Radiative leptonic decays

Davide Giusti

JÜLICH Forschungszentrum

Universität Regensburg

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OUTLINE

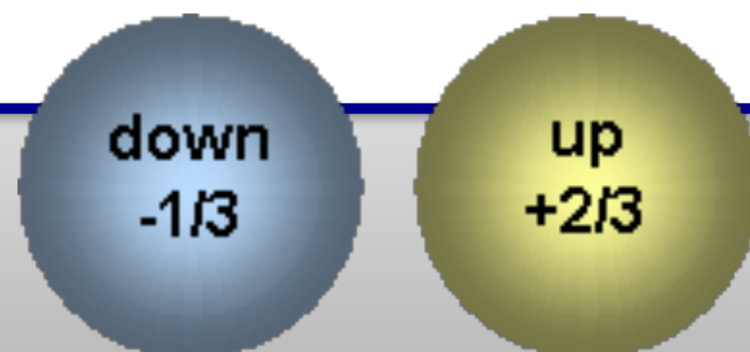
- Motivations
- Leptonic decays of hadrons $H \to \ell \nu_\ell (\gamma)$
- Outlook

In collaboration with

Christopher F. Kane, Christoph Lehner, Stefan Meinel, Amarjit Soni

arXiv:1907.00279, arXiv:2110.13196, arXiv:2302.01298

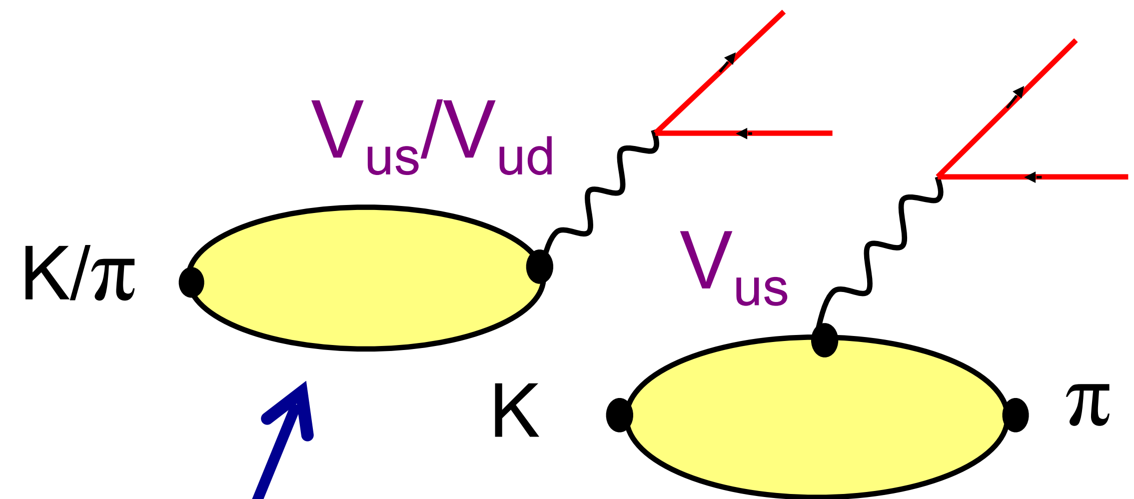
Phenomenological motivations



The determination of Vus and Vud

The relevant processes are leptonic and semileptonic

K and π decays

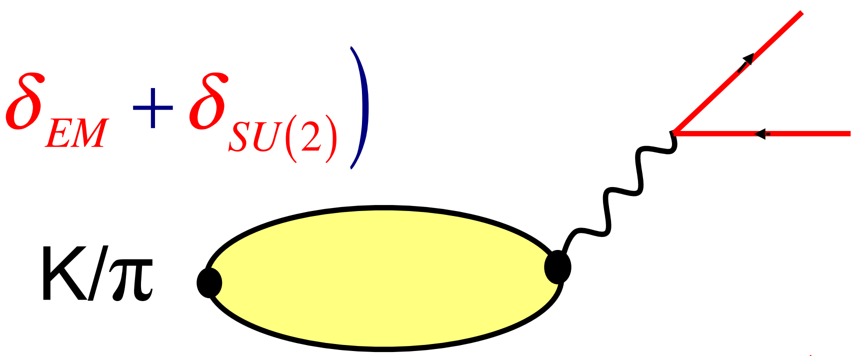


$$\frac{\Gamma\left(K^{+}\rightarrow \ell^{+}\nu_{\ell}(\gamma)\right)}{\Gamma\left(\pi^{+}\rightarrow \ell^{+}\nu_{\ell}(\gamma)\right)}=\left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$

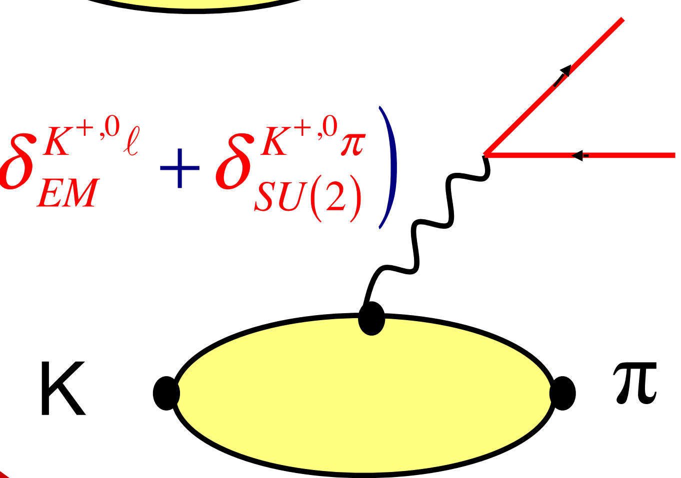
$$\Gamma\left(K^{+,0}\rightarrow \pi^{0,-}\ell^{+}\nu_{\ell}(\gamma)\right)=\frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3}C_{K^{+,0}}^2\left|V_{us}f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW}\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)$$

Vus and Vud: experimental results

$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$



$$\Gamma\left(K^{+,0} \to \pi^{0,-}\ell^+\nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW}\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)$$



$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi} = 0.27599(38)$$

$$|V_{us}| f_+(0) = 0.21654(41)$$

< 0.2%

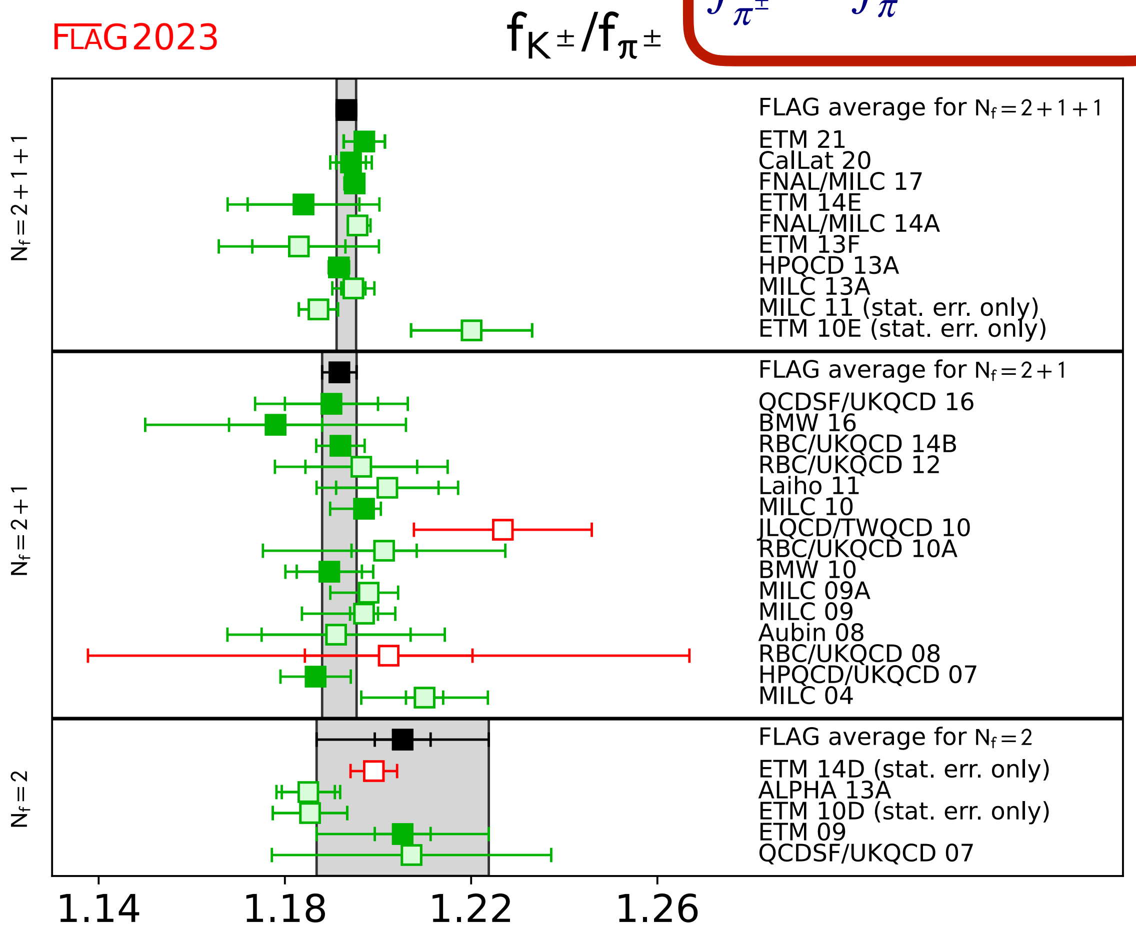
PDG

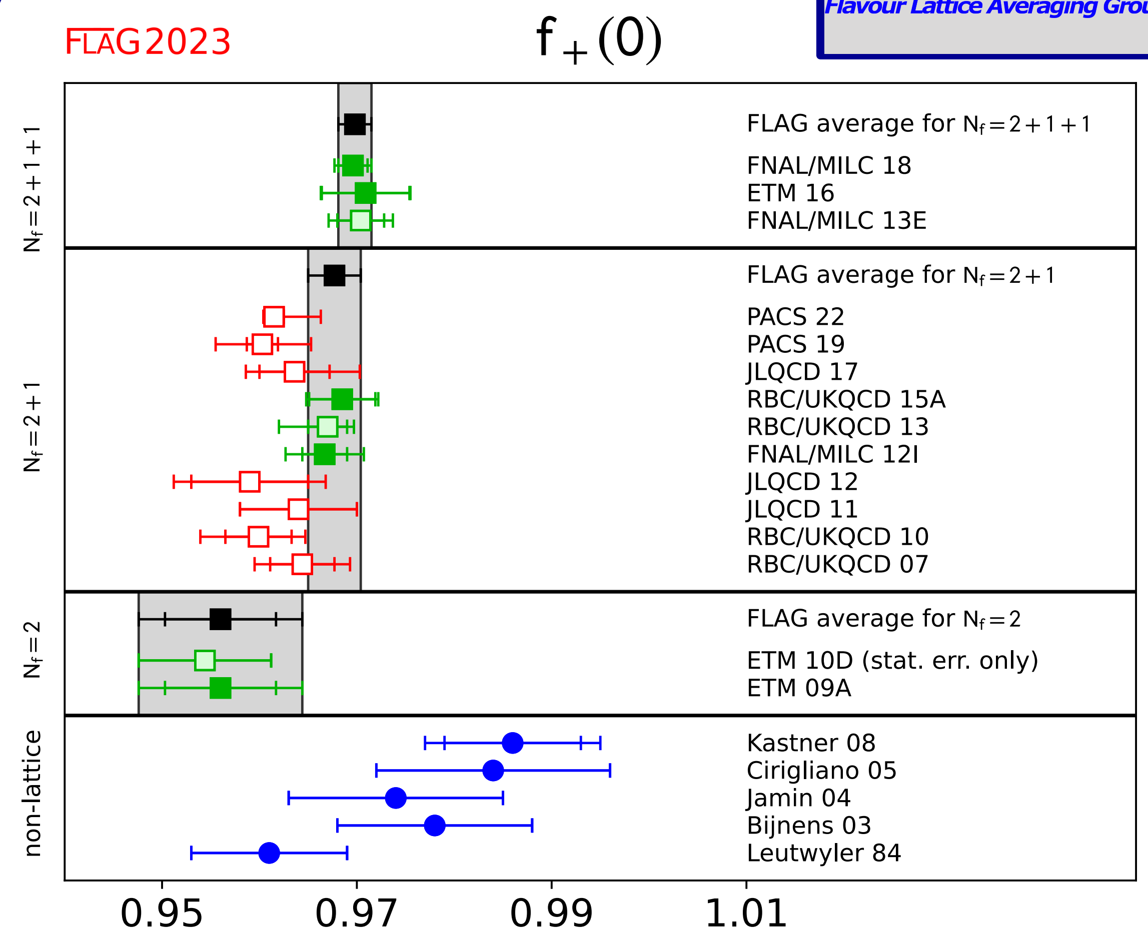
M. Moulson, arXiv:1704.04104

Vus and Vud: results from lattice QCD

$$\frac{f_{K^\pm}}{f_{\pi^\pm}} = \frac{f_K}{f_\pi}\sqrt{1+\delta_{SU(2)}}$$

FLAG 2021
Flavour Lattice Averaging Group





$f_{K^\pm} / f_{\pi^\pm} = 1.1934(19)$ Nf=2+1+1

$f_{K^\pm} / f_{\pi^\pm} = 1.1917(37)$ Nf=2+1

0.2%

$f_+(0) = 0.9698(17)$ Nf=2+1+1

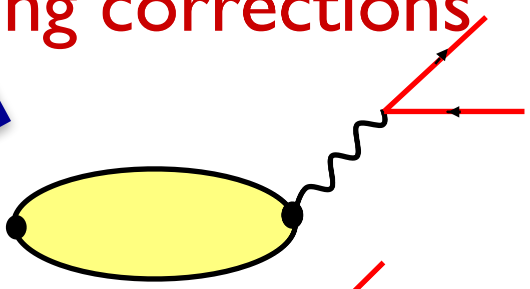
$f_+(0) = 0.9677(27)$ Nf=2+1

0.2%

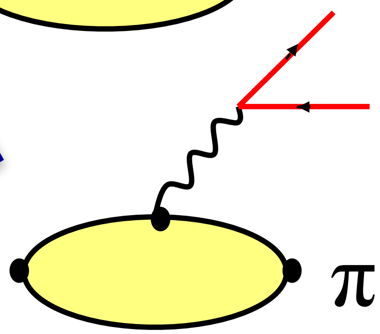
Electromagnetic and isospin-breaking effects

Given the present exper. and theor. (LQCD) accuracy, an important source of uncertainty are long distance electromagnetic and SU(2)-breaking corrections

$$\frac{\Gamma\left(K^+\to \ell^+\nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+\to\ell^+\nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$



$$\Gamma\left(K^{+,0}\to\pi^{0,-}\ell^+\nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW}\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)$$



For $\Gamma_{Kl2}/\Gamma_{\pi l2}$ At leading order in ChPT both δ_{EM} and $\delta_{SU(2)}$ can be expressed in terms of physical quantities (e.m. pion mass splitting, f_K/f_π, …)

- $\delta_{EM} = -0.0069(17)$ 25% of error due to higher orders ➡ 0.2% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  M.Knecht *et al.*, 2000; V.Cirigliano and H.Neufeld, 2011

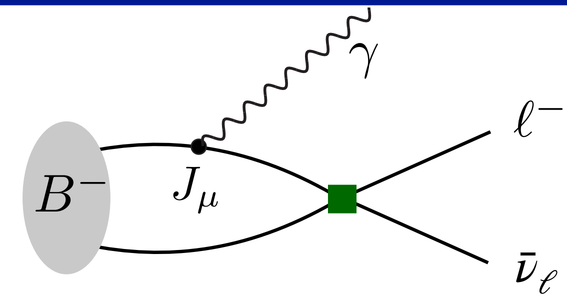
- $\delta_{SU(2)} = \left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}\right)^2 - 1 = -0.0044(12)$ 25% of error due to higher orders ➡ 0.1% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  J.Gasser and H.Leutwyler, 1985; V.Cirigliano and H.Neufeld, 2011

ChPT is not applicable to D and B decays

Radiative corrections to leptonic B-meson decays

$B^- \to \ell^- \bar{\nu}_\ell \gamma$



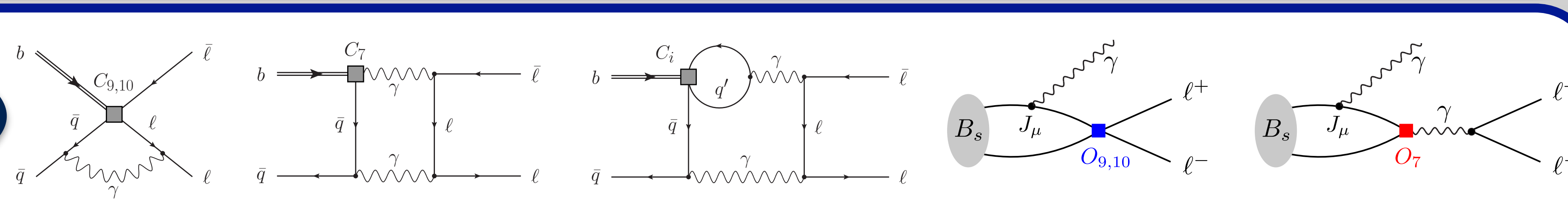
- The emission of a real hard photon removes the $(m_\ell/M_B)^2$ helicity suppression
- This is the simplest process that probes (for large E_γ) the first inverse moment of the B-meson LCDA

$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega} \Phi_{B+}(\omega, \mu)$$

λ_B is an important input in QCD-factorization predictions for non-leptonic B decays but is poorly known
M. Beneke, V. M. Braun, Y. Ji, Y.-B. Wei, 2018

- Belle 2018: $\mathscr{B}(B^- \to \ell^- \bar{\nu}_\ell \gamma, E_\gamma > 1 \text{ GeV}) < 3.0 \cdot 10^{-6}$ ⟶ $\lambda_B > 0.24$ GeV
- QCD sum rules in HQET: $\lambda_B(1 \text{ GeV}) = 0.46\,(11)$ GeV

$B_q \to \ell^+ \ell^- (\gamma)$



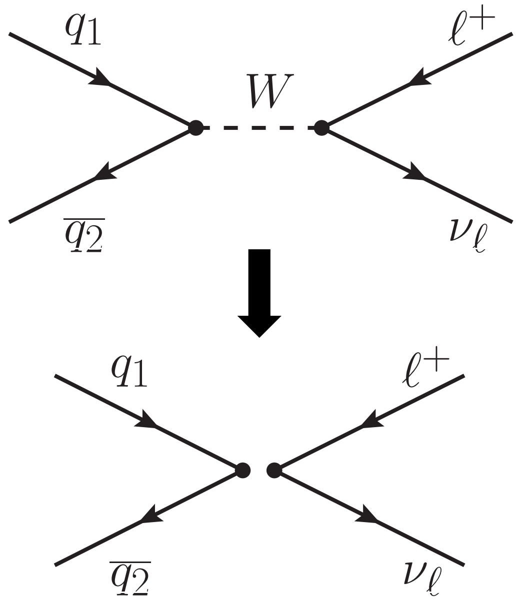
- Enhancement of the virtual corrections by a factor M_B/Λ_{QCD} and by large logarithms
M. Beneke, C. Bobeth, R. Szafron, 2019
- The real photon emission process is a clean probe of NP: sensitiveness to C_9, C_{10}, C_7

Radiative corrections to leptonic decay rates

$$H \to \ell \nu_\ell(\gamma)$$

Leptonic decays at tree level

Since the masses of the pion and kaon are much smaller than M_W we use the effective Hamiltonian

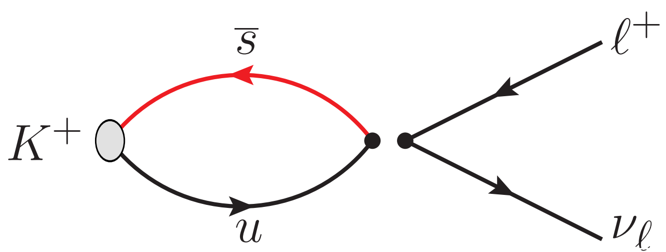


$$H_{eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(\overline{q_2}\gamma^\mu(1-\gamma_5)q_1\right)\left(\overline{\nu_\ell}\gamma_\mu(1-\gamma_5)\ell\right)$$

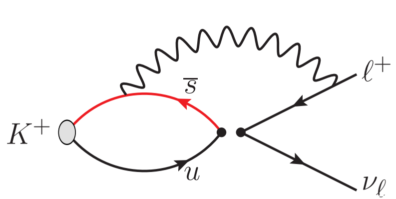
This replacement is necessary in a lattice calculation, since $1/a \ll M_W$

The rate is: $$\Gamma^{(tree)}_{P^\pm}\left(P^\pm \to \ell^\pm \nu_\ell\right) = \frac{G_F^2}{8\pi}\left|V_{q_1 q_2}\right|^2 \left[f_P^{(0)}\right]^2 M_{P^\pm} m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{P^\pm}^2}\right)^2$$

In the absence of electromagnetism, the non-perturbative QCD effects are contained in a single number, the pseudoscalar decay constant



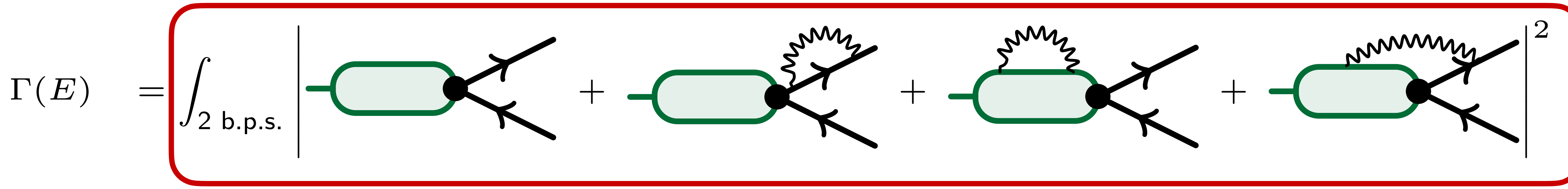
$$A_P^{(0)} \equiv \langle 0 | \overline{q}_2 \gamma_4 \gamma_5 q_1 | P^{(0)} \rangle = f_P^{(0)} M_P^{(0)}$$



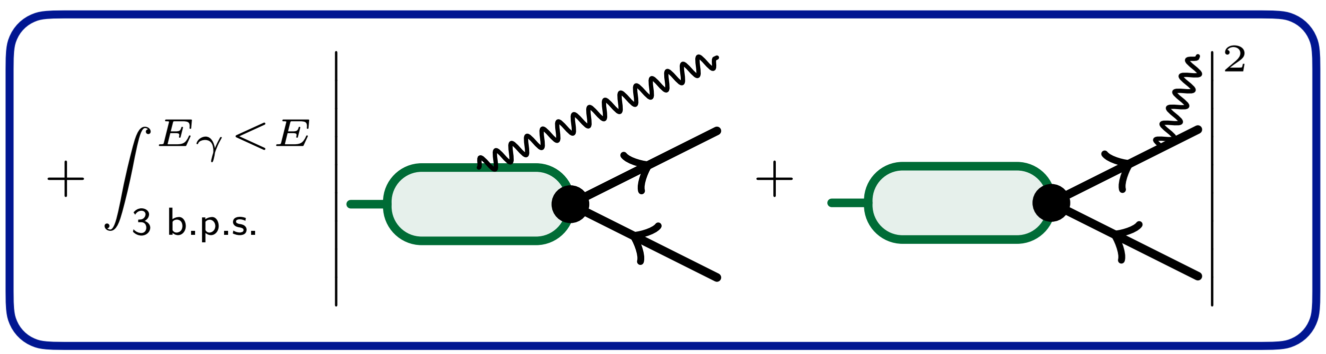
In the presence of electromagnetism it is not even possible to give a physical definition of f_P

