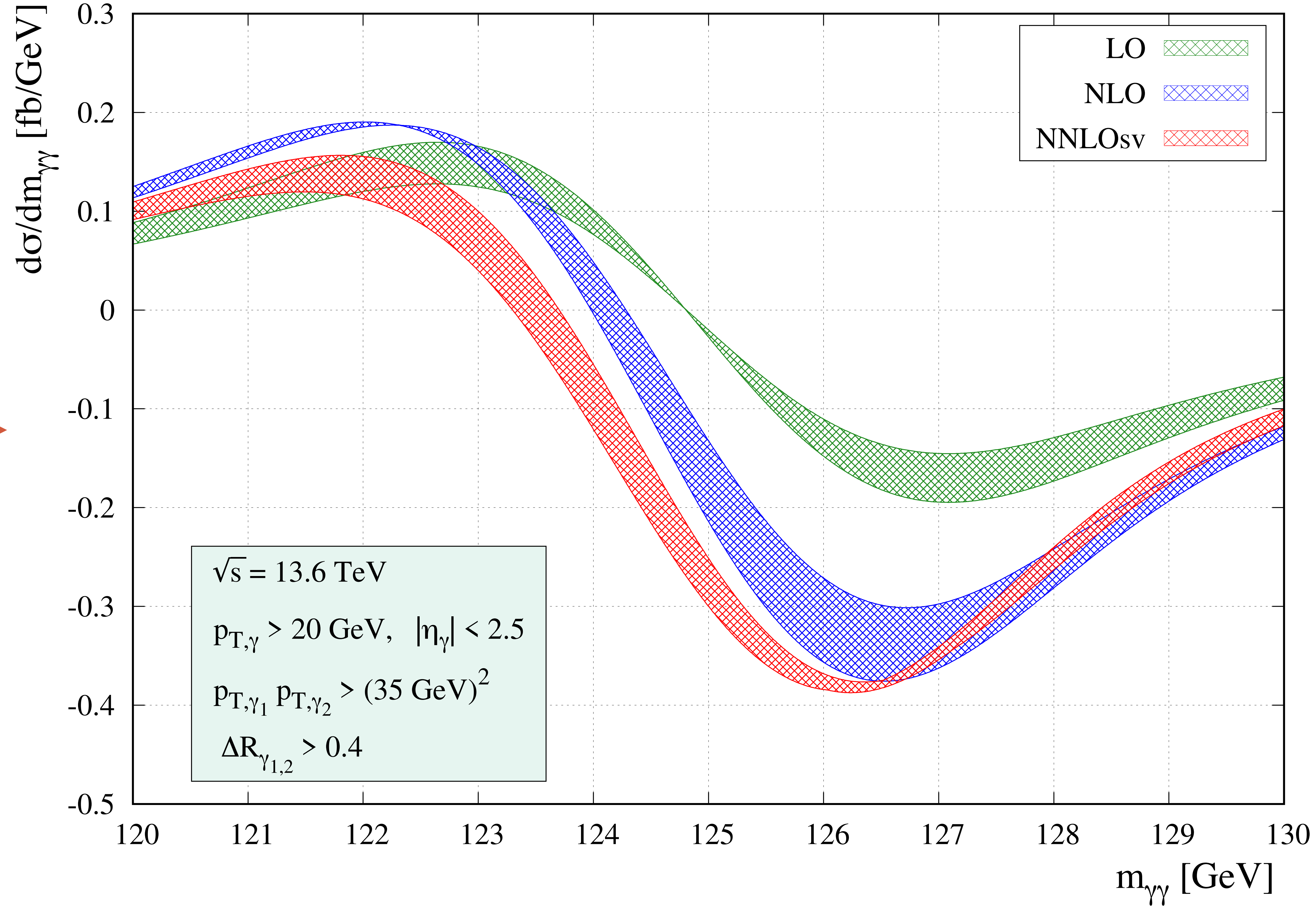
LHC **is reaching % level precision** for many of these observables, and much is still to come with **95% more data set at HL-LHC!**

% PRECISION, HOW DO WE GET THERE?

FROM THEORY TO THEORY PREDICTIONS IT'S A LONG WAY!



?

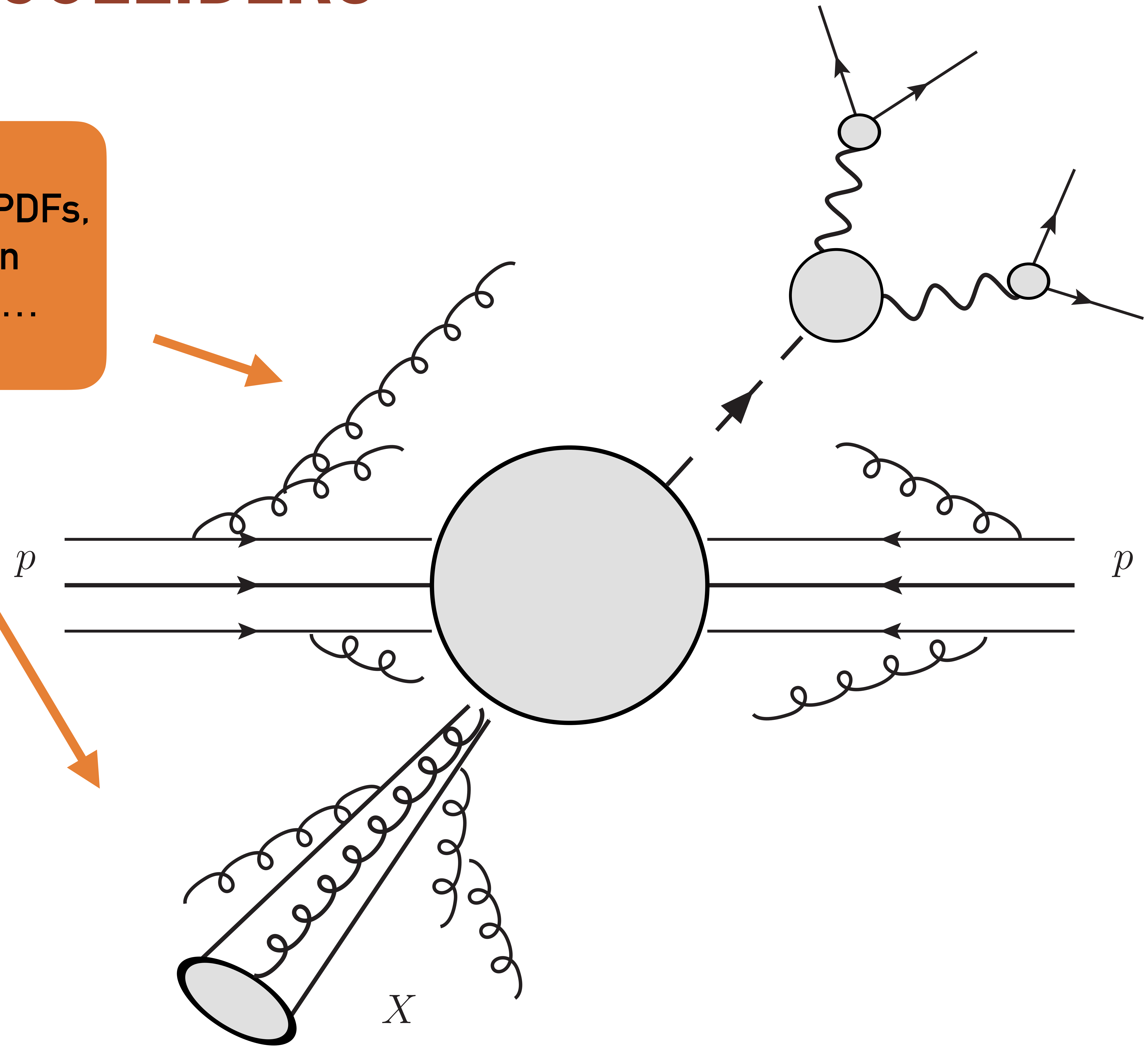


Signal to BKG interference for $gg \to H \to \gamma\gamma$

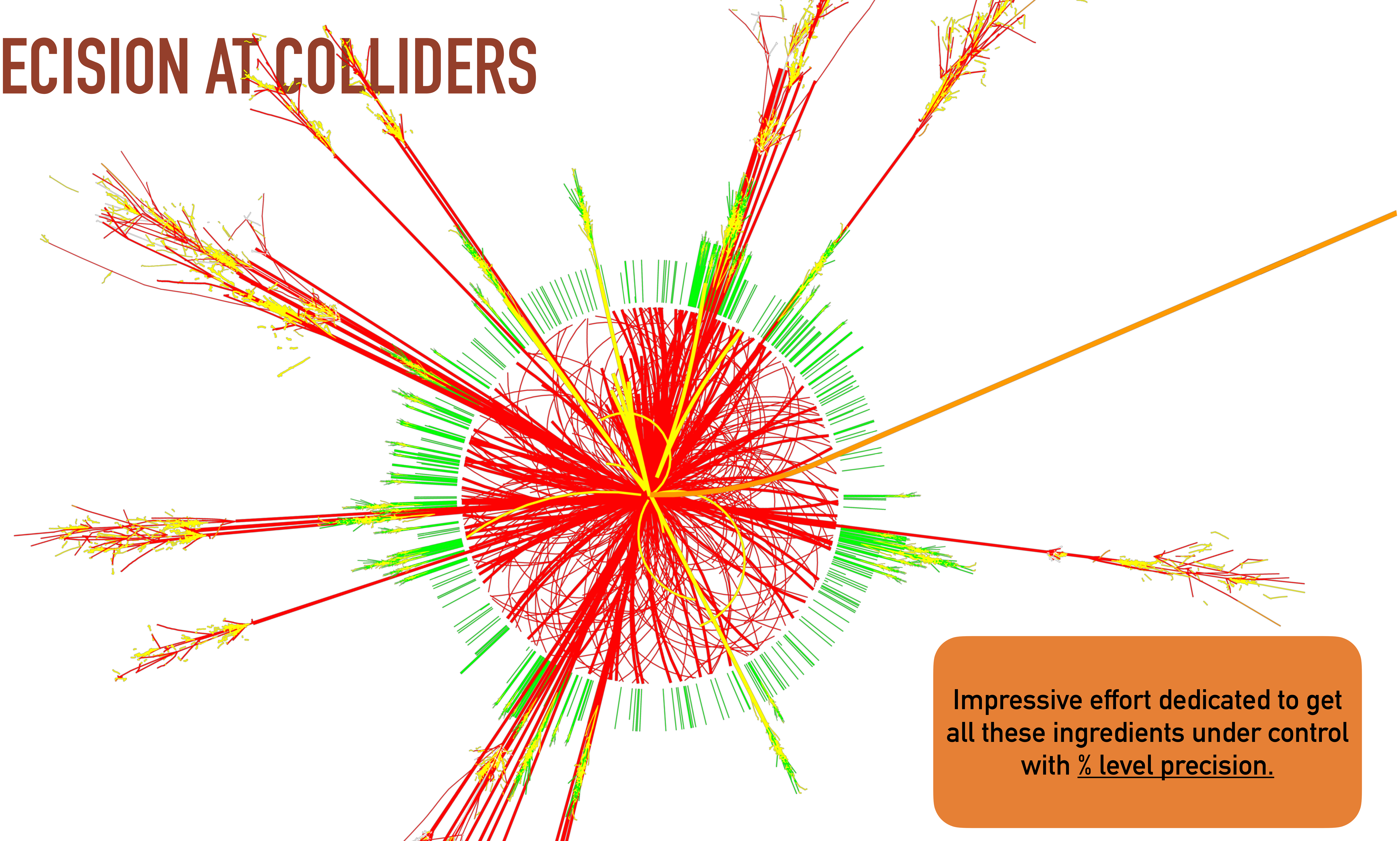
[Bargiela, Buccioni, Caola, Devoto, Manteuffel, Tancredi '22]

PRECISION AT COLLIDERS

“soft & collinear physics”: PDFs, jet substructure, parton showers, hadronization…



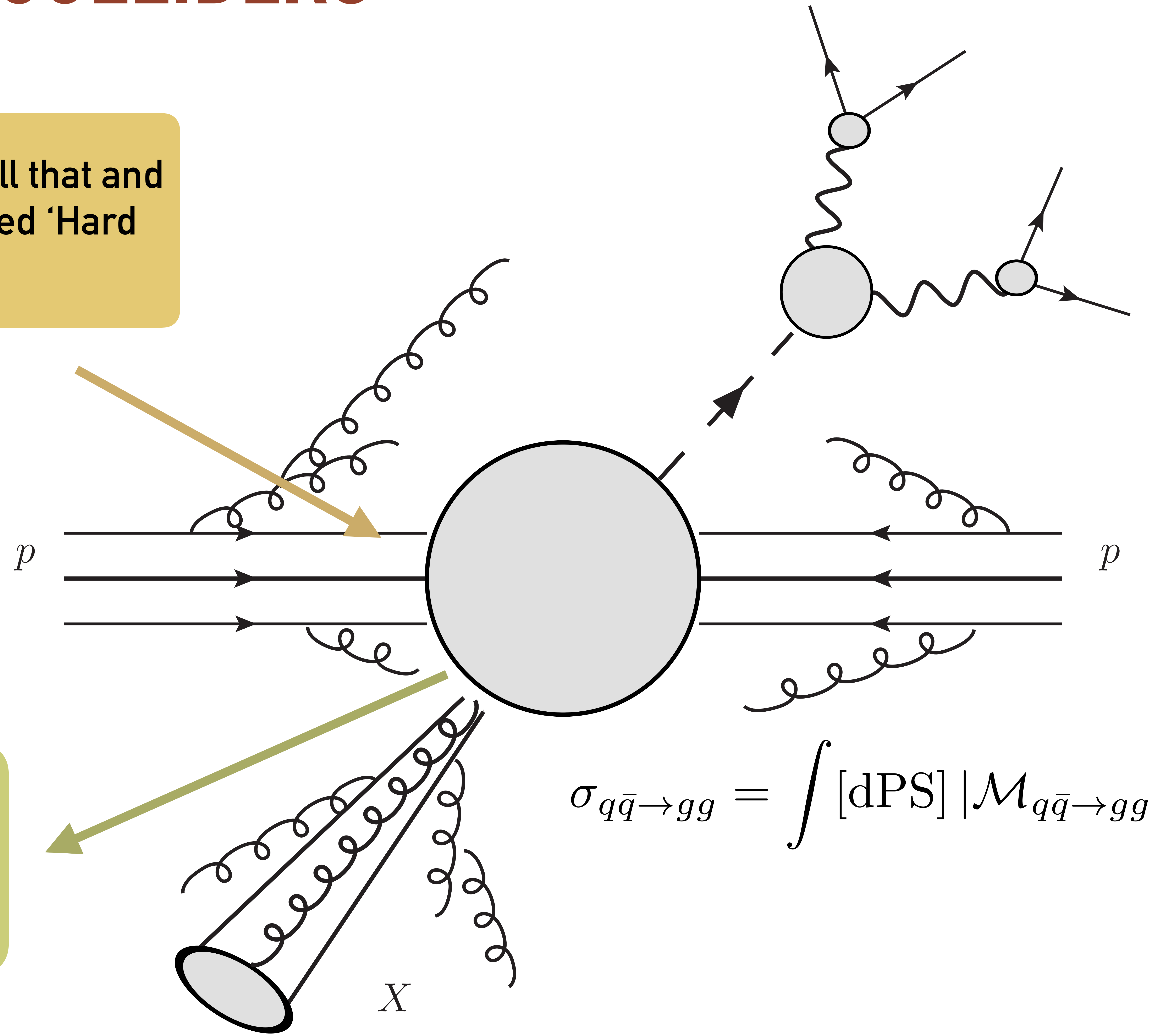
PRECISION AT COLLIDERS



Impressive effort dedicated to get all these ingredients under control with <u>% level precision.</u>

PRECISION AT COLLIDERS

For now, we ignore all that and zoom in the so-called 'Hard Scattering'



$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, |\mathcal{M}_{q\bar{q}\to gg}|^2$$

Building blocks are "Scattering Amplitudes"

% precision possible?!

HARD SCATTERING

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, |\mathcal{M}_{q\bar{q}\to gg}|^2$$

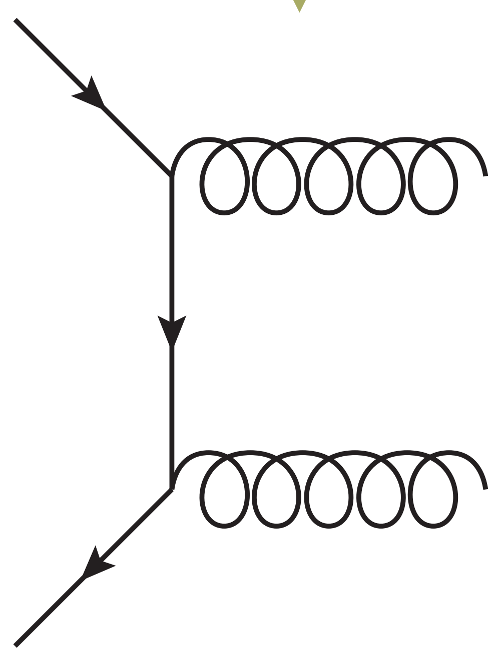
small "coupling constant" ~ 0.1

$$|\mathcal{M}_{q\bar{q}\to gg}|^2 = |\mathcal{M}^{LO}_{q\bar{q}\to gg}|^2 + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{NLO}_{q\bar{q}\to gg}|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}|^2 + \ldots$$

HARD SCATTERING

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

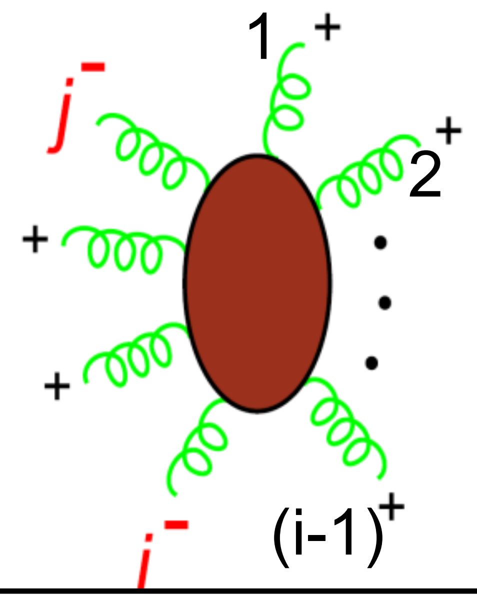
$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$



~ O(100%-50%) precision

$$A_n^{ij,\ \mathrm{MHV}} = A_n^{\mathrm{tree}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$$

=



$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

Parke-Taylor formula (1986)

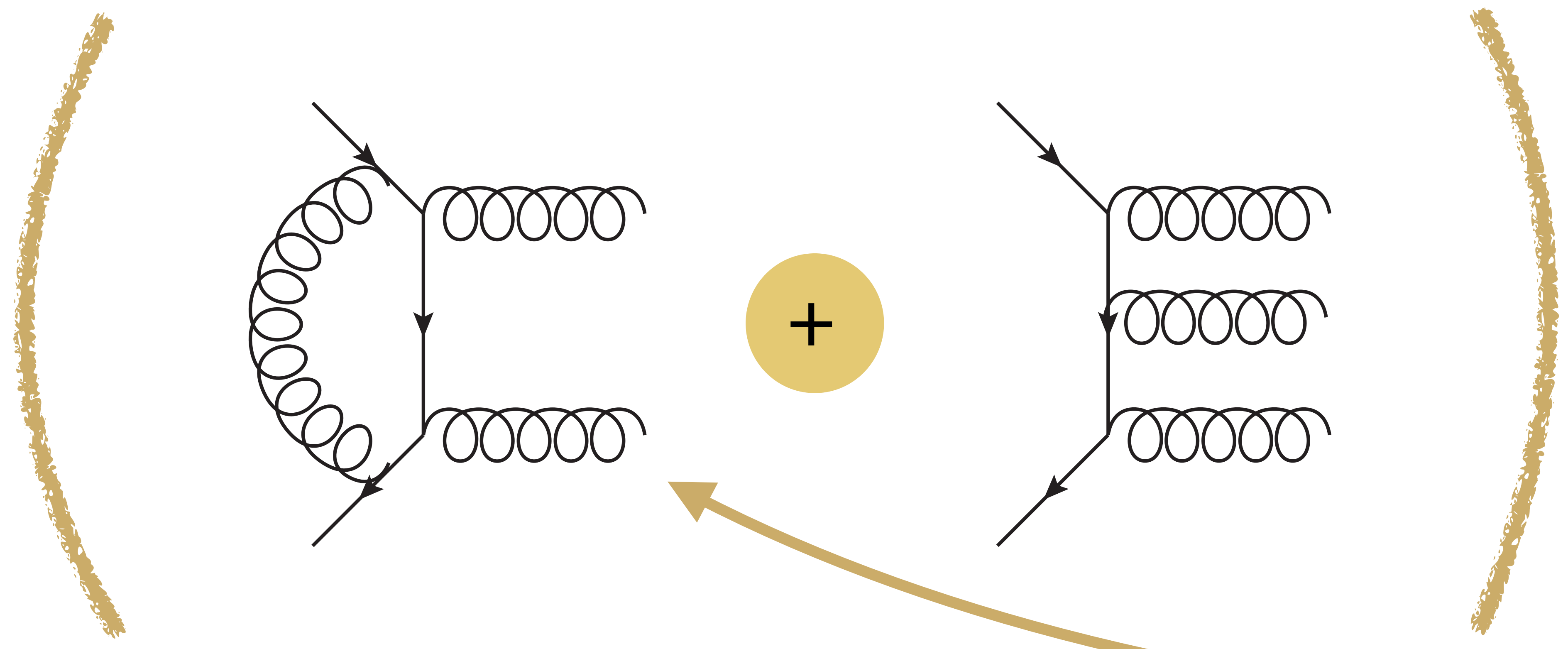
[slide from L. Dixon]

Tree-level Amplitudes

HARD SCATTERING

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \, |\mathcal{M}_{q\bar{q}\to gg}|^2$$

$$|\mathcal{M}_{q\bar{q}\to gg}|^2 = |\mathcal{M}^{LO}_{q\bar{q}\to gg}|^2 + \left(\frac{\alpha_s}{2\pi}\right) |\mathcal{M}^{NLO}_{q\bar{q}\to gg}|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2 |\mathcal{M}^{NNLO}_{q\bar{q}\to gg}|^2 + \ldots$$



Virtual

Real

~ O(30%-10%) precision

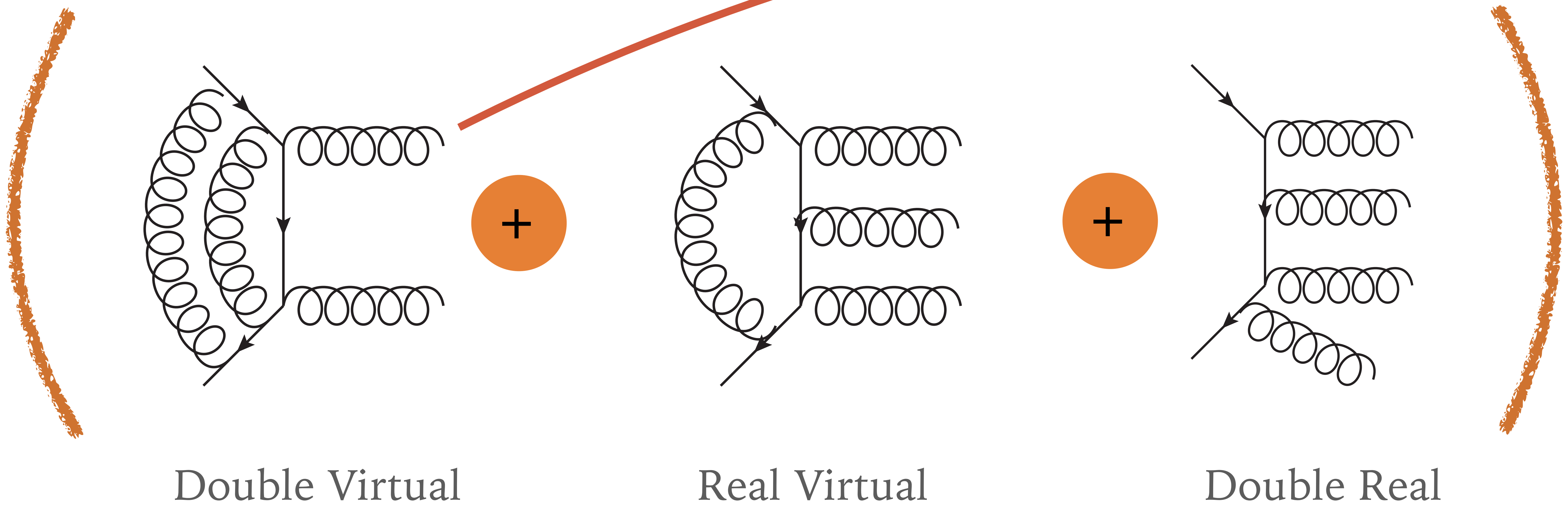
One-loop Amplitudes

HARD SCATTERING

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2 \left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$

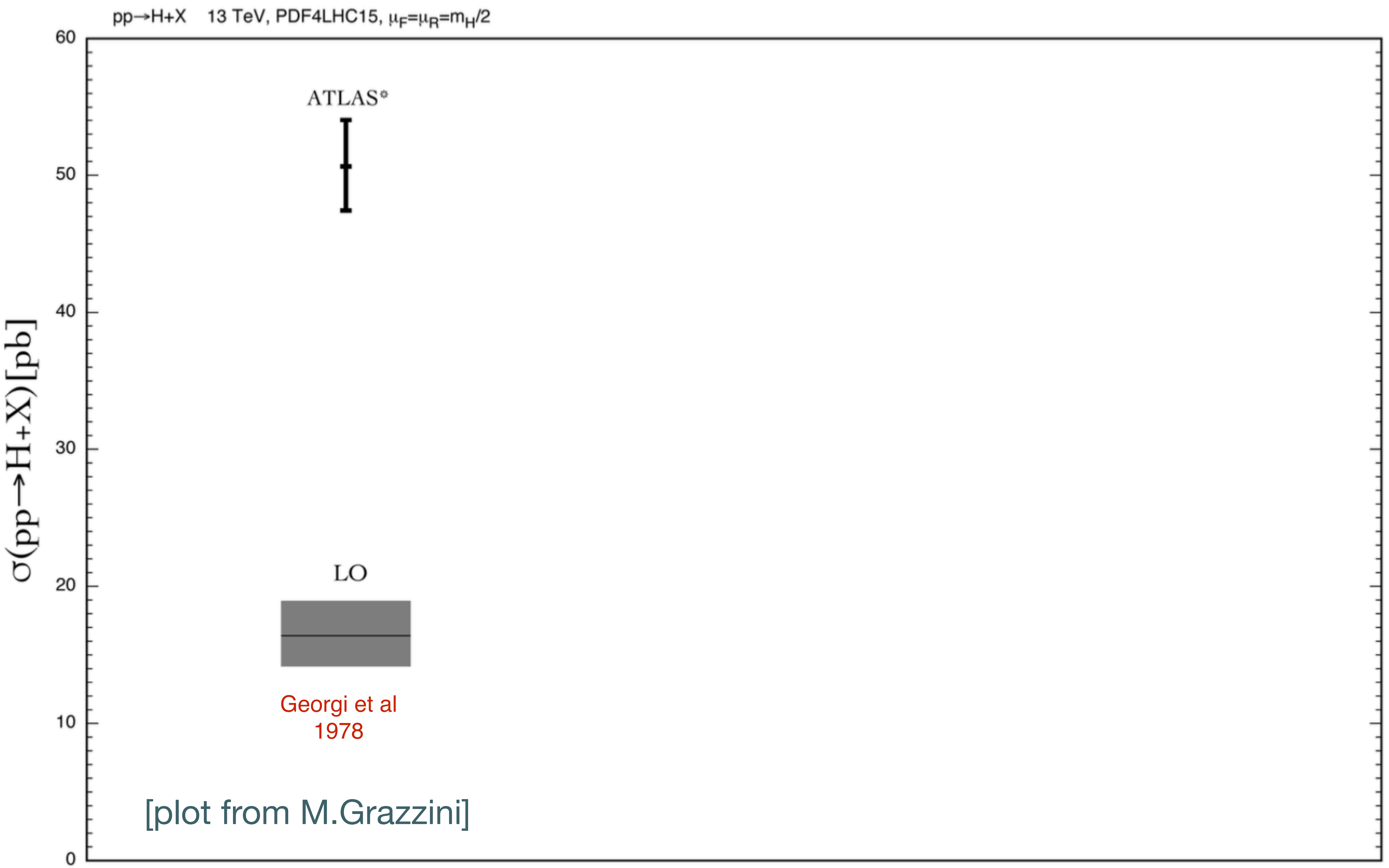
Two-loop amplitudes



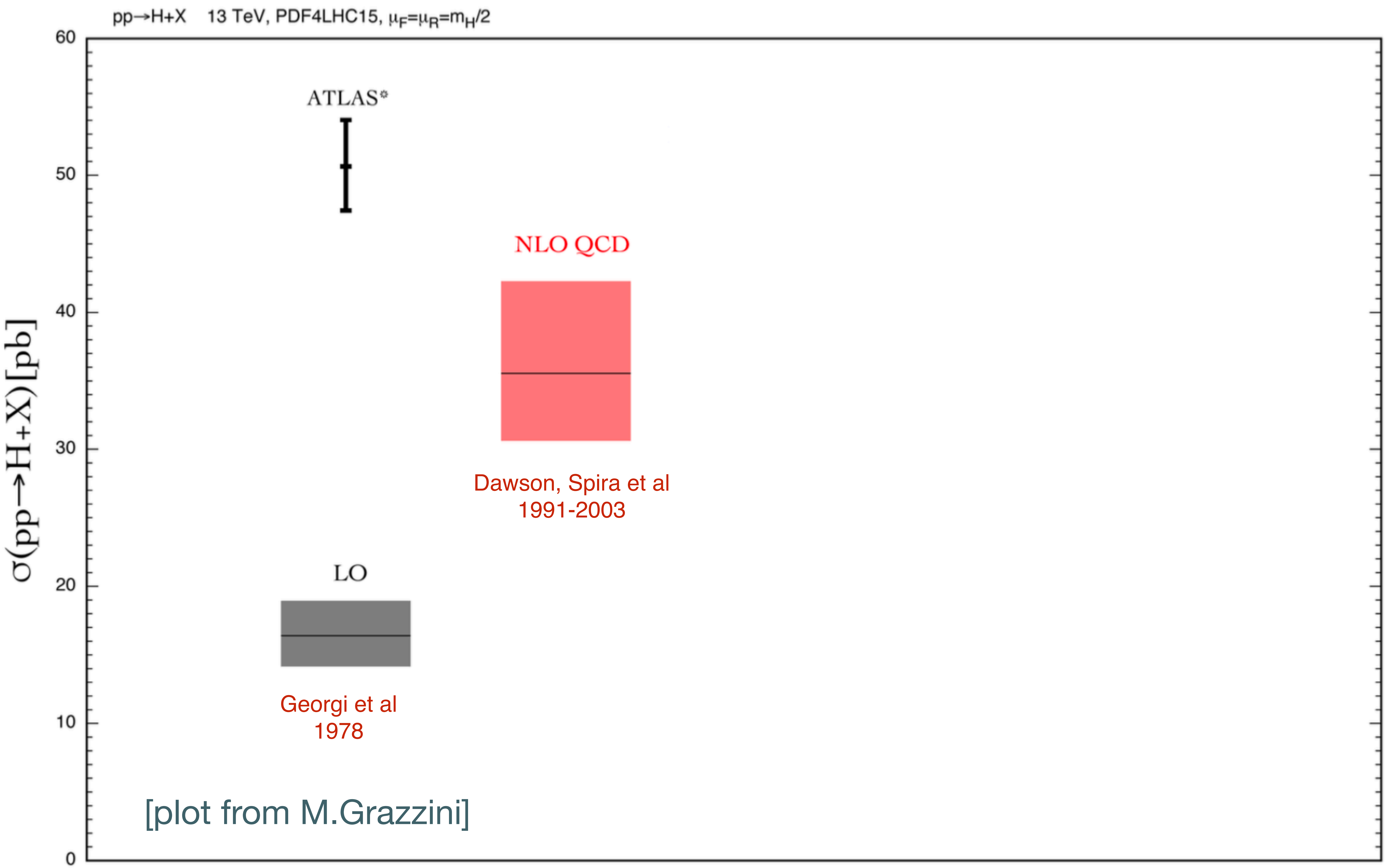
~ O(5%) precision

Often not enough!