J. Gasser and G.R.S. Zarnauskas, PLB 693 (2010) 122

Leptonic decays at O(α)



Γ_0



$\Gamma_1(E)$

$\Gamma(E) = \Gamma_0 + \Gamma_1(E)$ with $0 \leq E_\gamma \leq E$ is infrared finite

F. Bloch and A. Nordsieck, 1937

- Both Γ_0 and $\Gamma_1(E)$ can be evaluated in a fully non-perturbative way in lattice simulations
- The first lattice calculations of $\Gamma[\pi, K \to \ell\nu(\gamma)]$ have been finalized

N. Carrasco *et al.* arXiv:1502.00257 V. Lubicz *et al.* arXiv:1611.08497 DG *et al.* arXiv:1711.06537 M. Di Carlo *et al.* arXiv:1904.08731 P. Boyle *et al.* arXiv:2211.12865

Real photon emission amplitude

$$J_\mu^{em} = \sum_q Q_q \bar{q}\gamma_\mu q \qquad J_\nu^{weak} = \bar{q}_1\gamma_\nu(1-\gamma_5)\,q_2$$

$$\mathcal{A}(H^- \to \gamma\ell\bar{\nu}) = \frac{G_F V_{q_1 q_2}}{\sqrt{2}}\left[e(\epsilon^*)^\mu \bar{\ell}\gamma^\nu(1-\gamma_5)\nu \cdot T_{\mu\nu}(p_H, p_\gamma) - ieQ_\ell f_H \cdot \bar{\ell}\not{\epsilon}^*(1-\gamma_5)\nu\right]$$

$$T_{\mu\nu}(p_H, p_\gamma) = -i\int \mathrm{d}t_{\mathrm{em}} \int \mathrm{d}^3x\ e^{ip_\gamma\cdot x}\,\langle 0|\,\mathbf{T}\big(J_\mu^{\mathrm{em}}(t_{\mathrm{em}},\vec{x})J_\nu^{\mathrm{weak}}(0)\big)\,|H(\vec{p}_H)\rangle$$

- The hadronic tensor can be written as the sum $T_{\mu\nu} = T_{\mu\nu}^< + T_{\mu\nu}^>$ of the contributions from the two time orderings of the currents

Real photon emission amplitude

By setting $p_\gamma^2 = 0$, at fixed meson mass, the form factors depend on $p_H \cdot p_\gamma$ only.
Moreover, by choosing a *physical* basis for the polarization vectors, *i.e.* $\epsilon_r(\mathbf{p}_\gamma) \cdot p_\gamma = 0$, one has

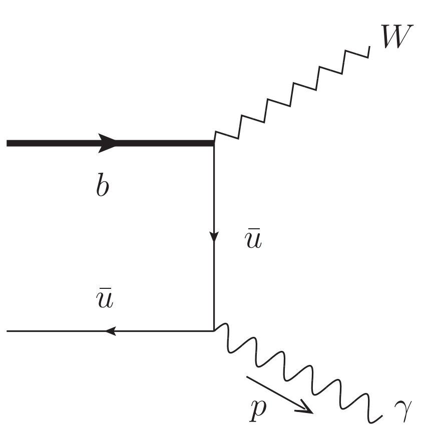
$$\epsilon_\mu^r(\mathbf{p}_\gamma)\, T^{\mu\nu}(p_\gamma, p_H) = \epsilon_\mu^r(\mathbf{p}_\gamma) \left\{ \varepsilon^{\mu\nu\tau\rho}(p_\gamma)_\tau v_\rho F_V + i\left[-g^{\mu\nu}(p_\gamma \cdot v) + v^\mu p_\gamma^\nu\right] F_A - i \frac{v^\mu v^\nu}{p_\gamma \cdot v} m_H f_H \right\}$$

In the case of off-shell photons $(p_\gamma^2 \neq 0) \longrightarrow \Gamma[H \to \ell \nu_\ell \ell^+ \ell^-]$ expressed in terms of 4 form factors

For large photon energies and in the B-meson rest frame the form factors can be written as

$$F_V(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) + \Delta\xi(E_\gamma)$$

$$F_A(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) - \Delta\xi(E_\gamma)$$



M. Beneke and J. Rohrwild, 2011

Form factors: results

PHYSICAL REVIEW D **103,** 014502 (2021)

arXiv:2006.05358

First lattice calculation of radiative leptonic decay rates of pseudoscalar mesons

A. Desiderio,[1] R. Frezzotti,[1] M. Garofalo,[2] D. Giusti,[3,4] M. Hansen,[5] V. Lubicz,[2] G. Martinelli,[6] C. T. Sachrajda,[7] F. Sanfilippo,[4] S. Simula,[4] and N. Tantalo[1]

$$F^P_{A,V}(x_\gamma) = C^P_{A,V} + D^P_{A,V} x_\gamma$$

F_A

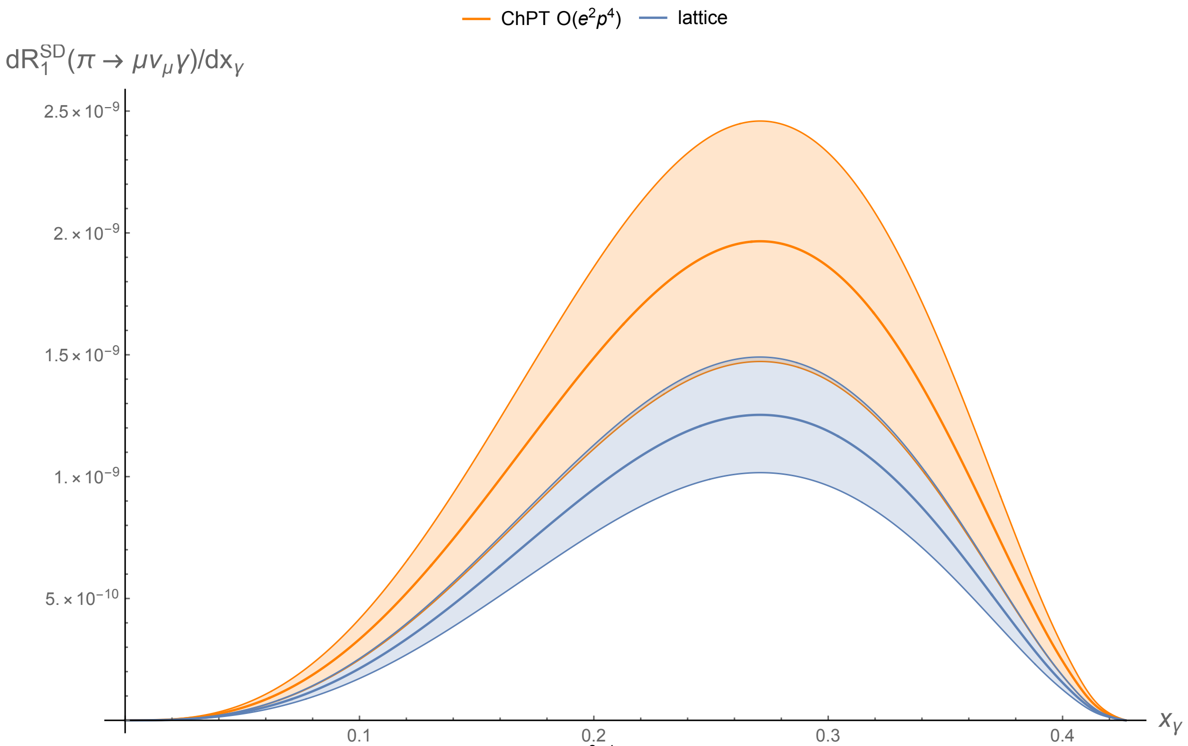
$$C_A^\pi = 0.010 \pm 0.003\,; \quad D_A^\pi = 0.0004 \pm 0.0006\,; \quad \rho_{C_A^\pi, D_A^\pi} = -0.419\,;$$
$$C_A^K = 0.037 \pm 0.009\,; \quad D_A^K = -0.001 \pm 0.007\,; \quad \rho_{C_A^K, D_A^K} = -0.673\,;$$
$$C_A^D = 0.109 \pm 0.009\,; \quad D_A^D = -0.10 \pm 0.03\,; \quad \rho_{C_A^D, D_A^D} = -0.557\,;$$
$$C_A^{D_s} = 0.092 \pm 0.006\,; \quad D_A^{D_s} = -0.07 \pm 0.01\,; \quad \rho_{C_A^{D_s}, D_A^{D_s}} = -0.745\,.$$

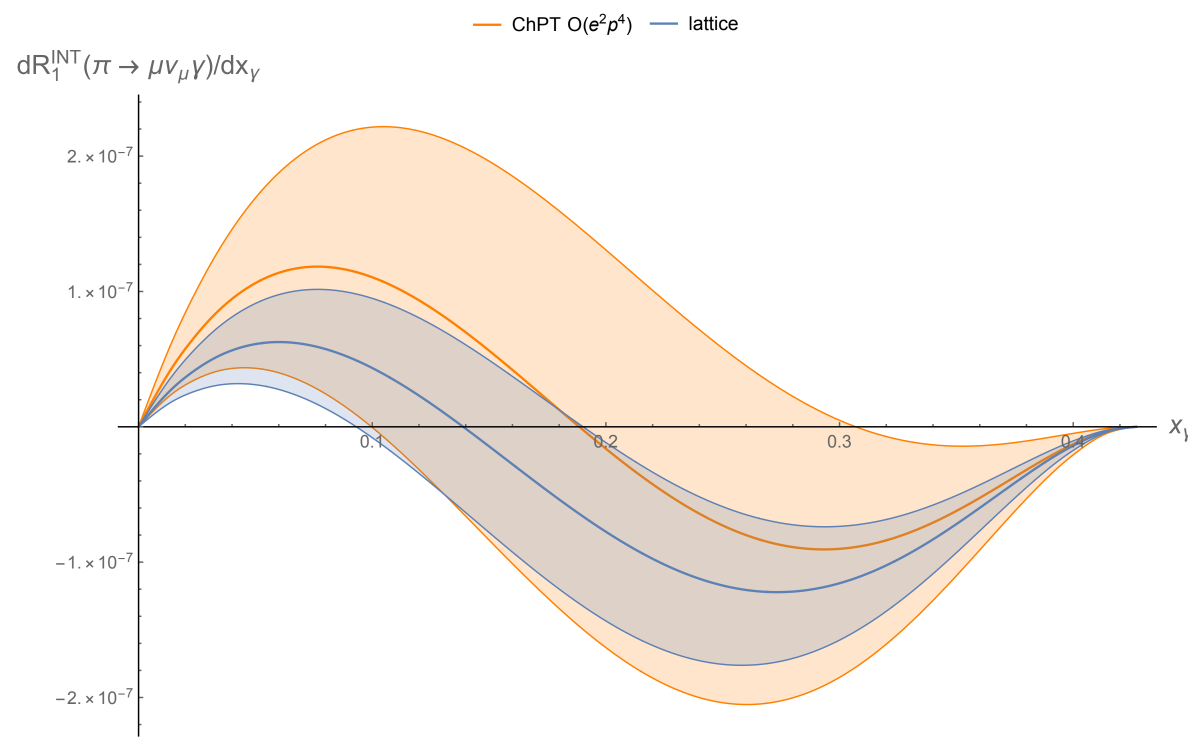
F_V

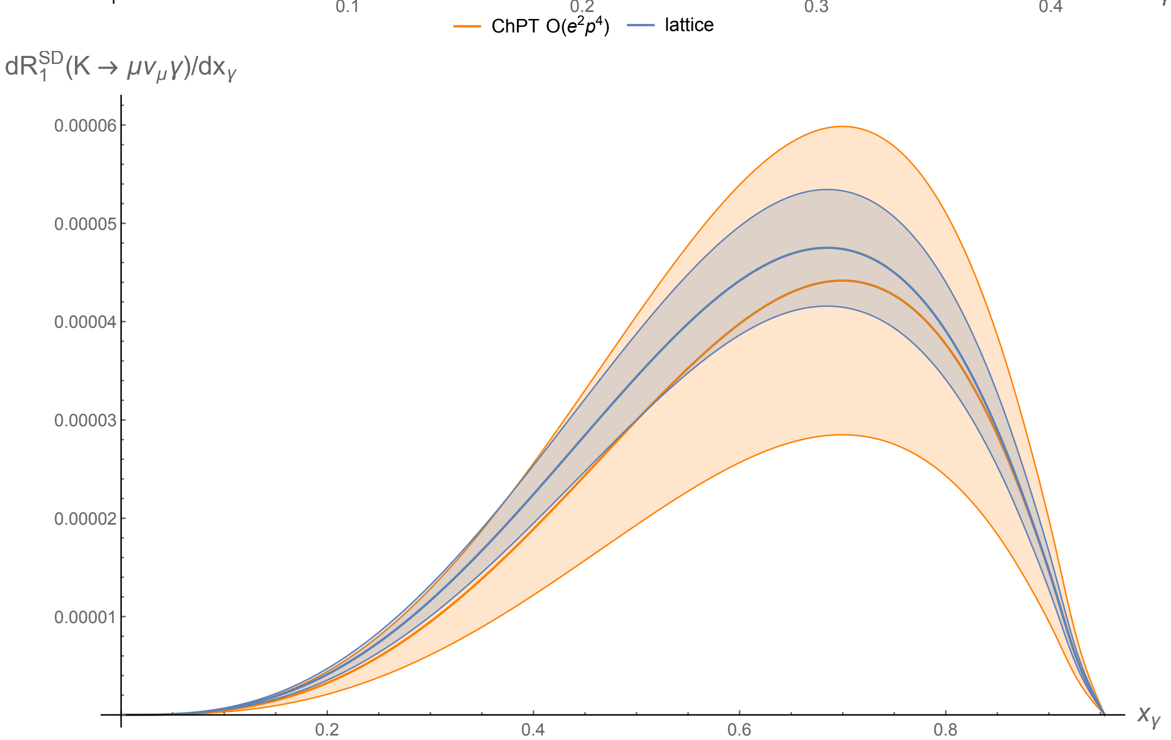
$$C_V^\pi = 0.023 \pm 0.002\,; \quad D_V^\pi = -0.0003 \pm 0.0003\,; \quad \rho_{C_V^\pi, D_V^\pi} = -0.570\,;$$
$$C_V^K = 0.12 \pm 0.01\,; \quad D_V^K = -0.02 \pm 0.01\,; \quad \rho_{C_V^K, D_V^K} = -0.714\,;$$
$$C_V^D = -0.15 \pm 0.02\,; \quad D_V^D = 0.12 \pm 0.04\,; \quad \rho_{C_V^D, D_V^D} = -0.580\,;$$
$$C_V^{D_s} = -0.12 \pm 0.02\,; \quad D_V^{D_s} = 0.16 \pm 0.03\,; \quad \rho_{C_V^{D_s}, D_V^{D_s}} = -0.900\,.$$

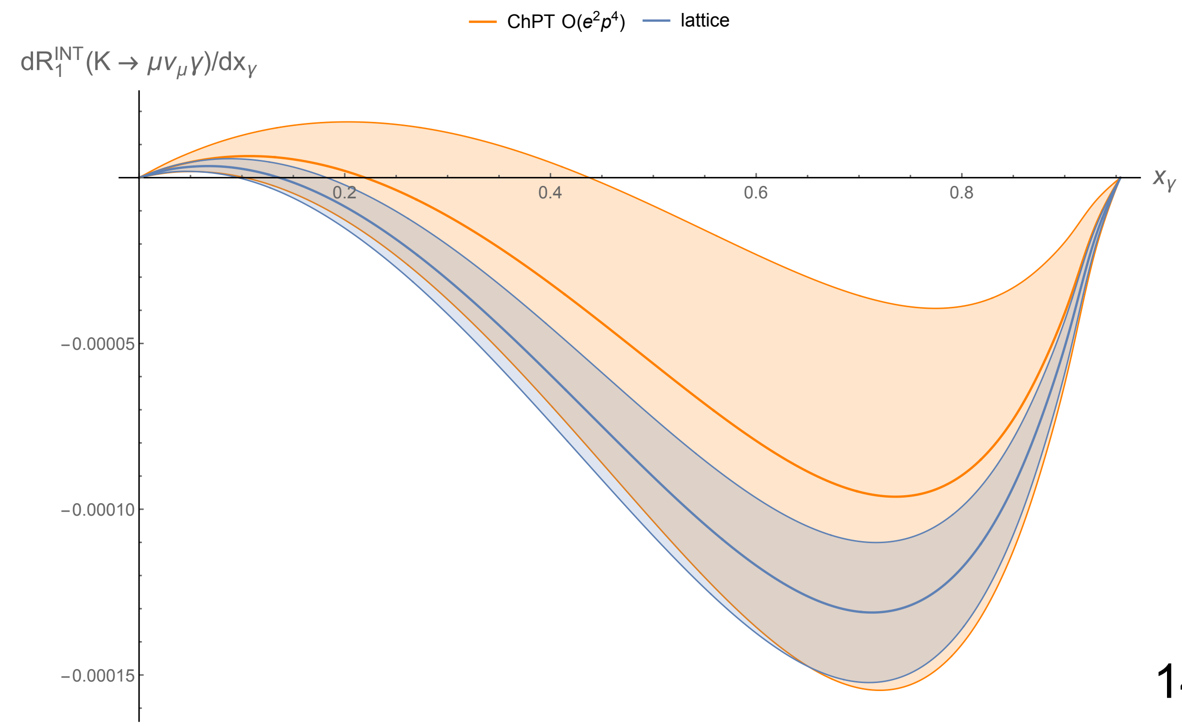
$$\frac{4\pi}{\alpha\,\Gamma_0^{\text{tree}}}\,\frac{d\Gamma_1^{\text{SD}}}{dx_\gamma}=\frac{m_P^2}{6 f_P^2\, r_\ell^2\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)^2+F_A(x_\gamma)^2\right] f^{\text{SD}}(x_\gamma)$$

$$\frac{4\pi}{\alpha\,\Gamma_0^{\text{tree}}}\,\frac{d\Gamma_1^{\text{INT}}}{dx_\gamma}=-\frac{2\,m_P}{f_P\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)\,f_V^{\text{INT}}(x_\gamma)+F_A(x_\gamma)\,f_A^{\text{INT}}(x_\gamma)\right]$$









Leptonic decays at O(α): RESULTS

$$\Gamma(\Delta E) = \Gamma^{(tree)} \left[1 + \delta R_0 + \delta R_{pt}(\Delta E) + \delta R_1^{SD}(\Delta E) + \delta R_1^{INT}(\Delta E) \right]$$

	$\pi_{e2[\gamma]}$	$\pi_{\mu 2[\gamma]}$	$K_{e2[\gamma]}$	$K_{\mu 2[\gamma]}$
δR_0	(*)	0.0411 (19)	(*)	0.0341 (10)
$\delta R_{\rm pt}(\Delta E_\gamma^{max})$	-0.0651	-0.0258	-0.0695	-0.0317
$\delta R_1^{\rm SD}(\Delta E_\gamma^{max})$	$5.4\ (1.0) \times 10^{-4}$	$2.6\ (5) \times 10^{-10}$	1.19 (14)	$2.2\ (3) \times 10^{-5}$
$\delta R_1^{\rm INT}(\Delta E_\gamma^{max})$	$-4.1\ (1.0) \times 10^{-5}$	$-1.3\ (1.5) \times 10^{-8}$	$-9.2\ (1.3) \times 10^{-4}$	$-6.1\ (1.1) \times 10^{-5}$
ΔE_γ^{max} (MeV)	69.8	29.8	246.8	235.5

(*) *Not yet evaluated by numerical lattice QCD+QED simulations.*

Large SD effects →

$\Gamma^{(tree)} \propto (m_\ell / m_P)^2$ helicity suppression

$\delta R_1^{SD} \propto (m_P / m_\ell)^2$ remove the suppression

Lattice calculation

PHYSICAL REVIEW D **107,** 074507 (2023)

arXiv:2302.01298

Methods for high-precision determinations of radiative-leptonic decay form factors using lattice QCD

Davide Giusti,[1] Christopher F. Kane,[2] Christoph Lehner,[1] Stefan Meinel,[2] and Amarjit Soni[3]

[1]*Fakultät für Physik, Universität Regensburg, 93040, Regensburg, Germany*
[2]*Department of Physics, University of Arizona, Tucson, Arizona 85721, USA*
[3]*Brookhaven National Laboratory, Upton, New York 11973, USA*

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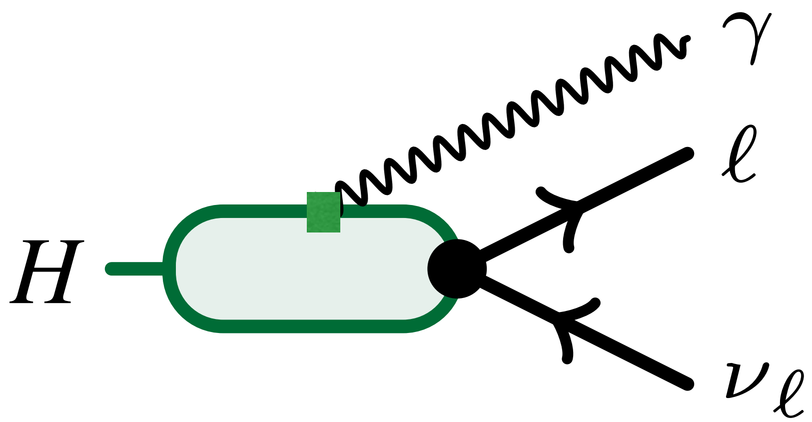
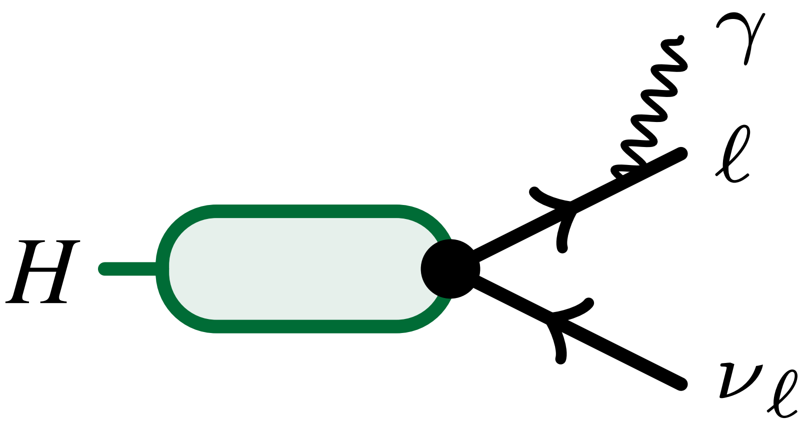
Hadronic tensor and form factors





$$T_{\mu\nu} = -i \int d^4x\, e^{ip_\gamma \cdot x} \langle 0 | \mathbf{T}\left(J^{em}_\mu(x)\, J^{weak}_\nu(0) \right) | H(\vec{p}_H) \rangle \qquad (p_H = m_H v)$$

$$= \varepsilon_{\mu\nu\tau\rho} p_\gamma^\tau v^\rho F_V + i \left[-g_{\mu\nu}(p_\gamma \cdot v) + v_\mu (p_\gamma)_\nu \right] F_A - i \frac{v_\mu v_\nu}{p_\gamma \cdot v} m_H f_H + (p_\gamma)_\mu - \text{terms}$$

$$F_A = F_{A,SD} + (-Q_\ell f_H / E_\gamma^{(0)}), \qquad E_\gamma^{(0)} = p_\gamma \cdot v$$