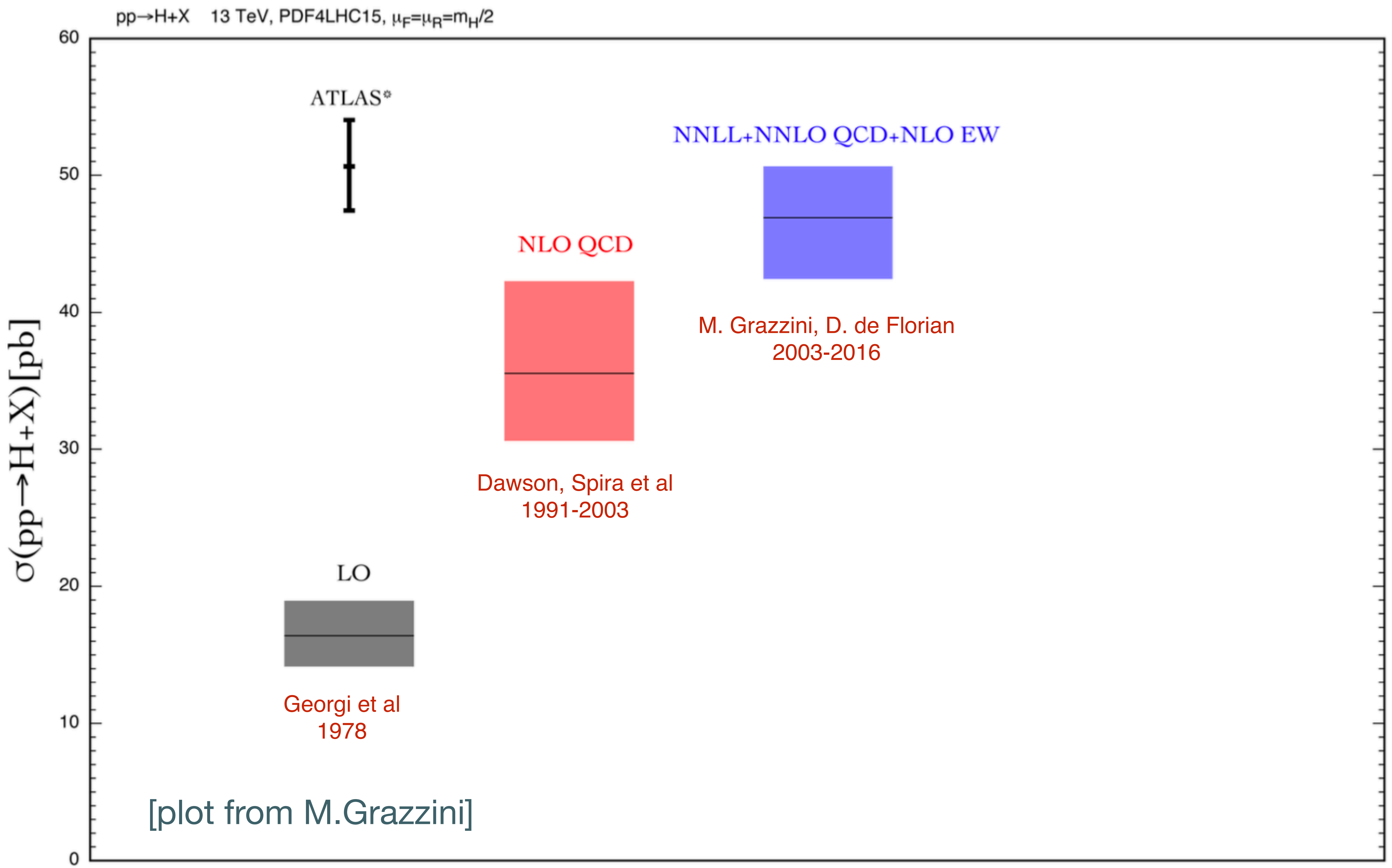
THE NEED OF PRECISION: TOWARDS THE % LEVEL



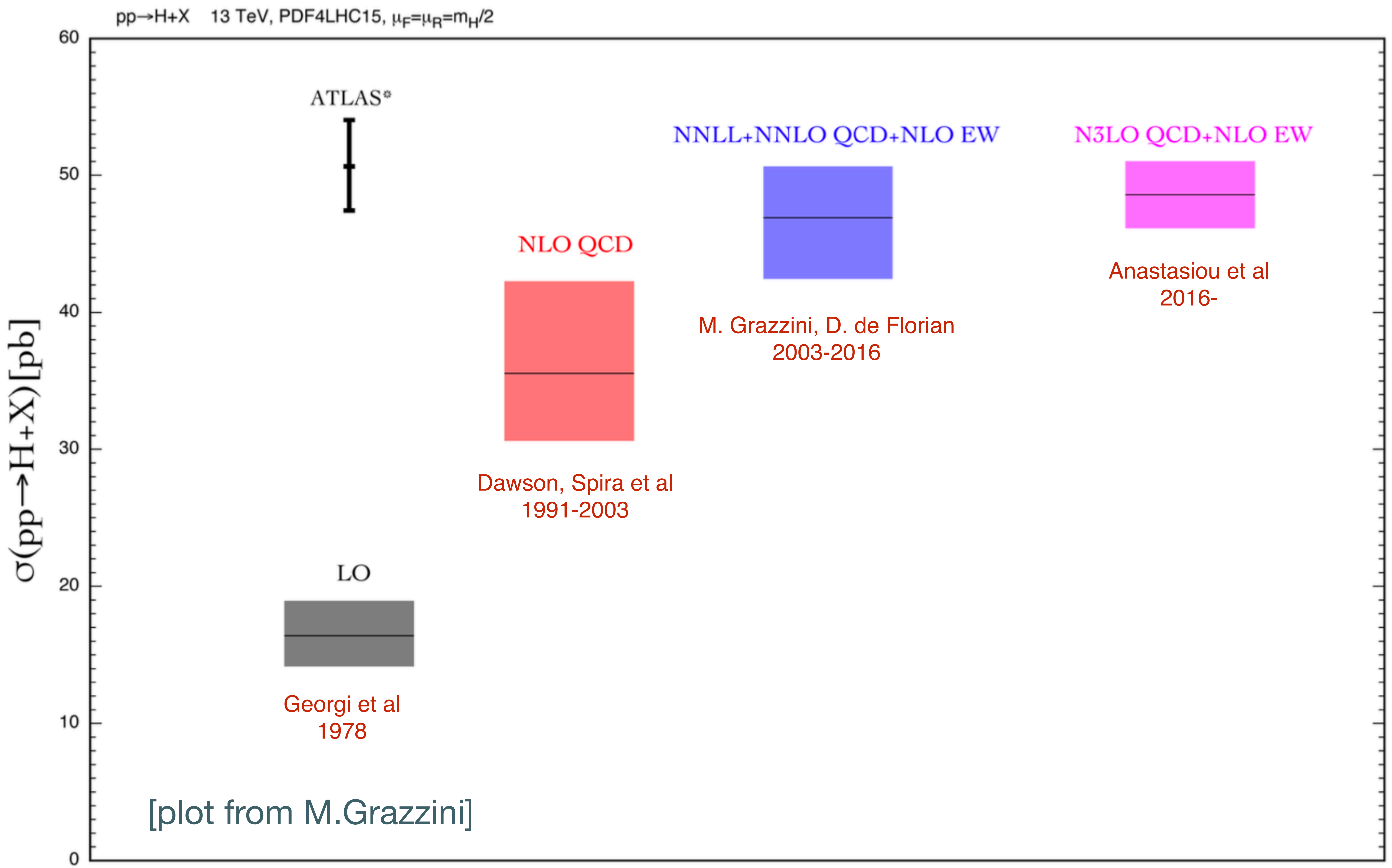
THE NEED OF PRECISION: TOWARDS THE % LEVEL



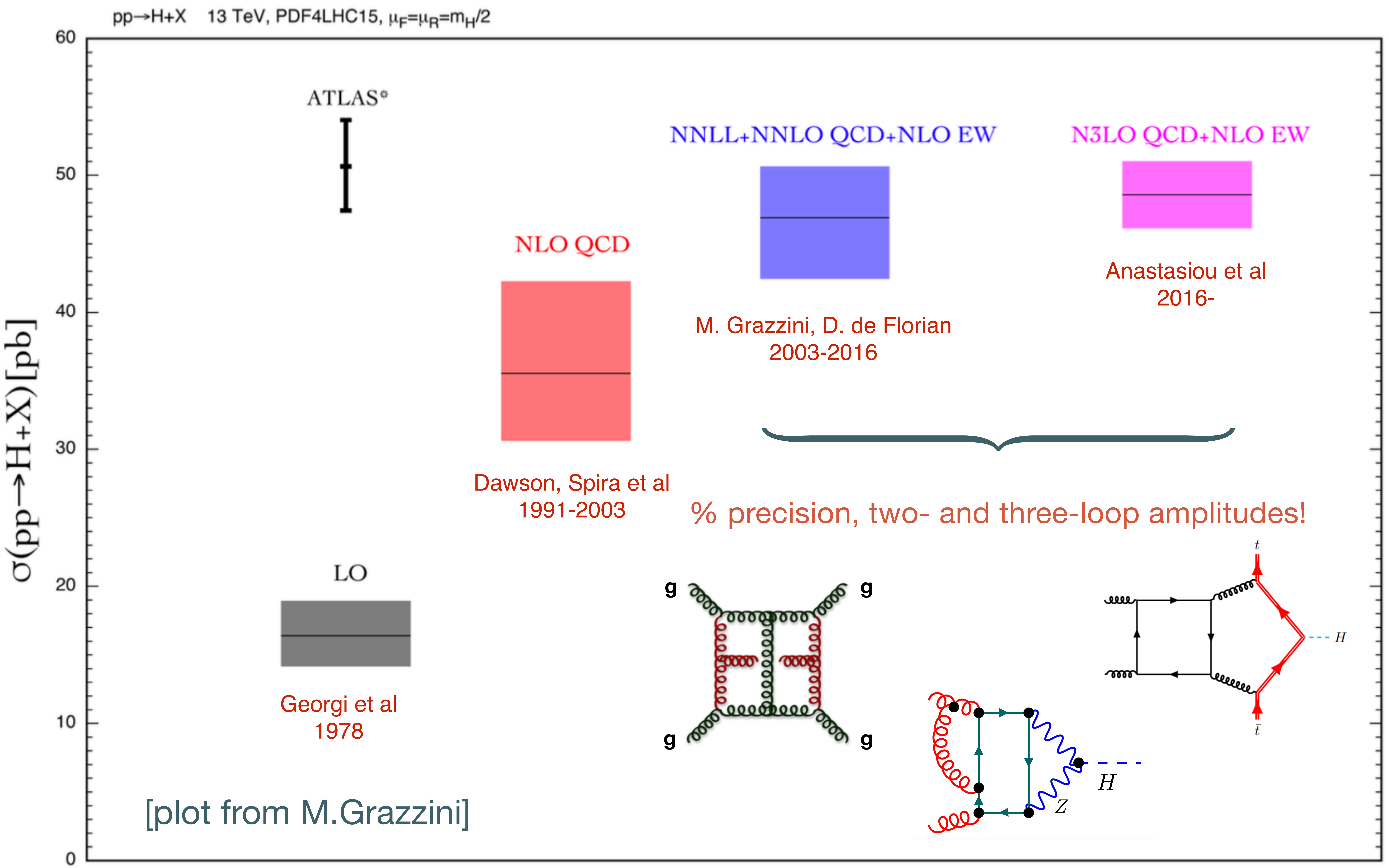
THE NEED OF PRECISION: TOWARDS THE % LEVEL



THE NEED OF PRECISION: TOWARDS THE % LEVEL



THE NEED OF PRECISION: TOWARDS THE % LEVEL



AMPLITUDES FOR COLLIDERS: HOW DO WE GO ABOUT THEM?

The integrand

$$\mathcal{A}$$

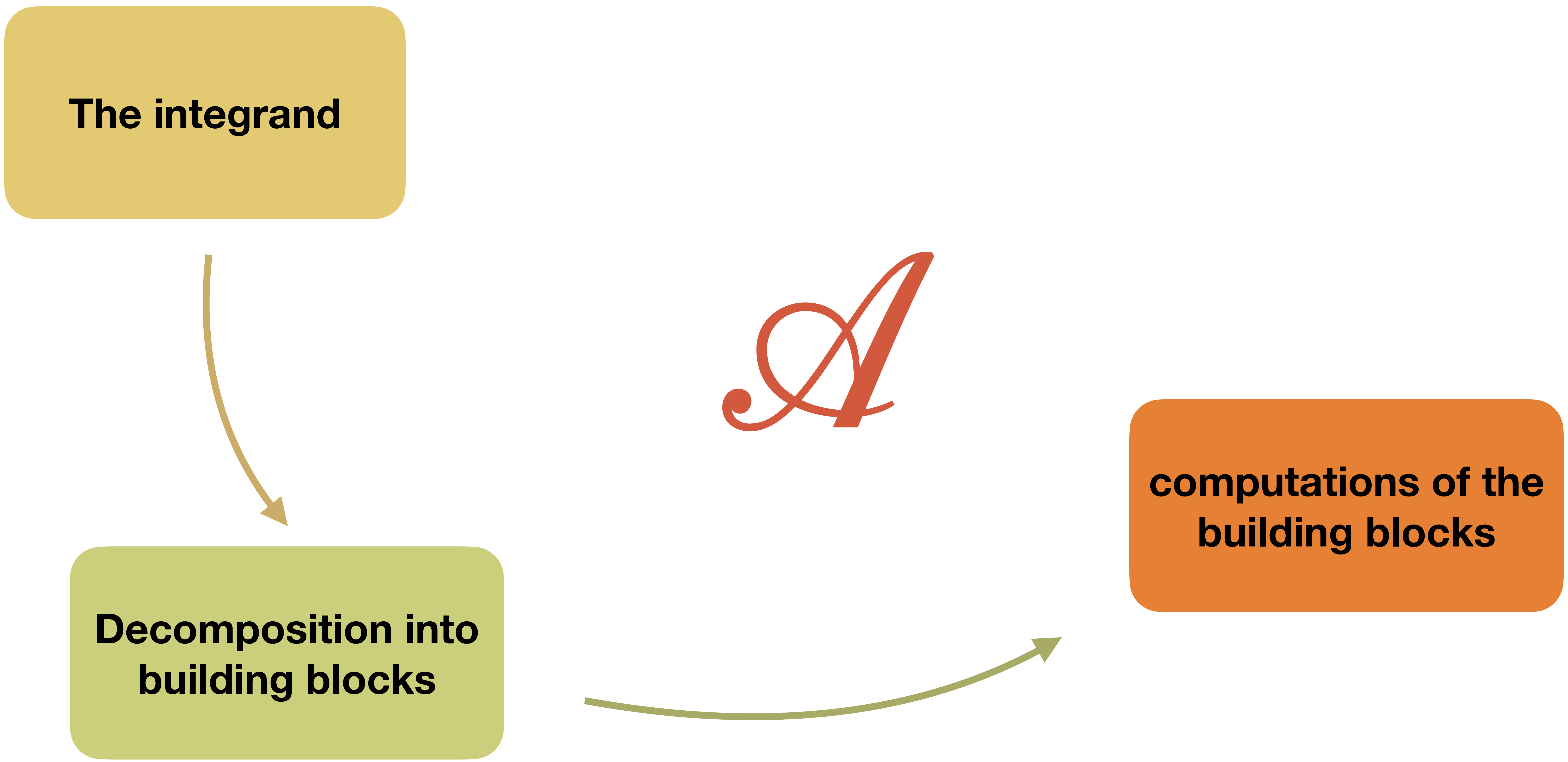
AMPLITUDES FOR COLLIDERS: HOW DO WE GO ABOUT THEM?

The integrand

Decomposition into building blocks

$\mathcal{A}$

AMPLITUDES FOR COLLIDERS: HOW DO WE GO ABOUT THEM?



AMPLITUDES FOR COLLIDERS: HOW DO WE GO ABOUT THEM?

2-loop electron g-2 in QED

$$\frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$$

Often: unexpected simplicity of final results

The integrand

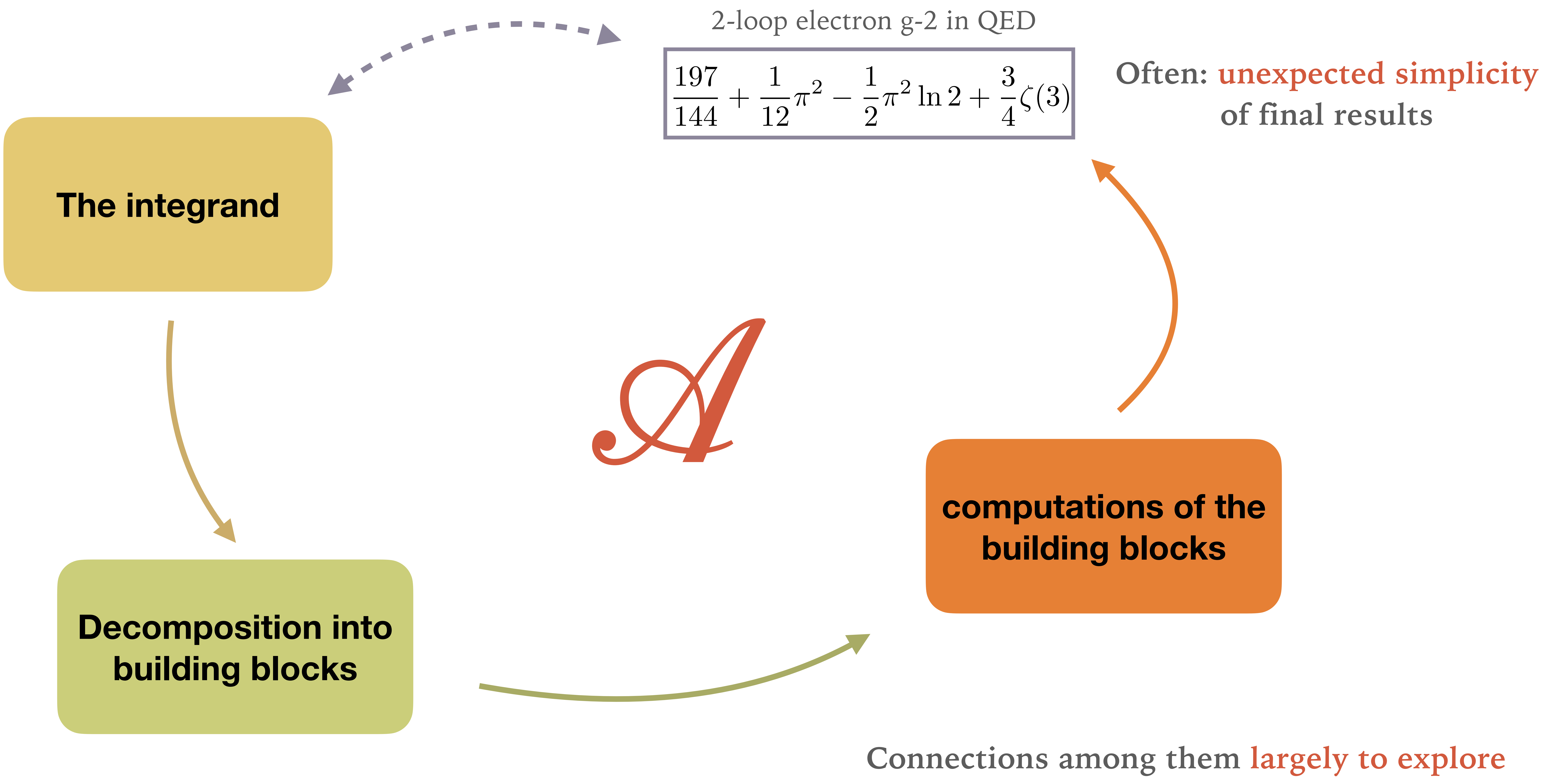
$\mathcal{A}$

computations of the building blocks

Decomposition into building blocks

Usually dealt with separately

AMPLITUDES FOR COLLIDERS: HOW DO WE GO ABOUT THEM?



MANY OPEN QUESTIONS AND SOME ANSWERS:

- What are general **numbers and functions** that can appear in the final result?
- How do physical constraints reflect in **mathematical properties** of the result?
- What is the **"shortest"** path to the **"simplest"** form of the result?
- ……..

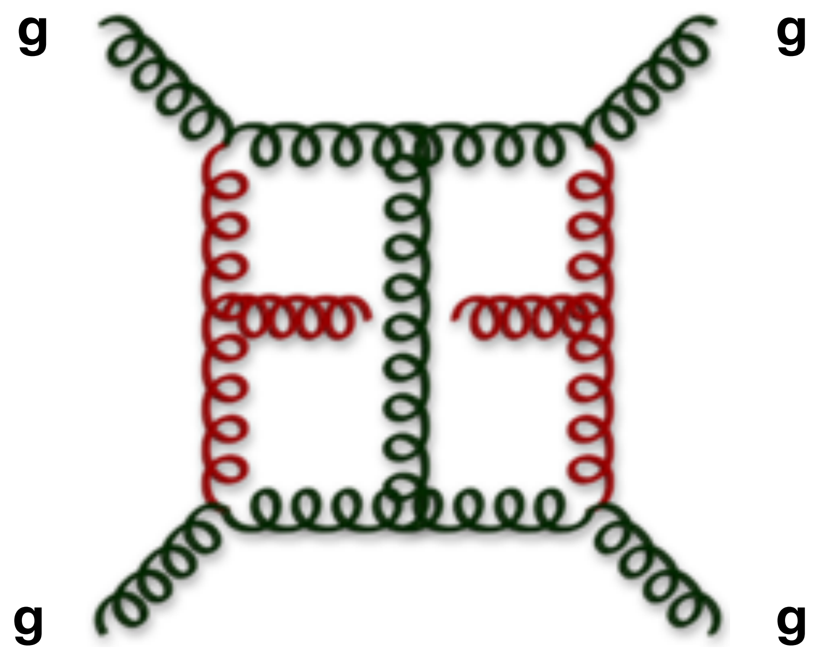
↓

A possible **key to understanding these questions**:

explore interplay between mathematics of scattering amplitudes (**geometry**) and their physical properties (**singularities, discontinuities, soft/collinear limits…**)

WHAT IS AN AMPLITUDE?

$\mathcal{A} \longrightarrow$



"just a sum of Feynman diagrams"

WHAT IS AN AMPLITUDE?

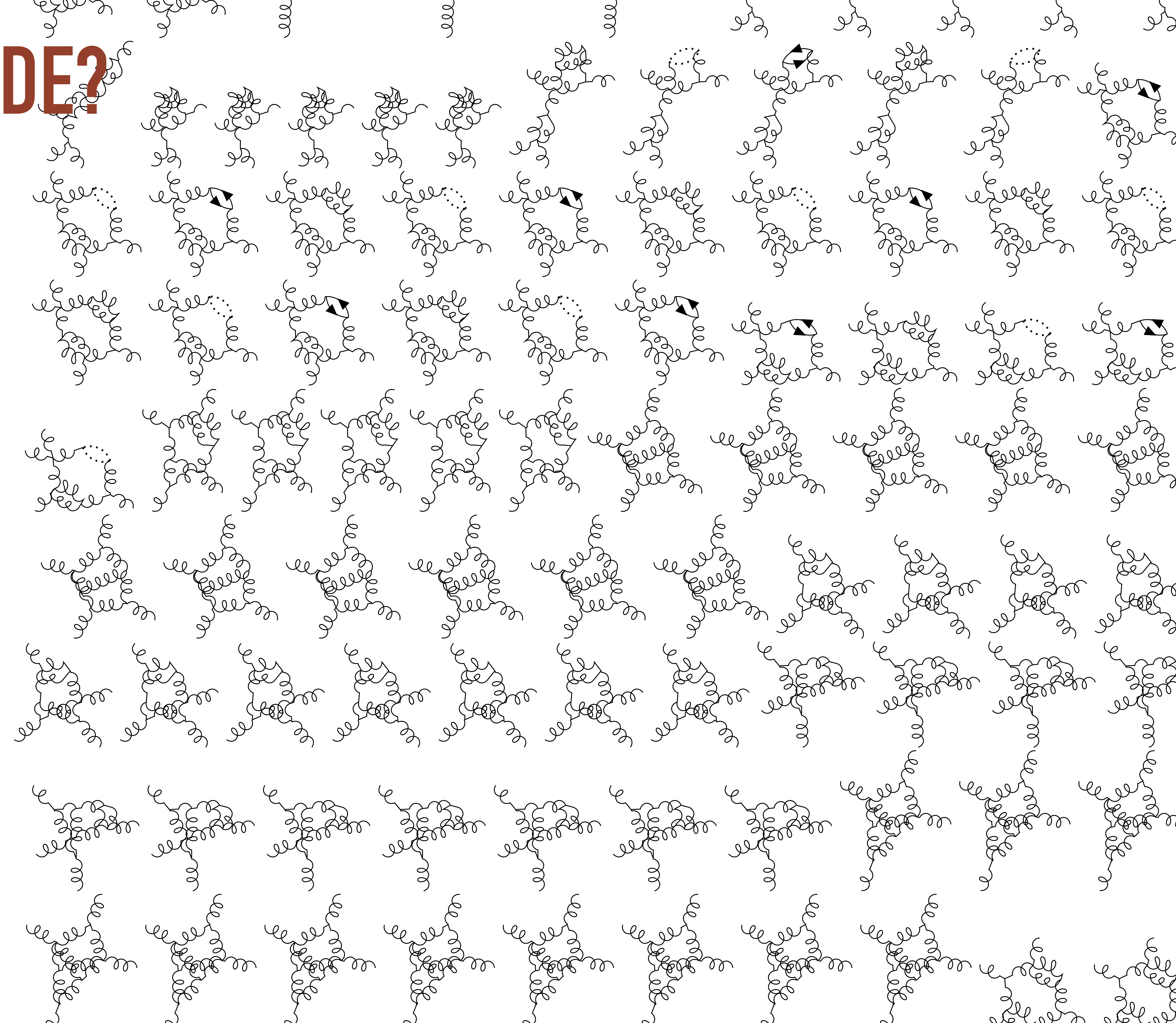
$\mathscr{A}$ →

gg → *gg* @ 3 loops in QCD

+ 500 more pages

= 50000 Feynman diagrams

= 10^7 Feynman integrals!



FROM INTEGRAND TO SPECIAL FUNCTIONS

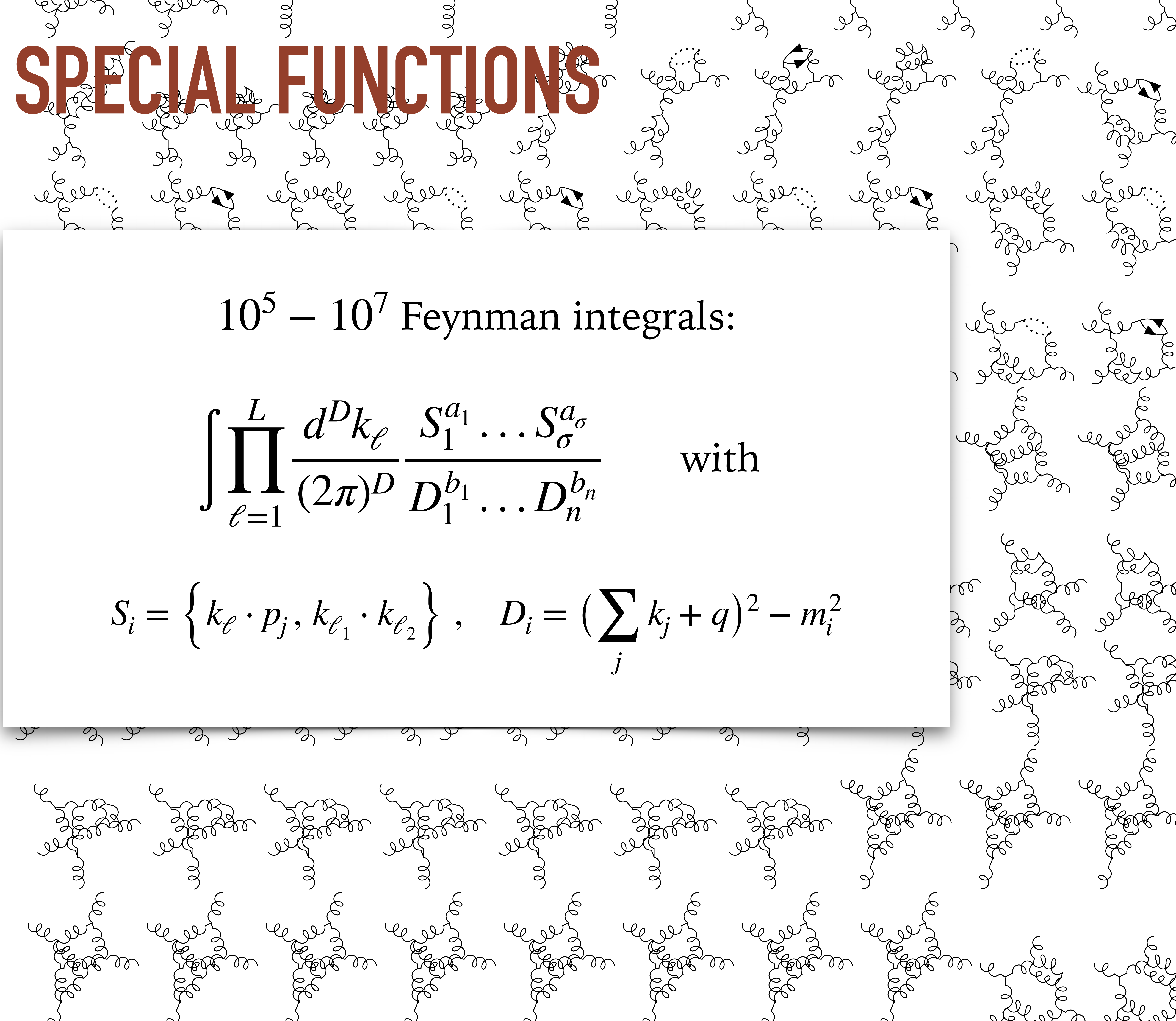
$\mathcal{A}$ ⟶

gg → *gg* @ 3 loops in QCD

+ 500 more pages

= 50000 Feynman diagrams

= 10^7 Feynman integrals!



$10^5 - 10^7$ Feynman integrals:

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}} \qquad \text{with}$$

$$S_i = \left\{ k_\ell \cdot p_j \, , k_{\ell_1} \cdot k_{\ell_2} \right\} \, , \quad D_i = \Big(\sum_j k_j + q \Big)^2 - m_i^2$$

FROM INTEGRAND TO SPECIAL FUNCTIONS

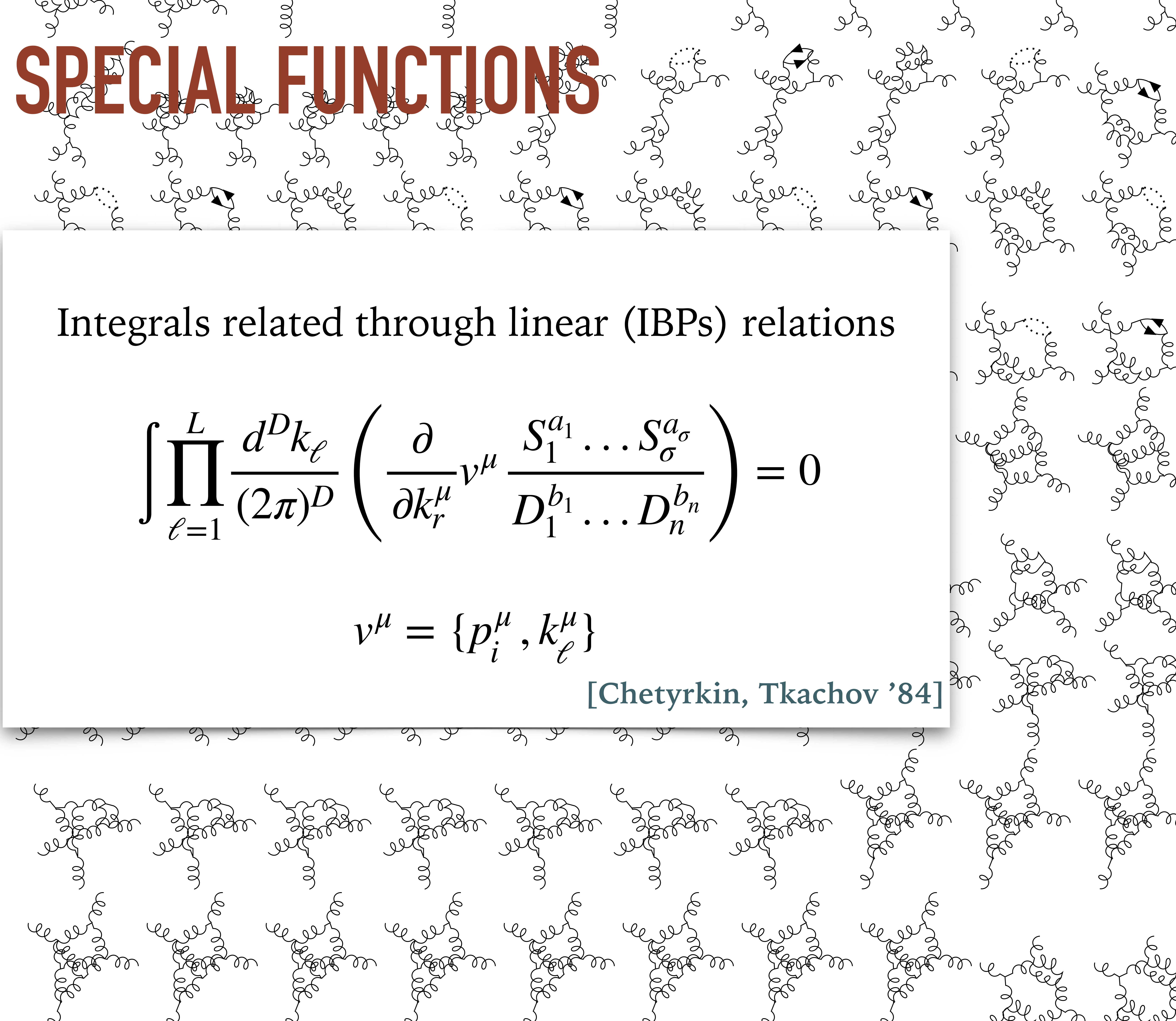
$\mathscr{A} \longrightarrow$

$gg \to gg$ @ 3 loops in QCD

\+ 500 more pages

= 50000 Feynman diagrams

= 10^7 Feynman integrals!