Goal: Calculate $F_V, F_{A,SD}$ as a function of $E_\gamma^{(0)}$

Euclidean correlation function

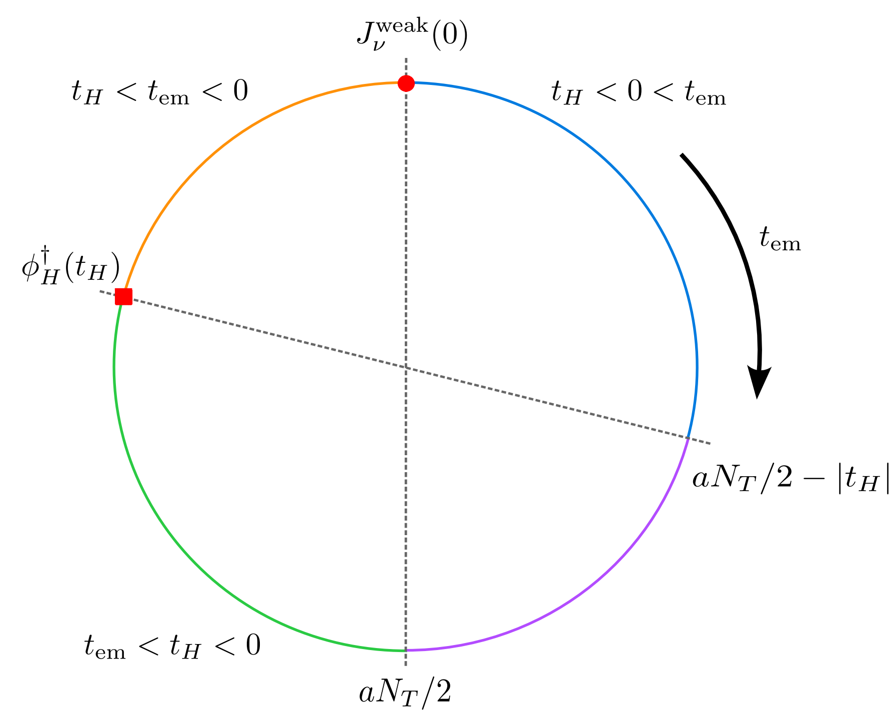
$$C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y\ e^{-i\vec{\mathbf{p}}_\gamma \cdot \vec{\mathbf{x}}} e^{i\vec{\mathbf{p}}_H \cdot \vec{\mathbf{y}}} \langle J^{em}_\mu(t_{em}, \vec{\mathbf{x}}) J^{weak}_\nu(0) \phi^\dagger_H(t_H, \vec{\mathbf{y}}) \rangle$$

$$\phi^\dagger_H \sim \bar{Q}\gamma_5 u$$

$$I^<_{\mu\nu}(T, t_H) = \int_{-T}^{0} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

$$I^>_{\mu\nu}(T, t_H) = \int_{0}^{T} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

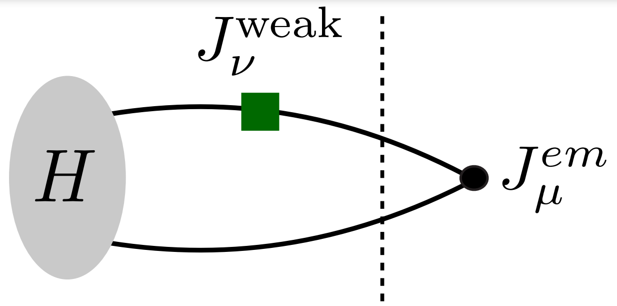
$$I_{\mu\nu}(T, t_H) = I^<(T, t_H) + I^>(T, t_H)$$



Show relation between $I_{\mu\nu}(T, t_H)$ and $T_{\mu\nu}$
$\rightarrow$ compare spectral decompositions of both time orderings of $I_{\mu\nu}$ and $T_{\mu\nu}$

Analytic continuation from Minkowski to Euclidean spacetime

Time ordering: $t_{em} > 0$



$$T^{>}_{\mu\nu} = -\sum_n \frac{\langle 0|\, J^{em}_\mu(0)\,|n(\vec{\mathbf{p}}_\gamma)\rangle\, \langle n(\vec{\mathbf{p}}_\gamma)|\, J^{weak}_\nu(0)\,|H(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})}$$

$$I^{>}_{\mu\nu}(t_H, T) = \int_0^T dt_{em}\; e^{E_\gamma t_{em}} C_{\mu\nu}(t_{em}, t_H)$$

$$= -\sum_m e^{E_m t_H} \frac{\langle m(\vec{\mathbf{p}}_H)|\, \phi^\dagger_H(0)\,|0\rangle}{2E_{m,\vec{\mathbf{p}}_H}}$$

$$\times \sum_n \frac{\langle 0|\, J^{em}_\mu(0)\,|n(\vec{\mathbf{p}}_\gamma)\rangle\, \langle n(\vec{\mathbf{p}}_\gamma)|\, J^{weak}_\nu(0)\,|m(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})} \left[1 - e^{(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})T}\right]$$

$t_H \to -\infty$ to achieve ground state saturation

$T \to \infty$ to remove unwanted exponentials that come with intermediate states

The unwanted exponential $e^{(E_\gamma - E_{n,\vec{p}_\gamma})\,T}$ goes to zero for large T if $E_{n,\vec{p}_\gamma} > E_\gamma$.

Because the states $|n(\vec{p}_\gamma)\rangle$ have a nonzero mass, this is always satisfied.

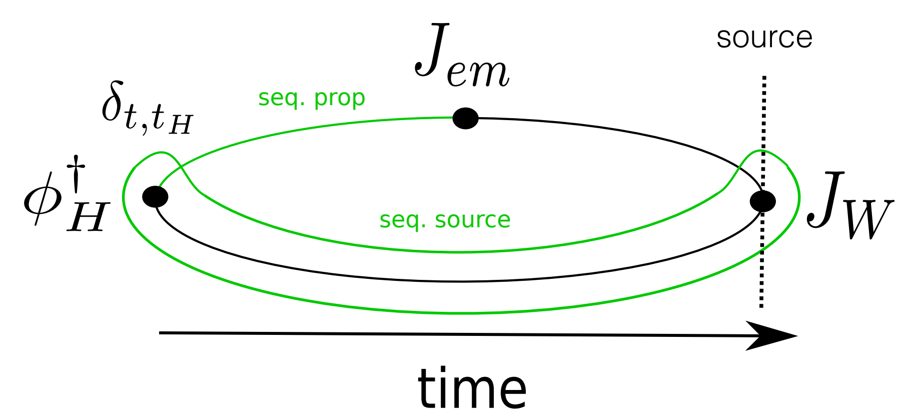
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_H\to-\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{\mathbf{p}}_H)| \phi_H^\dagger |0\rangle} \underbrace{\int_{-T}^{T} dt_{em}\ e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)}_{I_{\mu\nu}(T,t_H)}$$

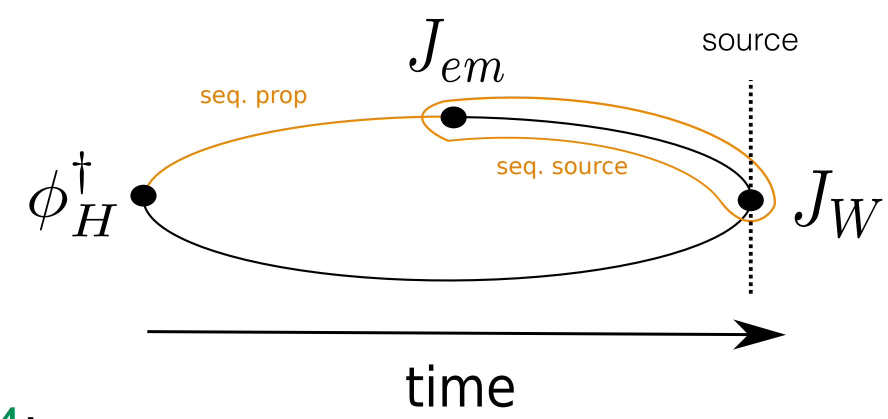
Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed t_H get all t_{em} for free

arXiv:1907.00279; arXiv:2110.13196 & arXiv:2302.01298



2: 4d sequential propagator through J_μ^{em} → calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed T get all t_H for free



RM123 & Soton Coll., arXiv:2006.05358 & arXiv:2306.05904:

Set $T = N_T/2$ and fit to constant in t_H where data has plateaued

Simulation details

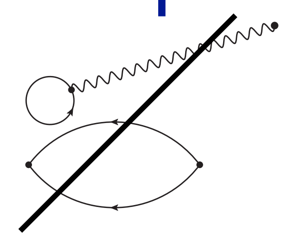
- $N_f = 2+1$ DWF, 3 RBC/UKQCD ensembles $M_\pi \simeq 139 \div 340$ MeV, $a \simeq 0.08 \div 0.11$ fm, charm valence quarks: Möbius DW with "stout" smear.

- 25 gauge configurations

Method	Source	Meson Momentum	Photon Momentum
3d	$\mathbb{Z}_2$-wall	$\vec{p}_{D_s} = (0,0,0)$	$\lvert\vec{p}_\gamma\rvert^2 \in (2\pi/L)^2\,\{1,2,3,4\}$
3d	point	$p_{D_s,z} \in 2\pi/L\,\{0,1,2\}$	all
4d	$\mathbb{Z}_2$-wall	$p_{D_s,z} \in 2\pi/L\,\{-1,0,1,2\}$	$p_{\gamma,z} = 2\pi/L$
$4d^{>,<}$	$\mathbb{Z}_2$-wall	$p_{D_s,z} \in 2\pi/L\,\{-1,0,1,2\}$	$p_{\gamma,z} = 2\pi/L$

- $\mathbb{Z}_2$ random wall sources & randomly placed point sources

- Disconnected diagrams are neglected



- Local electromagnetic current + mostly non-perturbative RCs

- Two datasets: $J^{weak}(0)$ or $J^{em}(0)$

- For point sources use translational invariance to fix em/weak operator at $\mathbf{0}$

use an "infinite-volume approximation" to generate data for arbitrary photon momenta (only exp. small FVEs are introduced)

$$C_{3,\mu\nu} = \int d^3x\, d^3y\, e^{-i\vec{p}_\gamma\cdot\vec{x}} \langle J^{em}_\mu(t_{em},\vec{x}) J^{weak}_\nu(0) \phi^\dagger_H(t_H,\vec{y})\rangle \qquad \vec{p}_H = 0, \text{ several } \vec{p}_\gamma$$

Fit form: 3d method

Include terms to fit

(1) unwanted exponential from first intermediate state

(2) first excited state

Fit form factors F_V and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

Time ordering $t_{em} < 0$:

$$F^<(t_H, T) = F^< + B_F^<(1 + B_{F,\mathrm{exc}}^< \overbrace{e^{\Delta E(T+t_H)}}) \overbrace{e^{-(E_\gamma - E_H + E^<)T}} + C_F^< \overbrace{e^{\Delta E t_H}}$$

■ fit parameters

Only have three values of t_H, fitting multiple exponentials not possible

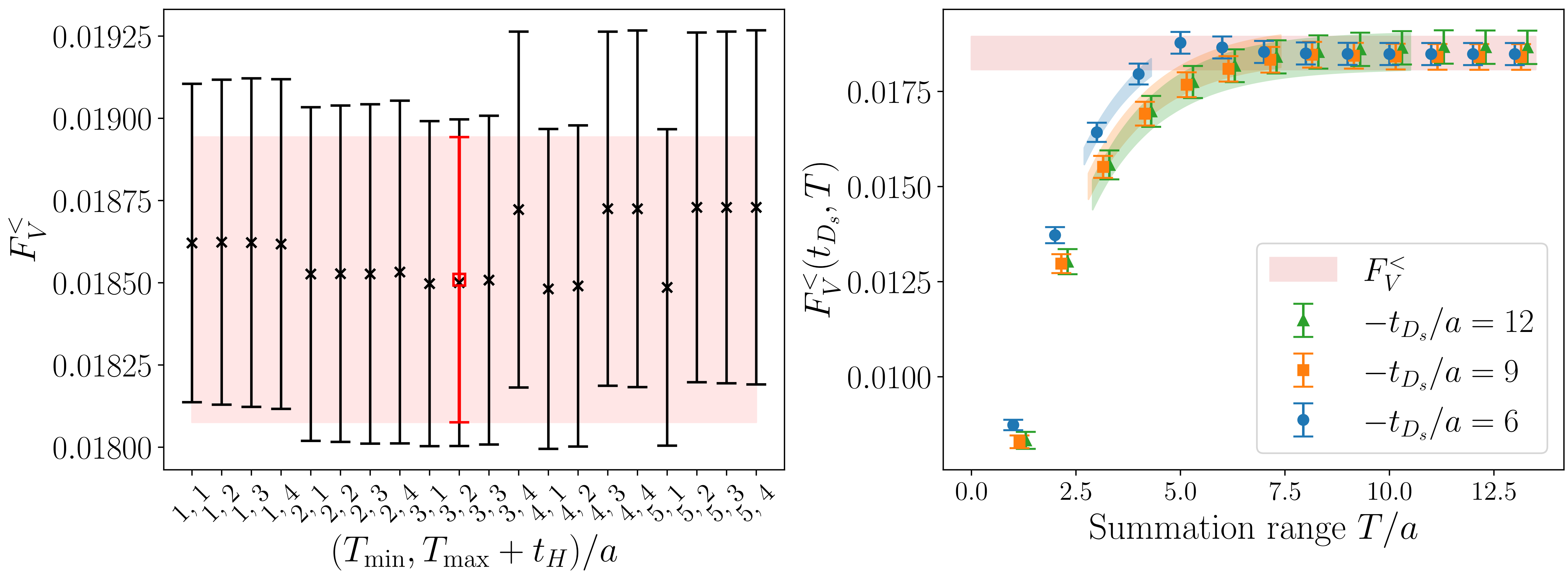
→ Determine ΔE from the pseudoscalar two-point correlation function

→ use result as Gaussian prior in form factor fits

$D_s \to \ell \nu_\ell \gamma$: 3d method

Time ordering $t_{em} < 0$:

$$F_V^<(t_H, T) = F_V^< + B_{F_V}^<(1 + B_{F_V,\text{exc}}^< \, e^{\Delta E(T+t_H)})\, e^{-(E_\gamma - E_H + E^<)T} + C_{F_V}^< \, e^{\Delta E t_H}$$



Fit form: 4d method

Use fit ranges where data has plateaued in t_H, i.e. $t_H \to -\infty$

Include terms to fit
(1) unwanted exponential from first intermediate state

Sum of both time orderings $I_{\mu\nu}(T, t_H) = I^{<}_{\mu\nu}(T, t_H) + I^{>}_{\mu\nu}(T, t_H)$

$$F(t_H, T) = F + B_F^{<} \underbrace{e^{-(E_\gamma - E_H + E^{<})T}}_{t_{em}<0} + B_F^{>} \underbrace{e^{(E_\gamma - E^{>})T}}_{t_{em}>0}$$

■ fit parameters

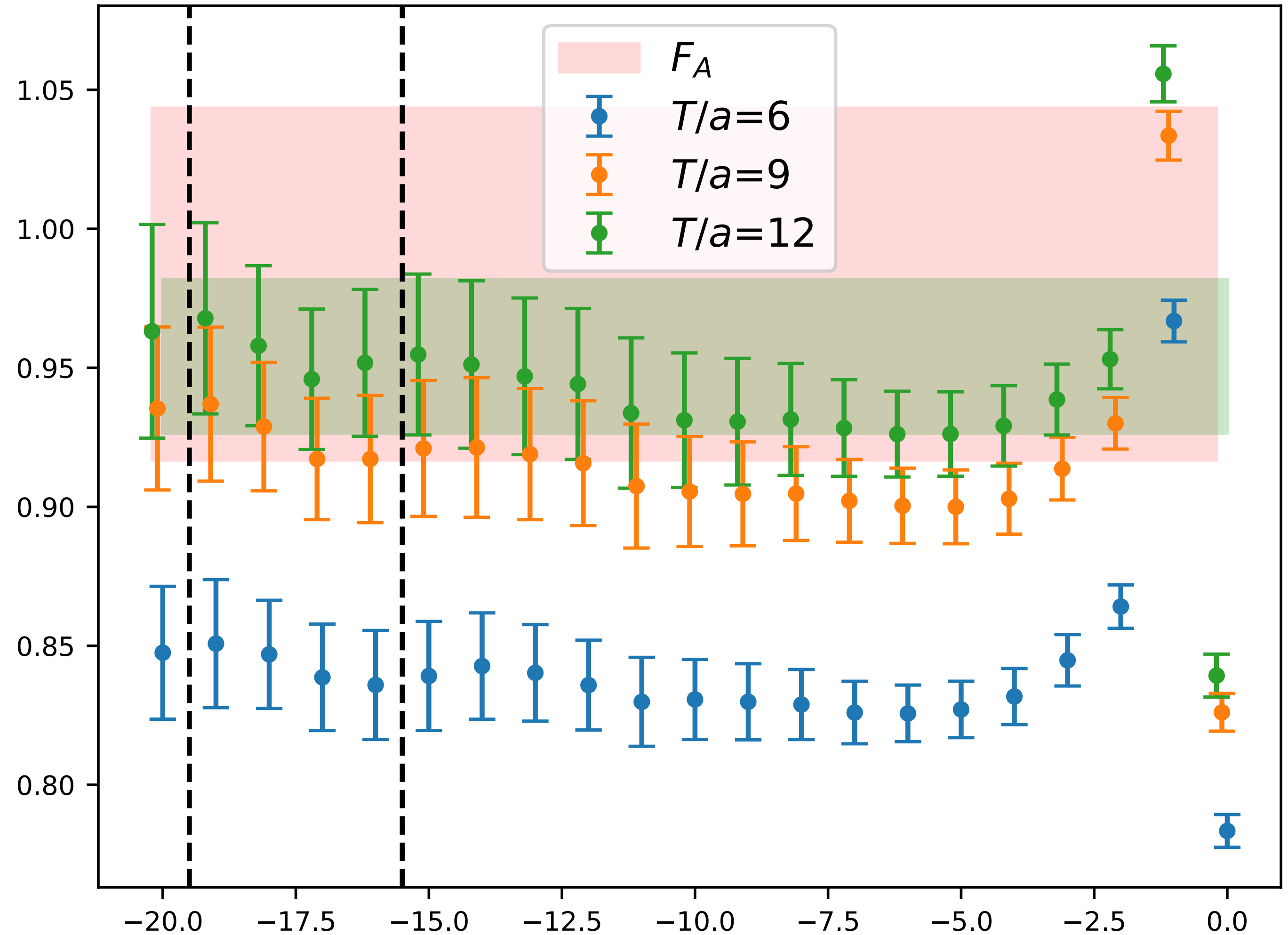
Only have three values of T, fitting multiple exponentials not possible
$\to$ Use broad Gaussian prior on $E^{>}$

$K \to \ell \nu_\ell \gamma$: 4d method

Sum of both time orderings $t_{em} < 0 + t_{em} > 0$:

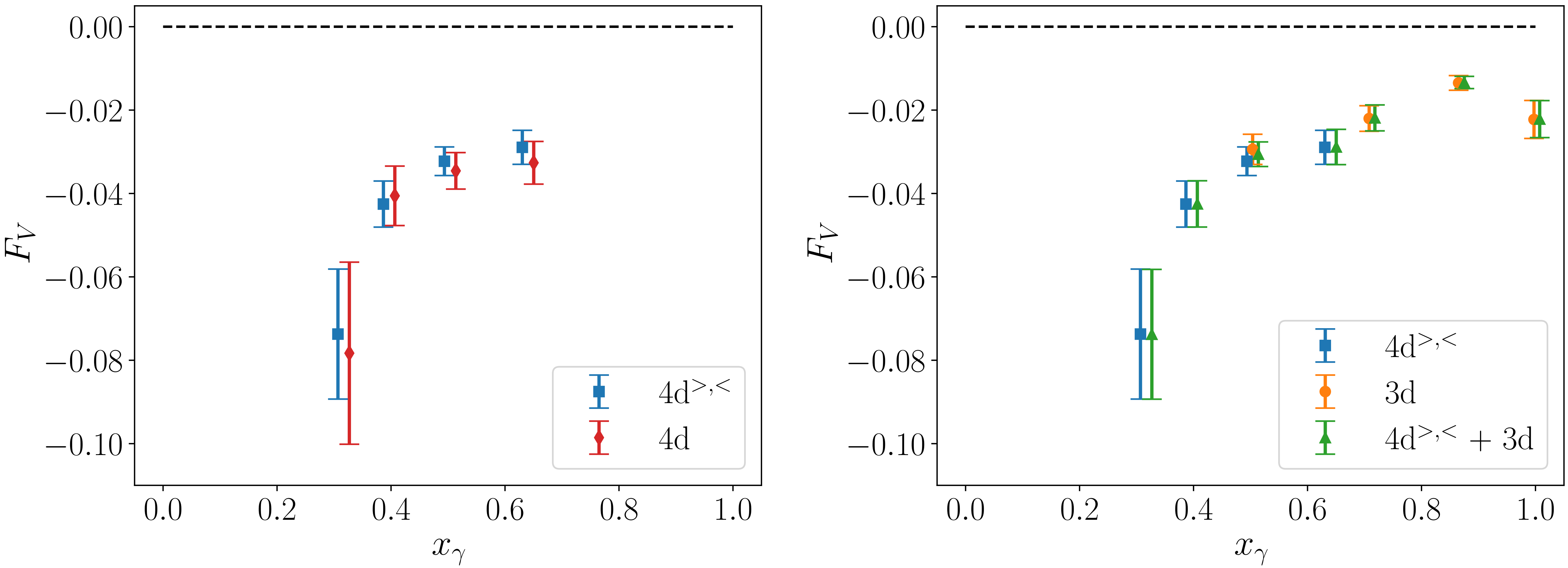
$$F_A(t_H, T) = F_A + B^{<}_{F_A} e^{-(E_\gamma - E_K + E^{<}_A)T} + B^{>}_{F_A} e^{(E_\gamma - E^{>}_A)T}$$