Integrals related through linear (IBPs) relations

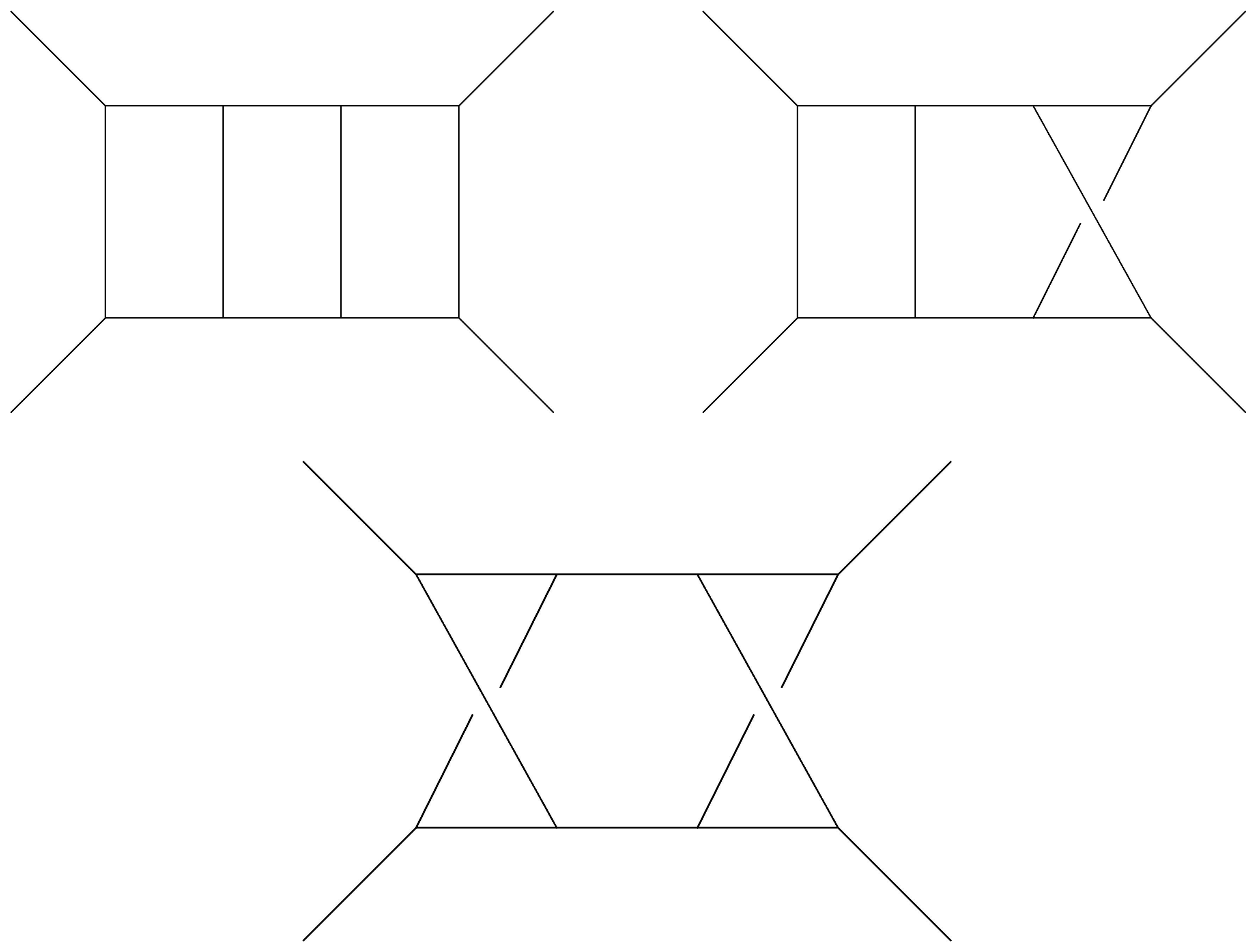
$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \left(\frac{\partial}{\partial k_r^\mu} v^\mu \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}} \right) = 0$$

$$v^\mu = \{p_i^\mu, k_\ell^\mu\}$$

[Chetyrkin, Tkachov ’84]

FROM INTEGRAND TO SPECIAL FUNCTIONS

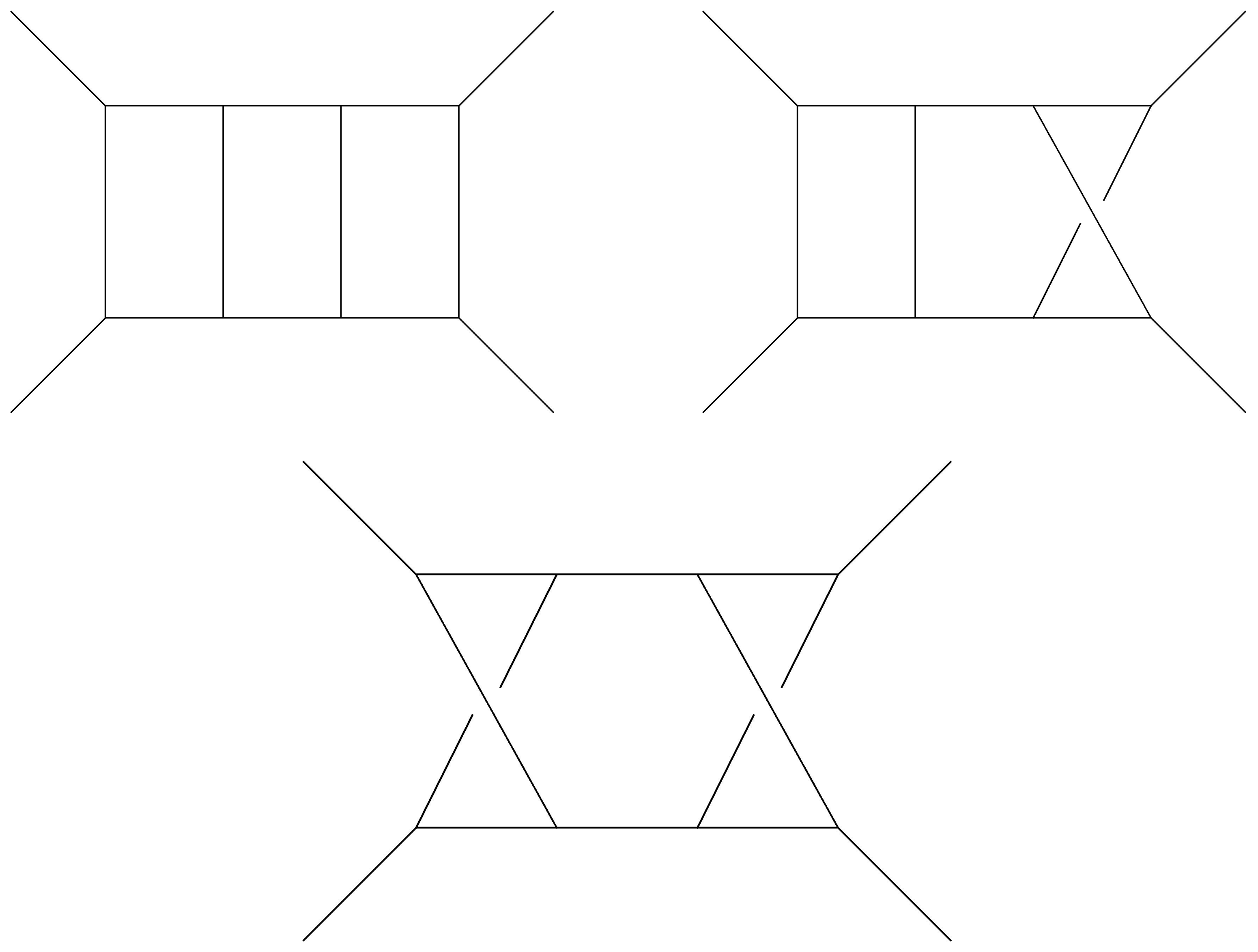
$$\mathscr{A} \longrightarrow \sum_i$$



$gg \to gg$ @ 3 loops in QCD   ~ "only" 500 master integrals

FROM INTEGRAND TO SPECIAL FUNCTIONS

$$\mathcal{A} \longrightarrow \sum_i$$



Coefficients in front of master integrals are just “rational functions”:

→ *poles from on-shell one-particle states !*

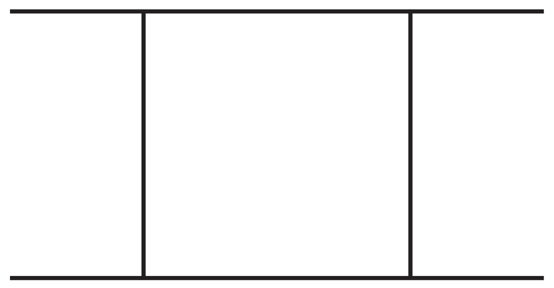
?

Master integrals contain all non-trivial functional dependence:

→ analytic dependence on kinematics, except *branch cuts from multiparticle states going on-shell*

FROM INTEGRAND TO SPECIAL FUNCTIONS

Master integrals can be conveniently ***organized in a "tree"*** depending on # of propagators



Box

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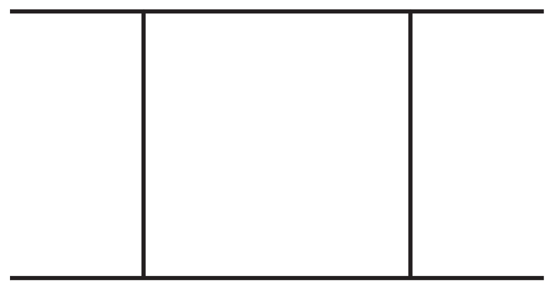

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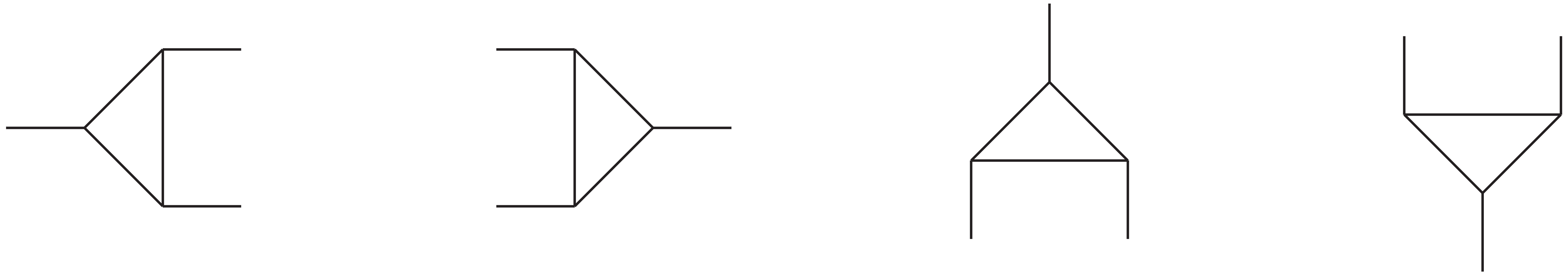

triangles

(pinching 1 propagator)

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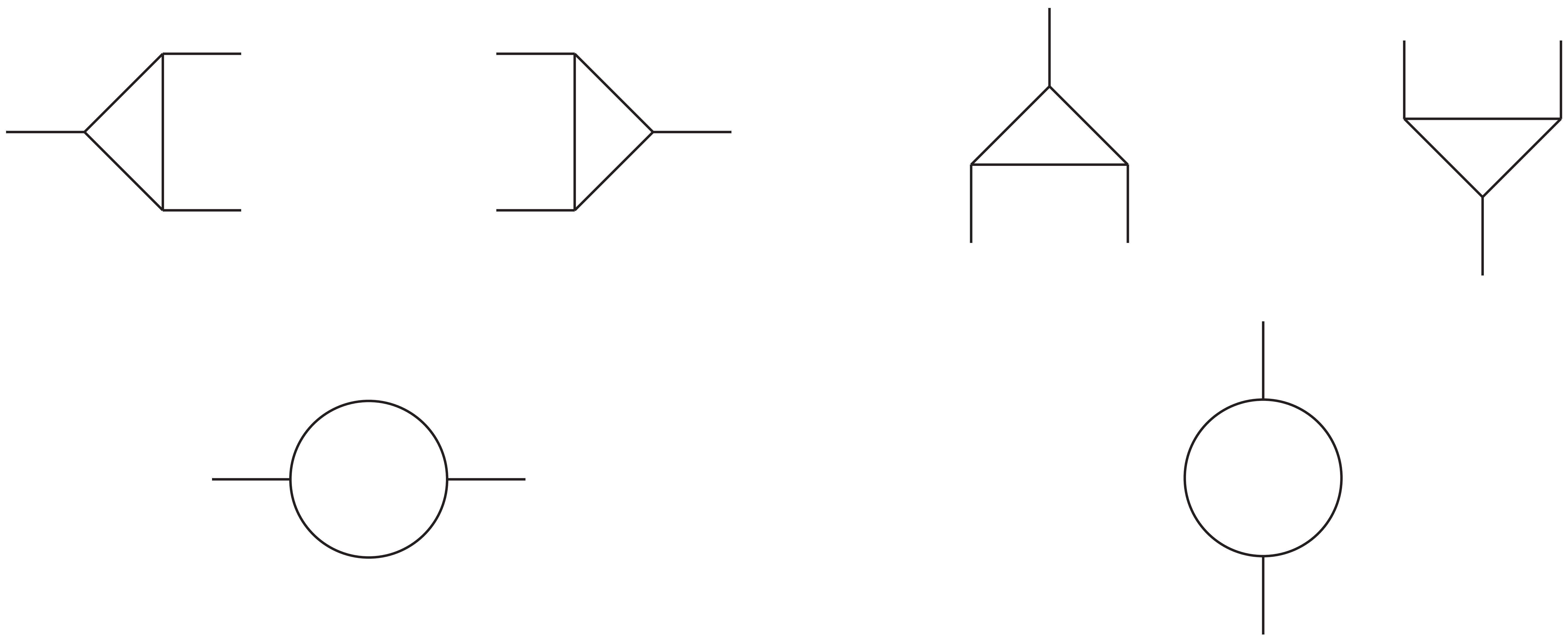

Box

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bubbles

(pinching 2 propagators)

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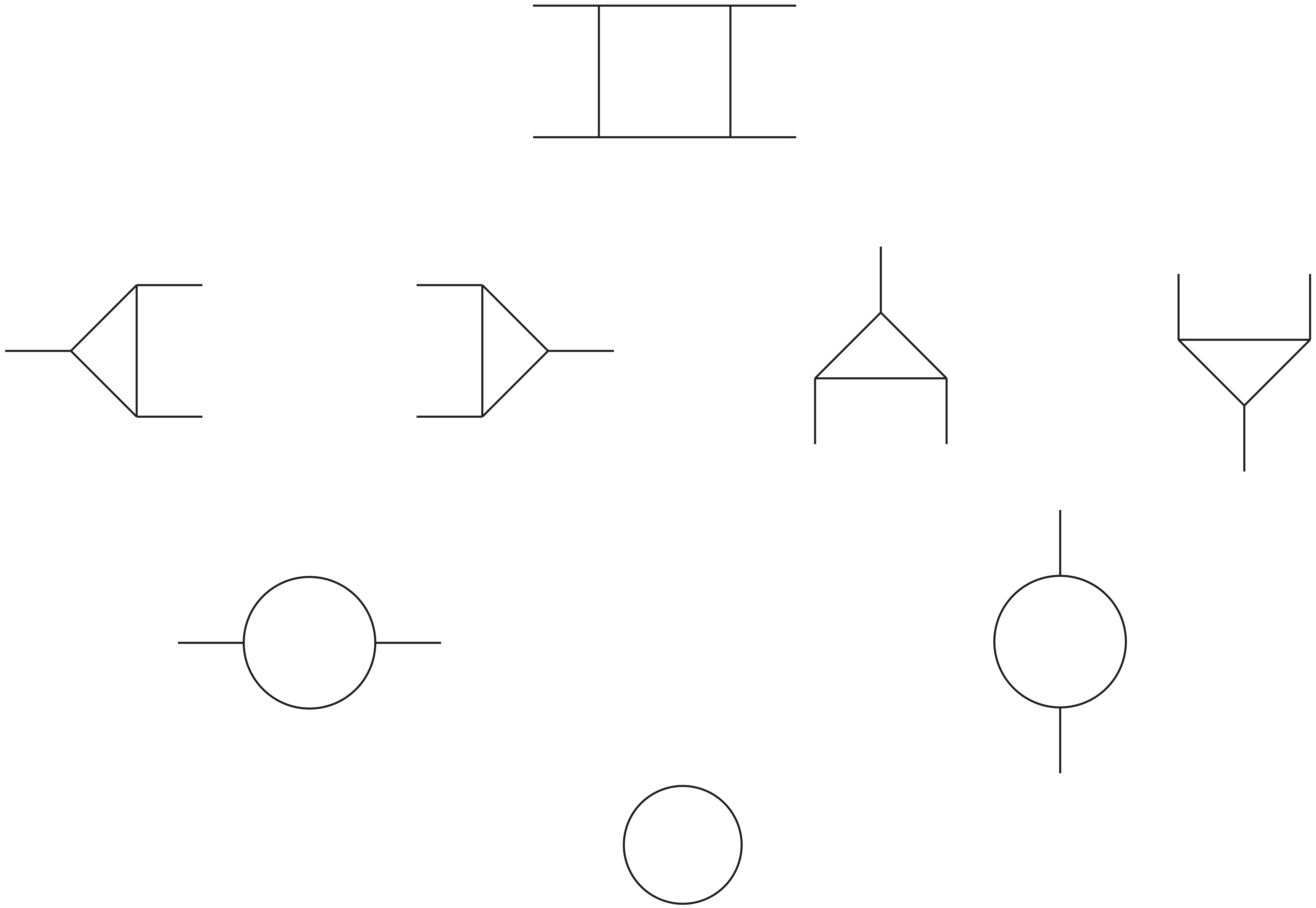

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bubbles

(pinching 2 propagators)

tadpoles

(pinching 3 propagators)

FROM INTEGRAND TO SPECIAL FUNCTIONS

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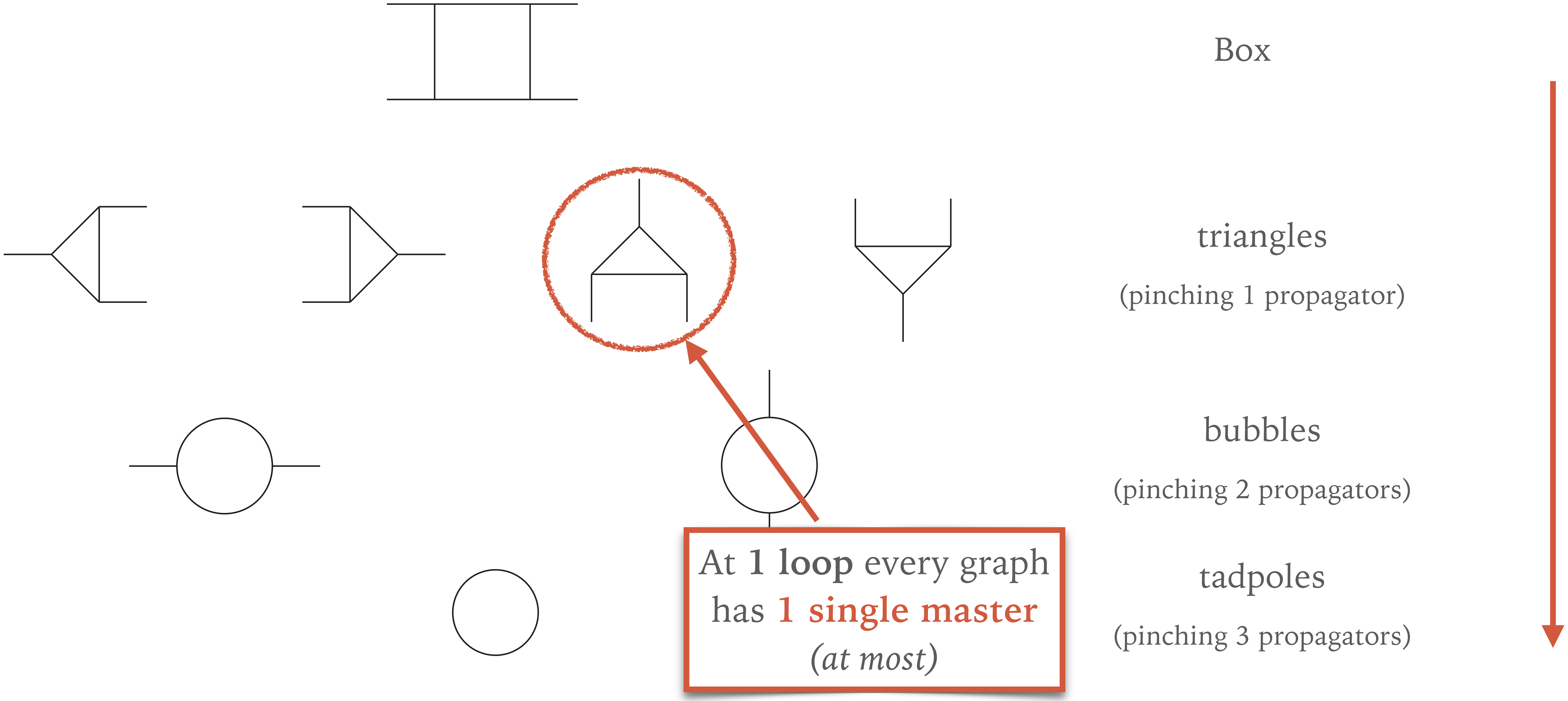
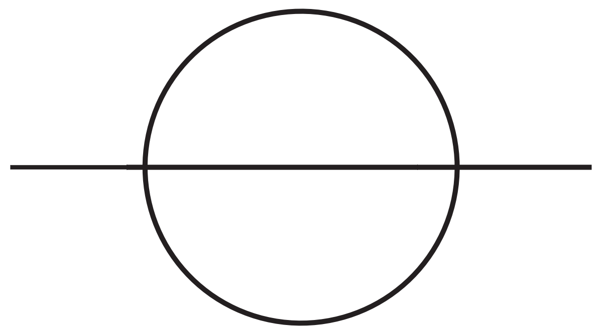
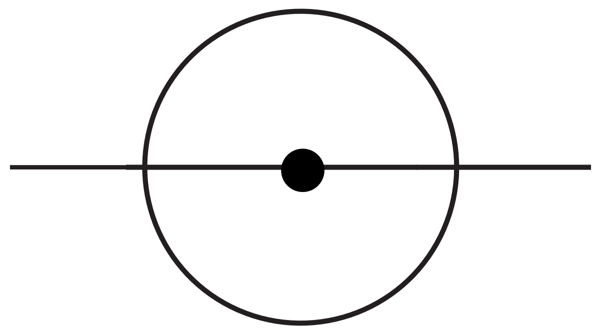


FROM INTEGRAND TO SPECIAL FUNCTIONS

At ***higher loops***: a "graph" can have more than one master integral





the equal-mass sunrise

FROM INTEGRAND TO SPECIAL FUNCTIONS

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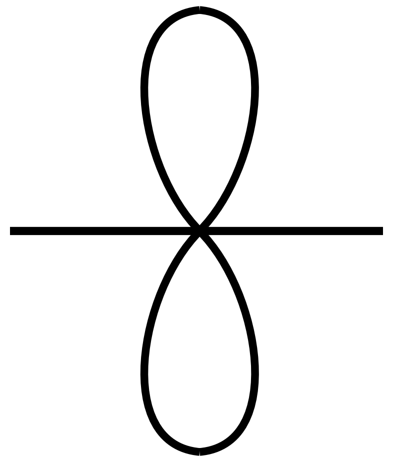

double tadpole

(pinching 1 propagator)

Fundamental difference with one loop → first hint that complexity of the problem jumps

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS

We can differentiate Feynman integrals w.r.t. the kinematical invariants

$$\frac{\partial}{\partial s_{ij}} \left[\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \ldots S_\sigma^{a_\sigma}}{D_1^{b_1} \ldots D_n^{b_n}} \right] = \qquad \forall s_{ij} = \{p_i \cdot p_j \,, m_k^2\}$$

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[Kotikov ’93; Remiddi ’97; Gehrmann, Remiddi ’99]

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↓

$$\frac{\partial}{\partial s_{ij}}\vec{I} = A(s_{ij}, D)\vec{I}\,, \qquad A(s_{ij}, D) \quad \text{Rational functions}$$

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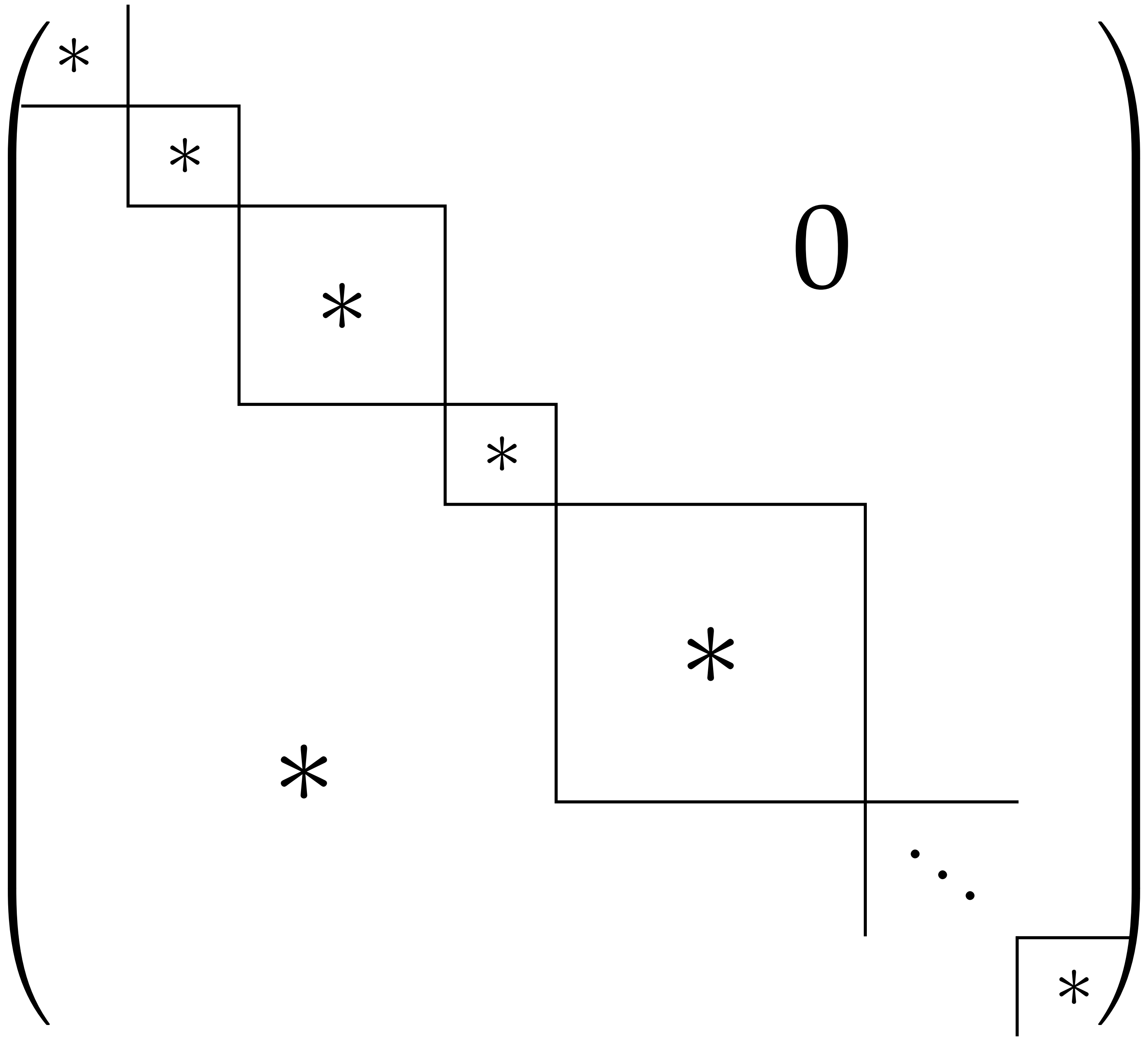
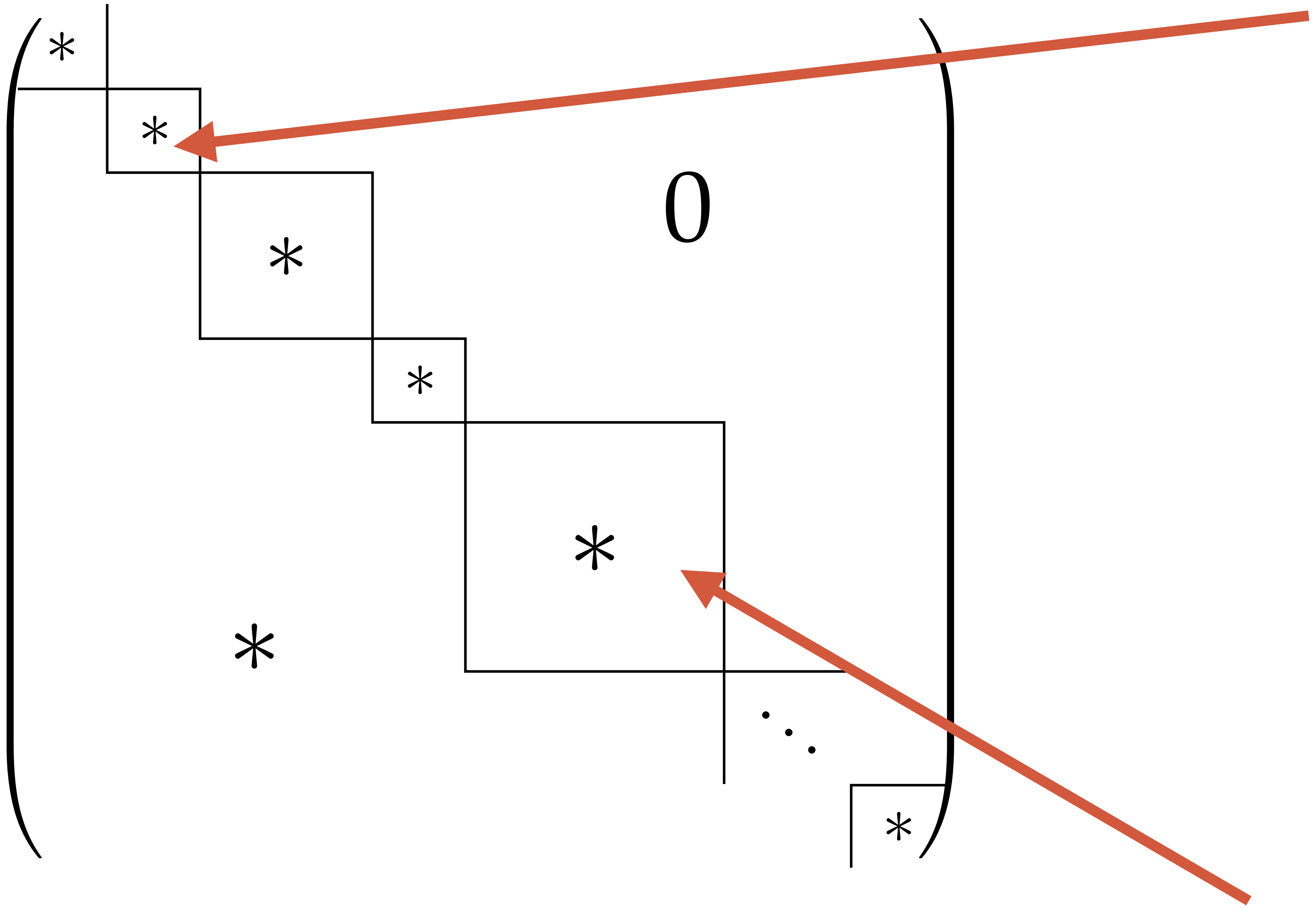
$$\frac{\partial}{\partial s_{ij}}\vec{I}=A(s_{ij},D)\vec{I}\,,\qquad A(s_{ij},D)=$$

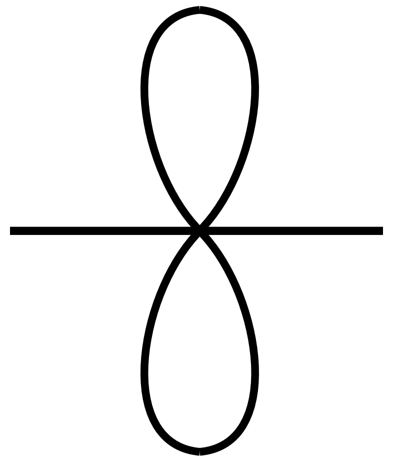


block-triangular:

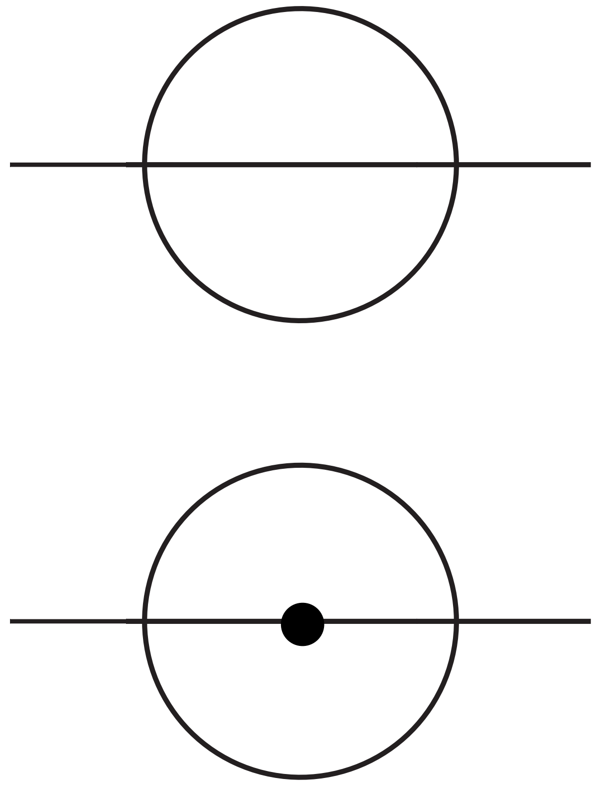
integrals with more propagators depend on ones with fewer

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS





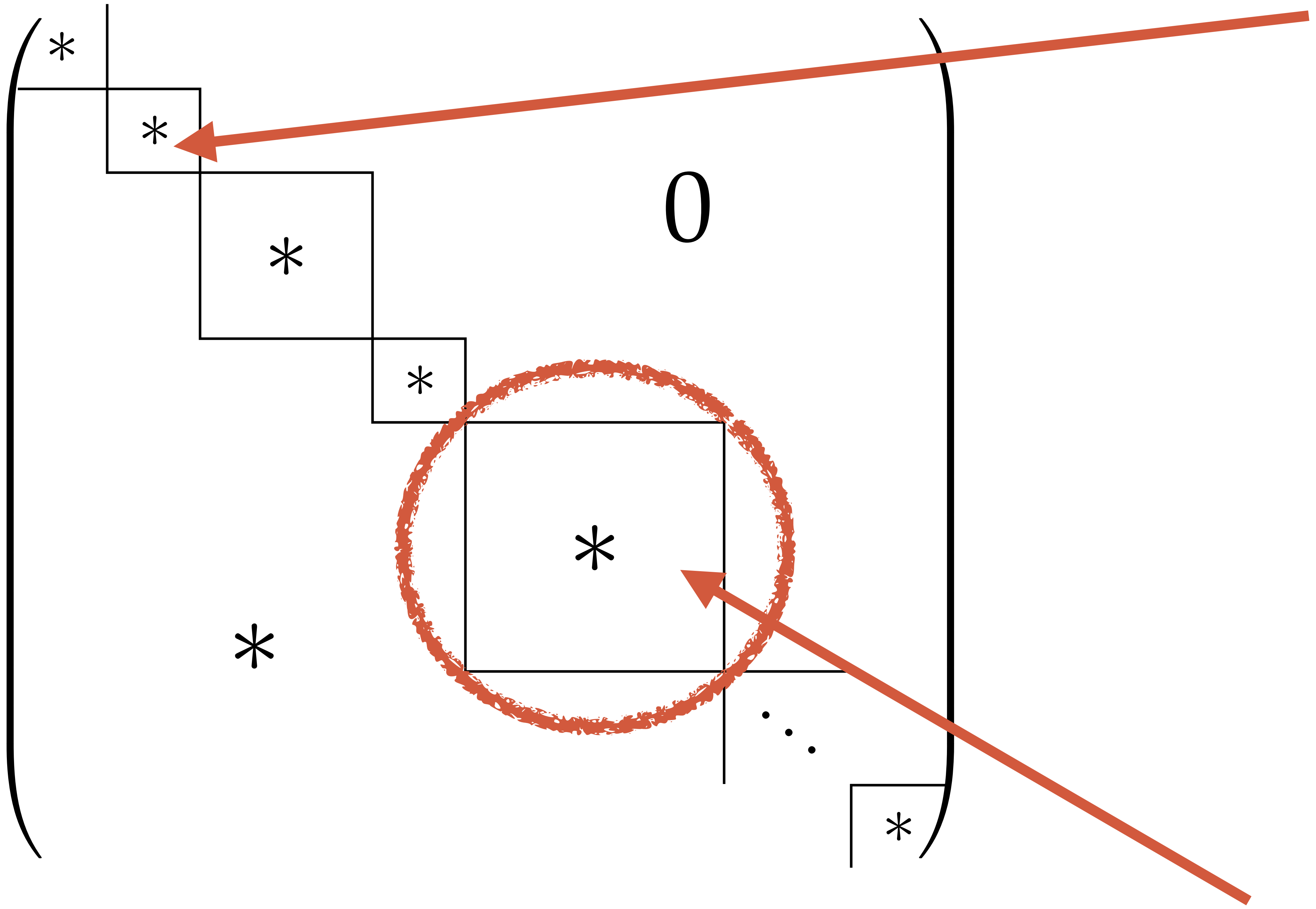
1×1 blocks correspond to single master integrals

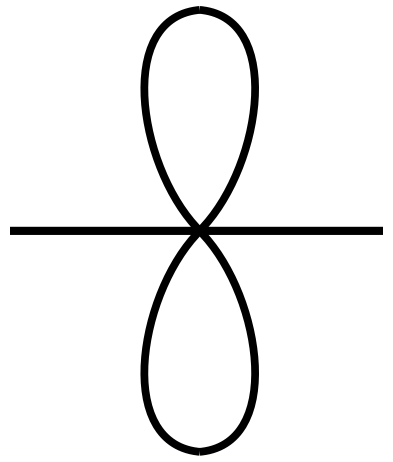


coupled $n \times n$ blocks correspond to graphs with n master integrals

True for general values of dimensions D !