$F_A(t_H, T)$



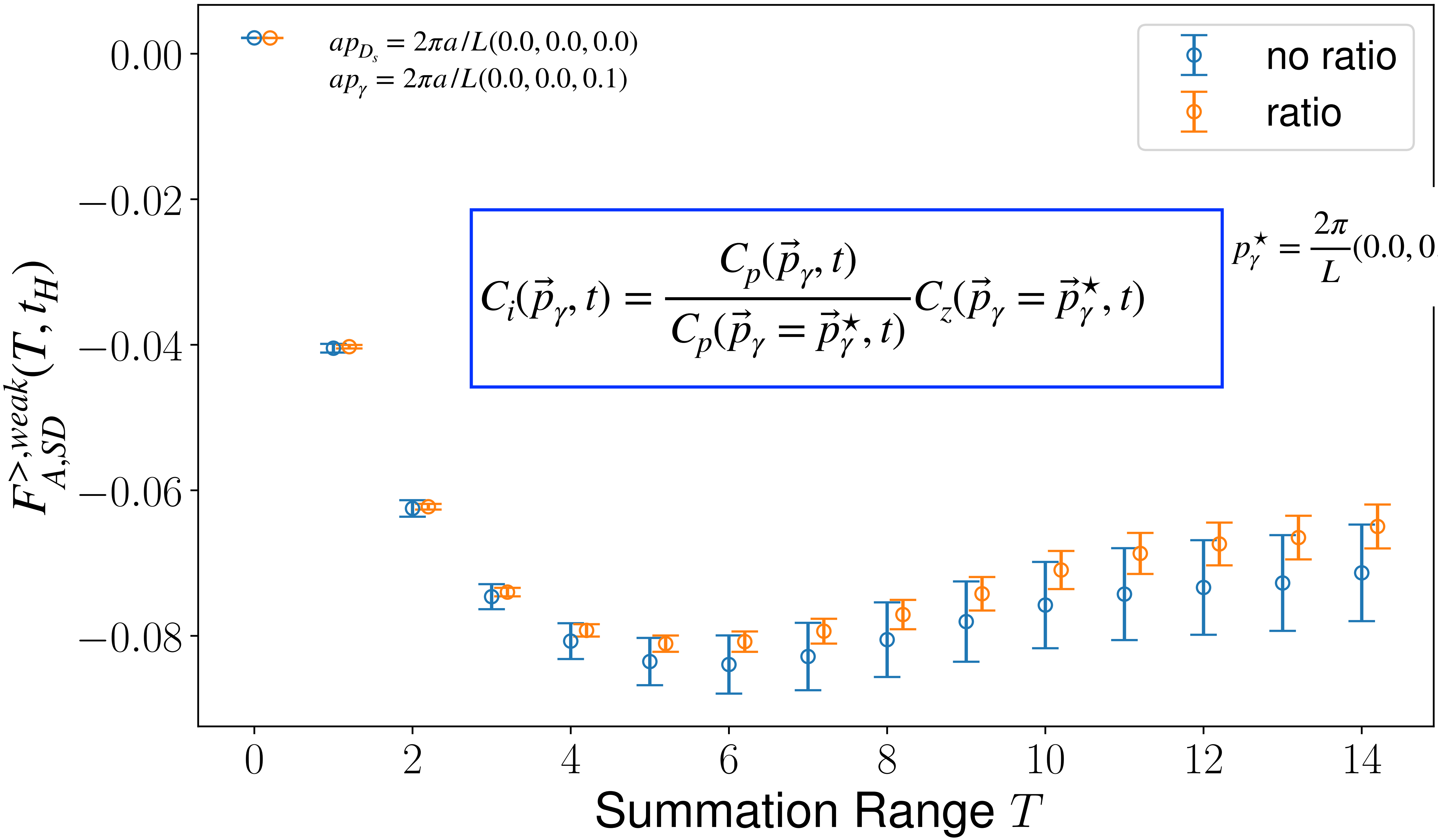
Source sink separation t_K/a

$D_s \to \ell \nu_\ell \gamma$: 3d vs 4d analysis results



- 4d method cannot resolve the sum of the unwanted exponentials of the separate time orderings
- 3d method offers good control over the unwanted exponentials for a significantly cheaper computational cost

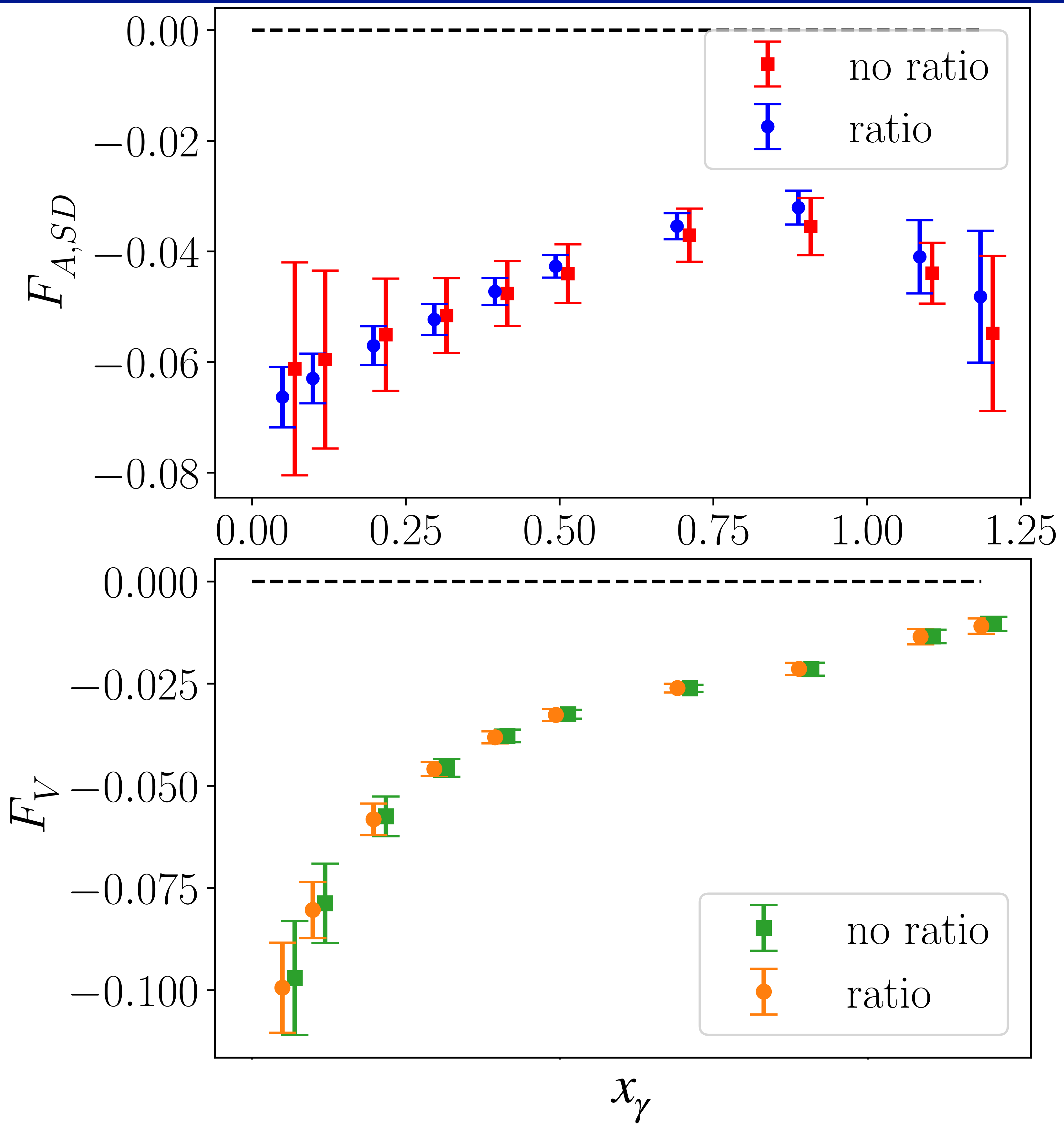
Improved form factors estimators



$$C_i(\vec{p}_\gamma, t) = \frac{C_p(\vec{p}_\gamma, t)}{C_p(\vec{p}_\gamma = \vec{p}^\star_\gamma, t)} C_z(\vec{p}_\gamma = \vec{p}^\star_\gamma, t)$$

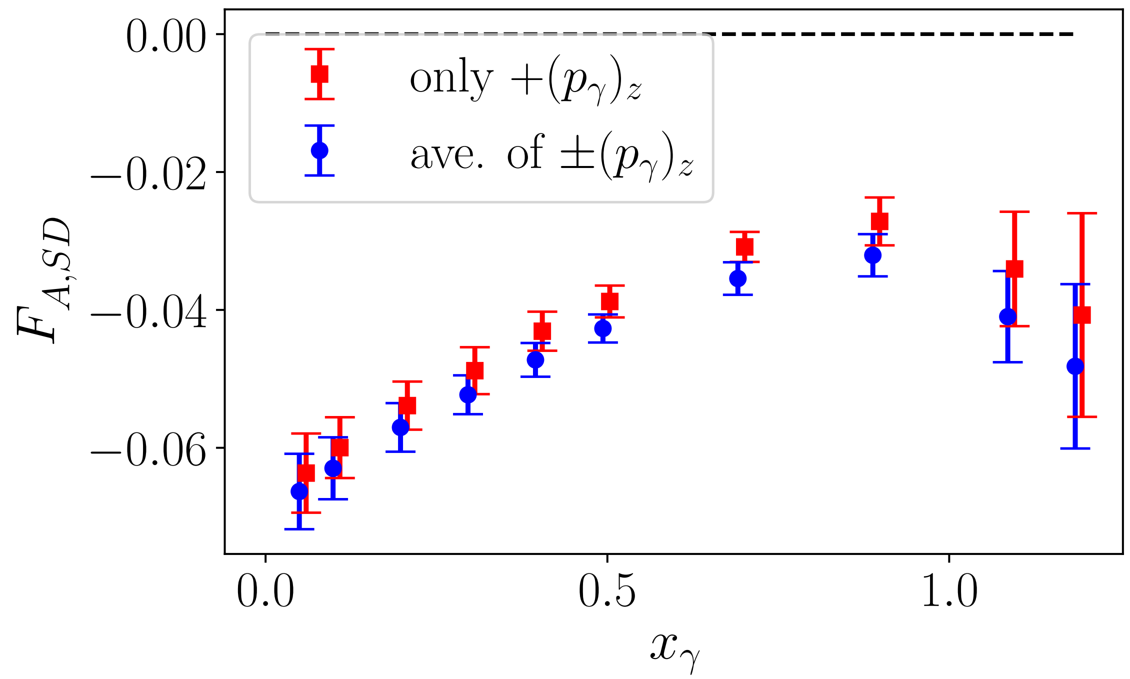
$$p^\star_\gamma = \frac{2\pi}{L}(0.0, 0.0, 0.0)$$

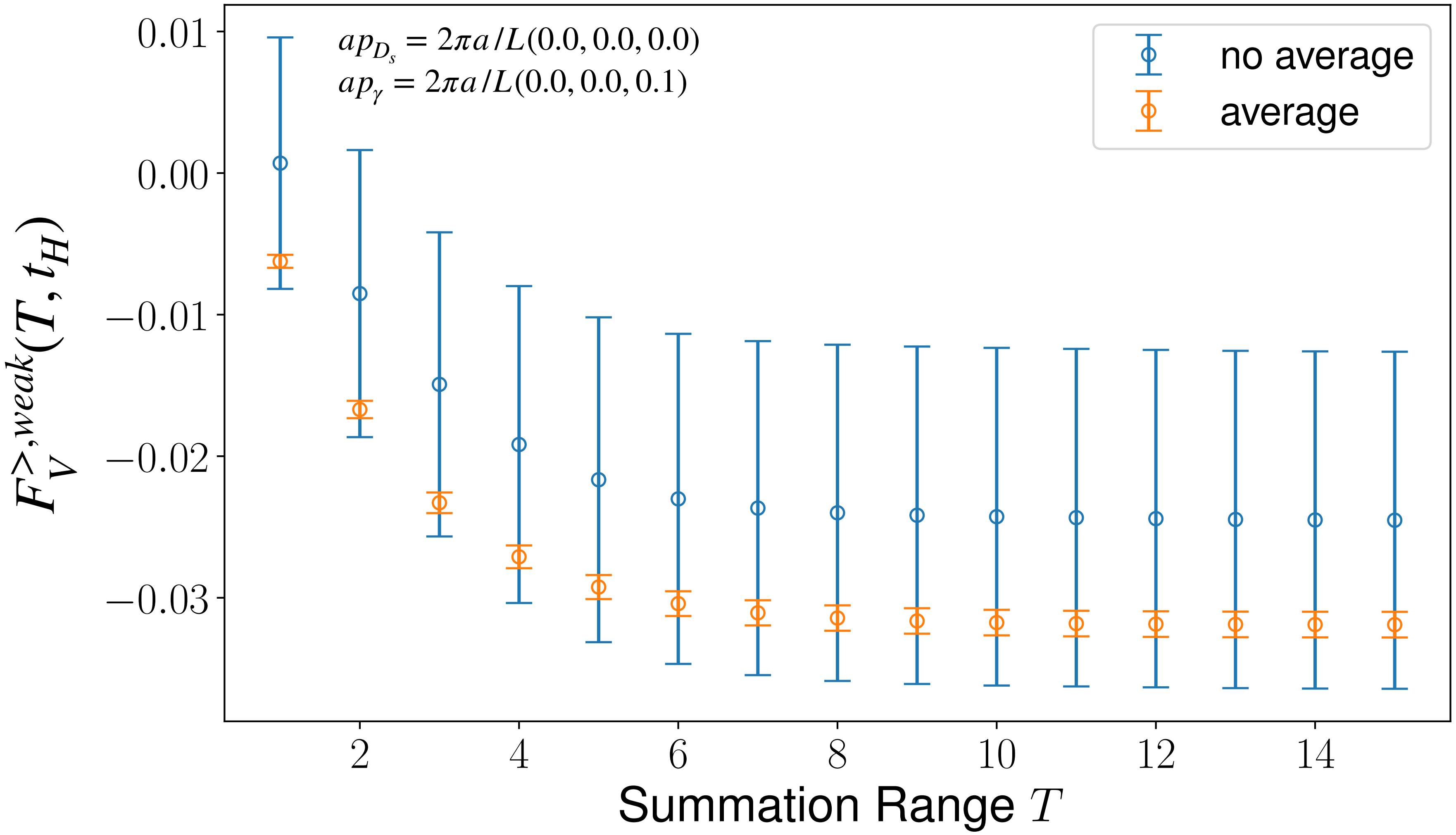
Improved form factors estimators



Improved form factors estimators [2]

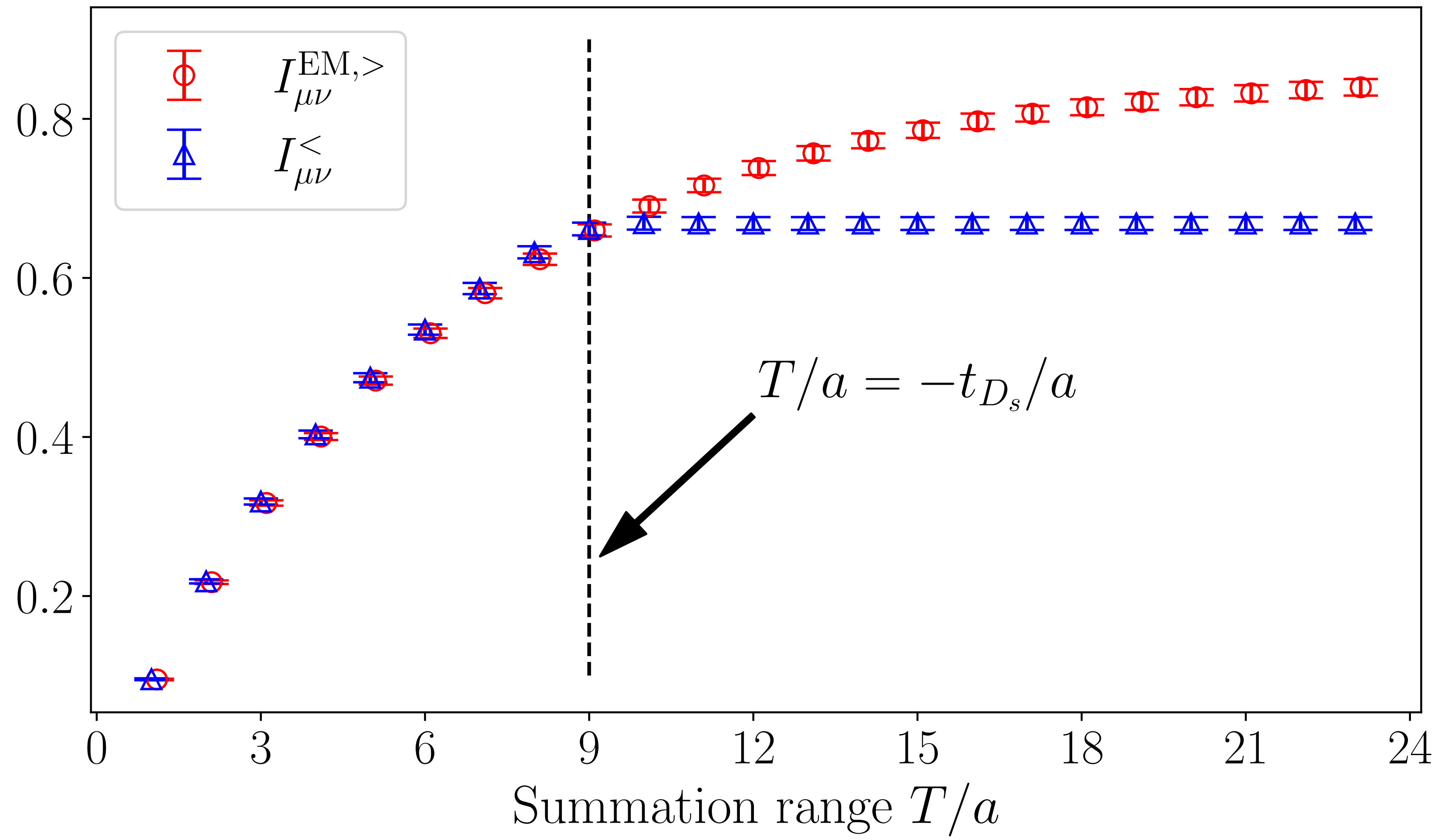
$\pm\vec{p}_\gamma$ average





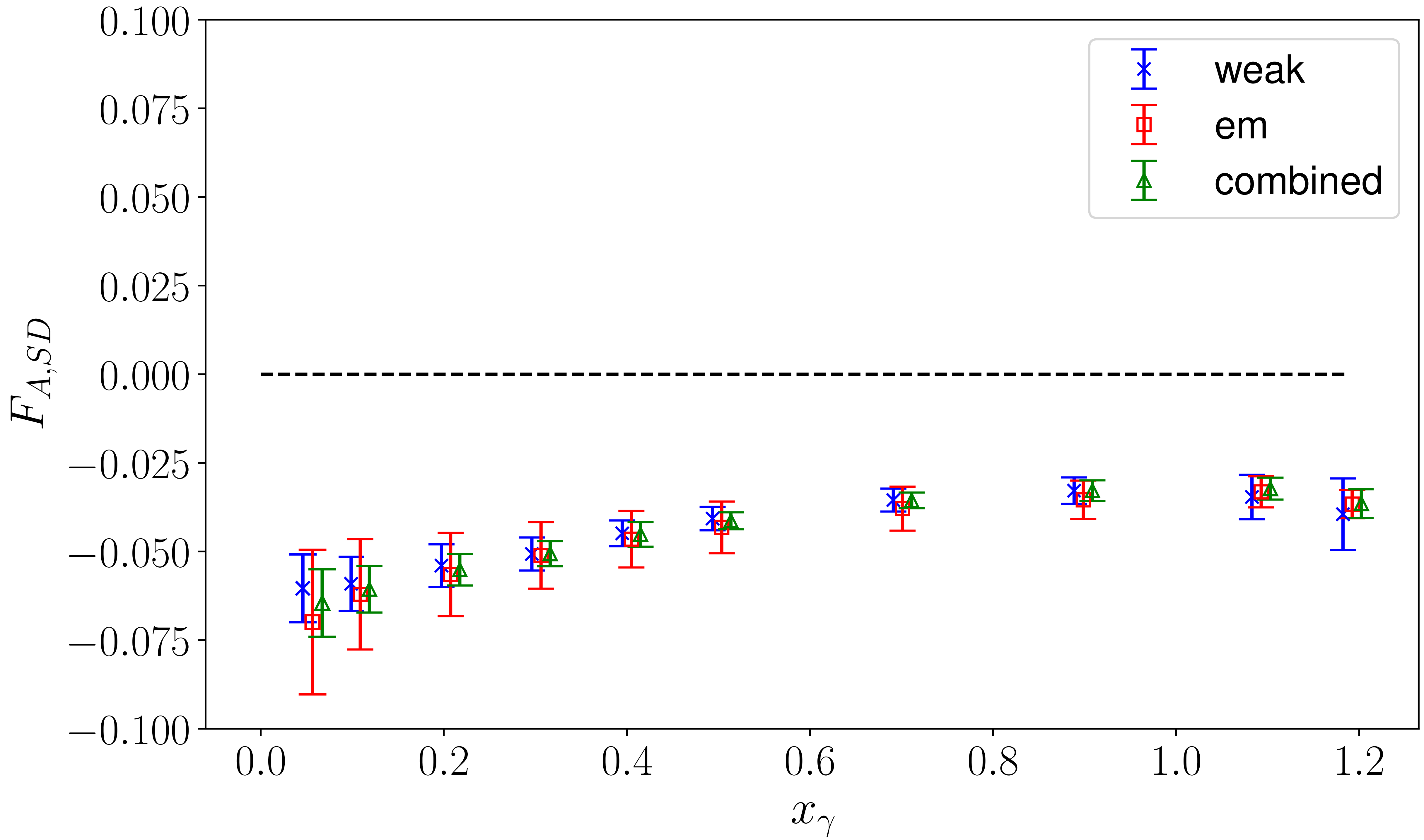
3pt function with e.m. current at origin

$$C^{\mathrm{EM}}_{3,\mu\nu}(t_W, t_H) = e^{E_H t_W} \int d^3x \int d^3y \, e^{i(\vec{p}_\gamma - \vec{p}_H)\cdot\vec{x}} e^{i\vec{p}_H\cdot\vec{y}} \langle J^{\mathrm{em}}_\mu(0) J^{\mathrm{weak}}_\nu(t_W, \vec{x}) \phi^\dagger_H(t_H, \vec{y}) \rangle$$



The spectral decomposition of the $t_W > 0$ time ordering of $I^{\mathrm{EM}}_{\mu\nu}$ and the $t_{em} < 0$ time ordering of $I_{\mu\nu}$ are equal up to excited state effects

$D_s \to \ell\nu_\ell\gamma$: 3d method

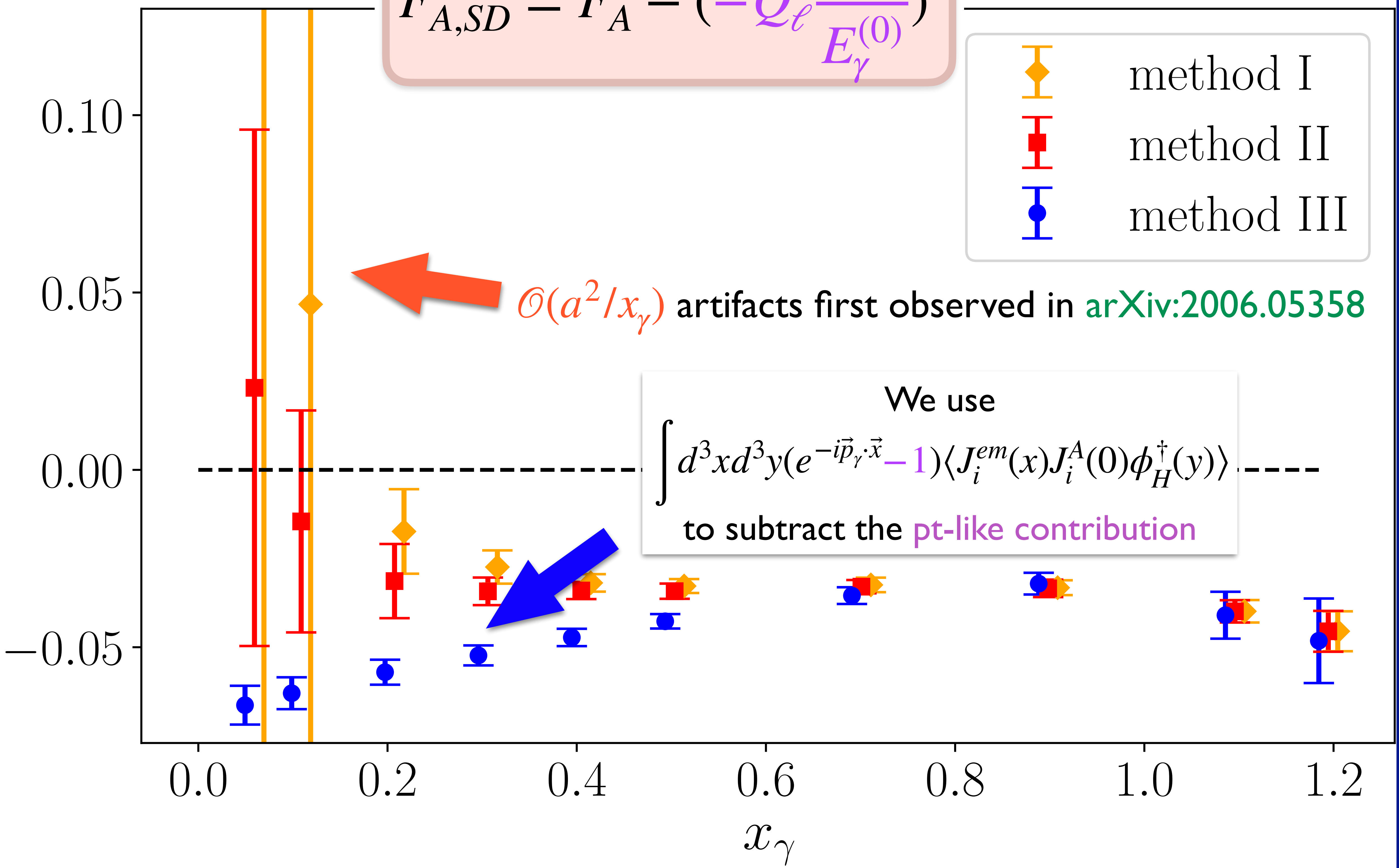


$$x_\gamma = \frac{2p_H \cdot p_\gamma}{m_H^2} \xrightarrow{\vec{p}_H = \mathbf{0}} x_\gamma = \frac{2E_\gamma^{(0)}}{m_H}$$

$$0 \leq x_\gamma \leq 1 - \frac{m_\ell^2}{m_H^2}$$

NP subtraction of IR-divergent discretization effects

$$F_{A,SD} = F_A - \left(-Q_\ell \frac{f_H}{E_\gamma^{(0)}}\right)$$



$\mathcal{O}(a^2/x_\gamma)$ artifacts first observed in arXiv:2006.05358

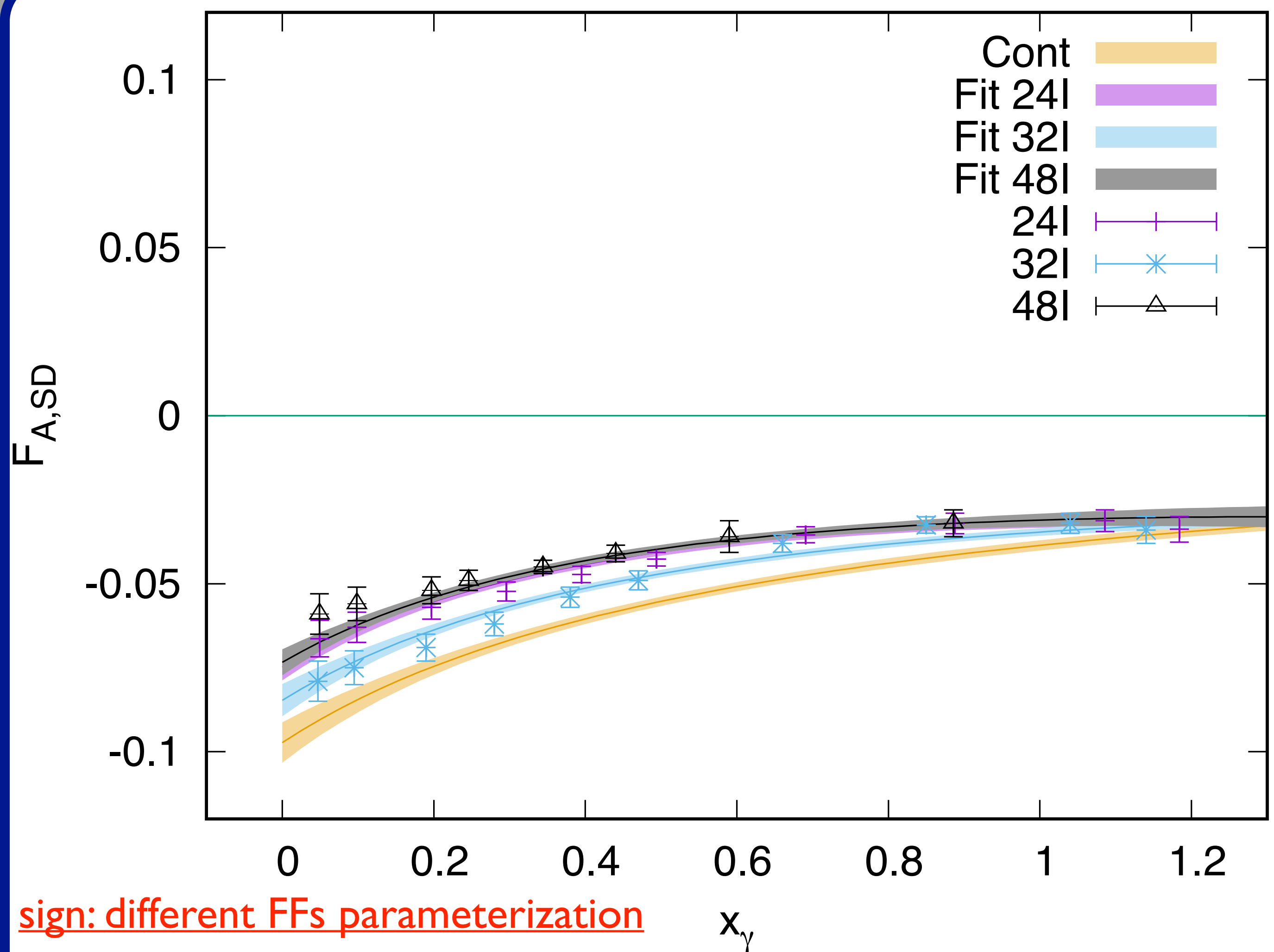
We use

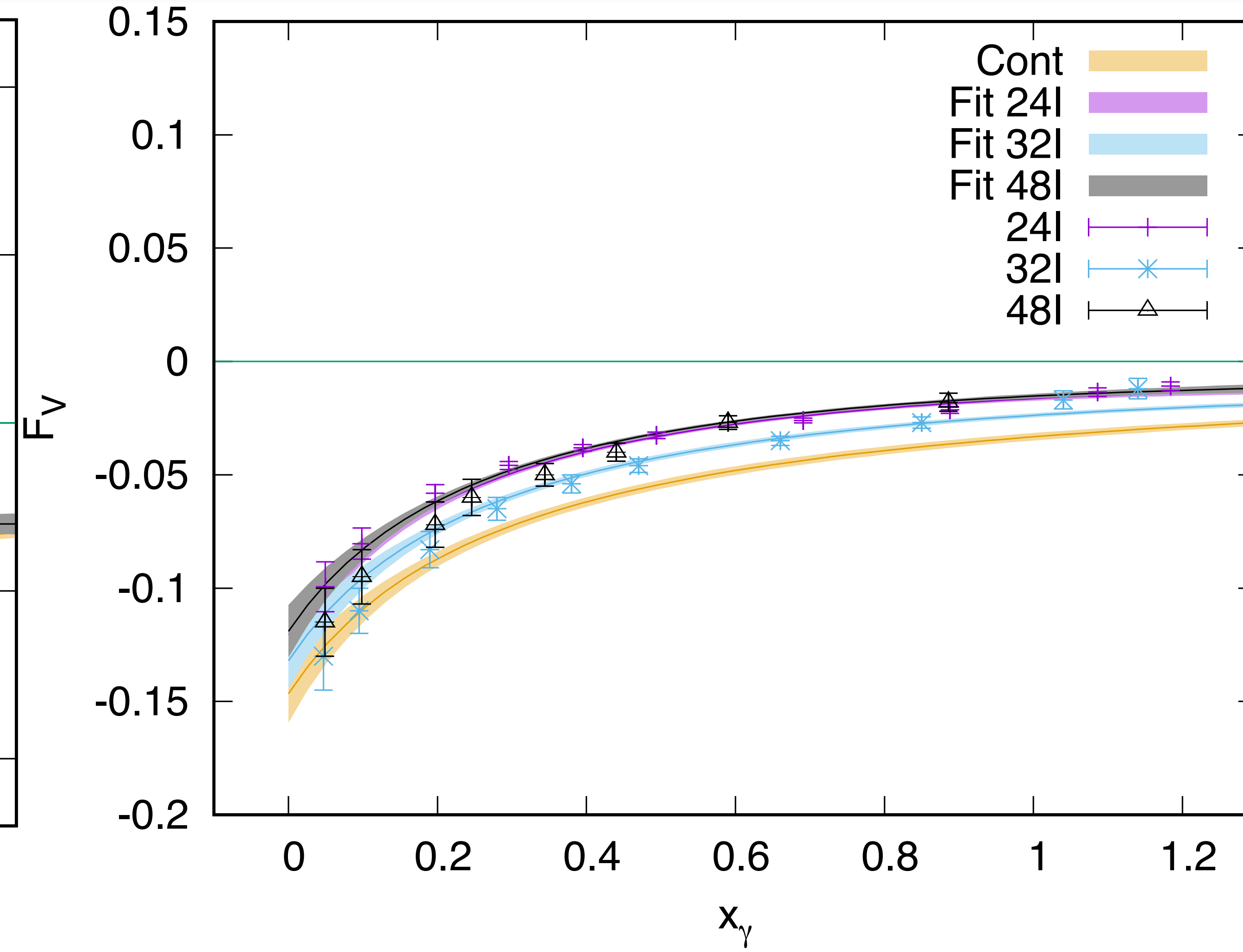
$$\int d^3x d^3y (e^{-i\vec{p}_\gamma \cdot \vec{x}} - 1) \langle J_i^{em}(x) J_i^A(0) \phi_H^\dagger(y) \rangle$$

to subtract the pt-like contribution

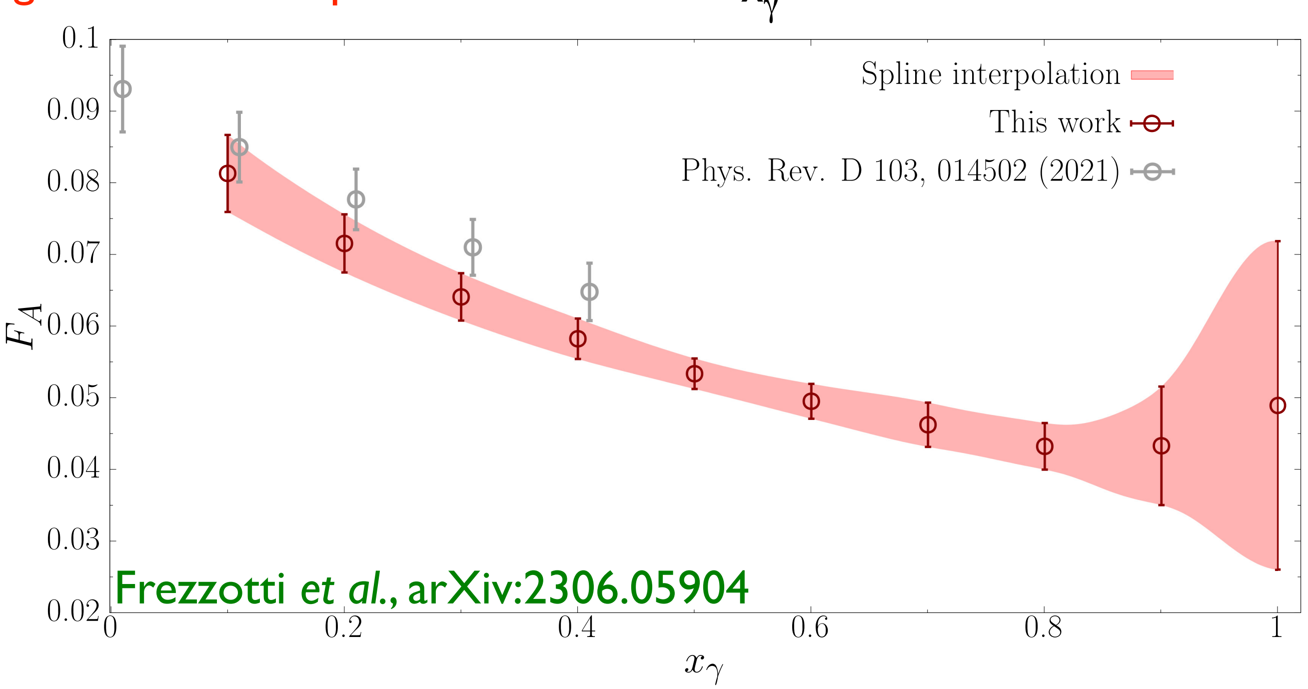
Blue data: improved subtraction of pt-like contribution

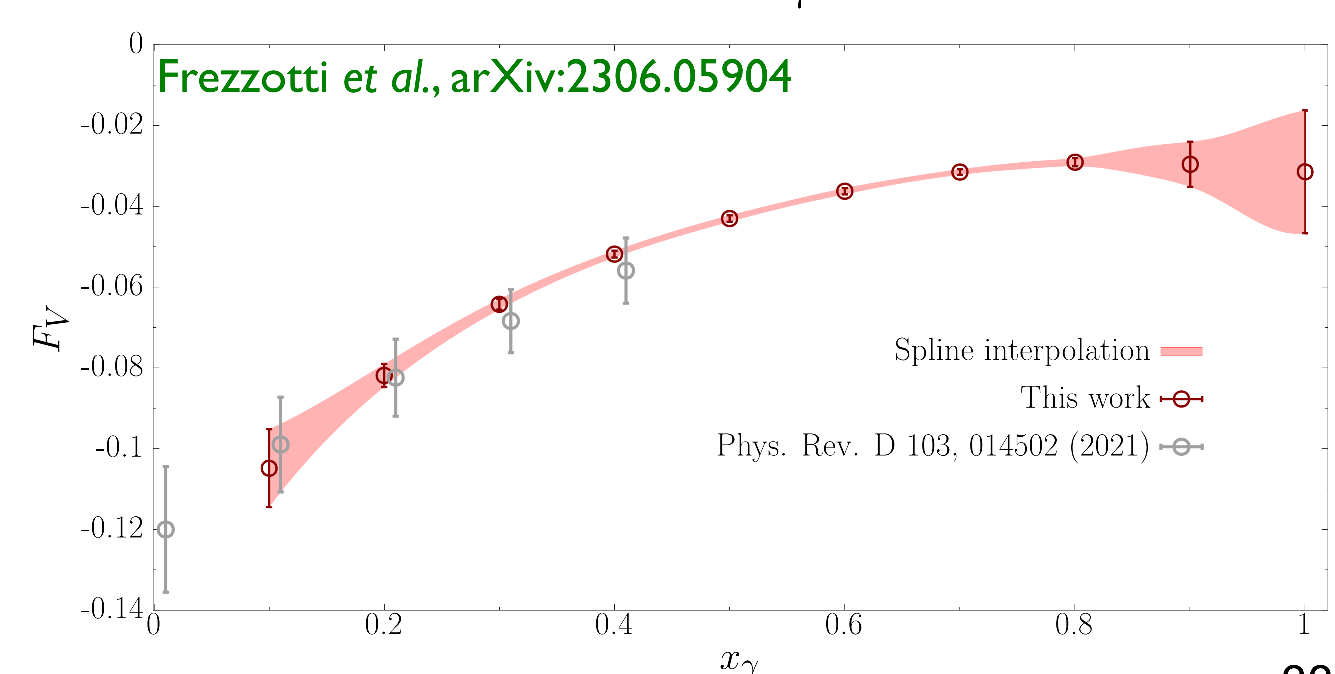
$D_s \to \ell \nu_\ell \gamma$: results on more ensembles



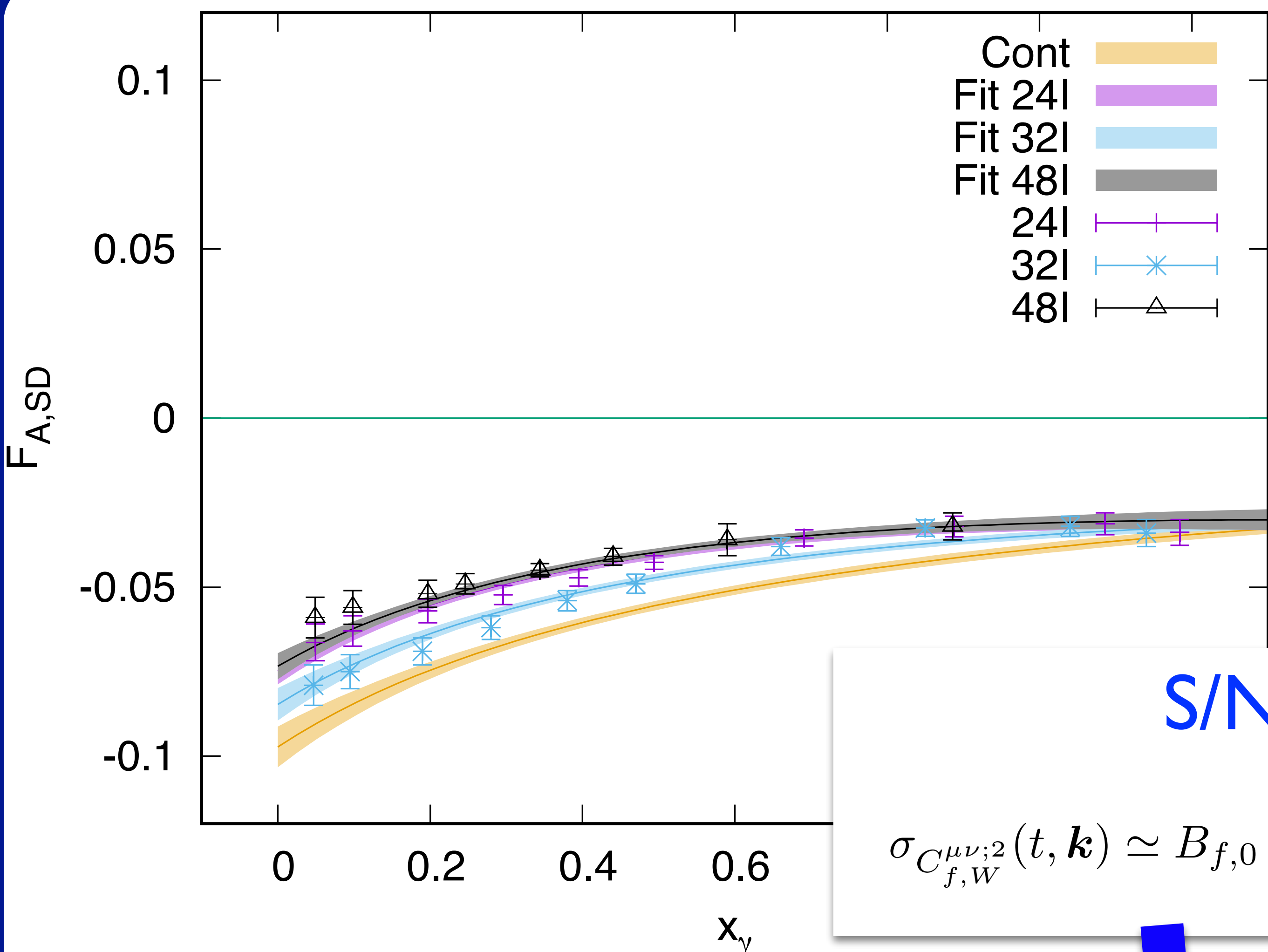


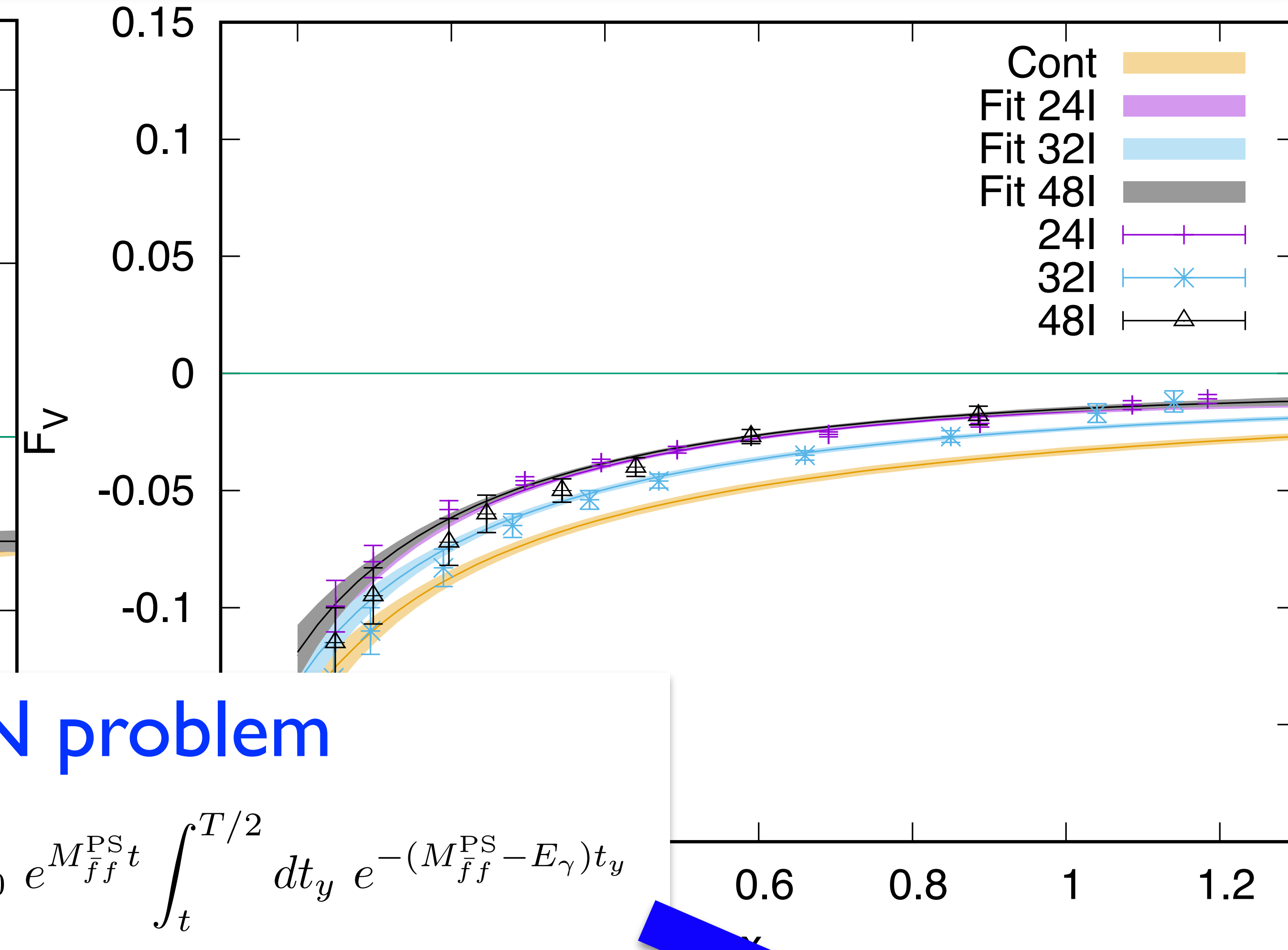
sign: different FFs parameterization





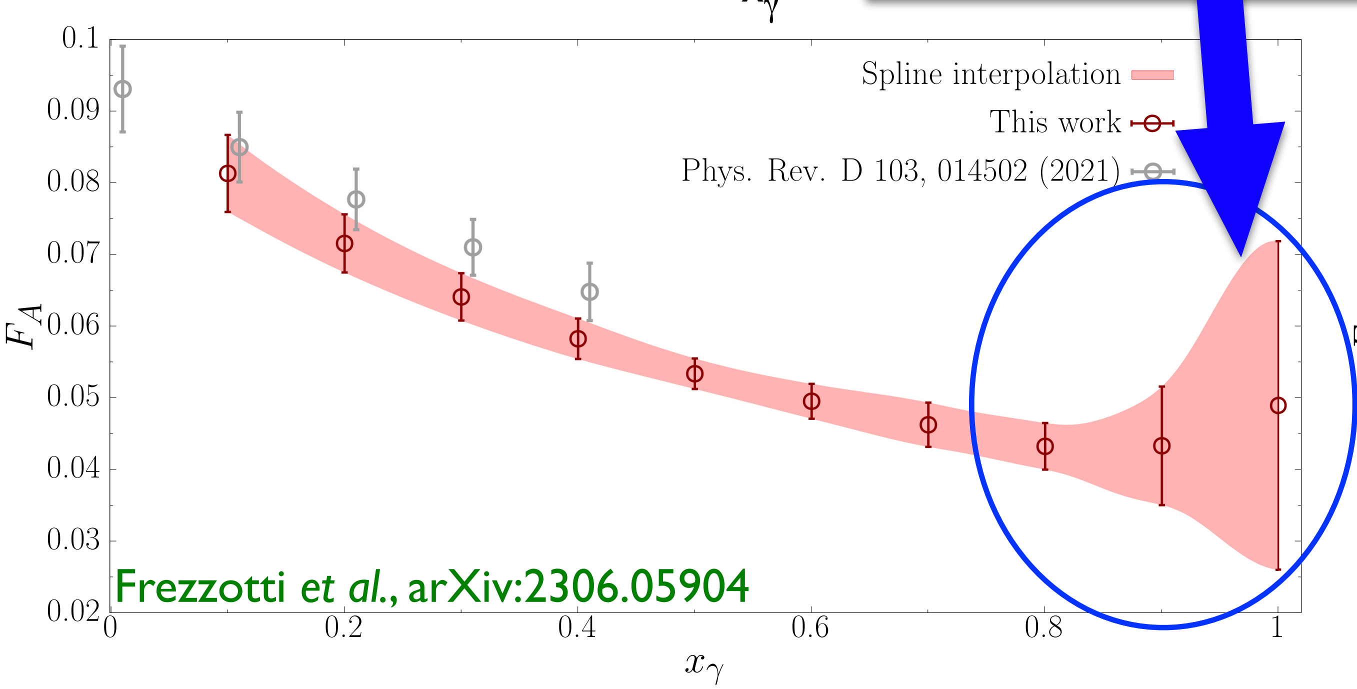
$D_s \to \ell \nu_\ell \gamma$: results on more ensembles