DIFFERENTIAL EQUATIONS FOR FEYNMAN INTEGRALS





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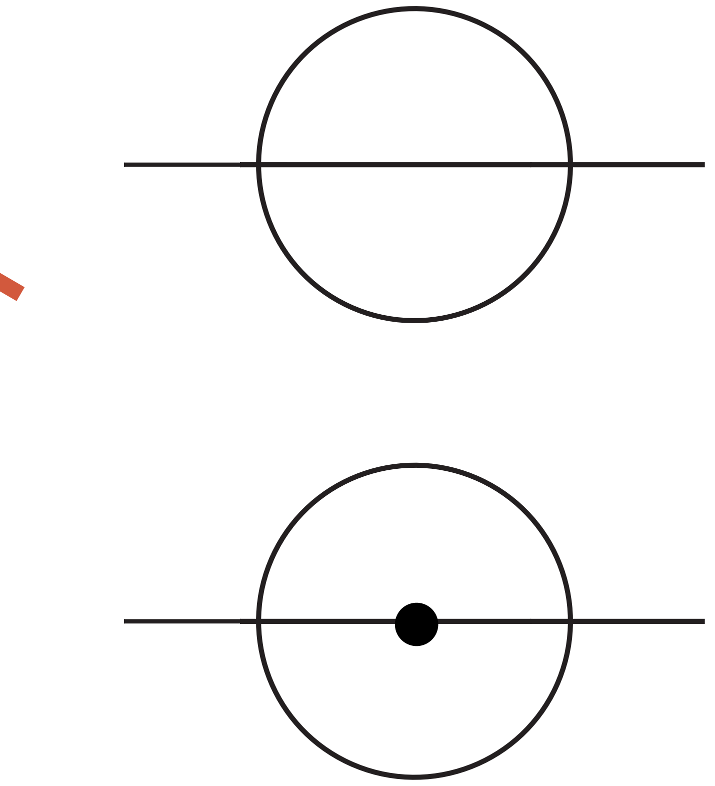

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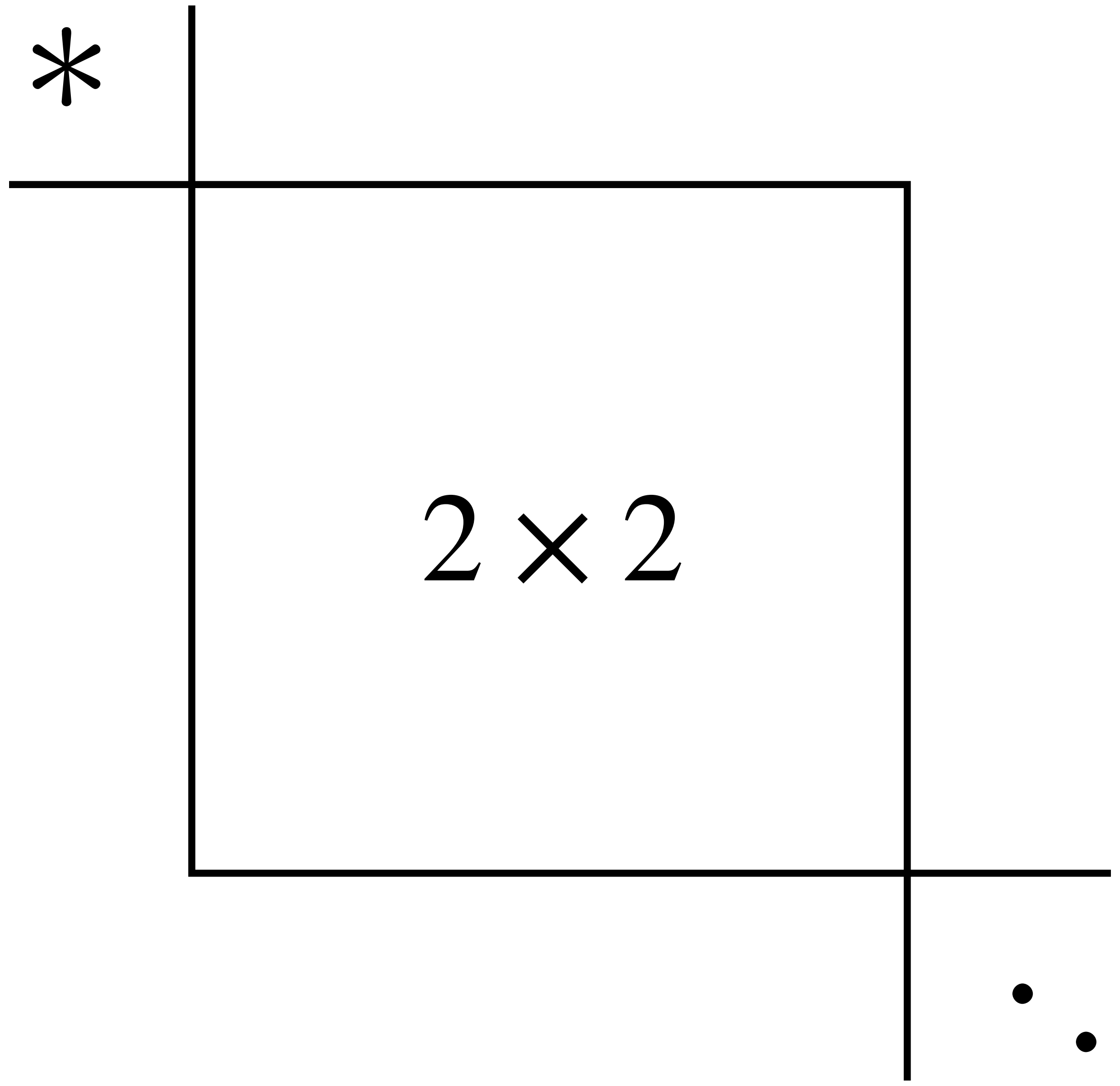
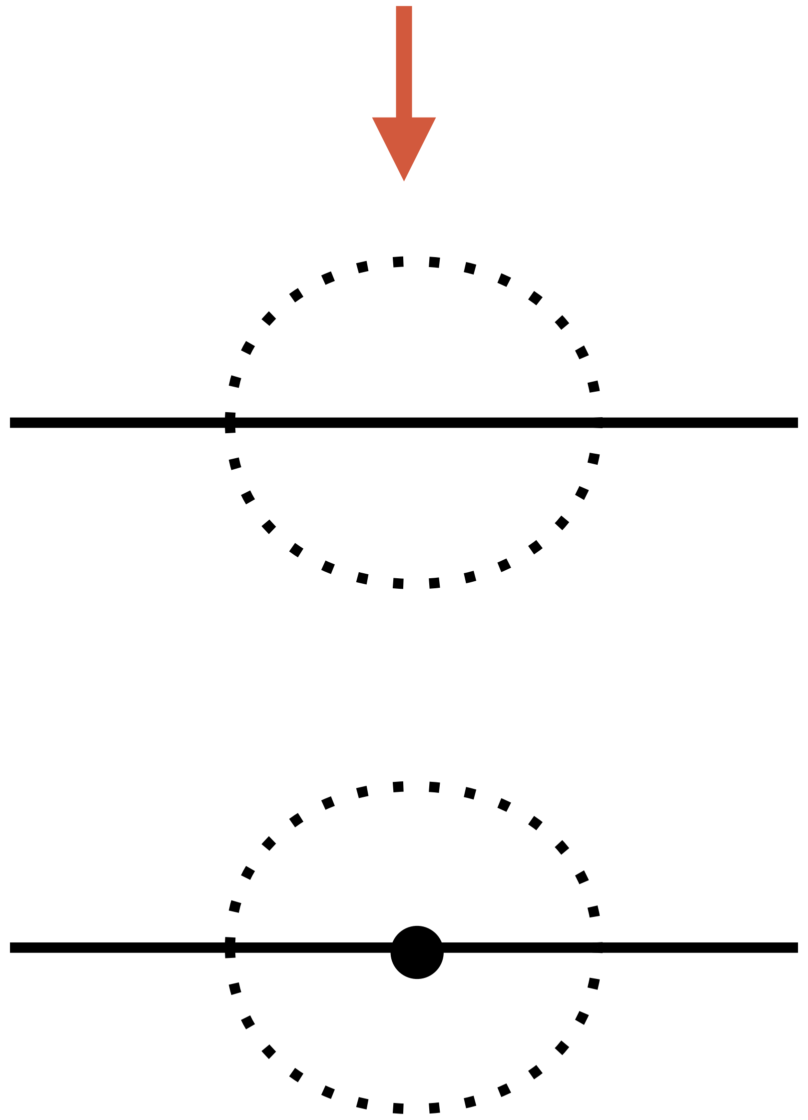
FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

The story might change for $D \to 4 - 2\epsilon$



$$\to \begin{bmatrix} a_{11}(s_{ij}) & a_{12}(s_{ij}) \\ 0 & a_{22}(s_{ij}) \end{bmatrix} + \mathcal{O}(\epsilon)$$

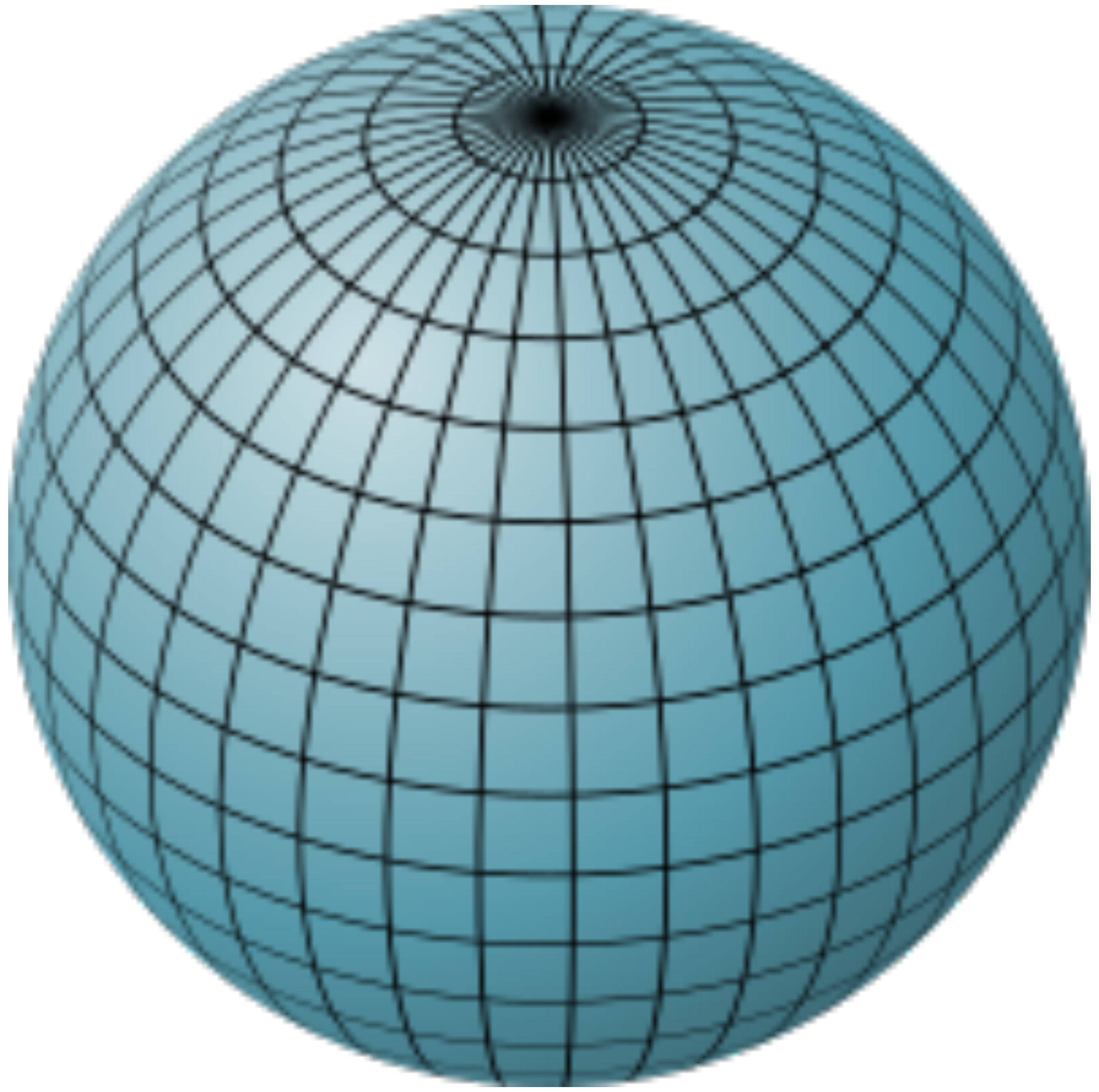
massless propagator: "photon"



Equations might "decouple" close to $D = 4$ space-time dimensions

a_{ij} are rational functions → solution written iteratively in ϵ as *iterated integrals of rational functions!*

MULTIPLE POLYLOGS AND THE RIEMANN SPHERE



If we integrate a rational function on $\mathbb{CP}^1$

Only non-trivial thing:

$$\log(1 - x/a) = \int_0^x \frac{dt}{t-a}$$

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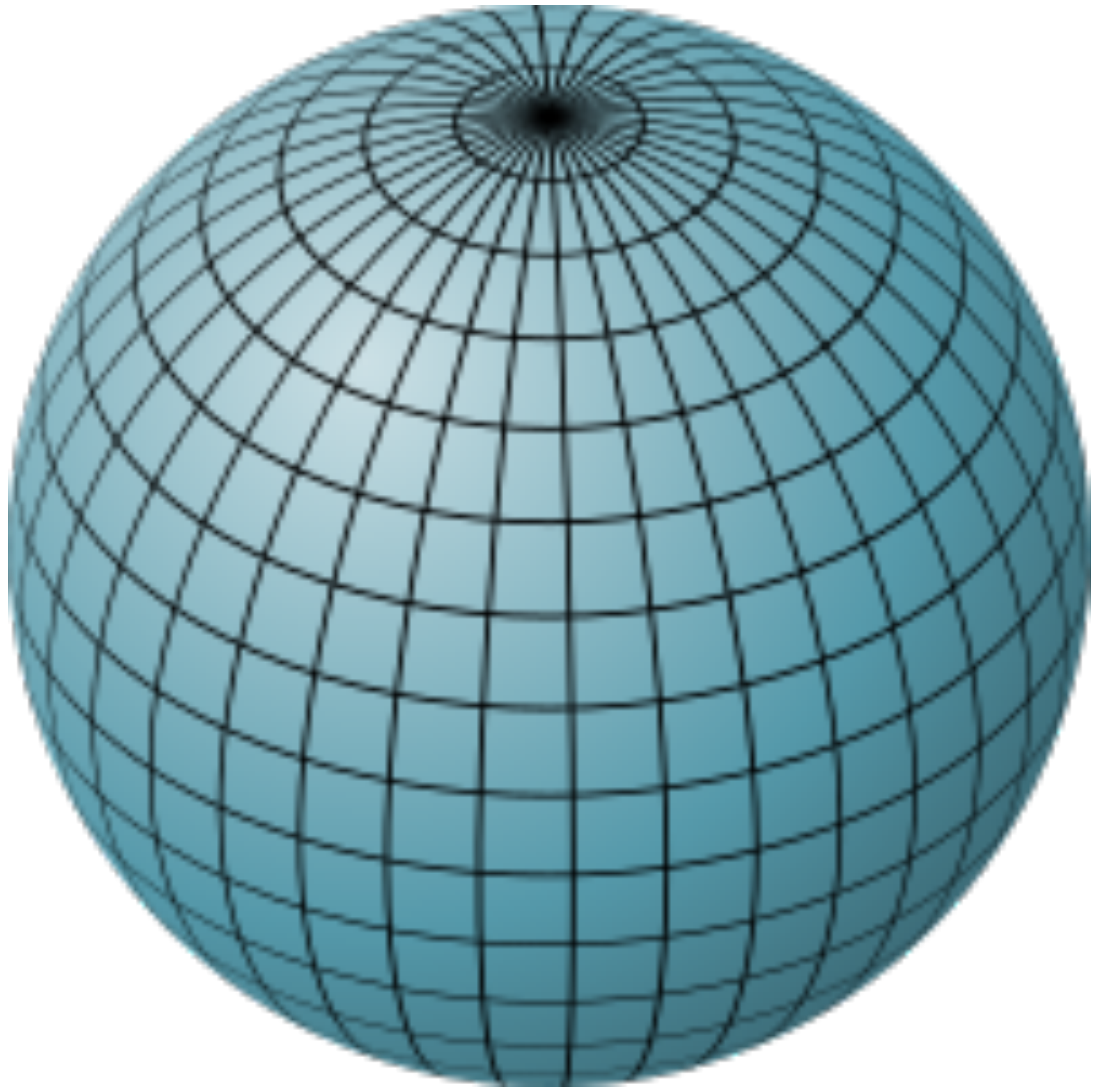

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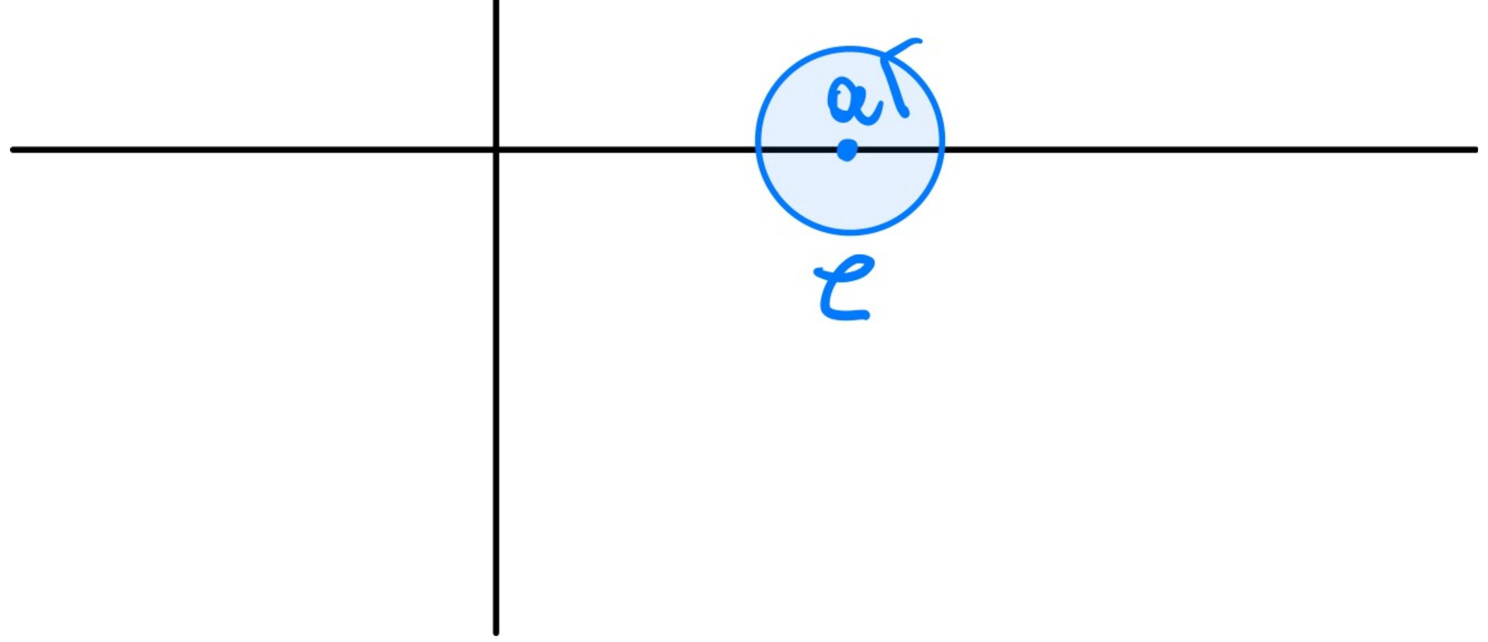
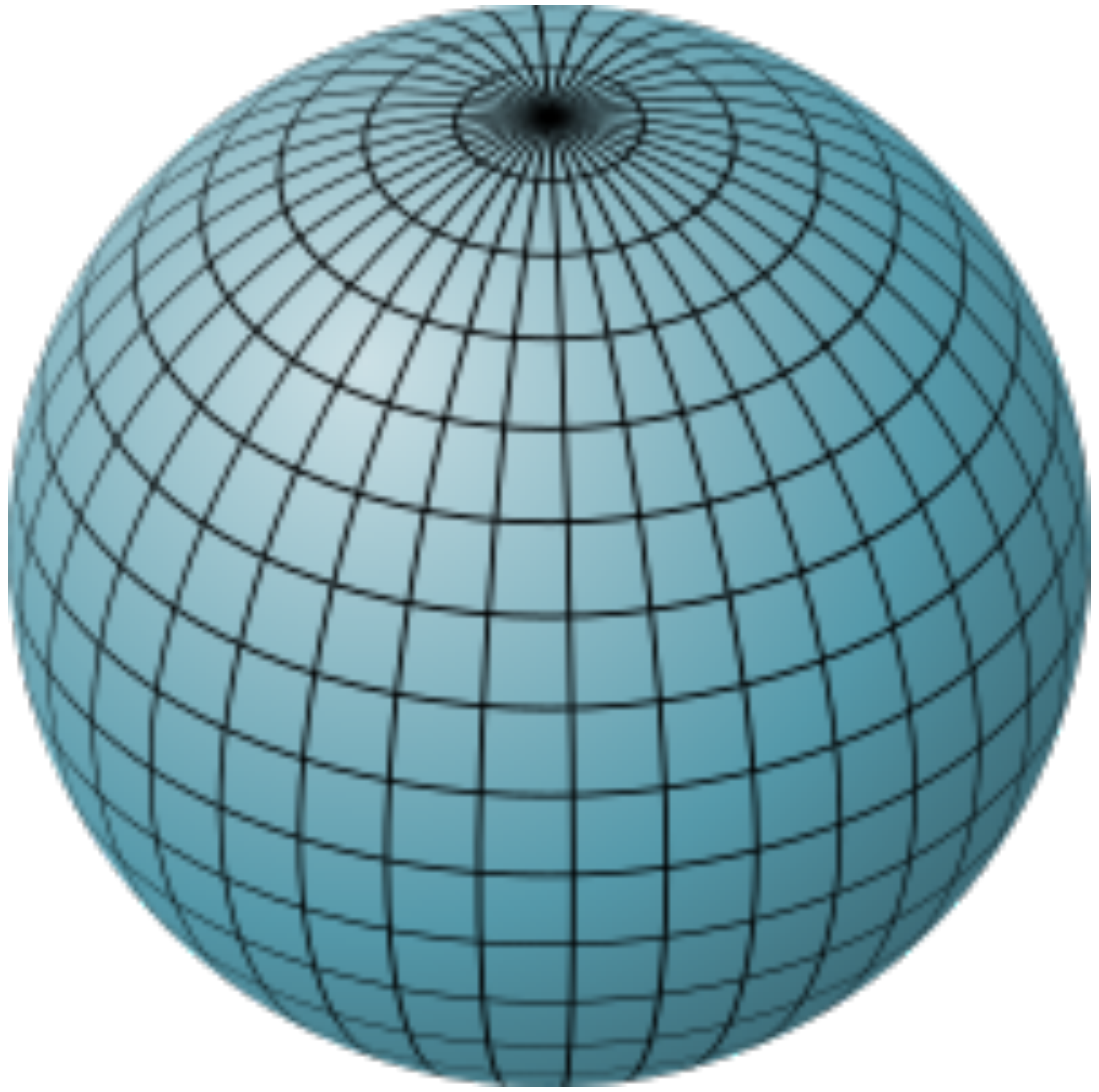
$$\oint_{\epsilon} \frac{dx}{x-a} = 2\pi i$$



MULTIPLE POLYLOGS AND THE RIEMANN SPHERE



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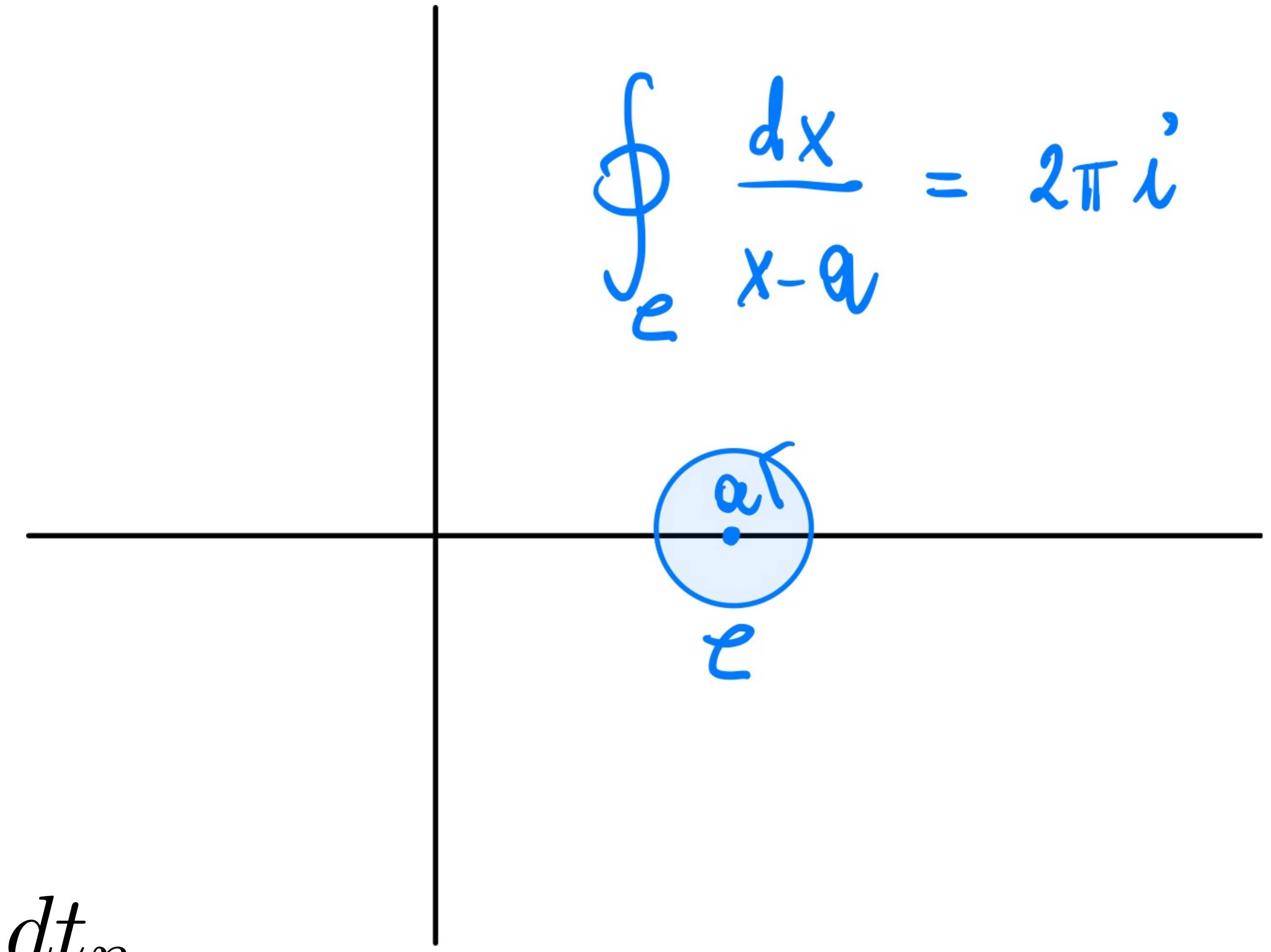

Generalisation: **Multiple PolyLogarithms (MPLs)**

$$G(c_1, c_2, \dots, c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, \dots, c_n, t_1)$$

$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \dots \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

CANONICAL DIFFERENTIAL EQUATIONS

MPLs bring non-trivial information (branch cuts), the rest (poles) are just rational functions

Can we choose MIs that evaluate directly to pure combinations of MPLs?

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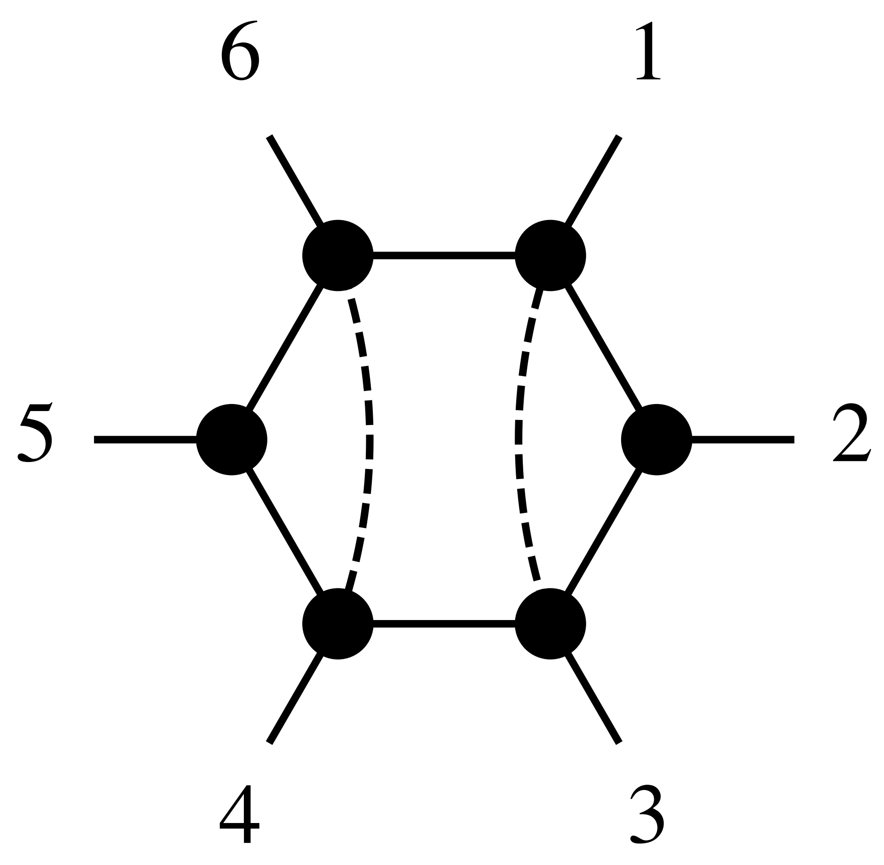

Local Integrals

$$\int_\gamma d\log f_1 \wedge d\log f_2 \wedge \ldots \wedge d\log f_n$$

[Arkani-Hamed et al '10]

↓

Diff form → oriented volume, **canonical form for integrand, amplituhedron** …

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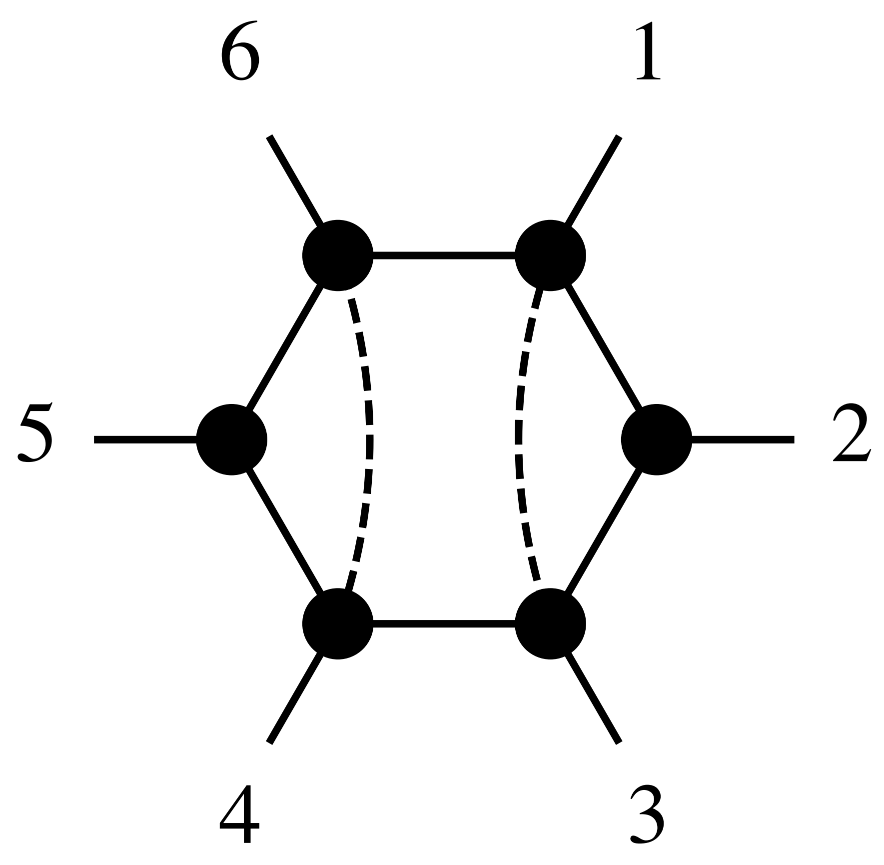

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[Arkani-Hamed et al '10]

Local integrals fulfil **canonical diff-equations** [Henn '13]

$$d\vec{I} = \epsilon \left[\quad \epsilon\text{-indep} \quad \right] \vec{I}, \qquad \rightarrow \qquad \left[\quad \epsilon\text{-indep} \quad \right] = \sum_i B_i \, \mathrm{d}\log f_i$$

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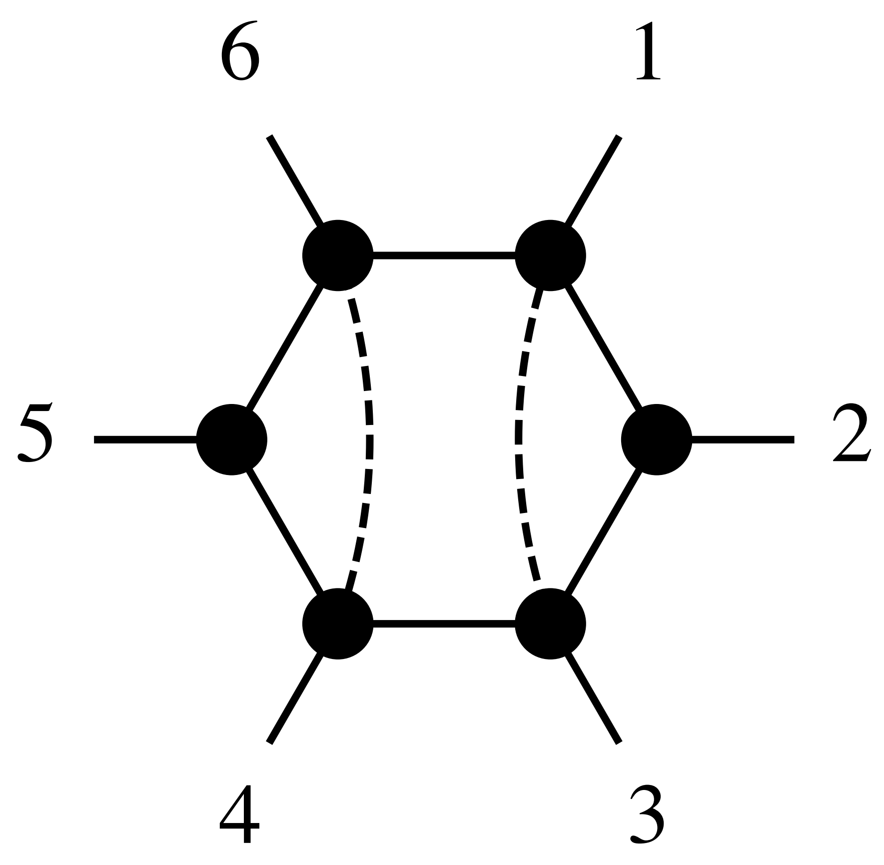

Local Integrals

$$\int_\gamma d\log f_1 \wedge d\log f_2 \wedge \ldots \wedge d\log f_n$$

[Arkani-Hamed et al '10]

Solution as **path-ordered exponential:** Their integration naturally produces MPLs if f_i are rational functions!

$$\vec{I} = \mathbb{P}\exp\left[\epsilon \sum_i B_i \int_\gamma \mathrm{d}\log f_i\right]\vec{I}_0$$

LEADING SINGULARITIES AND ALL THAT

Conjecturally: We can always choose a canonical basis if results are dlogs

Canonical integrals can be found studying the integral representation of their **generalized cuts**

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{1}{D_1 \dots D_n} \quad \longrightarrow \quad \text{generalized cuts = residues} \quad \longrightarrow \quad \oint_C \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{1}{D_1 \dots D_n}$$

deforms integration contour to circle some propagators

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generalized cuts = residues

deforms integration contour
to circle some propagators

Maximal iteration of residues ⟶ **Leading singularities** of the integral

Canonical integrals = integrals for which all cuts are in dlog form and all residues are normalized to 1

→ **Unit leading singularities**

DLOG FORMS AND AMPLITUDES

$$\mathcal{A} \longrightarrow \sum_i R_i(s_{ij}) \int_\gamma d\log f_n \wedge \ldots \wedge d\log f_1$$

Rational functions:

encode poles (single particles going on shell)

MPLs (or iterated integrals on dlogs):

encode branch cuts (multiple particles going on shell)

fully encoded in local / canonical integrals!

Not obvious!

DLOG FORMS AND AMPLITUDES

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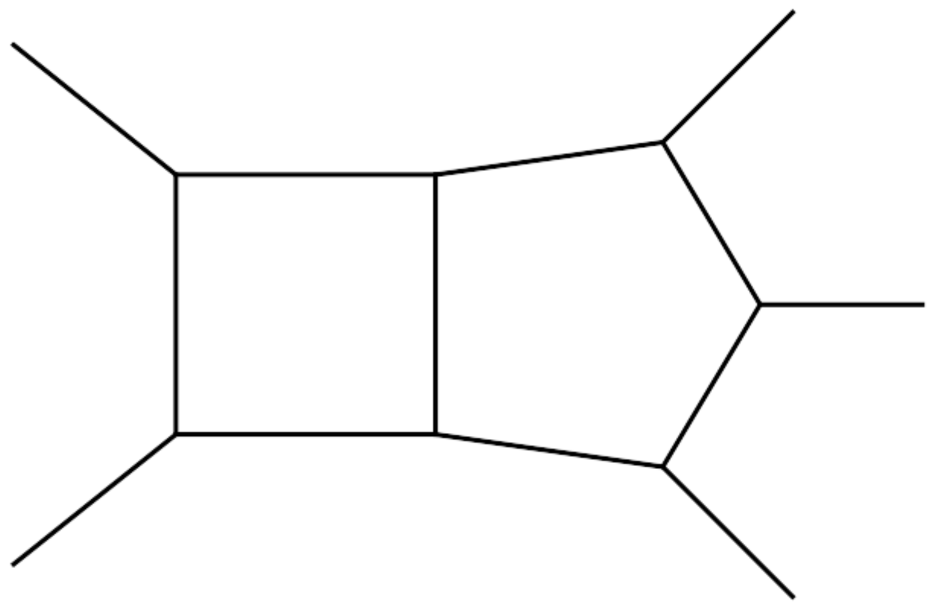
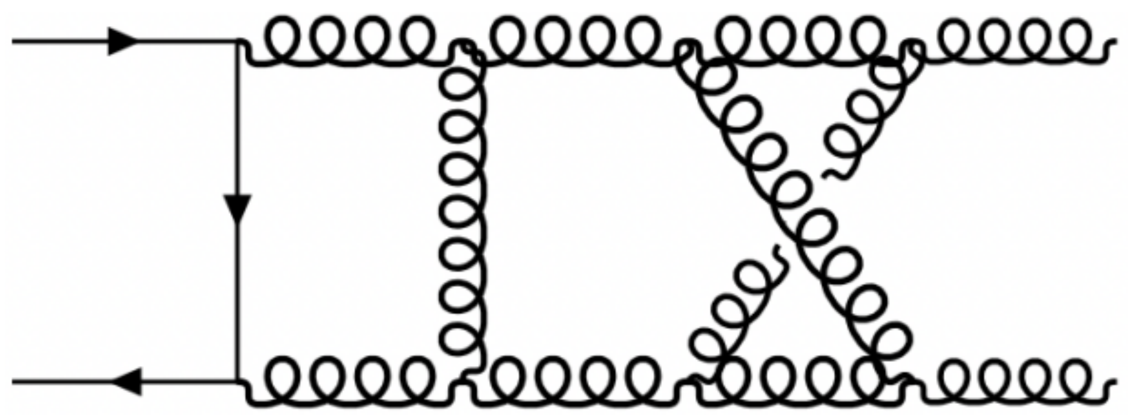
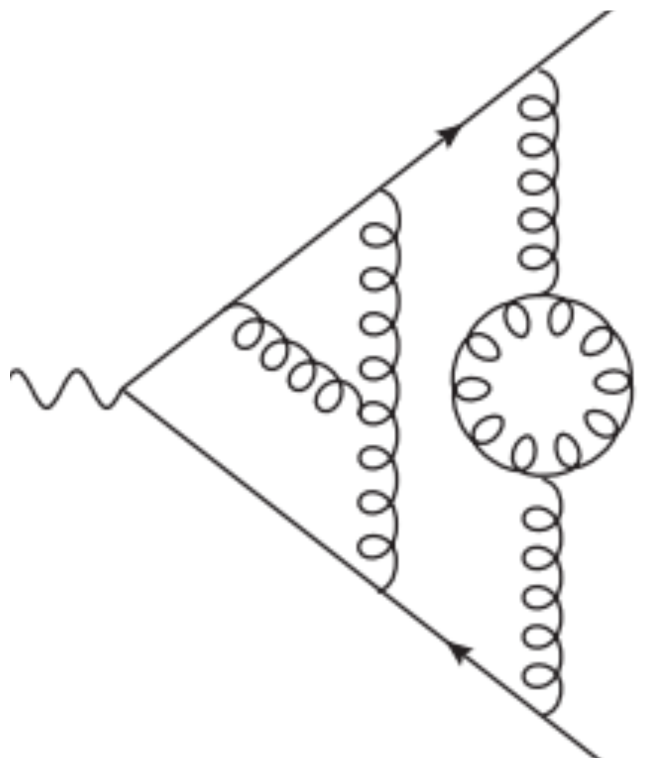






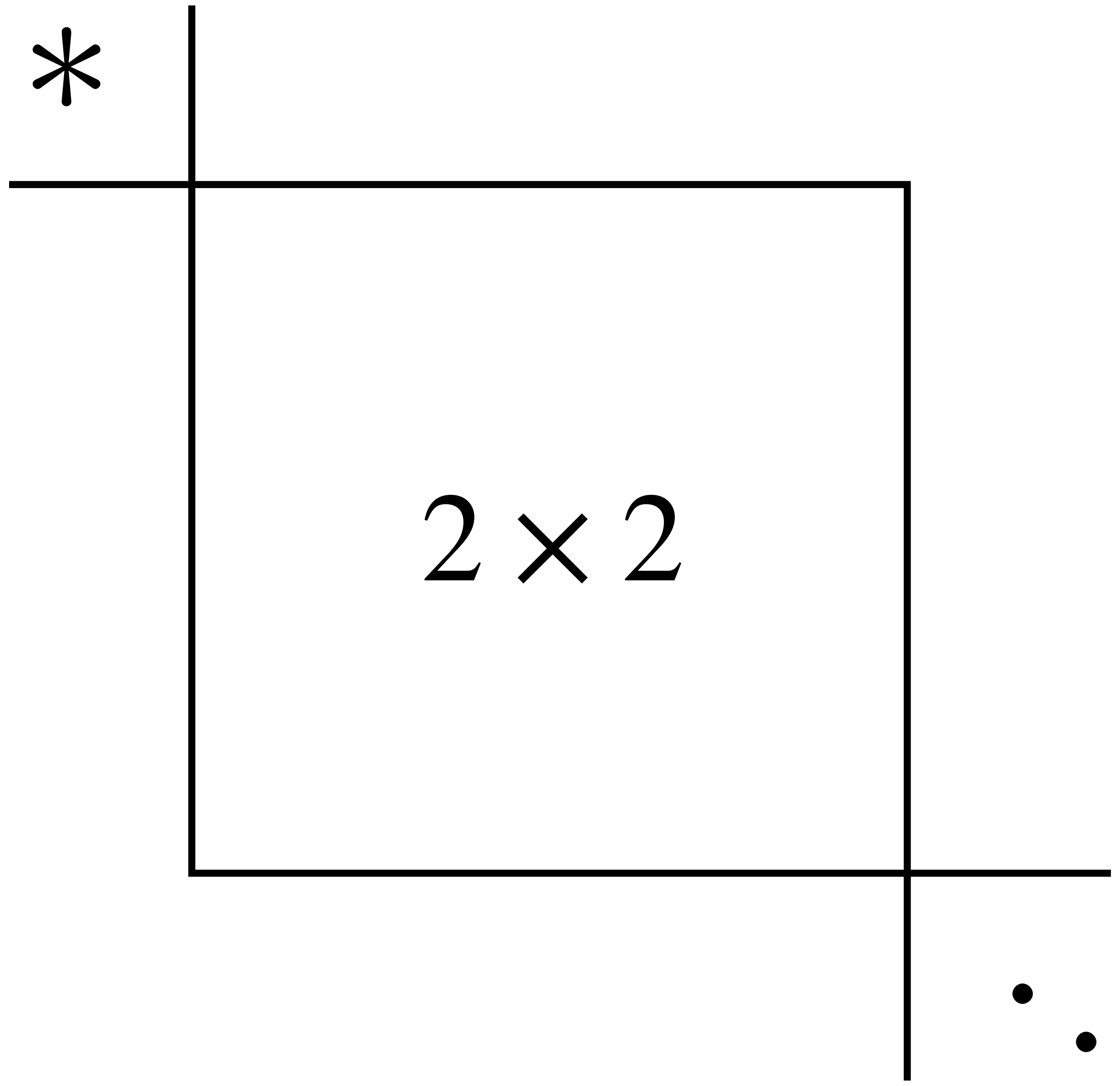
Massless QCD profited enormously from MPLs and "iterated integrals of dlog-forms"

HOW GENERAL IS THIS PICTURE?

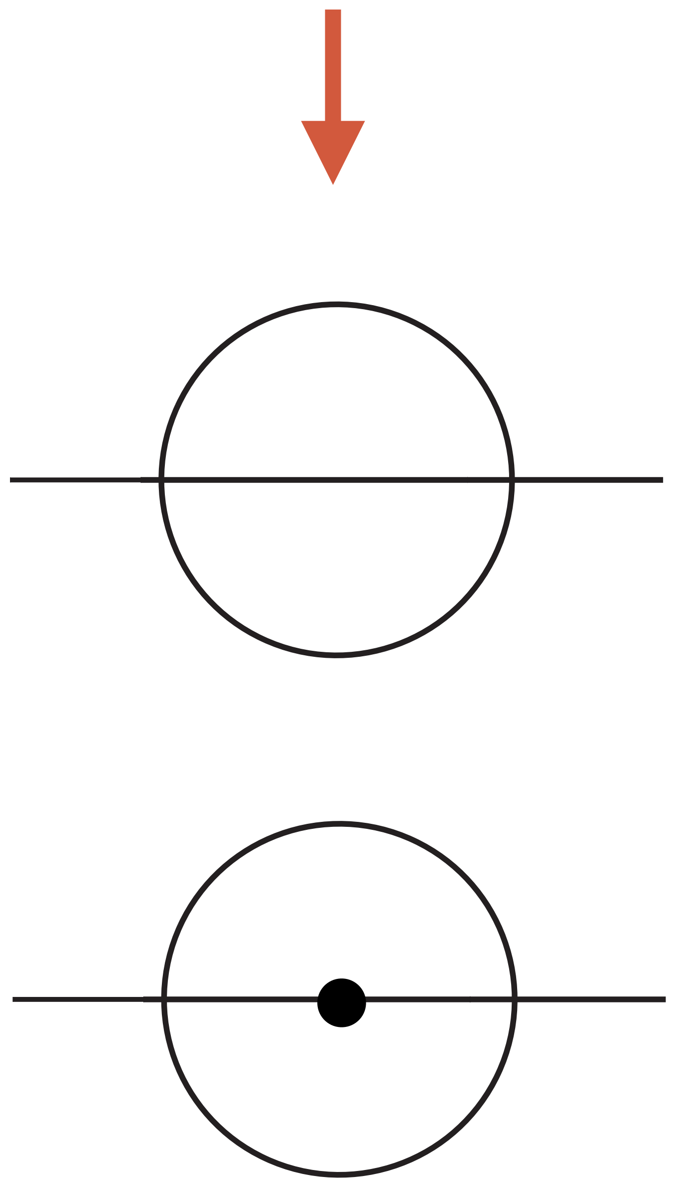
FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

The story might change for $D \to 4 - 2\epsilon$

all massive propagators



$$\to \begin{bmatrix} a_{11}(s_{ij}) & a_{12}(s_{ij}) \\ a_{21}(s_{ij}) & a_{22}(s_{ij}) \end{bmatrix} + \mathcal{O}(\epsilon)$$



If equation does not decouple, there is an intrinsic "higher order equation" *(2nd order Picard-Fuchs equation)*

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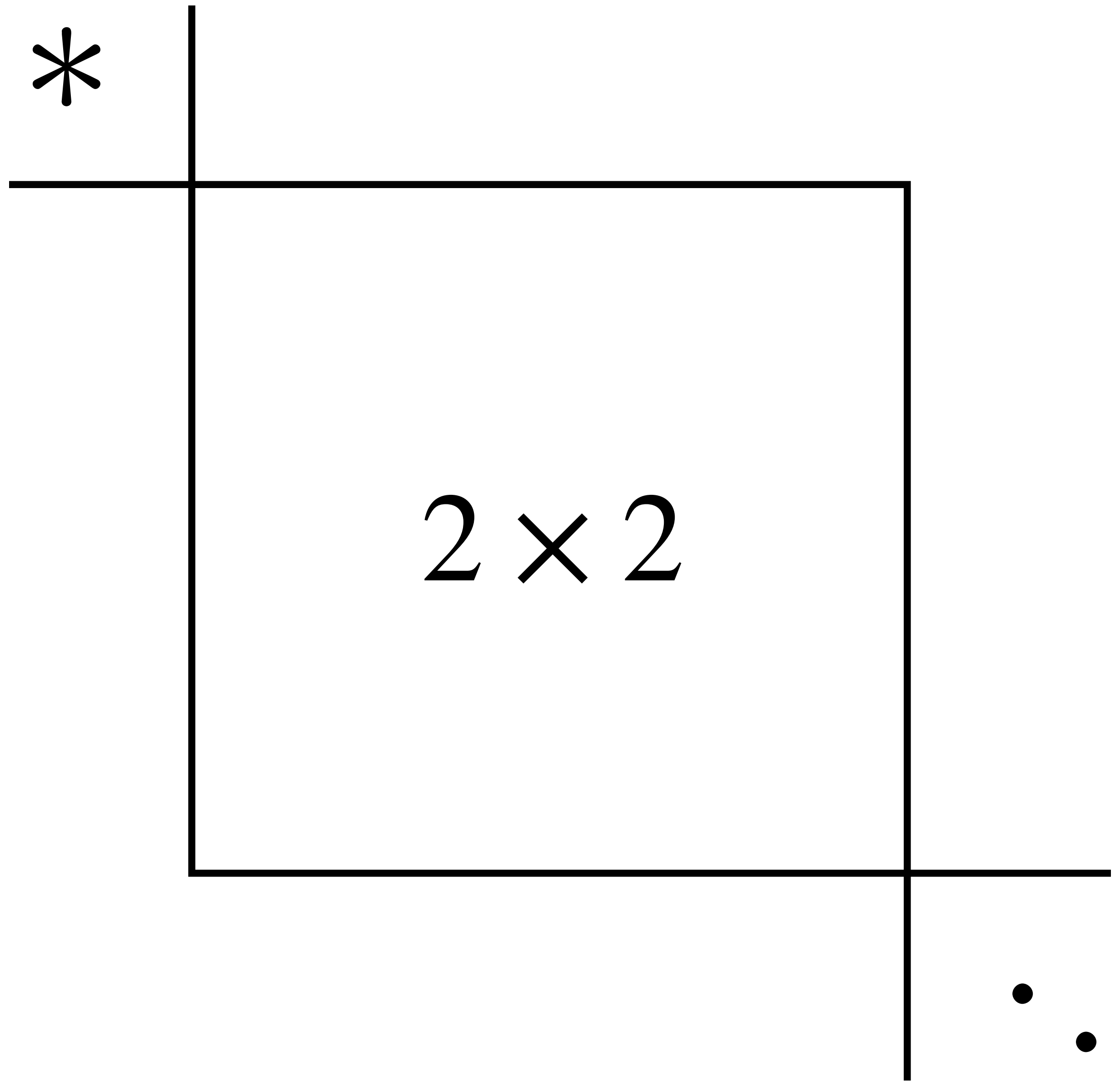

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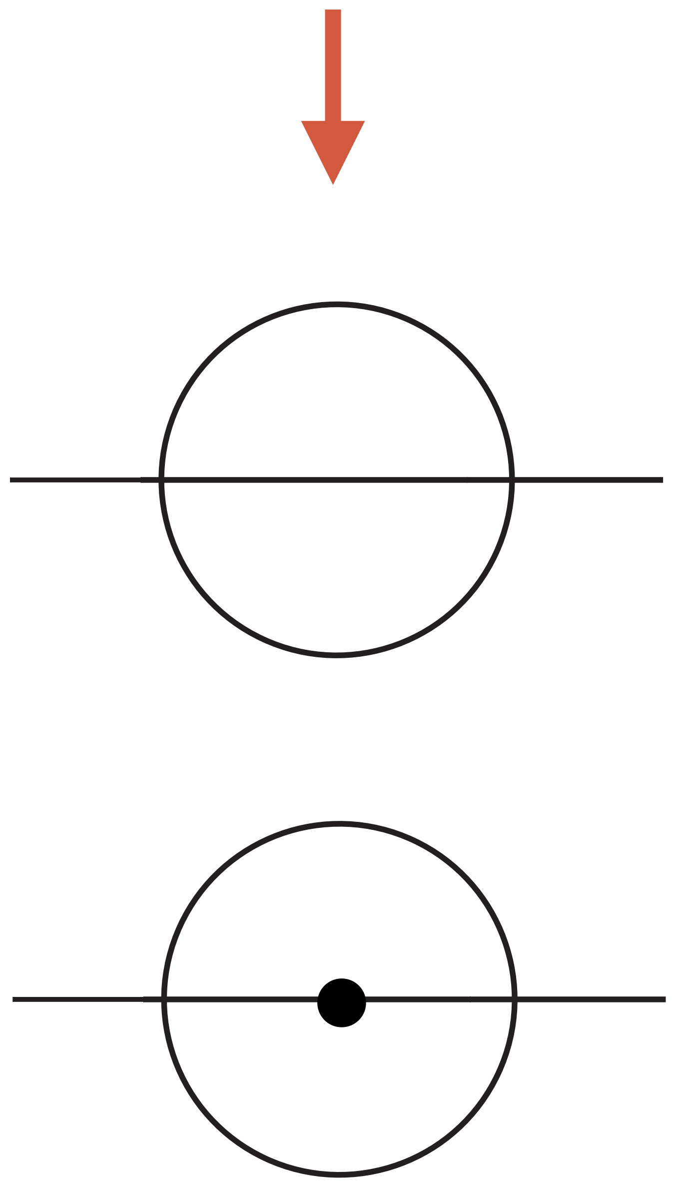
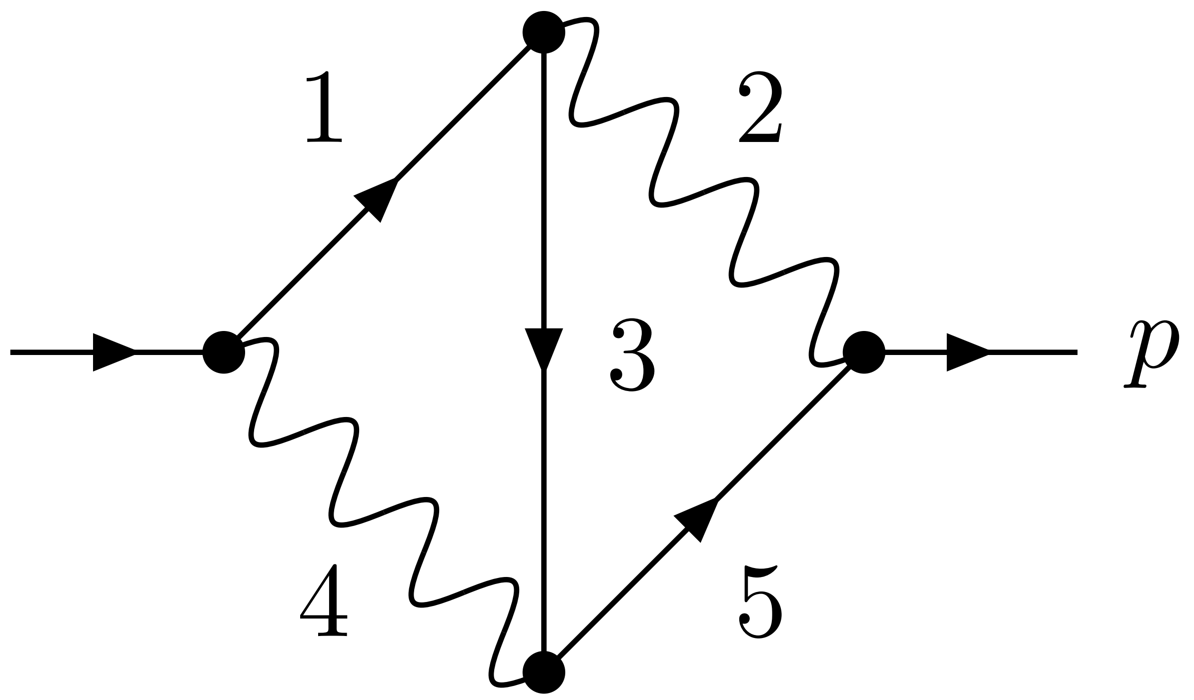


If equation does not decouple, there is an intrinsic "higher order equation" *(2nd order Picard-Fuchs equation)*

$$x = \frac{p^2}{m^2}$$

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]\varpi(x) = 0$$

THE TWO-LOOP ELECTRON PROPAGATOR



The electron propagator in QED; A. Sabri 1962

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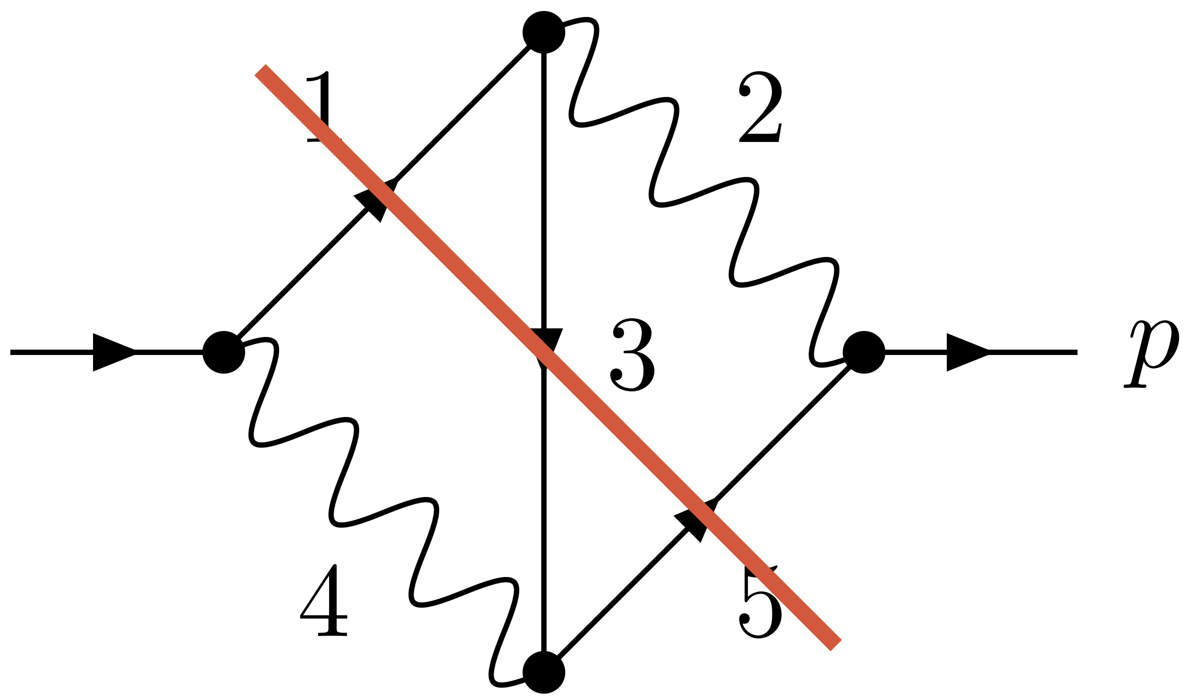

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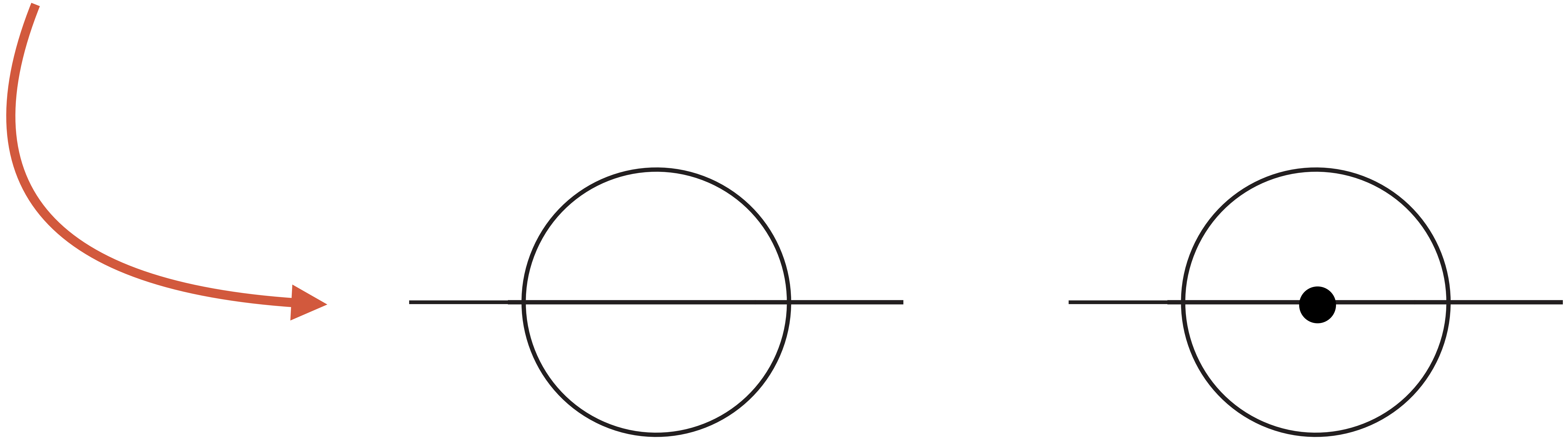
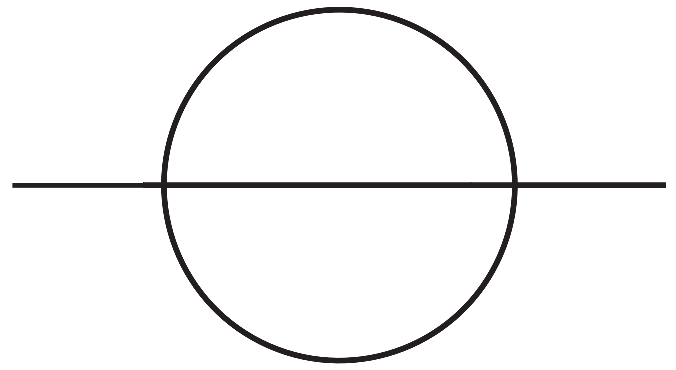
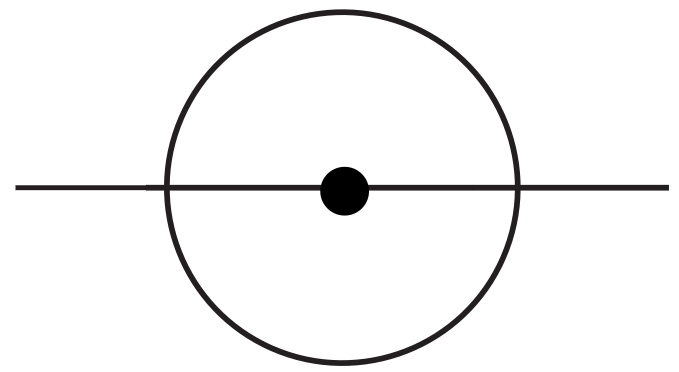


FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]\varpi(x) = 0$$



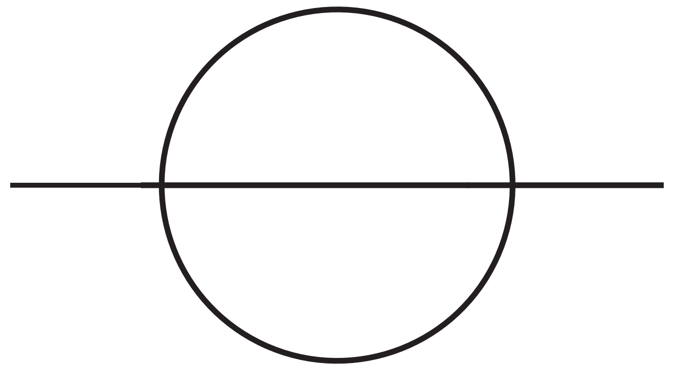


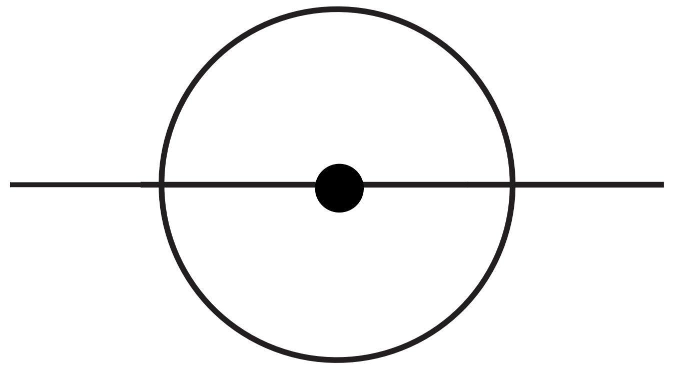
Solutions: **periods of an elliptic curve.** Obvious?