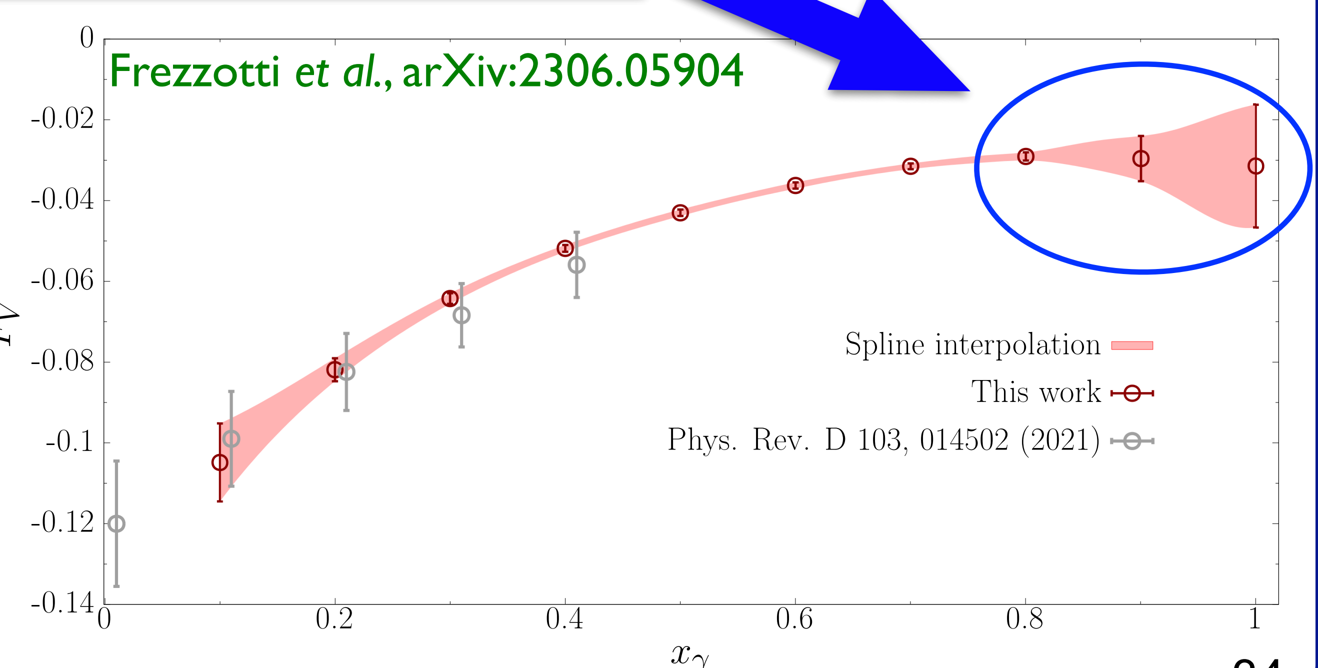




S/N problem

$$\sigma_{C^{\mu\nu;2}_{f,W}}(t, \boldsymbol{k}) \simeq B_{f,0}\; e^{M^{\rm PS}_{\bar{f}f} t} \int_t^{T/2} dt_y\; e^{-(M^{\rm PS}_{\bar{f}f} - E_\gamma) t_y}$$



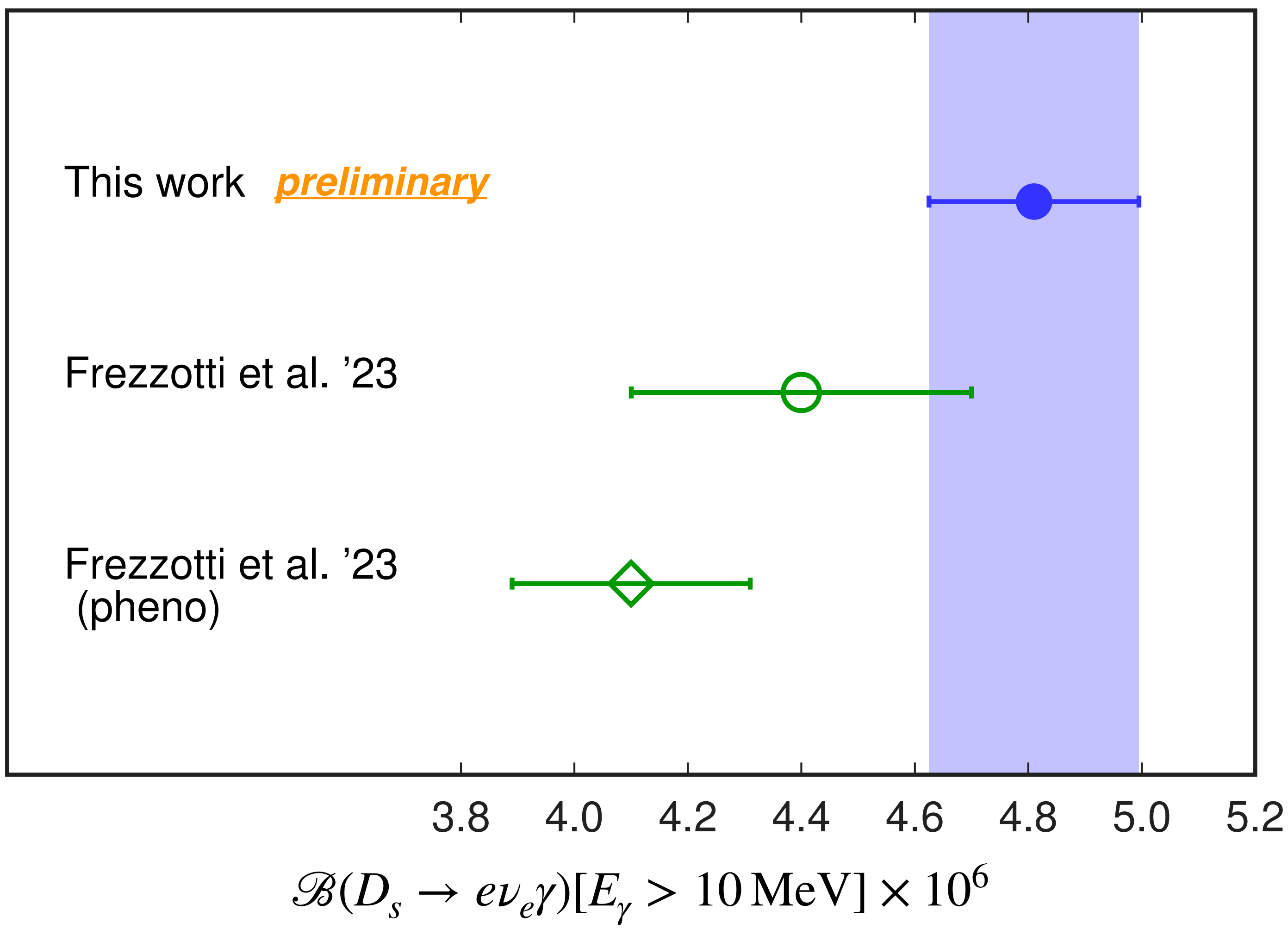


$\mathcal{B}(D_s \to e\nu_e\gamma)$

$$\frac{dR^{\rm pt}}{dx_\gamma} = -\frac{2}{(1-r_\ell^2)^2}\frac{1}{x_\gamma}\left\{\left[\frac{(2-x_\gamma)^2}{1-x_\gamma} - 4r_\ell^2\right](1-x_\gamma-r_\ell^2) - \left[2(1-r_\ell^2)(1+r_\ell^2-x_\gamma)+x_\gamma^2\right]\log\left(\frac{1-x_\gamma}{r_\ell^2}\right)\right\},$$

$$\frac{dR^{\rm int}}{dx_\gamma} = -\frac{2M_{D_s}}{f_{D_s}(1-r_\ell^2)^2}\left\{F_A\, x_\gamma\left[\frac{r_\ell^4}{1-x_\gamma} - 1 + x_\gamma + 2r_\ell^2\log\left(\frac{1-x_\gamma}{r_\ell^2}\right)\right] + (F_V - F_A)\, x_\gamma^2\left[\frac{r_\ell^2}{1-x_\gamma} - 1 + \log\left(\frac{1-x_\gamma}{r_\ell^2}\right)\right]\right\},$$

$$\frac{dR^{\rm SD}}{dx_\gamma} = \frac{M_{D_s}^2}{f_{D_s}^2}\left(F_V^2+F_A^2\right)\frac{x_\gamma^3}{r_\ell^2(1-r_\ell^2)^2}\frac{(2+r_\ell^2-2x_\gamma)(1-x_\gamma-r_\ell^2)^2}{6(1-x_\gamma)^2}.$$



Experiment

BESIII Collaboration upper bound

$$\mathcal{B}(D_s^+ \to e^+\nu_e\gamma)[E_\gamma > 10\,\text{MeV}] < 1.3\times 10^{-4}$$

arXiv:1902.03351

Conclusions and future perspectives

- The form factors for real emissions are accessible from Euclidean correlators

- We compared analysis methods using 3d and 4d data. 3d method results in smallest statistical uncertainties and allows to tame S/N problems at large photon energies.

- With moderate statistics we are able to provide rather precise, first-principles results for the form factors in the full kinematical (photon-energy) range

- Lattice calculations of radiative leptonic heavy-meson decays at high photon energy could provide useful information to better understand the internal structure of hadrons

- To extend the study to B-meson decays we will take advantage of new RBC/UKQCD ensembles at $a^{-1} \approx (3.5, 4.5)$ GeV

PRACE

	48I	64I	96I
$L^3 \cdot T/a^4$	$48^3 \cdot 96$	$64^3 \cdot 128$	$96^3 \cdot 192$
β	2.13	2.25	2.31
am_l	0.00078	0.000678	0.0054
am_h	0.0362	0.02661	0.02132
α	2.0	2.0	2.0
a^{-1} (GeV)	1.730(4)	2.359(7)	≈ 2.8
a (fm)	0.1141(3)	0.0837(3)	≈ 0.071
L (fm)	5.476(12)	5.354(16)	≈ 6.8
L_s/a	24	12	12
m_π (MeV)	139.2(4)	139.2(5)	≈ 135
$m_\pi L$	3.863(6)	3.778(8)	≈ 4.7
N_{conf}	120	160	20

PRACE

Supplementary slides

A strategy for Lattice QCD:

The isospin-breaking part of the Lagrangian is treated as a perturbation

Expand in: $m_d - m_u$ + α_{em}

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Isospin breaking effects due to the up-down mass difference in lattice QCD

RM123 collaboration
G.M. de Divitiis,[a,b] P. Dimopoulos,[c,d] R. Frezzotti,[a,b] V. Lubicz,[e,f] G. Martinelli,[g,d] R. Petronzio,[a,b] G.C. Rossi,[a,b] F. Sanfilippo,[c,d] S. Simula,[f] N. Tantalo[a,b] and C. Tarantino[e,f]

PHYSICAL REVIEW D 87, 114505 (2013)

Leading isospin breaking effects on the lattice

G. M. de Divitiis,[1,2] R. Frezzotti,[1,2] V. Lubicz,[3,4] G. Martinelli,[5,6] R. Petronzio,[1,2] G. C. Rossi,[1,2] F. Sanfilippo,[7] S. Simula,[4] and N. Tantalo[1,2]
(RM123 Collaboratio

arXiv:1303.4896

[1]*Dipartimento di Fisica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Rome, Italy*
[2]*INFN, Sezione di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Rome, Italy*
[3]*Dipartimento di Matematica e Fisica, Università Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy*
[4]*INFN, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy*
[5]*SISSA, Via Bonomea 265, 34136 Trieste, Italy*
[6]*INFN, Sezione di Roma, Piazzale Aldo Moro 5, I-00185 Rome, Italy*
[7]*Laboratoire de Physique Théorique (Bâtiment 210), Université Paris Sud, F-91405 Orasay-Cedex, France*
(Received 3 April 2013; published 7 June 2013)

RM123 Collaboration

① The (md-mu) expansion

- Identify the isospin-breaking term in the QCD action

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) - \frac{1}{2}(m_d - m_u)(\bar{u}u - \bar{d}d) \right] =$$

$$= \sum_x \left[m_{ud}(\bar{u}u + \bar{d}d) - \Delta m(\bar{u}u - \bar{d}d) \right] = S_0 - \Delta m \hat{S}$$

$\hat{S} = \Sigma_x(\bar{u}u - \bar{d}d)$

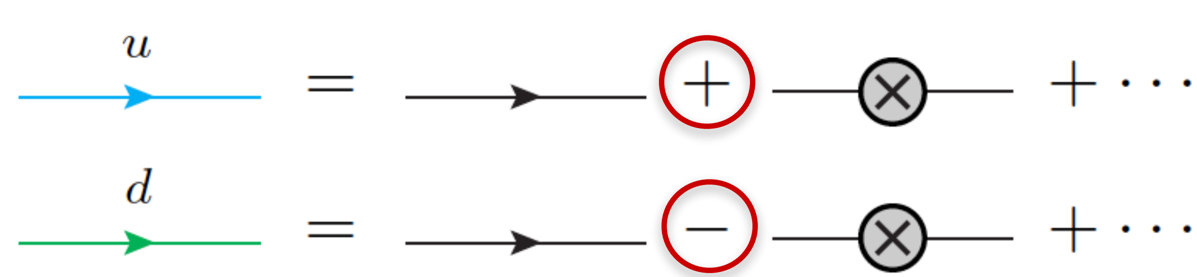
- Expand the functional integral in powers of Δm

Advantage: factorized out

$$\langle O \rangle = \frac{\int D\phi\, O e^{-S_0 + \Delta m \hat{S}}}{\int D\phi\, e^{-S_0 + \Delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\phi\, O e^{-S_0}(1 + \Delta m \hat{S})}{\int D\phi\, e^{-S_0}(1 + \Delta m \hat{S})} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0}{1 + \Delta m \langle \hat{S} \rangle_0} = \langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0$$

for isospin symmetry

- At leading order in Δm the corrections only appear in the valence-quark propagators:



(disconnected contractions of $\bar{u}u$ and $\bar{d}d$ vanish due to isospin symmetry)

② The QED expansion

- Non-compact QED: the dynamical variable is the gauge potential $A_\mu(x)$ in a fixed covariant gauge ($\nabla^-_\mu A_\mu(x) = 0$)

$$S_{QED} = \frac{1}{2}\sum_{x;\mu\nu} A_\nu(x)\left(-\nabla^-_\mu \nabla^+_\mu\right) A_\nu(x) \overset{(p.b.c.)}{=} \frac{1}{2}\sum_{k;\mu\nu} \tilde{A}^*_\nu(k)\left(2\sin(k_\mu/2)\right)^2 \tilde{A}_\nu(k)$$

- The photon propagator is IR divergent → subtract the zero momentum mode

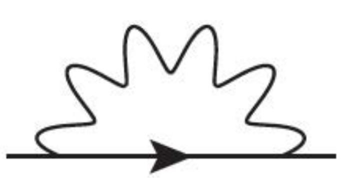
- Full covariant derivatives are defined introducing QED and QCD links:

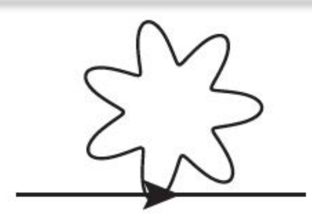
$$A_\mu(x) \rightarrow E_\mu(x) = e^{-iaeA_\mu(x)}$$

$$D^+_\mu q_f(x) = \left[E_\mu(x)\right]^{e_f} U_\mu(x)\, q_f(x+\hat{\mu}) - q_f(x)$$

QED ($\left[E_\mu(x)\right]^{e_f}$)   QCD ($U_\mu(x)$)

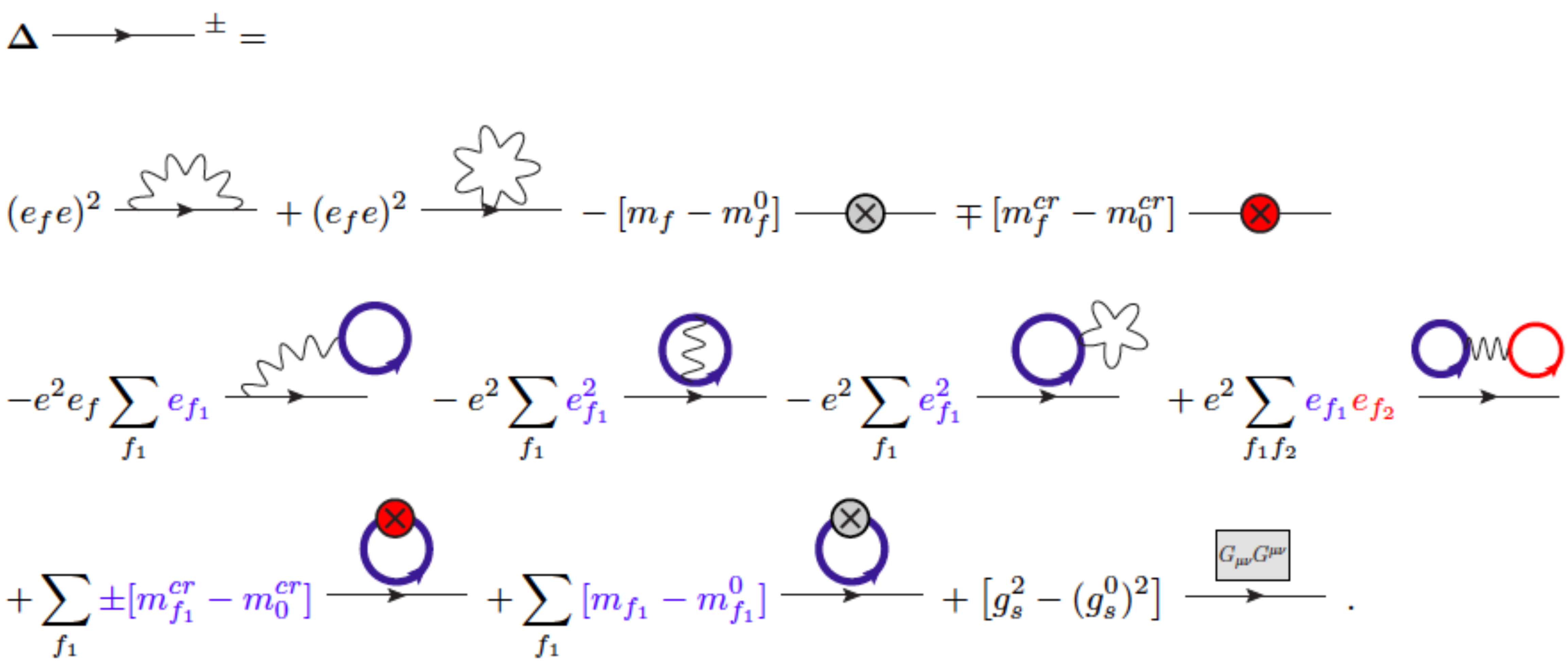
- Since $E_\mu(x) = e^{-ieA_\mu(x)} = 1 - ieA_\mu(x) - 1/2\, e^2 A^2_\mu(x) + \ldots$ the expansion leads to:

$(e_f e)^2$ 

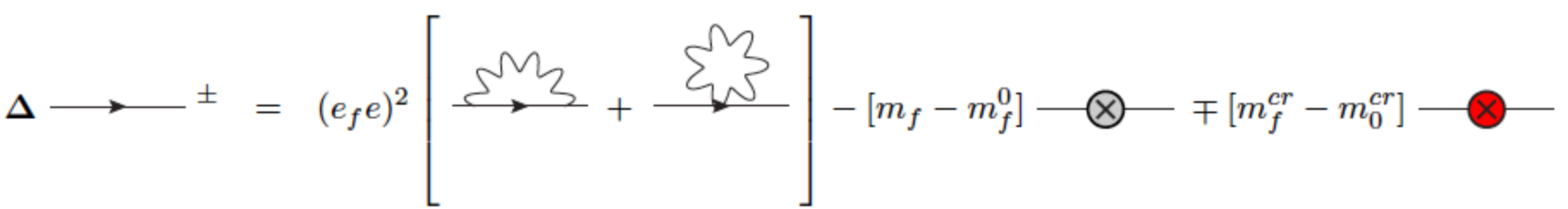
$(e_f e)^2$ 

\+ counterterms

The QED expansion for the quark propagator



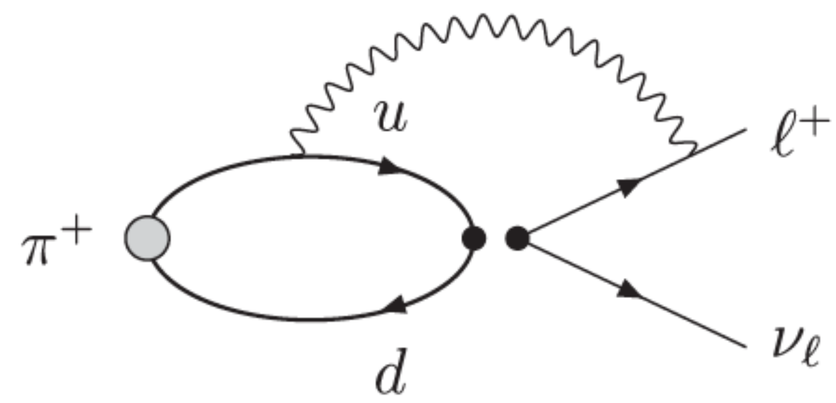
In the electro-quenched approximation:



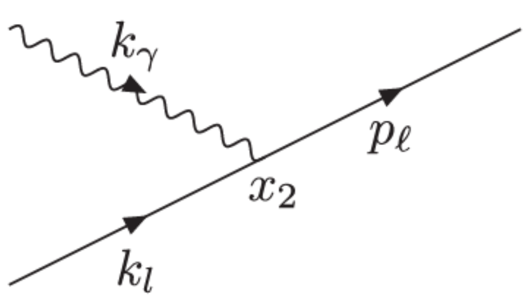
Lattice calculation of Γ_0 at $O(\alpha)$

- <u>A technical but important point</u>:

$$\delta C^{(q\ell)}(t)_{\alpha\beta} = -\int d^3\vec{x}\, d^4x_1\, d^4x_2\, \langle 0|T\left\{J_W^{\nu}(0)\, j_\mu(x_1)\, \phi^\dagger(\vec{x},-t)\right\}|0\rangle$$
$$\times\, \Delta(x_1,x_2)\left(\gamma_\nu(1-\gamma^5)S(0,x_2)\gamma_\mu\right)_{\alpha\beta} e^{E_\ell t_2 - i\,\vec{p}_\ell\cdot\vec{x}_2}$$



We need to ensure that the t_2 integration converges as $t_2 \to \infty$. The large t_2 behavior is given by the factor $\exp\left[\left(E_\ell - \omega_\ell - \omega_\gamma\right)t_2\right]$



$$E_\ell = \sqrt{\vec{p}_\ell^{\,2} + m_\ell^2} \qquad \omega_\ell = \sqrt{\vec{k}_\ell^{\,2} + m_\ell^2} \qquad \omega_\gamma = \sqrt{\vec{k}_\gamma^{\,2} + m_\gamma^2} \qquad \vec{k}_\ell + \vec{k}_\gamma = \vec{p}_\ell$$

$$\Rightarrow \quad \left(\omega_\ell + \omega_\gamma\right)_{\min} = \sqrt{\left(m_\ell^2 + m_\gamma^2\right) + \vec{p}_\ell^{\,2}} > E_\ell$$

The integral is convergent and the continuation from Minkowski to Euclidean space can be performed (same if we set $m_\gamma=0$ but remove the photon zero mode in FV).