In some cases, you might be lucky enough to find the diff equation in some list of known ones…

FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]\varpi(x) = 0$$





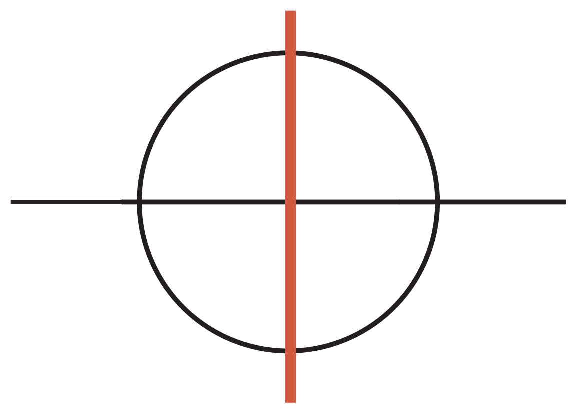
Solutions: **periods of an elliptic curve.** Obvious?

In some cases, you might be lucky enough to find the diff equation in some list of known ones…

By cutting all propagators (and continuing down to leading singularities) we can "expose" simplest integral which fulfils the **homogeneous differential equation**

[Primo, Tancredi '16,'17]

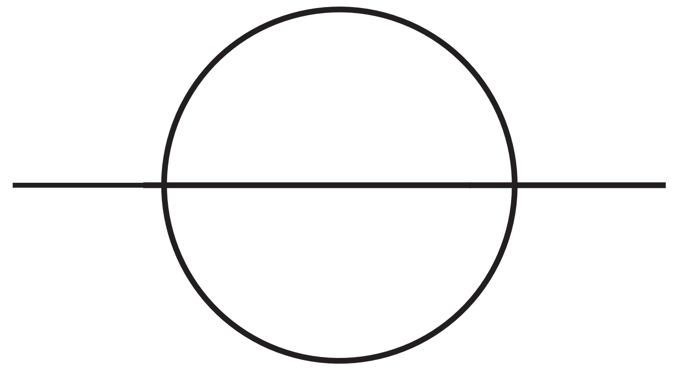
$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{1}{x-1} + \frac{9}{x-9} + 2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{27}{4(x-9)} + \frac{1}{4(x-1)} + 1\right]$$

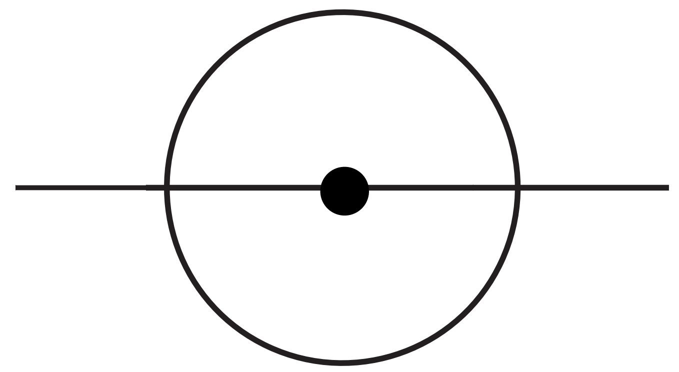


$= 0$

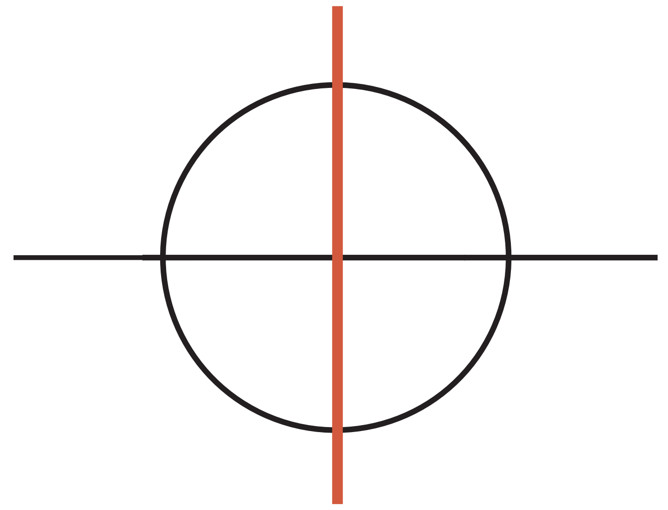
FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2+\left(\frac{1}{x-1}+\frac{9}{x-9}+2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)+\frac{27}{4(x-9)}+\frac{1}{4(x-1)}+1\right]\varpi(x)=0$$



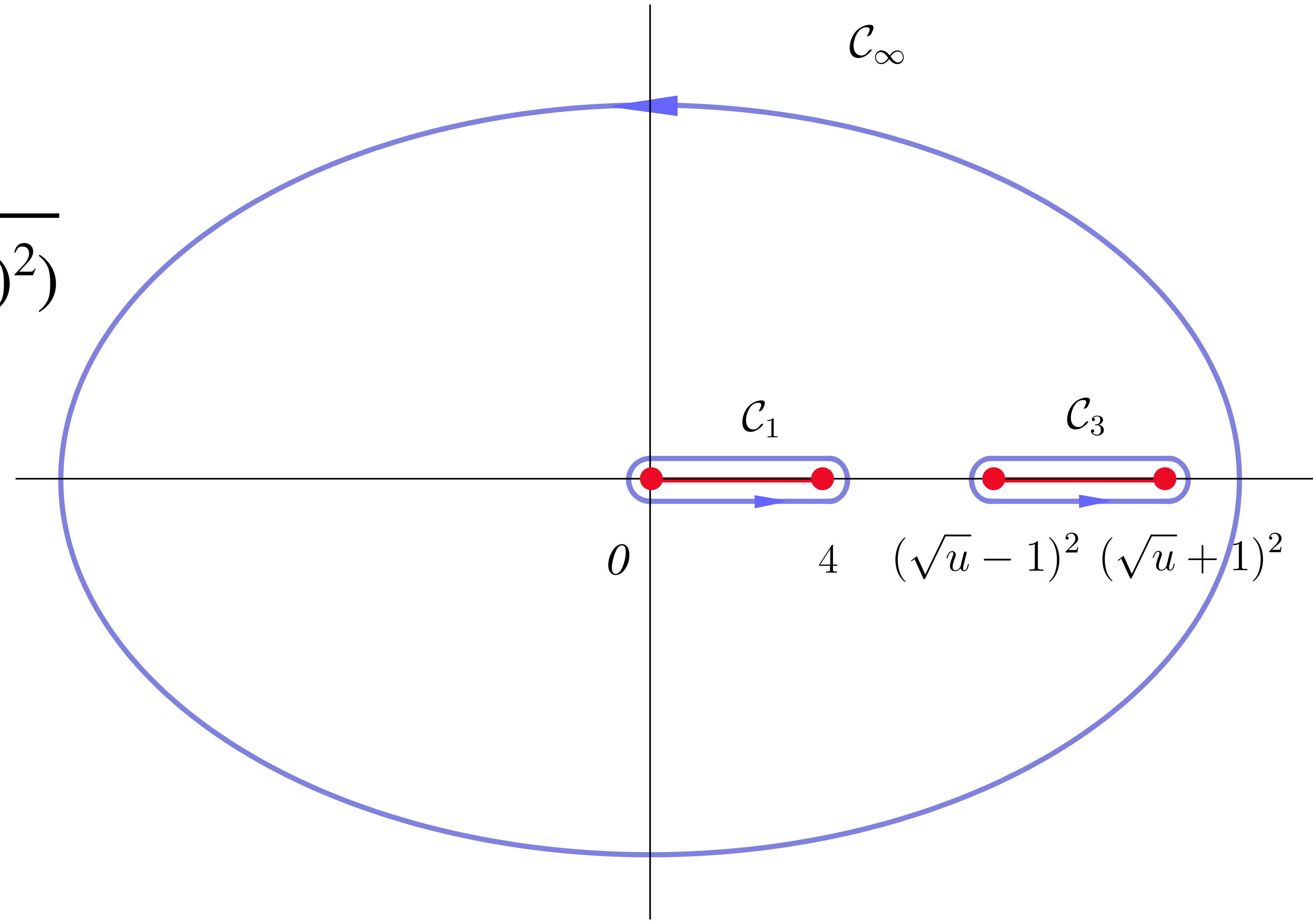


Solutions: **periods of an elliptic curve.** Obvious!!!



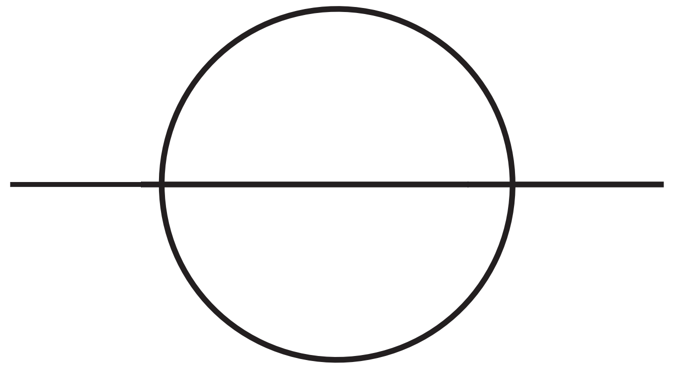
$$=\int_{4m^2}^{(\sqrt{s}-m)^2}\frac{dt}{Y}\qquad Y=\sqrt{t(t-4m^2)(t-(\sqrt{s}-m)^2)(t-(\sqrt{s}+m)^2)}$$

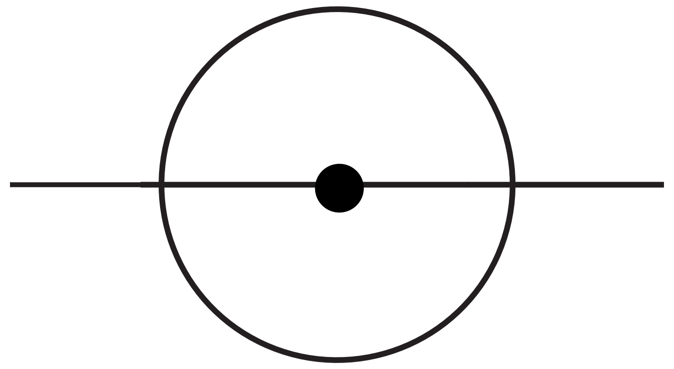
$$=\frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$



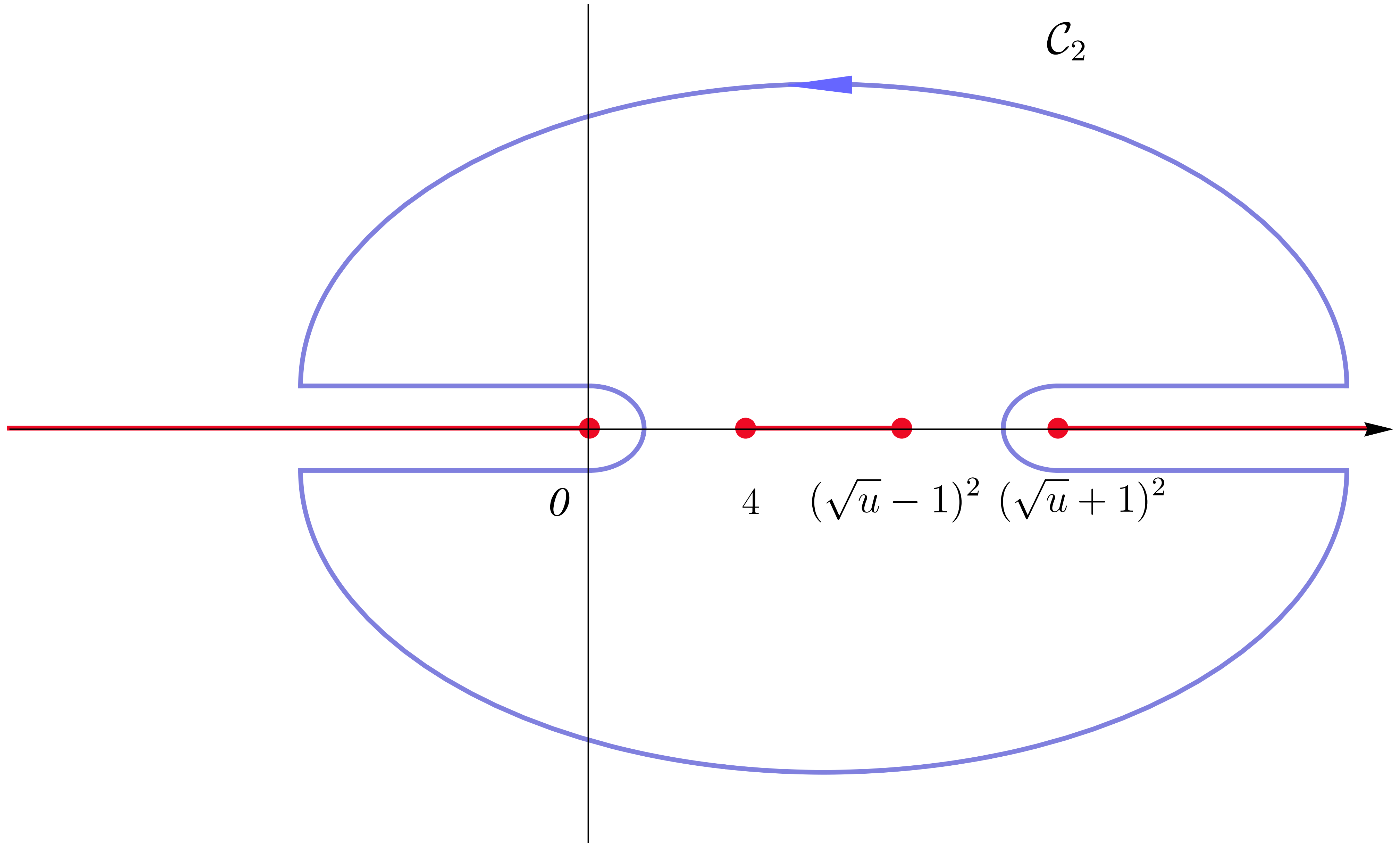
FROM DIFFERENTIAL EQUATIONS TO GEOMETRY

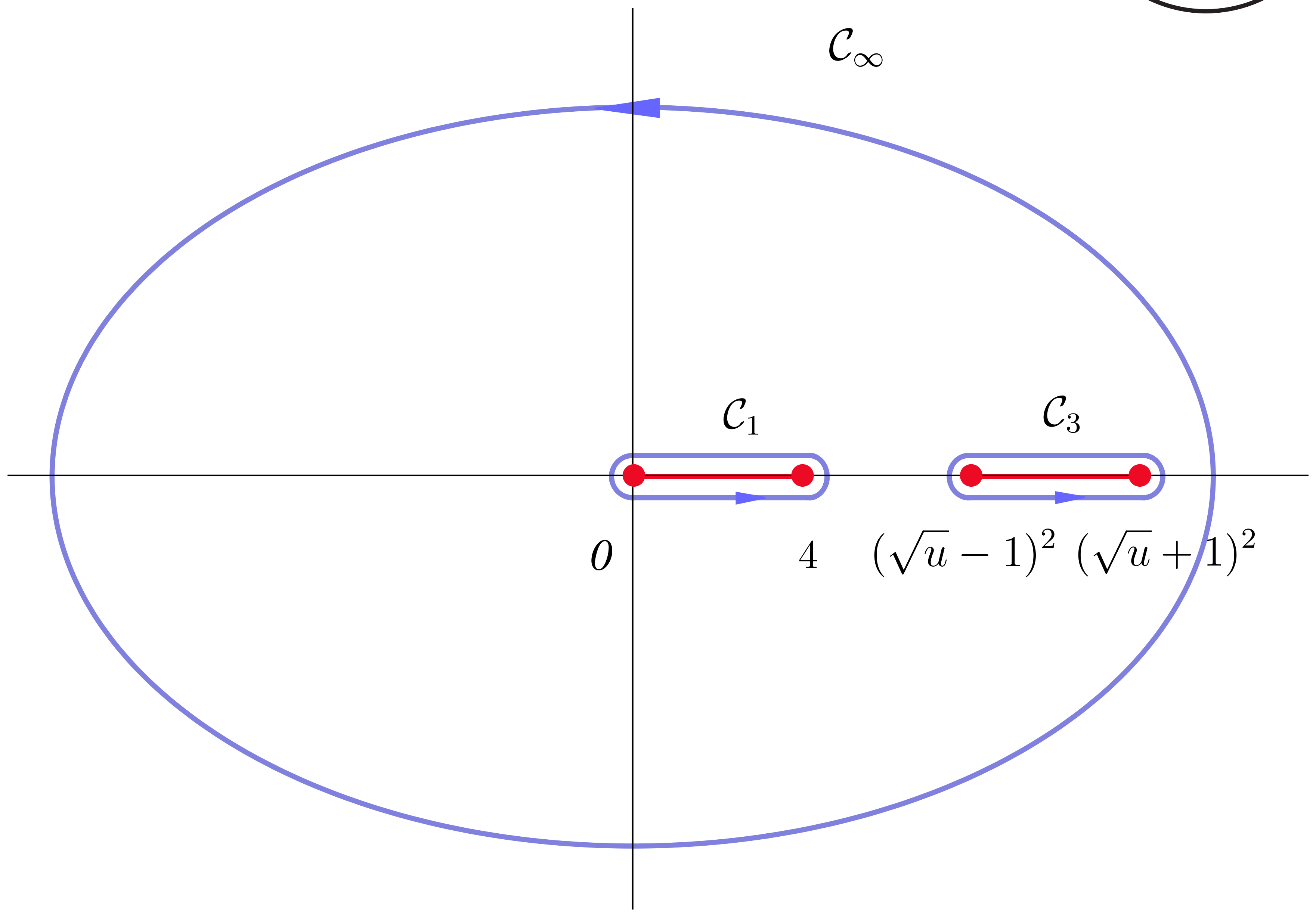
$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2+\left(\frac{1}{x-1}+\frac{9}{x-9}+2\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)+\frac{27}{4(x-9)}+\frac{1}{4(x-1)}+1\right]\varpi(x)=0$$





Solutions: **periods of an elliptic curve.** Obvious!!!





second independent solution from second integration contour

[Primo, Tancredi ’16,’17]

ELLIPTIC CURVES AND COMPLEX TORI

Elliptic curve given by an algebraic equation

$$y = \pm\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}$$

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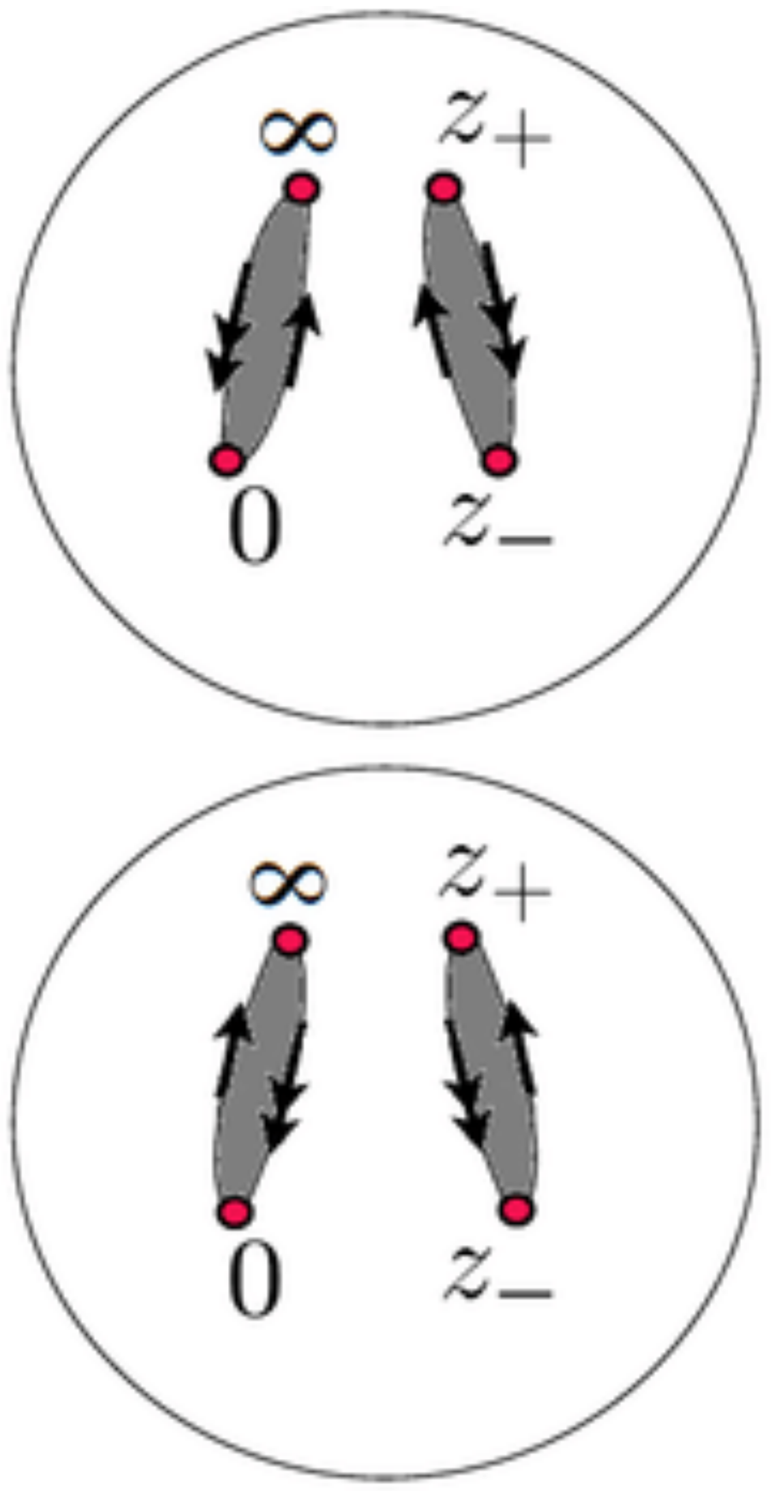
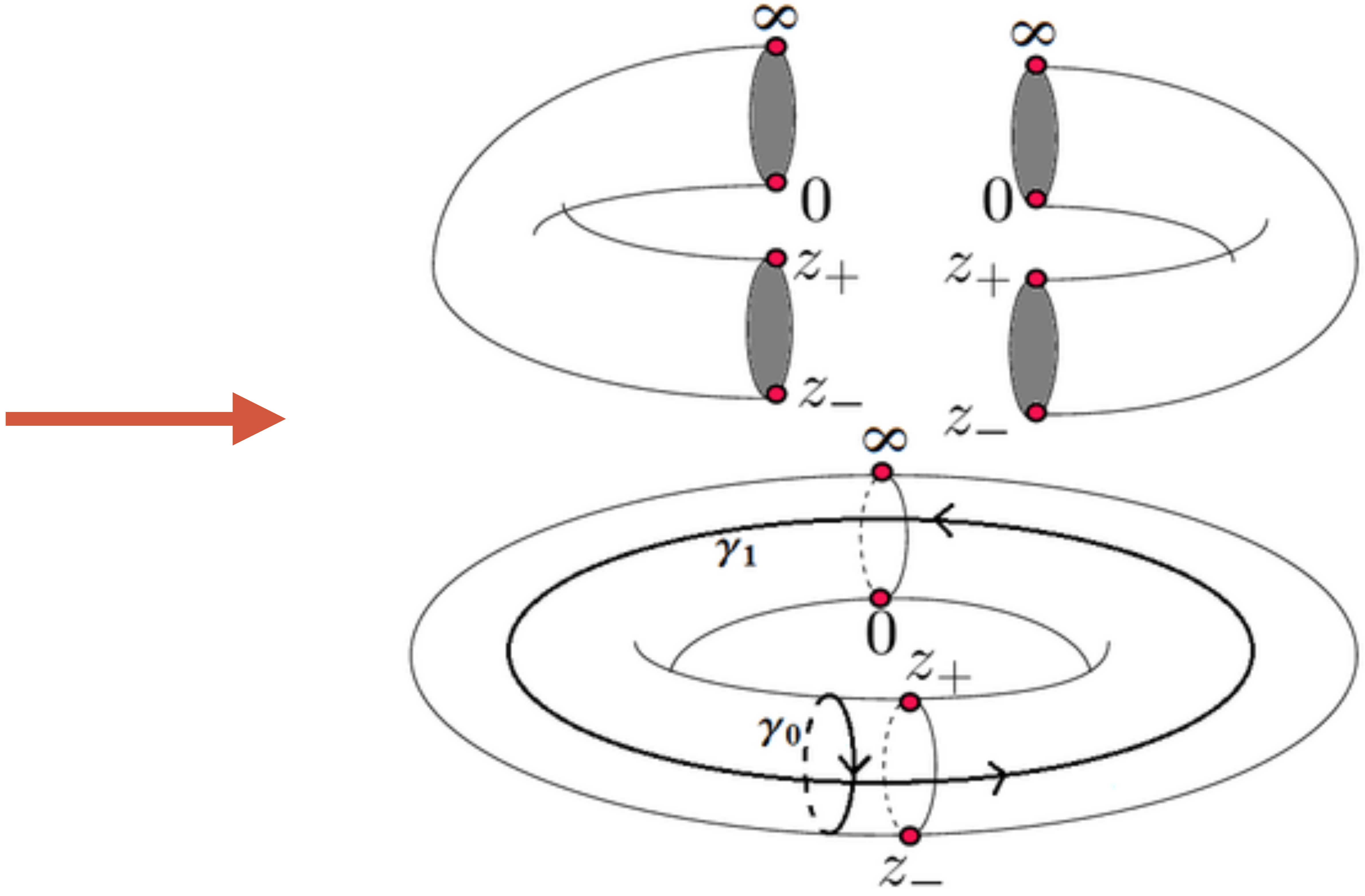
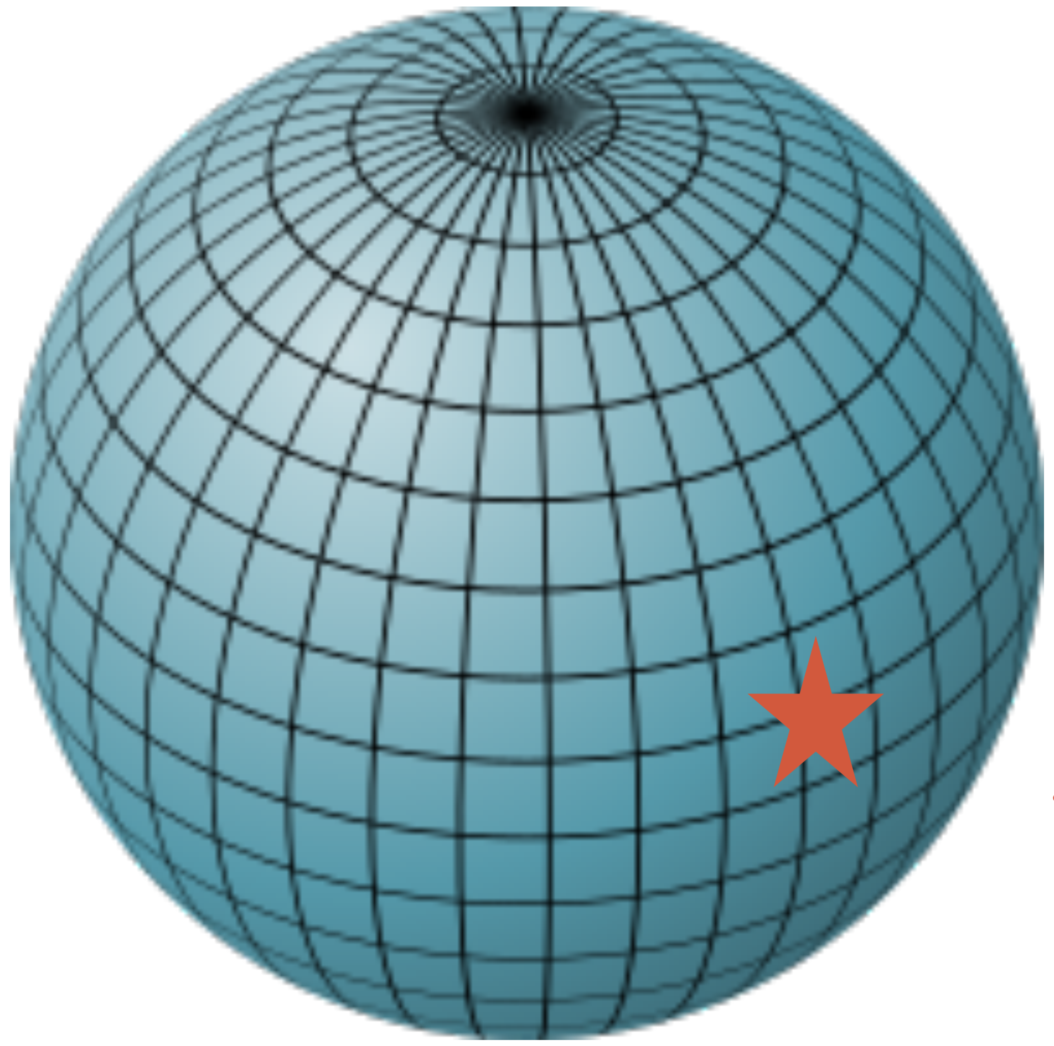
Torus is the **Riemann surface** associated to the square root with 3 or 4 branching points





HIGHER POLES ON ELLIPTIC GEOMETRIES

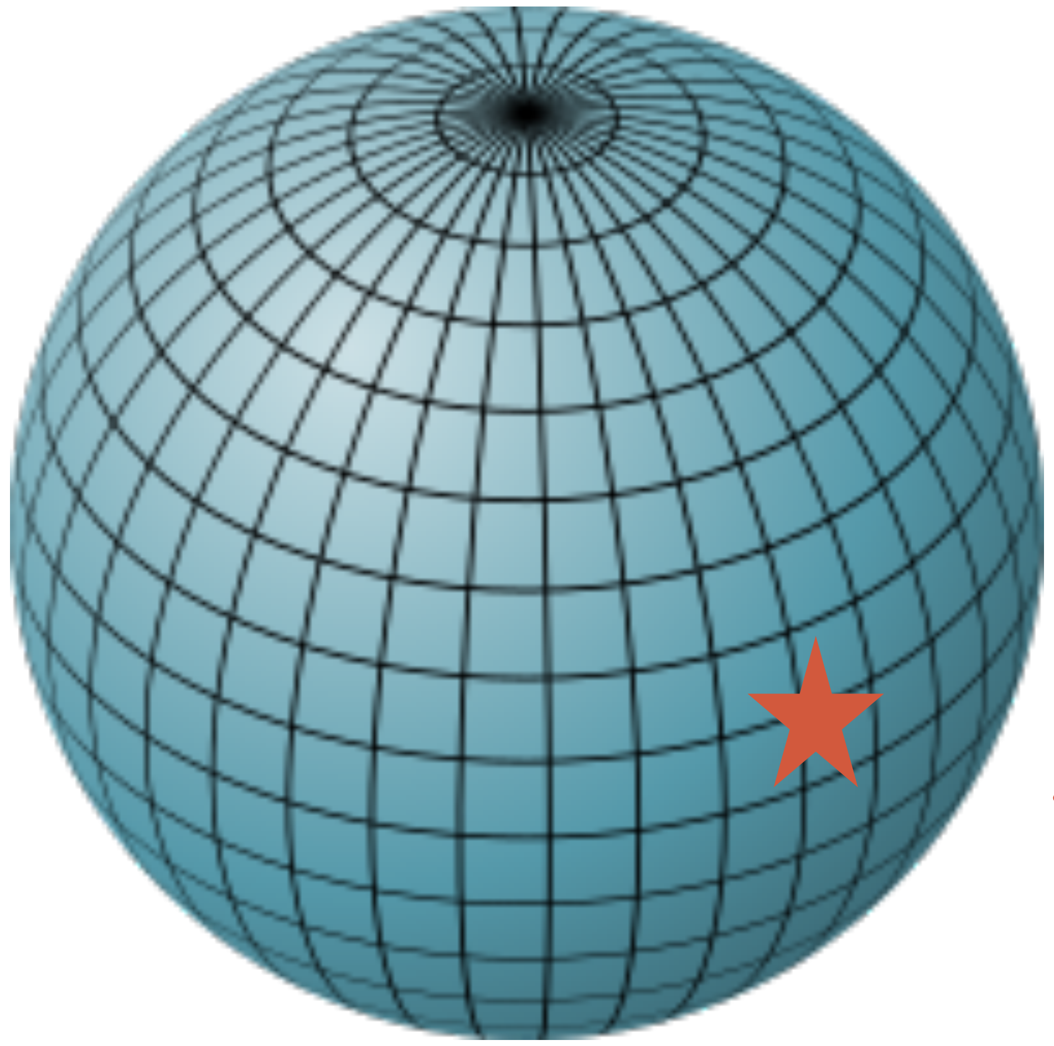


$x = a$

entire space of functions spanned by single poles

$$\log(1 - x/a) = \int_0^x \frac{dt}{t - a}$$

HIGHER POLES ON ELLIPTIC GEOMETRIES



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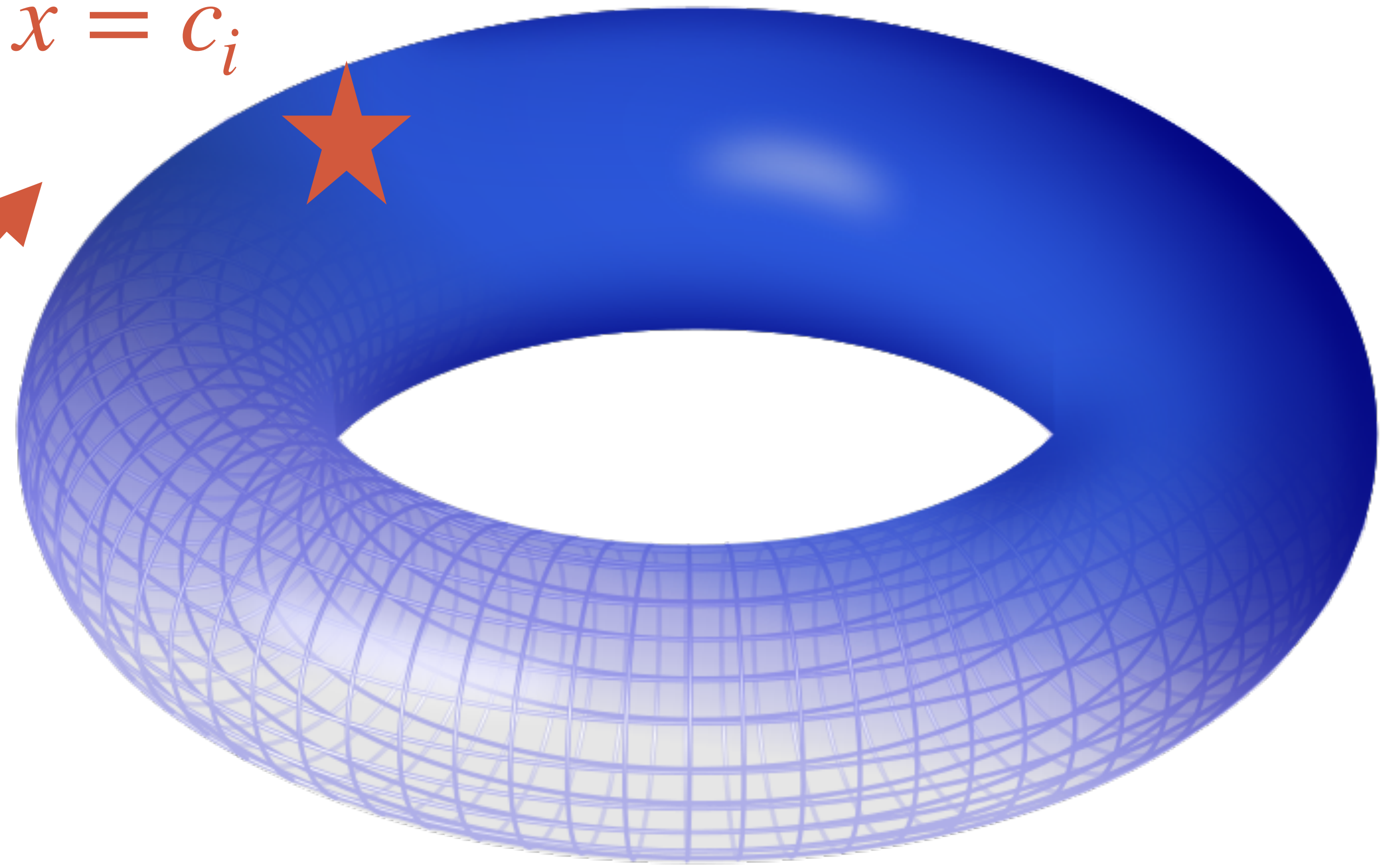
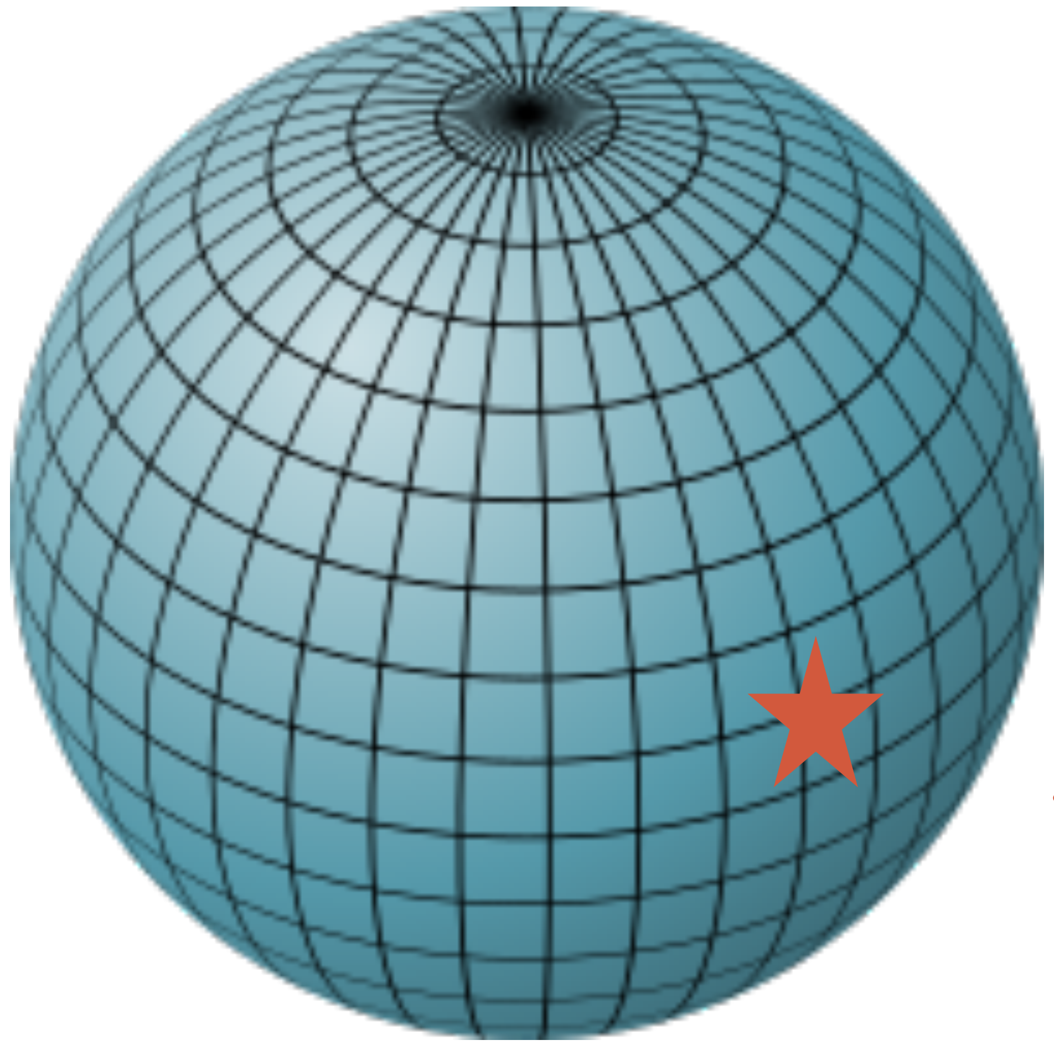


genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

Third kind

single poles $\quad g \sim \int \frac{dx}{(x-c_i)y}$

HIGHER POLES ON ELLIPTIC GEOMETRIES



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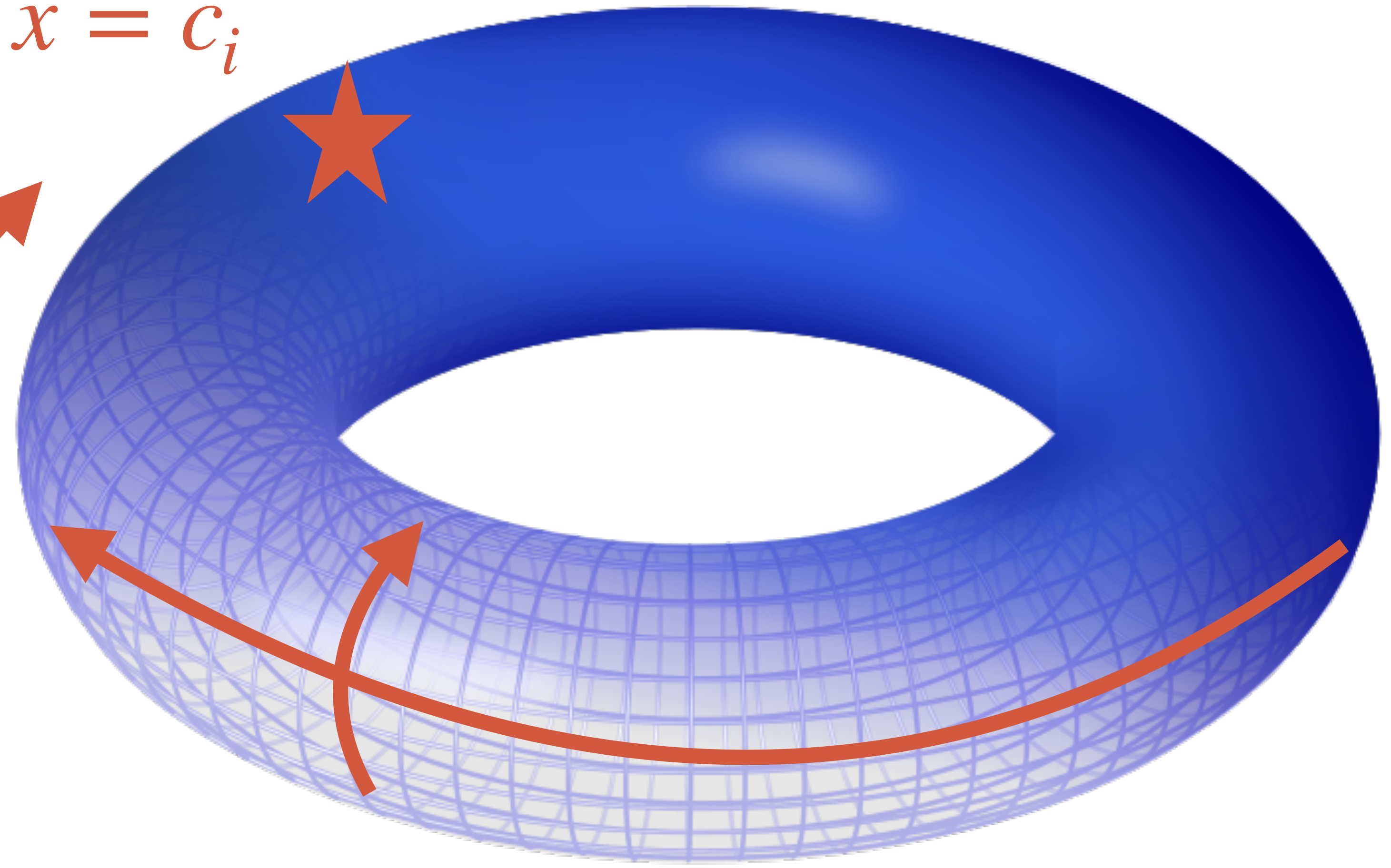


genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

First kind

No poles $\omega \sim \int \frac{dx}{y}$

Second kind

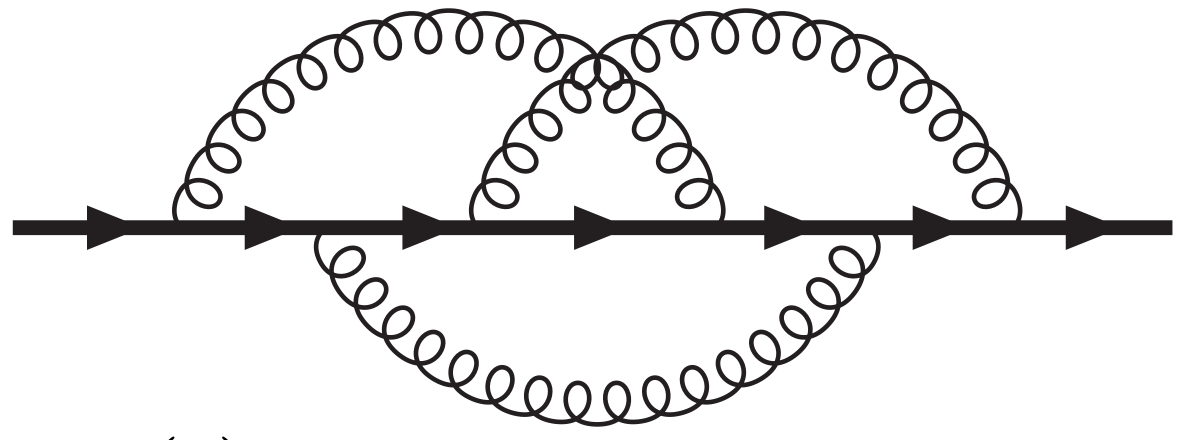
double poles $\eta \sim \int \frac{dx\, x}{y}$

Third kind

single poles $g \sim \int \frac{dx}{(x - c_i)y}$

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

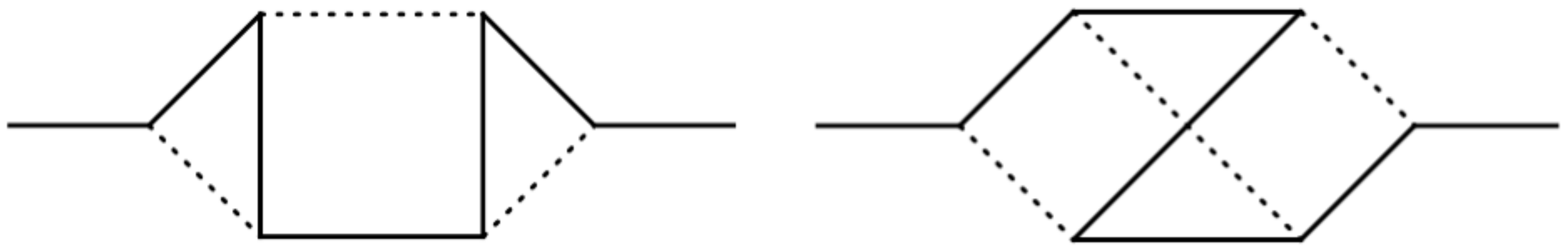
[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



$$\hat{p}\ \Sigma_V(p^2, m^2) + m\ \Sigma_S(p^2, m^2)$$

Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

2 "top graphs"



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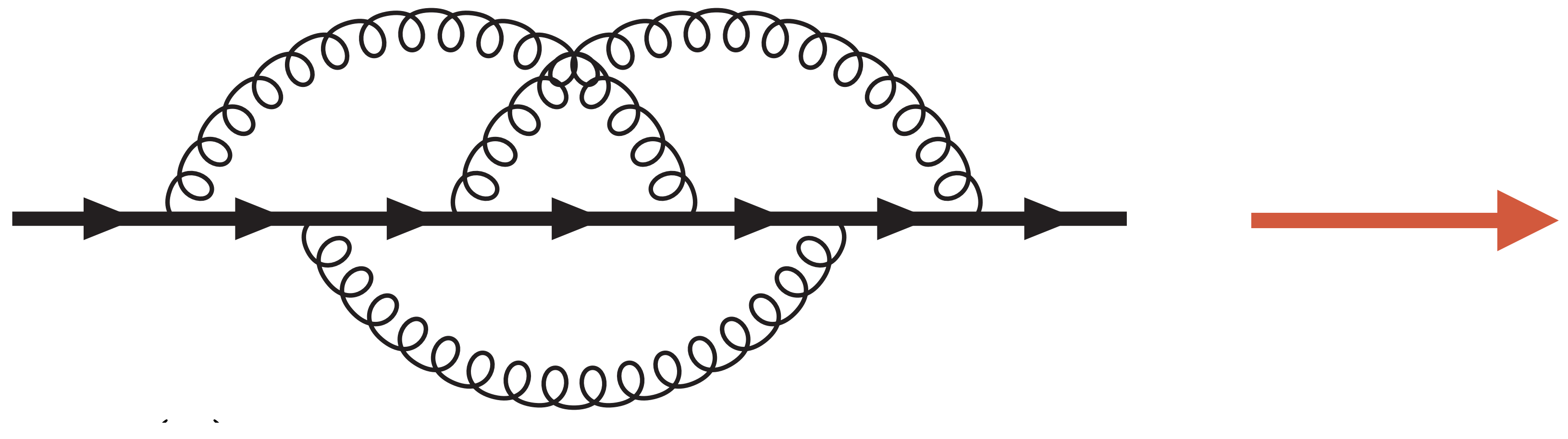

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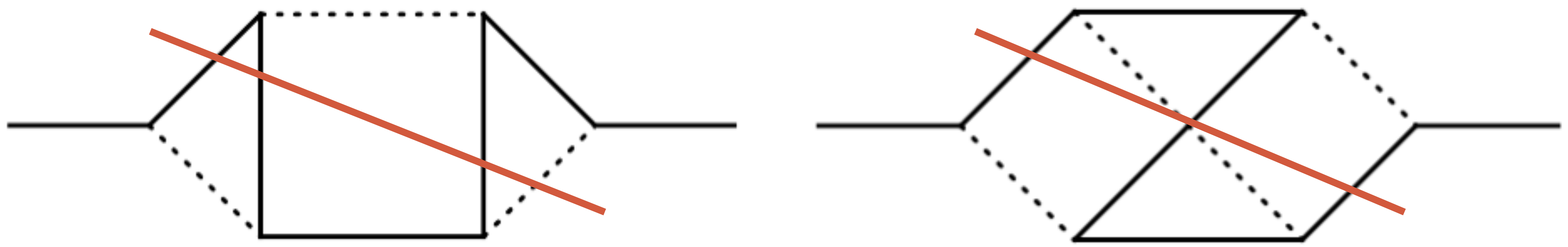
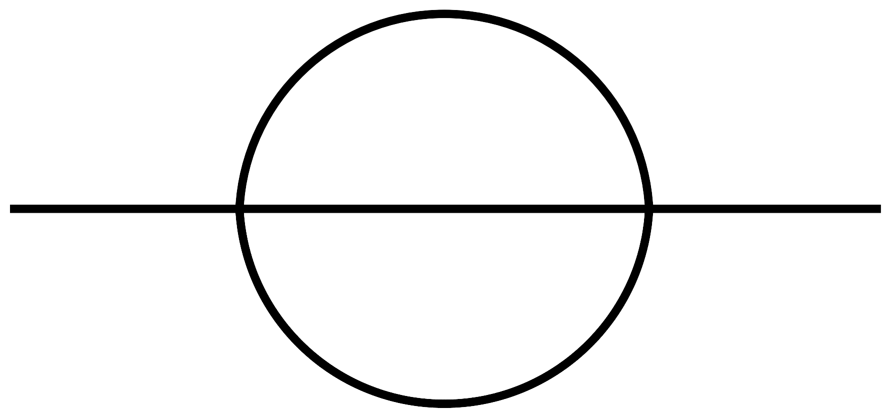


mix of elliptic and polylogarithmic sectors

same elliptic curve as 2loop sunrise graph



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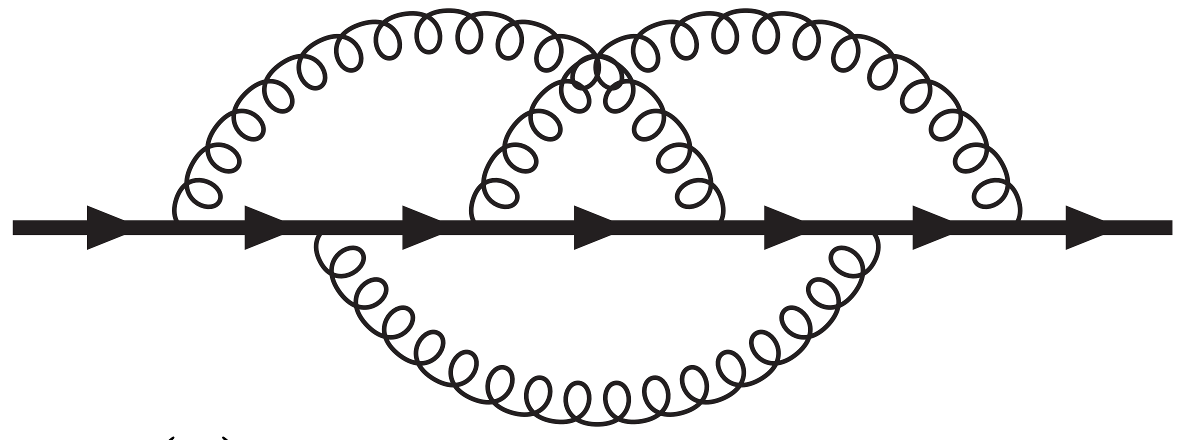


$\hat{p}\,\Sigma_V(p^2, m^2) + m\,\Sigma_S(p^2, m^2)$

Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

Geometrical picture allows us to find also in this case a **canonical basis**

Various delicate points due to double poles on elliptic curve

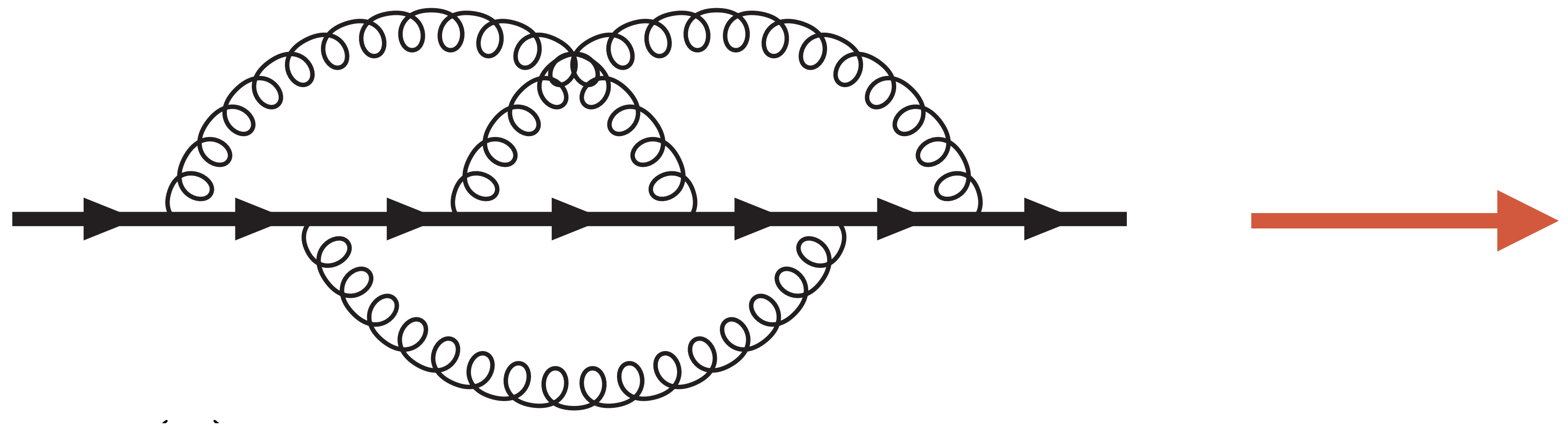
[Görges, Nega, Tancredi, Wagner '23]

$$\mathrm{d}\vec{J} = \epsilon \left(\sum_i G_i\,\omega_i \right) \vec{J} \quad \longleftrightarrow \quad f_i(x)\mathrm{d}x = \omega_i$$

Progress also from: **Dlapa, Henn, Wagner; Pögel, Wang, Weinzierl; Frellesvig; ...**

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



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Σ_V & Σ_S expressed in terms of $\mathcal{O}(50)$ Masters Integrals $\vec{J}$

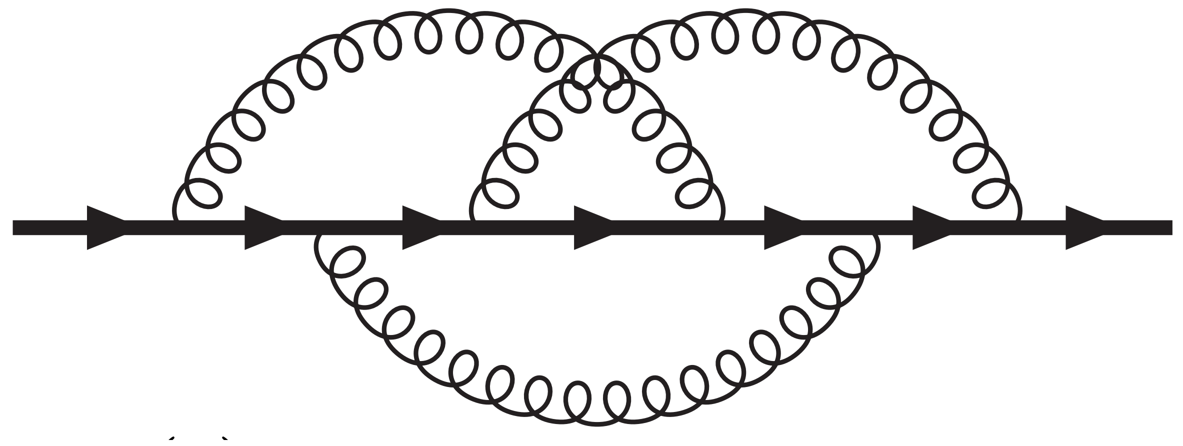
7 (independent) elliptic differential forms: full analytic control over **iterated integrals** over these forms

$$f_i \in \left\{ \frac{1}{x(x-1)(x-9)\varpi_0(x)^2}, \varpi_0(x), \frac{\varpi_0(x)}{x-1}, \frac{(x-3)\varpi_0(x)}{\sqrt{(1-x)(9-x)}}, \frac{(x+3)^4\varpi_0(x)^2}{x(x-1)(x-9)}, \frac{(x+3)(x-1)\varpi_0(x)^2}{x(x-9)}, \frac{\varpi_0(x)^2}{(x-1)(x-9)} \right\} \quad \text{for } i = 10, \ldots, 16\,,$$

$\varpi_0(x)$ is the first elliptic period

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



$\longrightarrow \quad \hat{p}\, \Sigma_V(p^2, m^2) + m\, \Sigma_S(p^2, m^2)$

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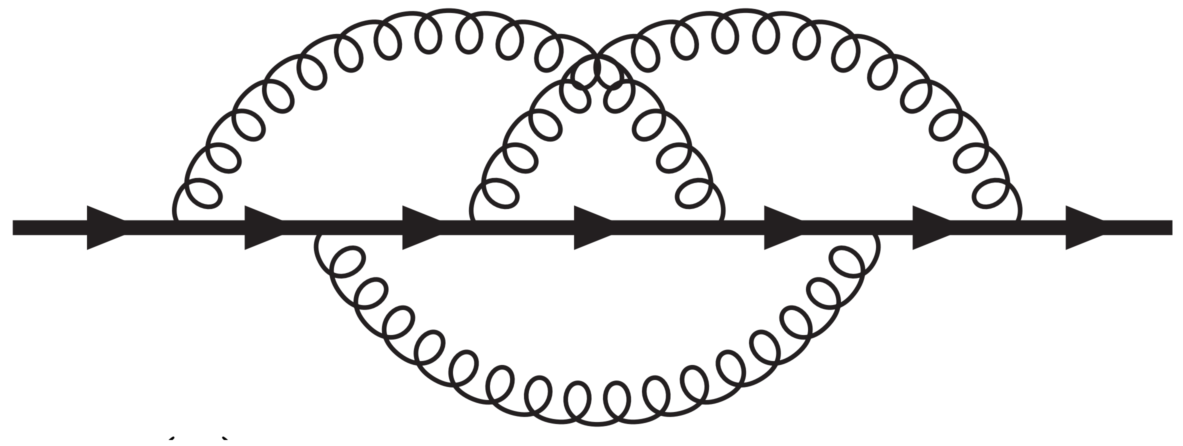
3 of the kernels drop in the physical amplitude: they are related to forms of the second kind with "double poles" → a hint for bootstrap program?

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl ’24]



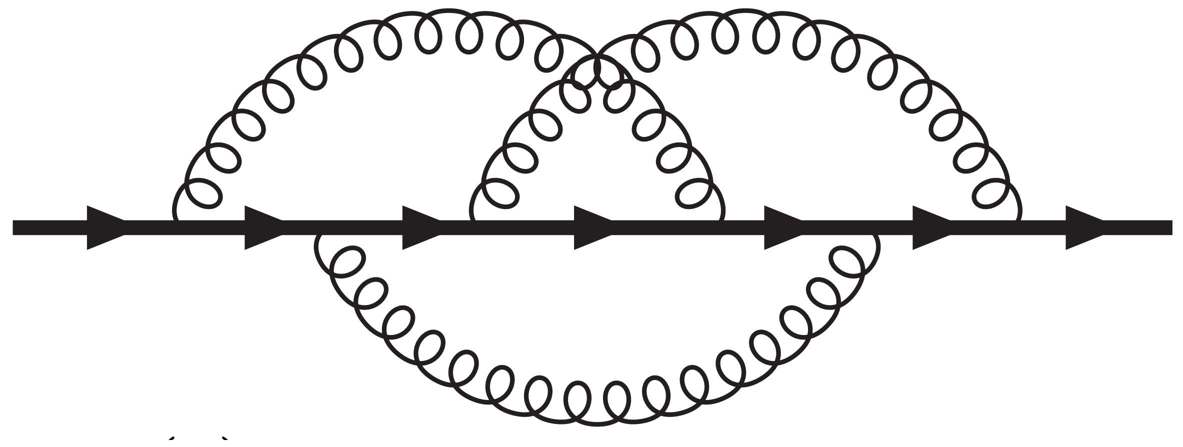
$$\hat{p}\,\Sigma_V(p^2, m^2) + m\,\Sigma_S(p^2, m^2)$$

One can obtain resummed results close to on-shell limit $p^2 = m^2$ required for UV renormalization

$$\Sigma^{(3)}_{V,\text{res}} = \left[-\frac{27}{128\epsilon^3} - \frac{673}{1152\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + (1-x)^{-2\epsilon}\left[\frac{27}{64\epsilon^3} + \frac{27}{32\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right]$$
$$+ (1-x)^{-4\epsilon}\left[-\frac{27}{128\epsilon^3} - \frac{27}{128\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + \mathcal{O}(1-x)\,,$$
$$\Sigma^{(3)}_{S,\text{res}} = \left[-\frac{653}{1152\epsilon^3} + \frac{1447}{6912\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + (1-x)^{-2\epsilon}\left[\frac{1}{x-1}\left(\frac{27}{32\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right)\right.$$
$$\left. + \frac{9}{16\epsilon^3} - \frac{91}{256\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + (1-x)^{-4\epsilon}\left[\frac{27}{128\epsilon^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right] + \mathcal{O}(1-x)\,,$$

AN EXAMPLE CALCULATION: THE THREE-LOOP QED SELF-ENERGY

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]



$$\to \quad \hat{p}\, \Sigma_V(p^2, m^2) + m\, \Sigma_S(p^2, m^2)$$

Similarly, one can easily expand results and obtain **fast converging series expansions** for numerics

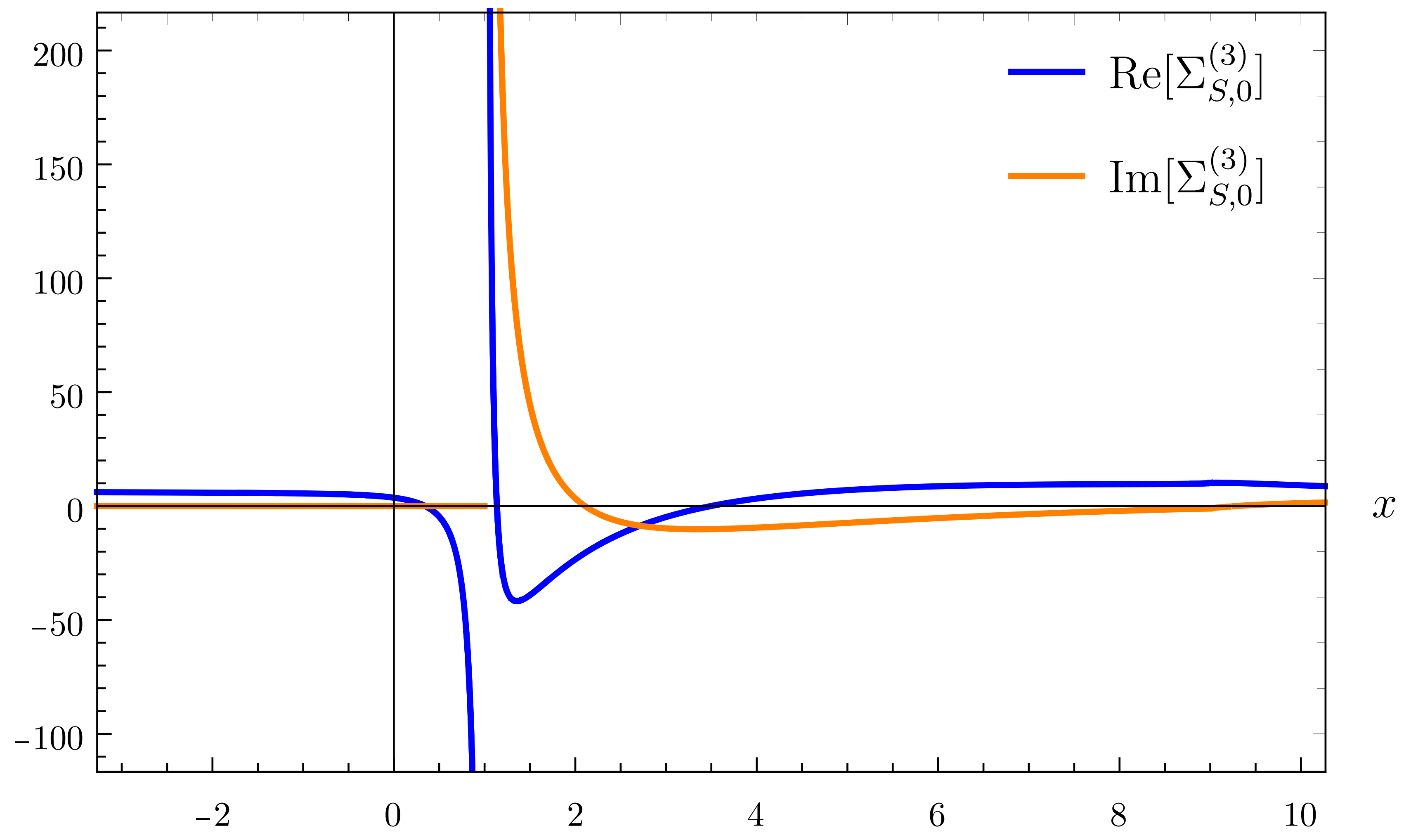


Figure 4: Real and imaginary part of $\Sigma^{(3)}_{S,0}$ (for $\xi = 0$).