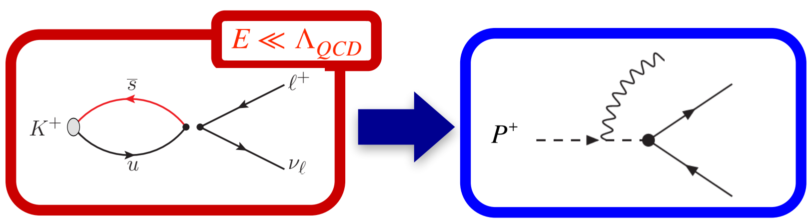
<u>CONDITIONS</u>: - mass gap between the decaying particle and the intermediate states
- absence of lighter intermediate states

The strategy

$$\Gamma[P_{\ell 2}] = (\Gamma_0 - \Gamma_0^{pt}) + (\Gamma_0^{pt} + \Gamma_1^{pt}(E))$$

- The contributions from soft virtual photon to Γ_0 and Γ_0^{pt} in the first term are exactly the same and the IR divergence cancels in the difference $\Gamma_0 - \Gamma_0^{\mathrm{pt}}$.

- The sum $\Gamma_0^{pt} + \Gamma_1^{pt}(E)$ in the second term is also IR finite since it is a physically well defined quantity. This term can be thus calculated in perturbation theory with a different IR cutoff.

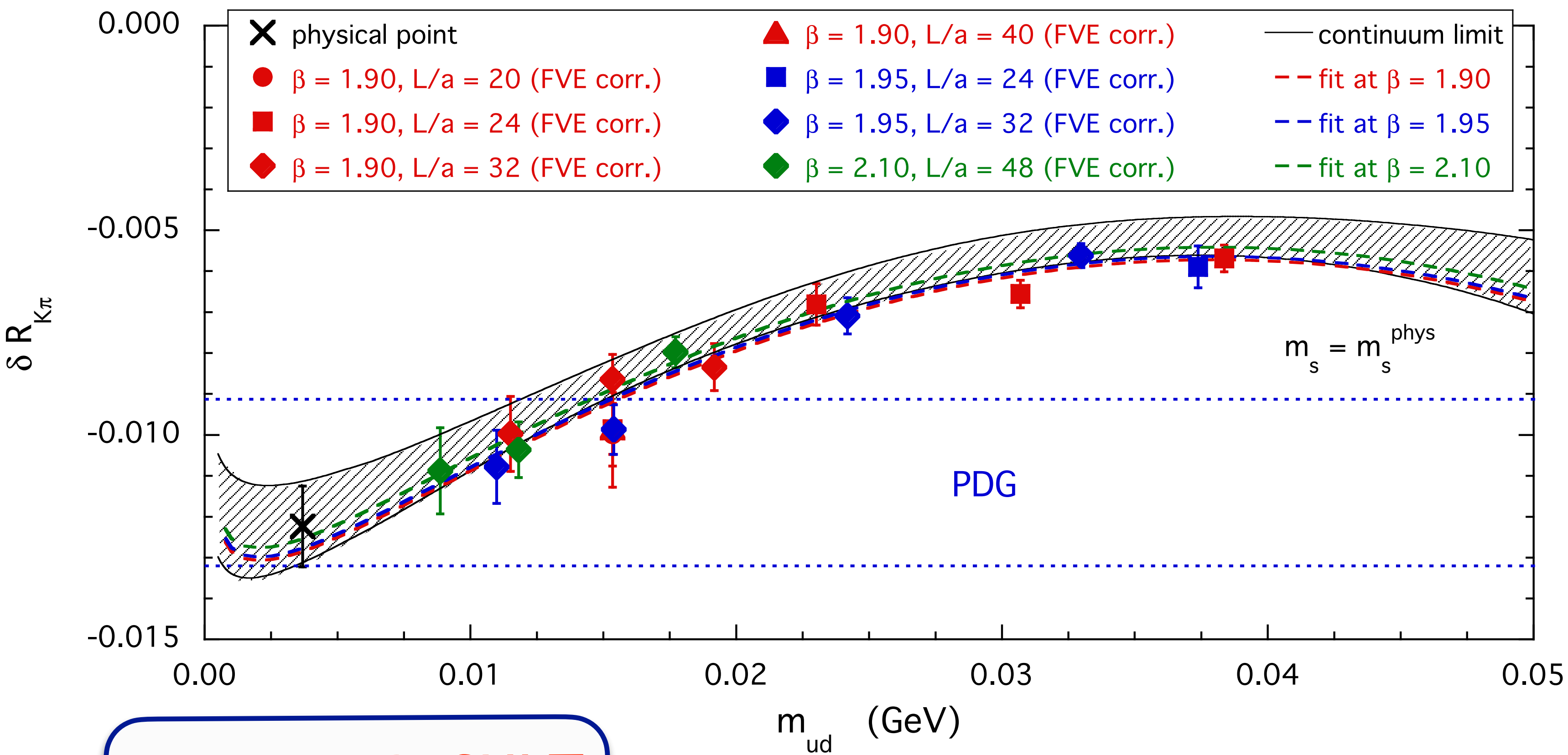


- The two terms are also separately gauge invariant.

$$\Delta\Gamma_0(L) = \Gamma_0(L) - \Gamma_0^{pt}(L)$$

$$\Gamma^{pt}(E) = \lim_{m_\gamma \to 0} \left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(E, m_\gamma)\right]$$

Leptonic decays at O(α): RESULTS



RM123 & Soton Coll., 2017

$$\delta R_{K\pi} = C_0 + C_\chi \log(m_{ud}) + C_1 m_{ud} + C_2 m_{ud}^2 + Da^2$$
$$+\frac{K_2}{L^2}\left[\frac{1}{M_K^2} - \frac{1}{M_\pi^2}\right] + \frac{K_2^\mu}{L^2}\left[\frac{1}{(E_\mu^K)^2} - \frac{1}{(E_\mu^\pi)^2}\right]$$
$$+\delta\Gamma^{pt}(\Delta E_\gamma^{\max,K}) - \delta\Gamma^{pt}(\Delta E_\gamma^{\max,\pi})$$

LATTICE RESULT

$$\delta R_{K\pi} = -0.0126(10)_{stat}(2)_{input}(5)_{chir}(5)_{FVE}(4)_{disc}(6)_{qQED}$$
$$= -0.0126(14)$$

ChPT

$$\delta R_K - \delta R_\pi = -0.0112(21)$$

V.Cirigliano and H.Neufeld, PLB 700 (2011) 7

$$\left|\frac{V_{us}}{V_{ud}}\right| \frac{f_K^{(0)}}{f_\pi^{(0)}} = 0.27683(29)_{exp}(20)_{th}$$

$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23135(24)_{exp}(39)_{th}$$

FLAG(2019) Nf=2+1+1 $\frac{f_K^{(0)}}{f_\pi^{(0)}} = 1.1966(18)$

$|V_{ud}|$ from

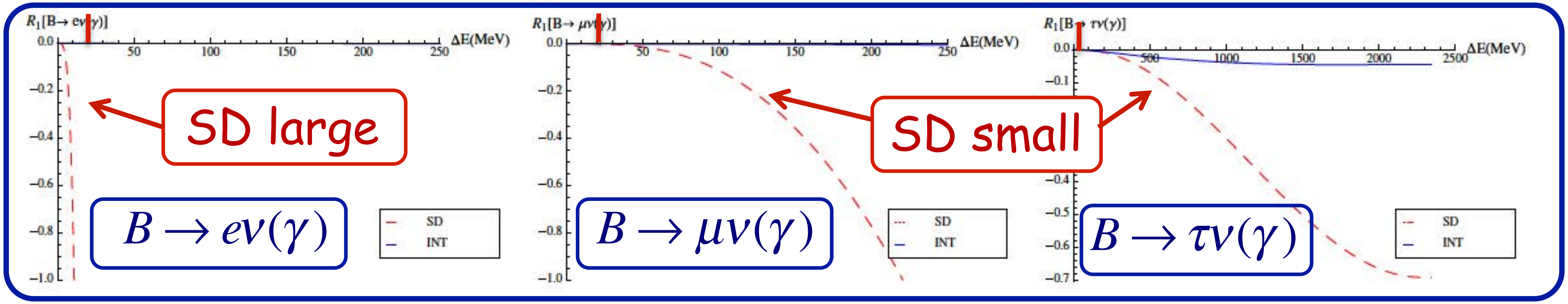
$|V_{us}| = 0.22538(46)$ Hardy and Towner, 2016

$|V_{us}| = 0.22526(46)$ Seng *et al.*, 2018

Structure dependent contributions to decays of D and B mesons

- For the studies of D and B mesons decays we cannot apply ChPT
- For B mesons in particular we have another small scale, $m_{B^*} - m_B \simeq 45\ \text{MeV}$
  ➡ the radiation of a soft photon may still induce sizeable SD effects
- A phenomenological analysis based on a simple pole model for F_V and F_A confirms this picture

D. Becirevic *et al.*, PLB 681 (2009) 257



$$F_V \simeq \frac{\tilde{C}_V}{1-(p_B-k)^2/m_{B^*}^2}$$

$$F_A \simeq \frac{\tilde{C}_A}{1-(p_B-k)^2/m_{B_1}^2}$$

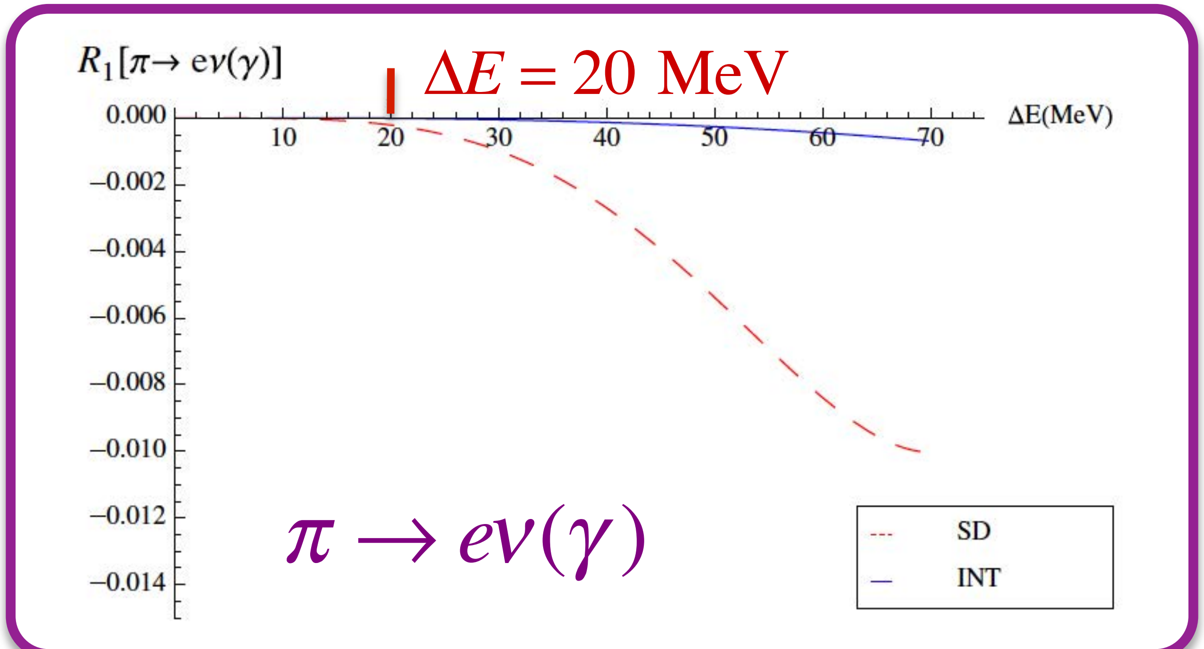
Under this assumption the SD contributions to $B \to e\nu(\gamma)$ for $E_\gamma \simeq 20$ MeV can be very large, but are small for $B \to \mu\nu(\gamma)$ and $B \to \tau\nu(\gamma)$

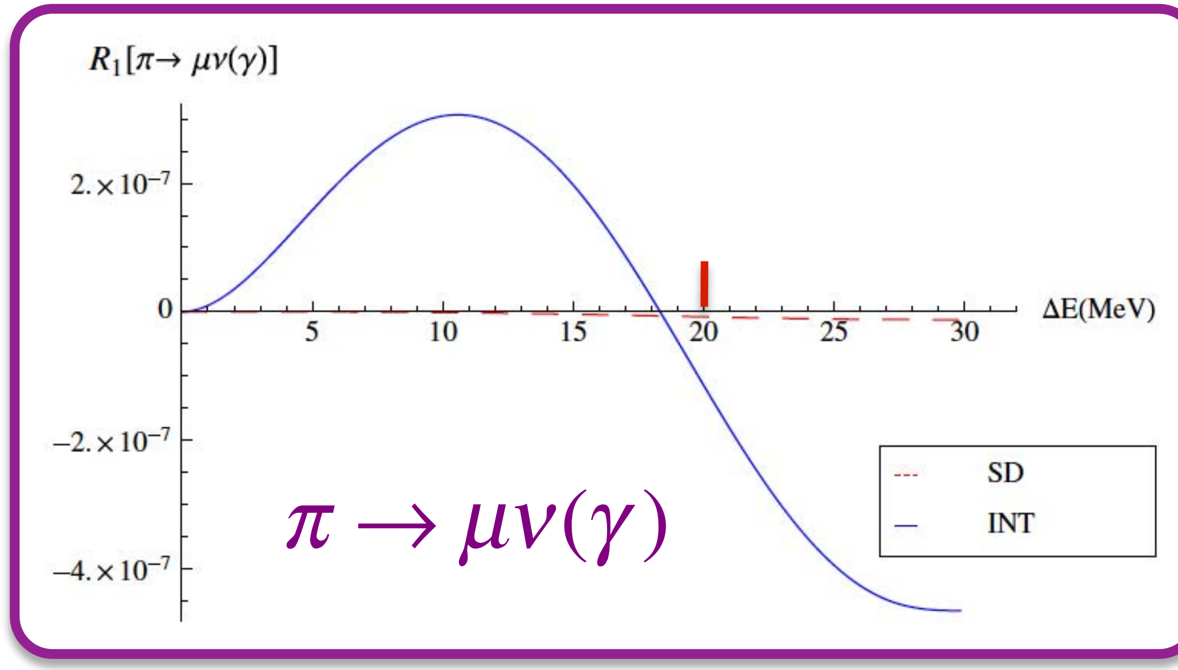
A lattice calculation of F_V and F_A would be very useful

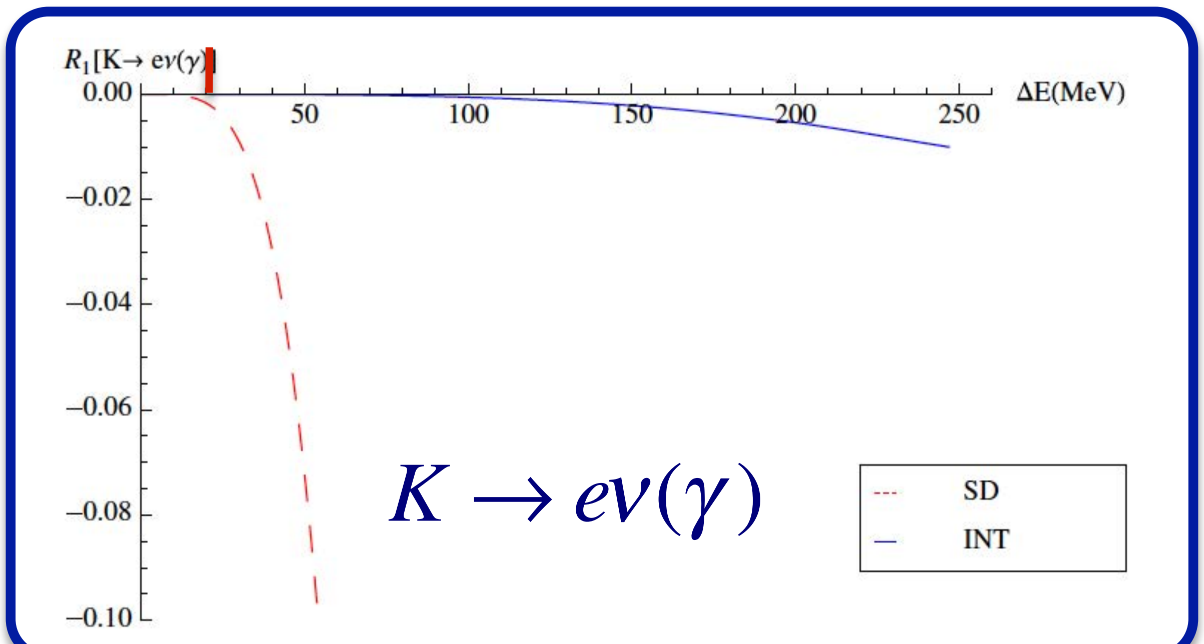
$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,pt} + \Gamma_1^{pt}(\Delta E)} \quad , \quad A = \{SD, INT\}$$

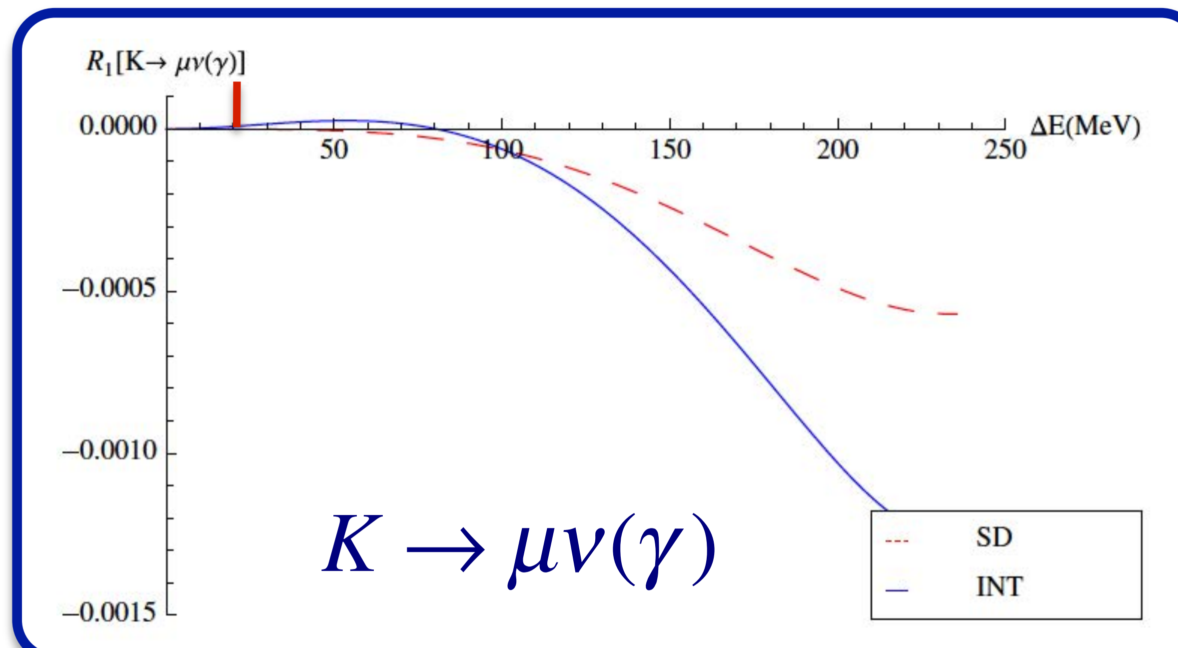
SD = structure dependent

INT = interference









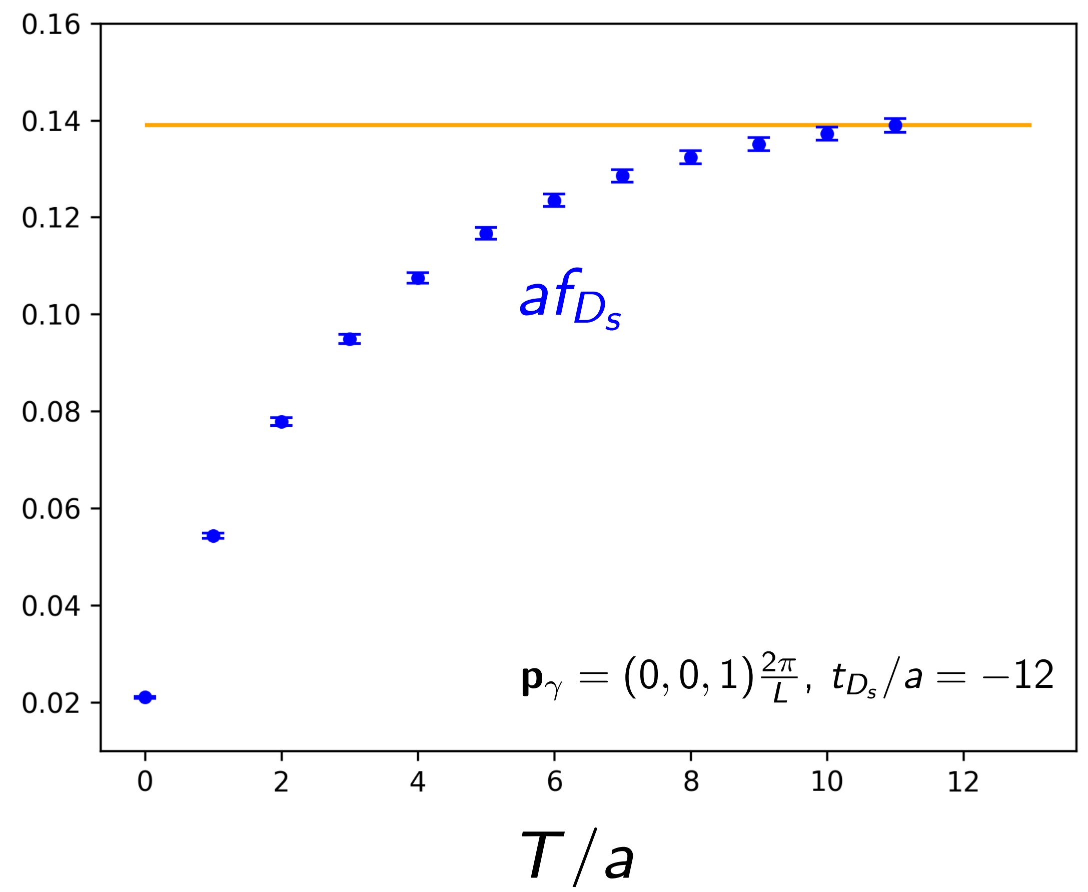
- Interference contributions are negligible in all the decays
- Structure-dependent contributions can be sizable for $K \to e\nu(\gamma)$ but they are negligible for $\Delta E < 20$ MeV (which is experimentally accessible)