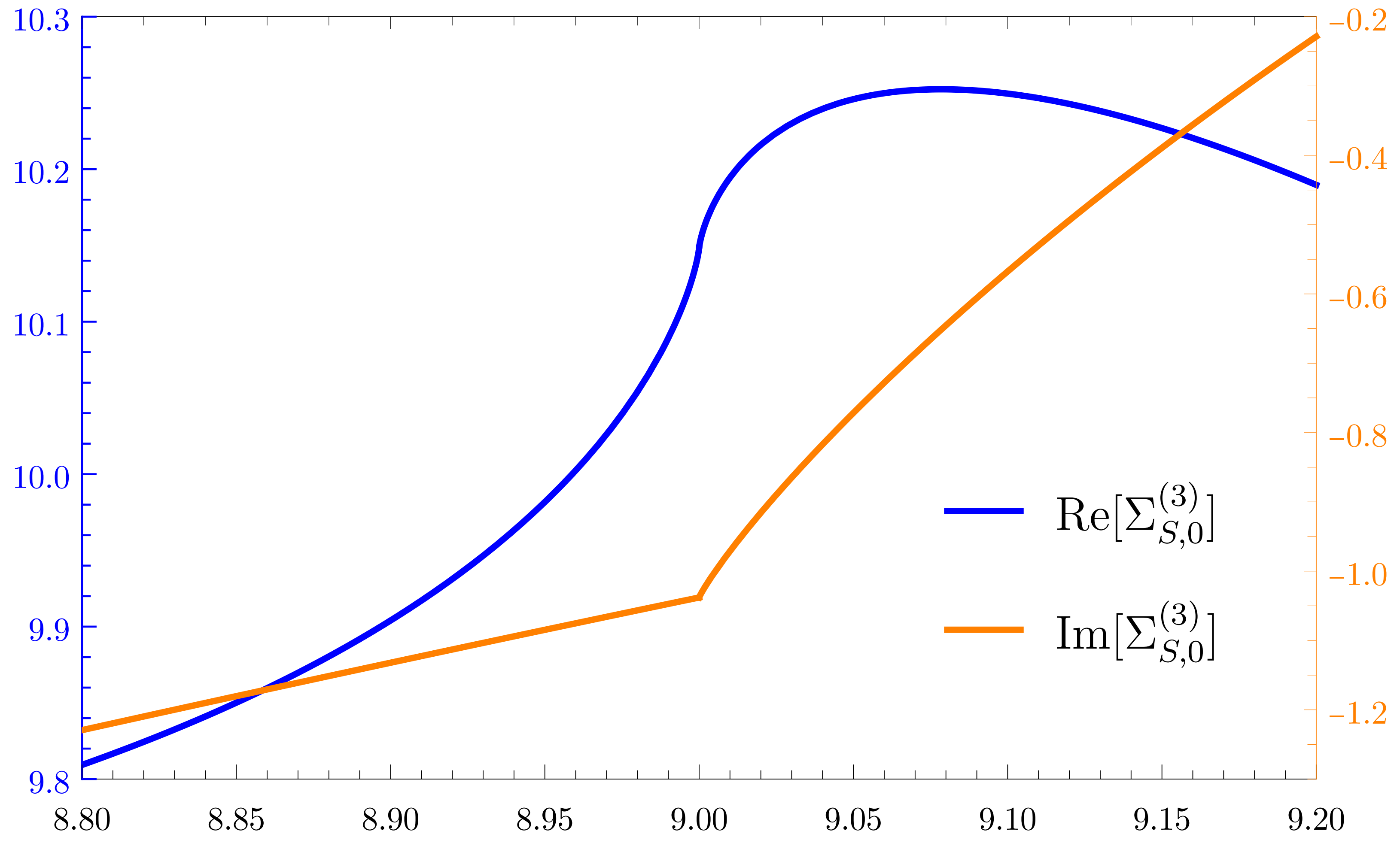
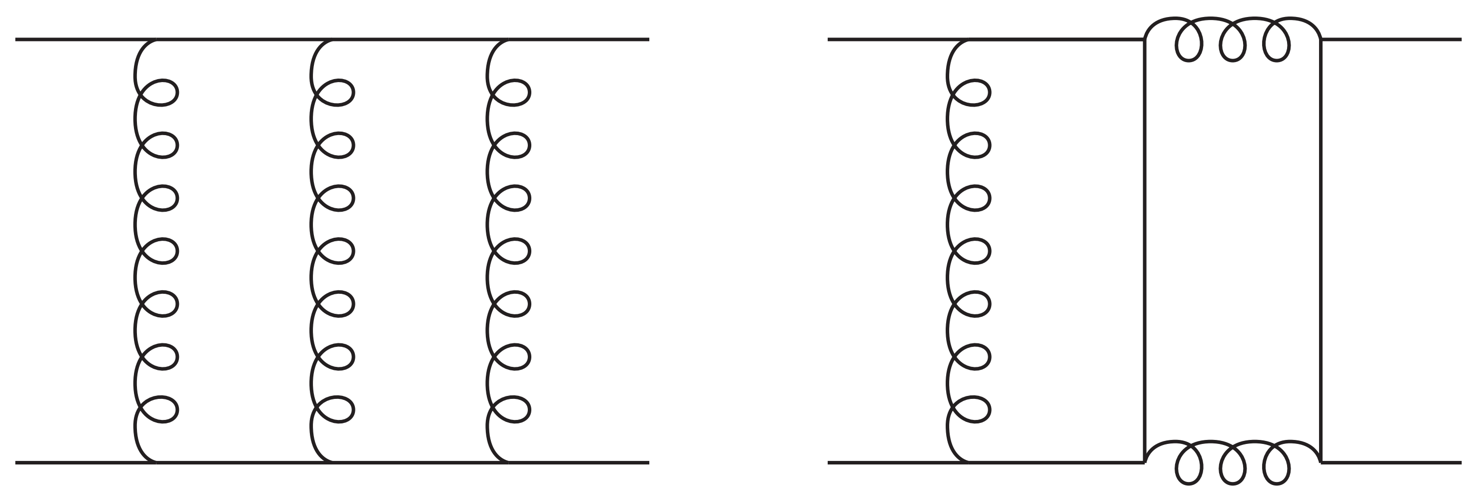


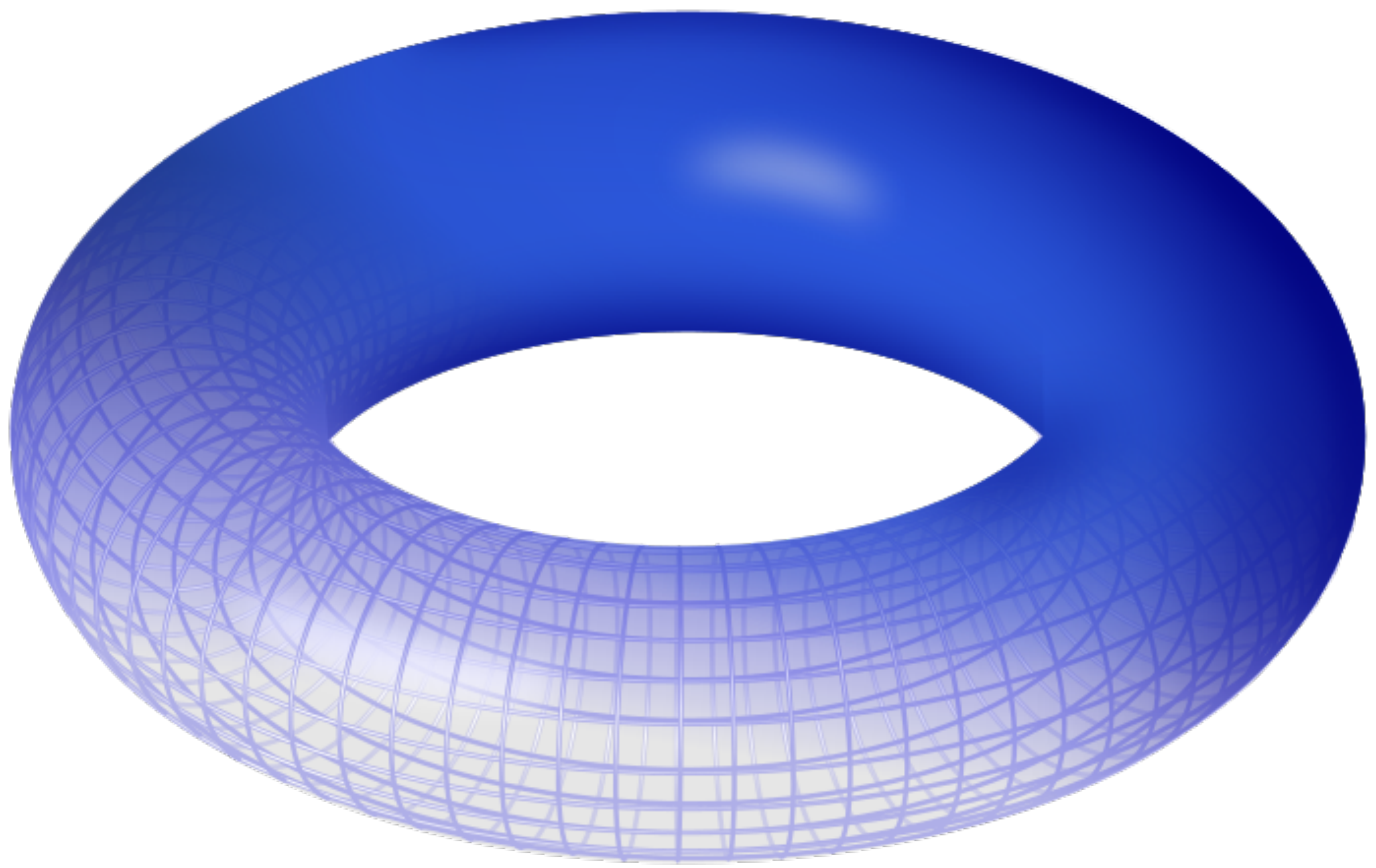
Figure 6: Real and imaginary part of $\Sigma^{(3)}_{S,0}$ close to $x_0 = 9$ (for $\xi = 0$).

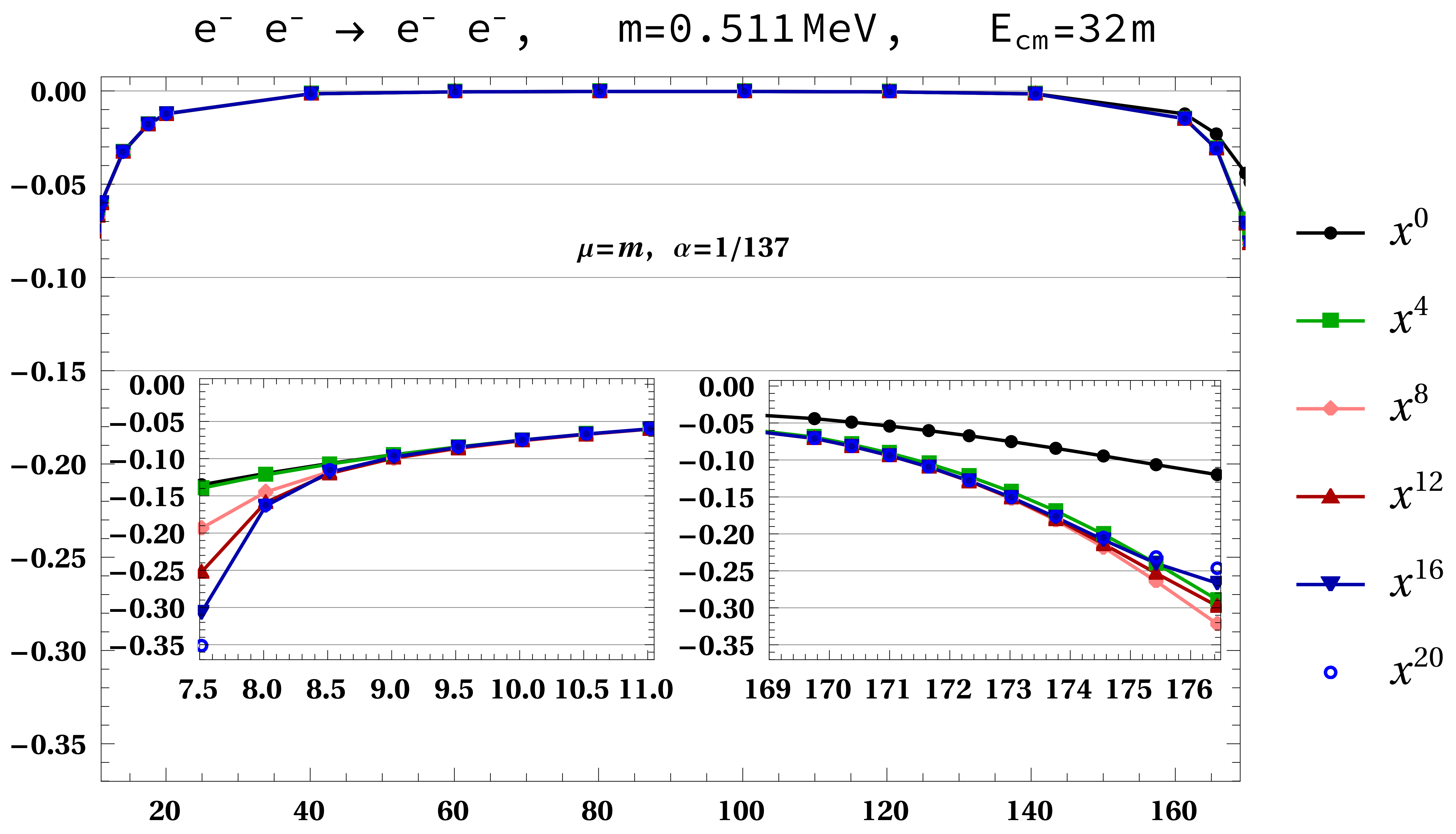
BEYOND ELLIPTICS IN GENUS AND IN DIMENSION

This picture holds for **many other elliptic cases:** recently, 2loop Bhabha scattering

[Delto, Duhr, Zhu, Tancredi '23]



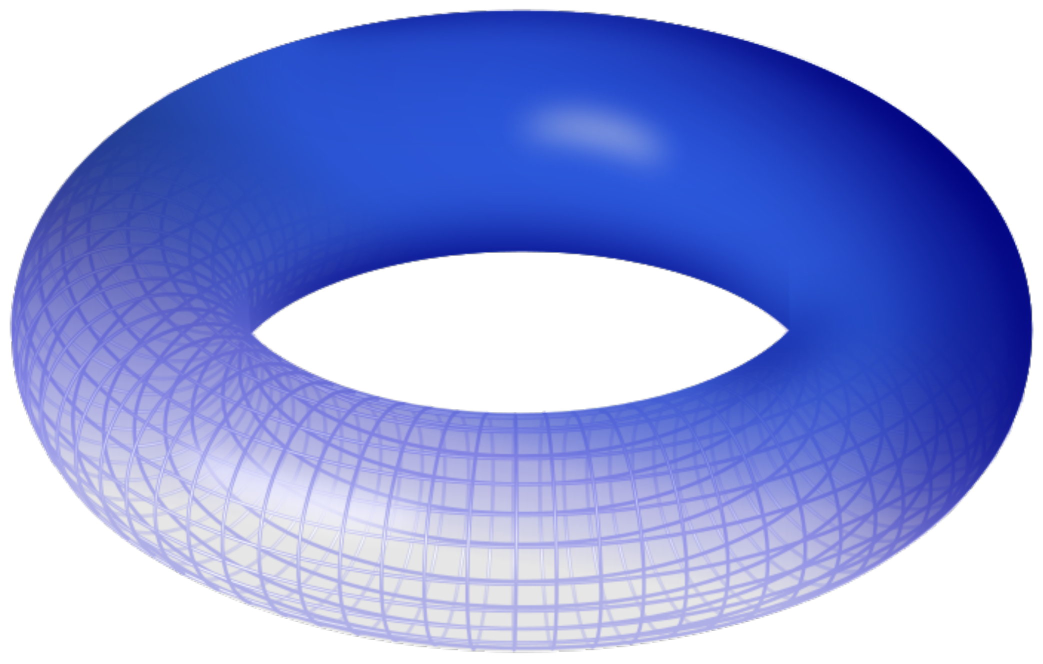


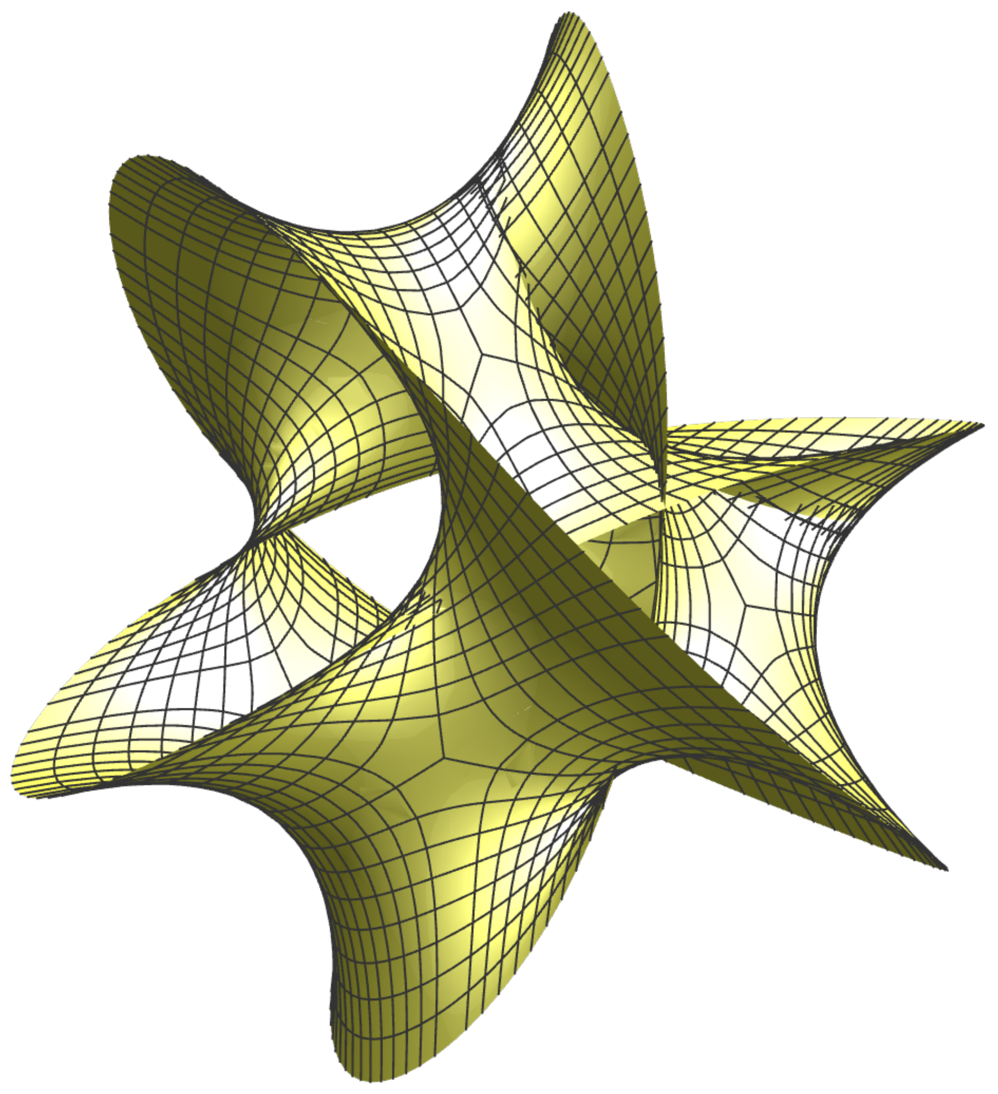


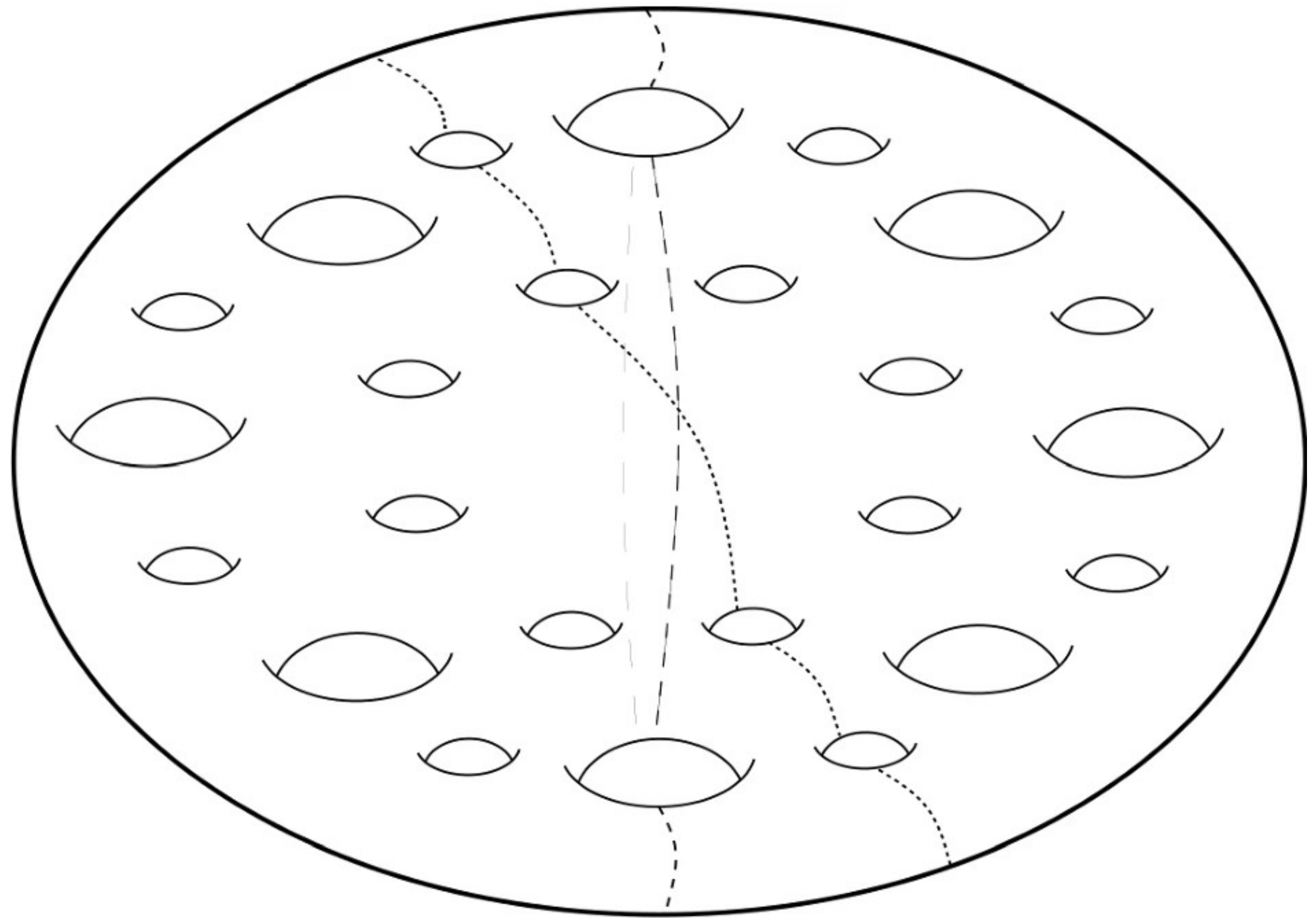
BEYOND ELLIPTICS IN GENUS AND IN DIMENSION

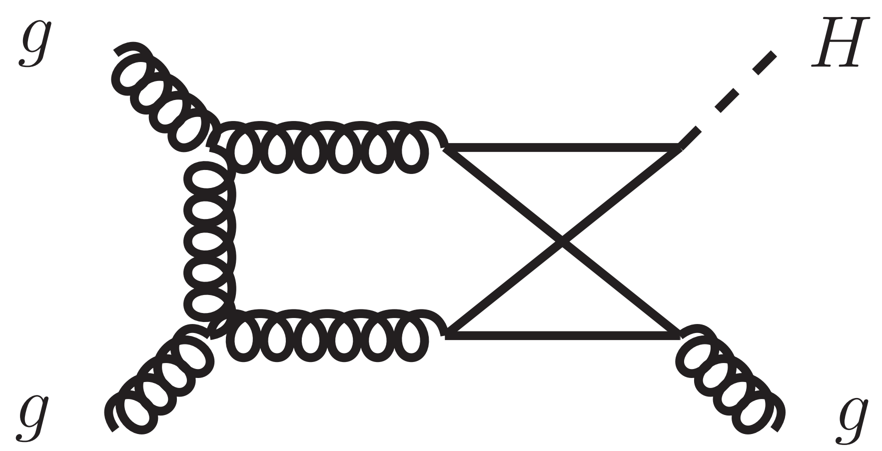
This picture holds for **many other elliptic cases** (more to appear hopefully soon :-))

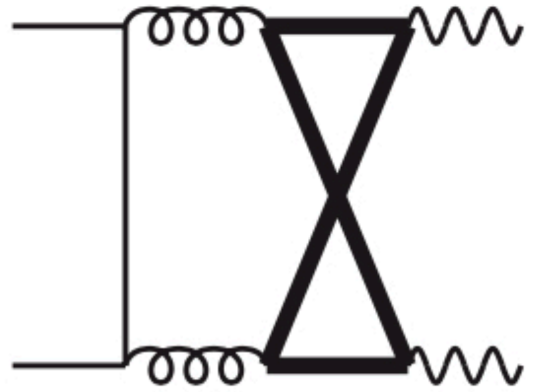
AND for **more general geometries**, with obvious generalizations: higher order eqs, more “solutions”…

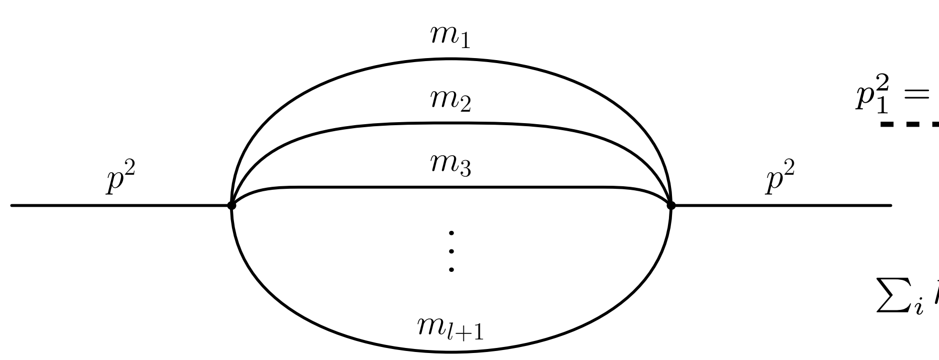


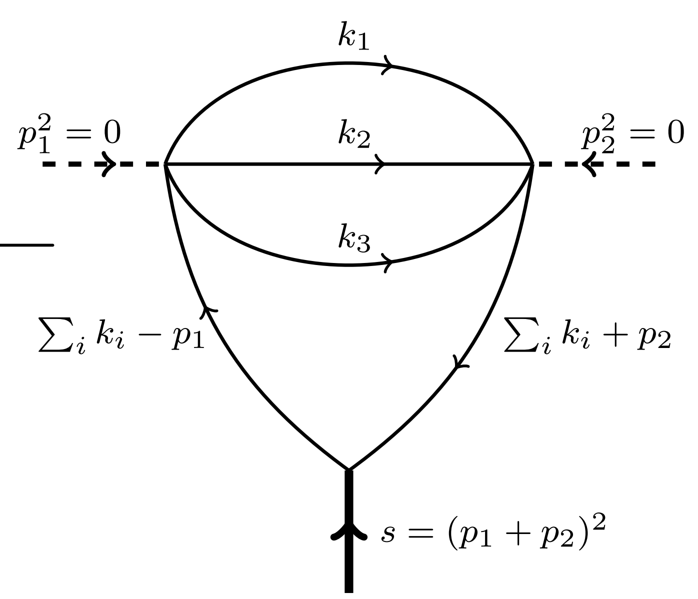


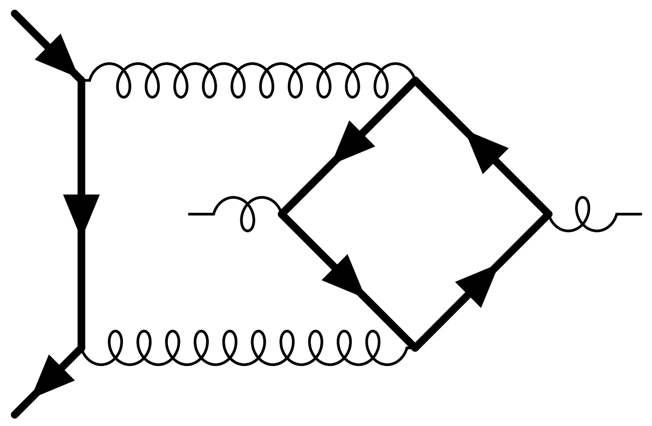


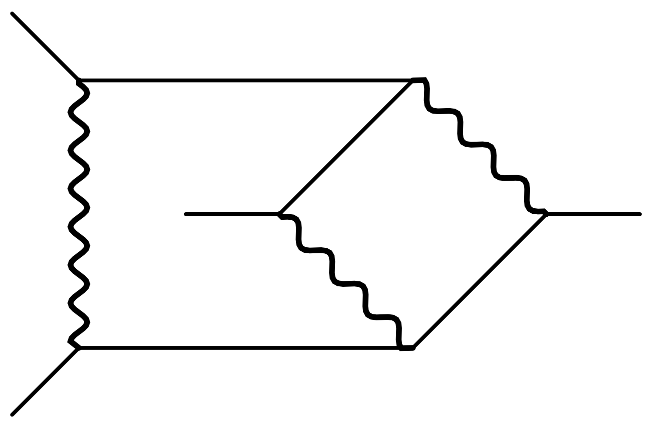












CONCLUSIONS AND OUTLOOK

- amplitudes are *fundamental building blocks in QFT*, for precision collider physics and beyond
- complexity of the calculations is often matched by **unexpected simplicity in final results**
- searching for a way to make **simplicity manifest**, informs on how to compute amplitudes more efficiently ***(language of differential forms on complex varieties is an example!)***
- what we learnt in past 10 years is finally **bearing fruit**: first realistic "correlators" and amplitudes under ***analytic and numerical control***
- same structures have been observed in *gravitational waves calculations* and *cosmological corr.*
- will this be competitive with purely numerical methods? for now, no idea, **but it's fun!**

THANK YOU VERY MUCH!