Cross-checks

Recall

$$T_{\mu\nu} = \epsilon_{\mu\nu\tau\rho} p_\gamma^\tau v^\rho F_V + i[-g_{\mu\nu}(p_\gamma \cdot v) + v_\mu (p_\gamma)_\nu] F_A - i \frac{v_\mu v_\nu}{p_\gamma \cdot v} m_{D_s} f_{D_s}$$
$$+(p_\gamma)_\mu\text{-terms}$$

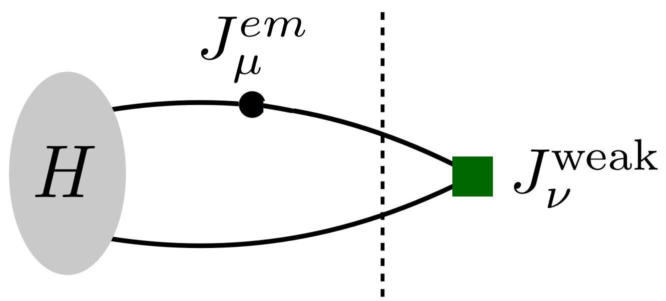
$\longrightarrow$ also extract f_{D_s} as a cross-check



Yellow line = FLAG 2021 average

Analytic continuation from Minkowski to Euclidean spacetime [2]

Time ordering: $t_{em} < 0$



$$T^{<}_{\mu\nu} = -\sum_n \frac{\langle 0|\, J_\nu^{\rm weak}(0)\,|n(\vec{p}_H-\vec{p}_\gamma)\rangle\,\langle n(\vec{p}_H-\vec{p}_\gamma)|\, J_\mu^{\rm em}(0)\,|H(\vec{p}_H)\rangle}{2E_{n,\vec{p}_H-\vec{p}_\gamma}(E_\gamma + E_{n,\vec{p}_H-\vec{p}_\gamma} - E_{H,\vec{p}_H})}$$

$$I^{<}_{\mu\nu}(t_H,T) = \int_{-T}^{0} dt_{\rm em}\; e^{E_\gamma t_{\rm em}} C_{3,\mu\nu}(t_{\rm em},t_H)$$

$$= \sum_{l,n} \frac{\langle 0|\, J_\nu^{\rm weak}(0)\,|n(\vec{p}_H-\vec{p}_\gamma)\rangle\,\langle n(\vec{p}_H-\vec{p}_\gamma)|\, J_\mu^{\rm em}(0)\,|l(\vec{p}_H)\rangle\,\langle l(\vec{p}_H)|\,\phi_H^\dagger(0)\,|0\rangle}{2E_{n,\vec{p}_H-\vec{p}_\gamma}2E_{l,\vec{p}_H}(E_\gamma + E_{n,\vec{p}_H-\vec{p}_\gamma} - E_{l,\vec{p}_H})}$$

$$\times\, e^{E_{l,\vec{p}_H}t_H}\left[1 - e^{-(E_\gamma - E_{l,\vec{p}_H} + E_{n,\vec{p}_H-\vec{p}_\gamma})T}\right]$$

Since the electromagnetic current operator cannot change the flavor quantum numbers of a state, the lowest-energy state appearing in the sum over n is the meson H.

The unwanted exponential vanishes if $|\vec{p}_\gamma| + \sqrt{m_H^2 + (\vec{p}_H - \vec{p}_\gamma)^2} > \sqrt{m_H^2 + \vec{p}_H^2}$, which is always true for $|\vec{p}_\gamma| > 0$

Infinite-volume approximation

We assume there exist $c, d, \Lambda, \Lambda' \in \mathbb{R}^+$ and $L_0 \in \mathbb{N}$ for which

$$\tilde{C}^L(q) \equiv \sum_{x=-L/2}^{L/2-1} C^L(x) e^{iqx}$$

for all x with $-L/2 \leq x \leq L/2$ and $L \geq L_0$ and

$$|C^\infty(x)| \leq d e^{-\Lambda'|x|}$$

for all x with $|x| > L/2$. We now define

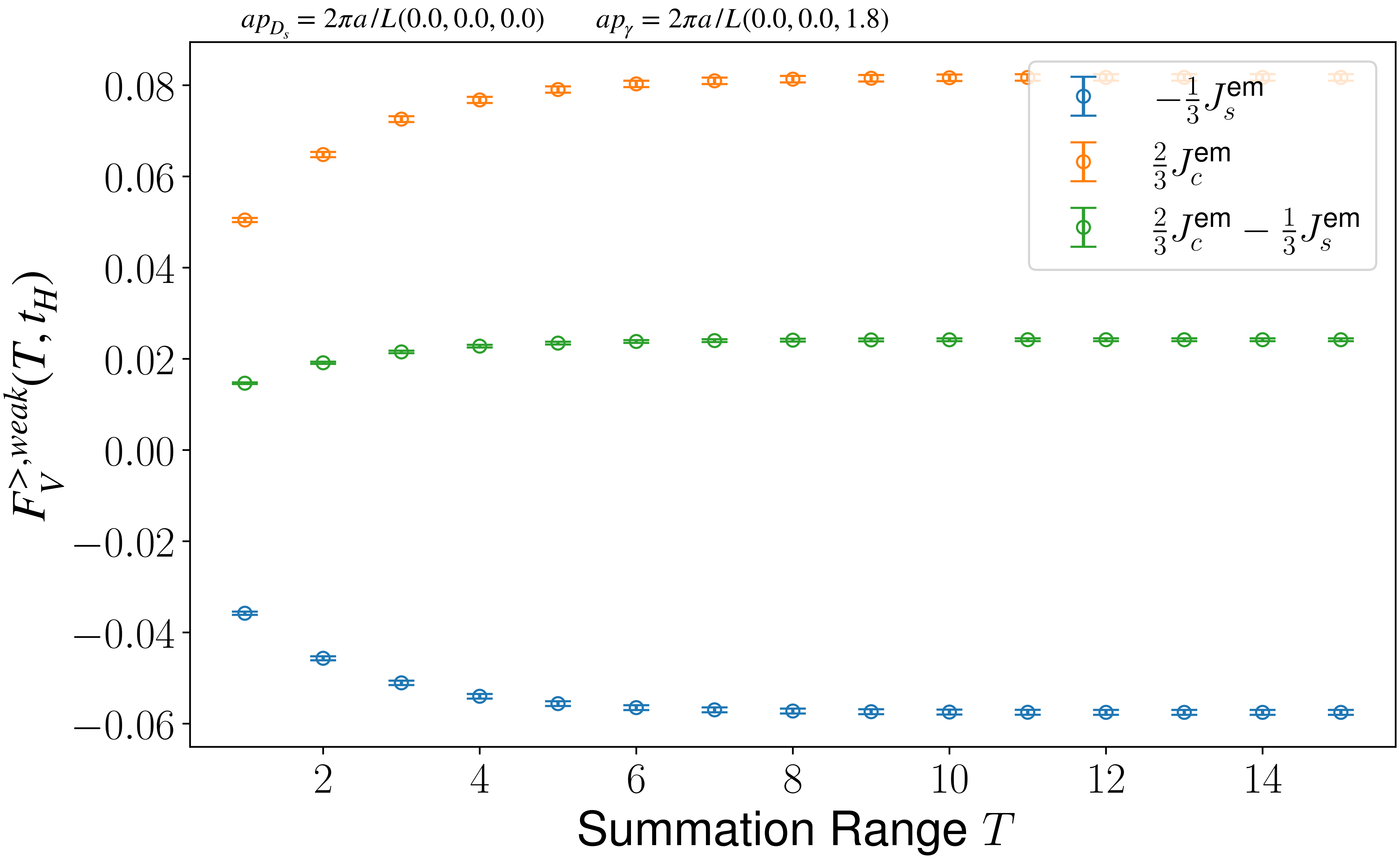
$$|C^\infty(x) - C^L(x)| \leq c e^{-\Lambda L} \qquad \text{and} \qquad \tilde{C}^\infty(q) \equiv \sum_{x=-\infty}^{\infty} C^\infty(x) e^{iqx}\,.$$

Under the above assumptions, it then follows that there is a $\tilde{c} \in \mathbb{R}^+$ for which

$$|\tilde{C}^\infty(q) - \tilde{C}^L(q)| \leq \tilde{c} e^{-\Lambda_0 L}$$

for all $q \in [-\pi, \pi]$ and all $L \geq L_0$, with $\Lambda_0 \equiv \min(\Lambda, \Lambda'/2)$.

Cancellation between quark components



Fit form: 3d method

Include terms to fit

(1) unwanted exponential from first intermediate state
(2) first excited state

Fit form factors F_V and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

$t_H < t_{em} < 0 \quad t_H < 0 < t_W$

$$F^{weak}_{<}(t_H, T) = F^{<} + B^{<}_{F}\left(1 + B^{<}_{F,exc}\overbrace{e^{\Delta E(T+t_H)}}\right)\overbrace{e^{-(E_\gamma - E_H + E^{<})T}} + C^{<}_{F}\overbrace{e^{\Delta E t_H}}$$

$$F^{em}_{>}(t_H, T) = F^{<} + B^{<}_{F}\left[1 + B^{<}_{F,exc}\frac{E_\gamma + E^{<} - (\Delta E + E_H)}{E_\gamma + E^{<} - E_H}e^{\Delta E t_H}\right]e^{-(E_\gamma - E_H + E^{<})T} + \tilde{C}^{<}_{F}e^{\Delta E t_H}$$

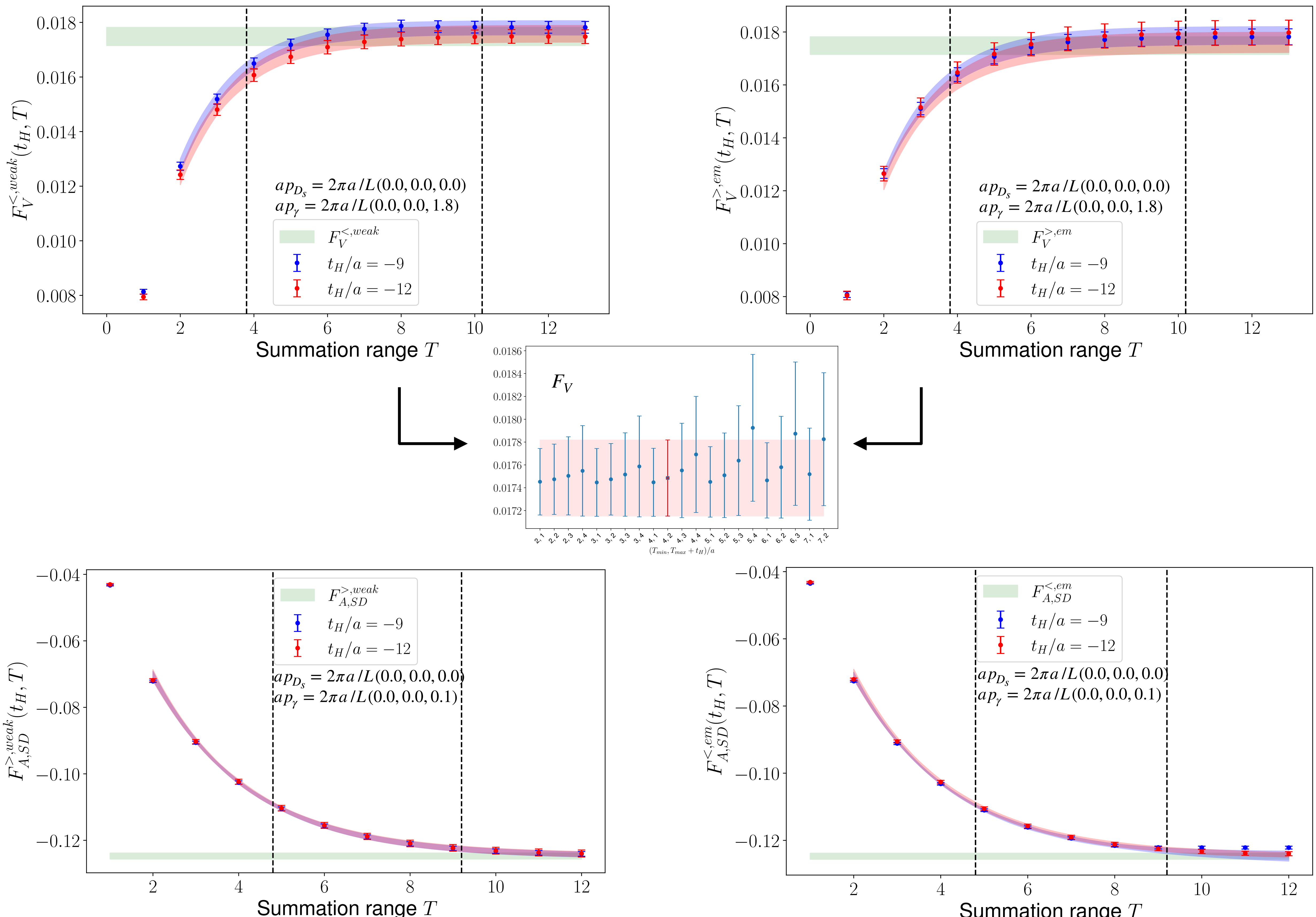
$t_H < 0 < t_{em} \quad t_H < t_W < 0$

$$F^{weak}_{>}(t_H, T) = F^{>} + B^{>}_{F}\left(1 + B^{>}_{F,exc}e^{\Delta E t_H}\right)e^{(E_\gamma - E^{>})T} + C^{>}_{F}e^{\Delta E t_H}$$

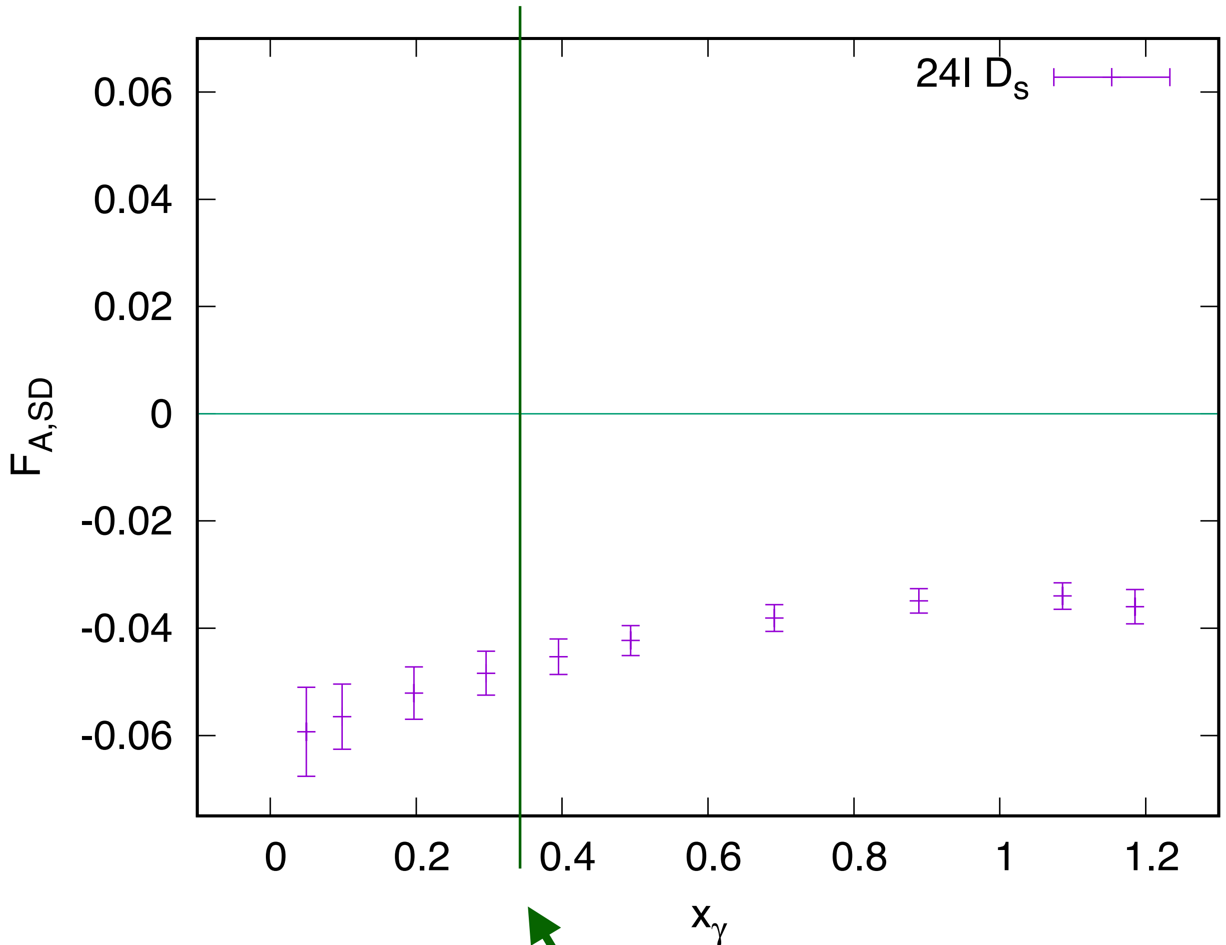
$$F^{em}_{<}(t_H, T) = F^{>} + B^{>}_{F}\left[1 + B^{>}_{F,exc}\frac{E_\gamma - E^{>}}{E_\gamma - E^{>} + \Delta E}e^{\Delta E(T+t_H)}\right]e^{(E_\gamma - E^{>})T} + \tilde{C}^{>}_{F}e^{\Delta E t_H}$$

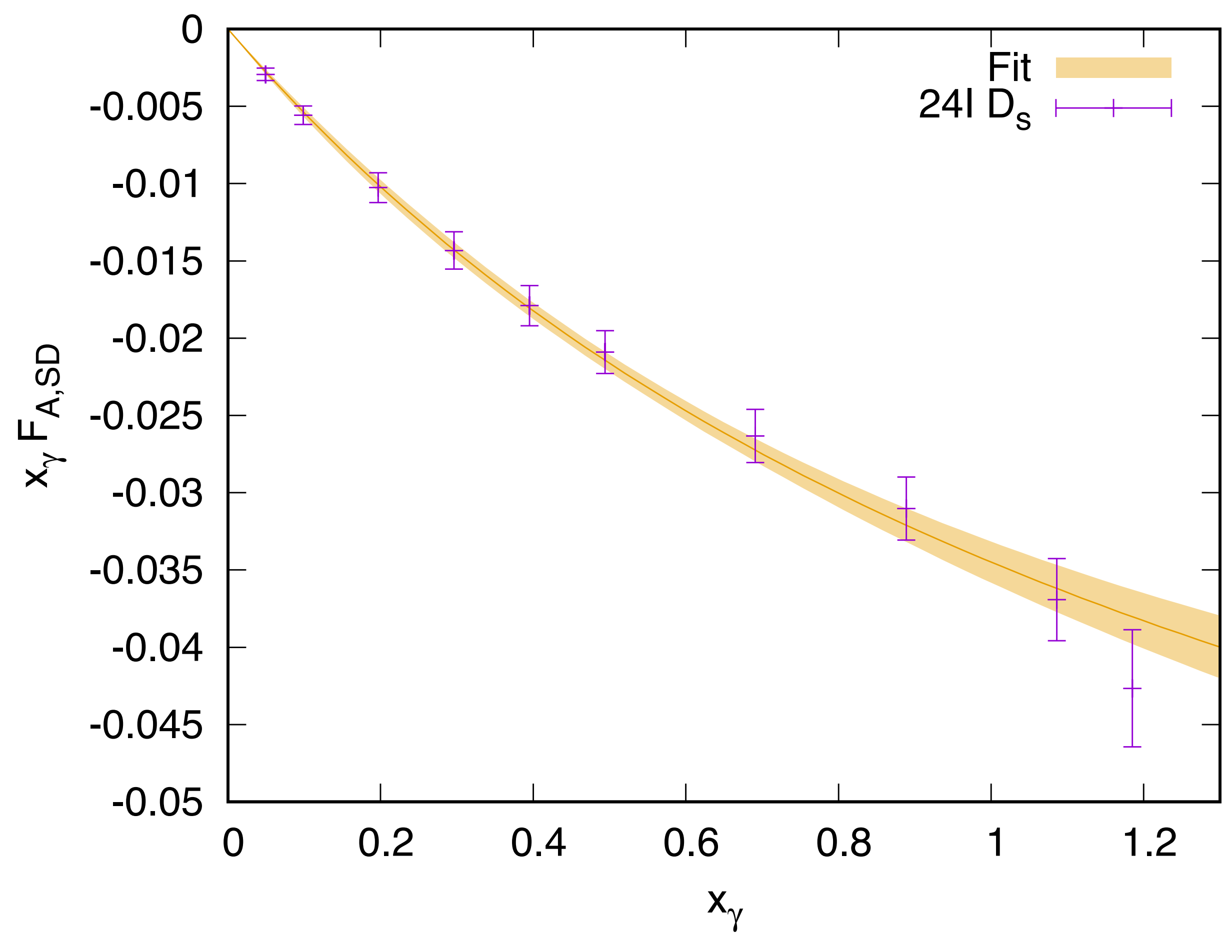
Only have two values of t_H, fitting multiple exponentials not possible
$\rightarrow$ Determine ΔE from the pseudoscalar two-point correlation function
$\rightarrow$ use result as Gaussian prior in form factor fits

$D_s \to \ell\nu_\ell\gamma$: 3d method

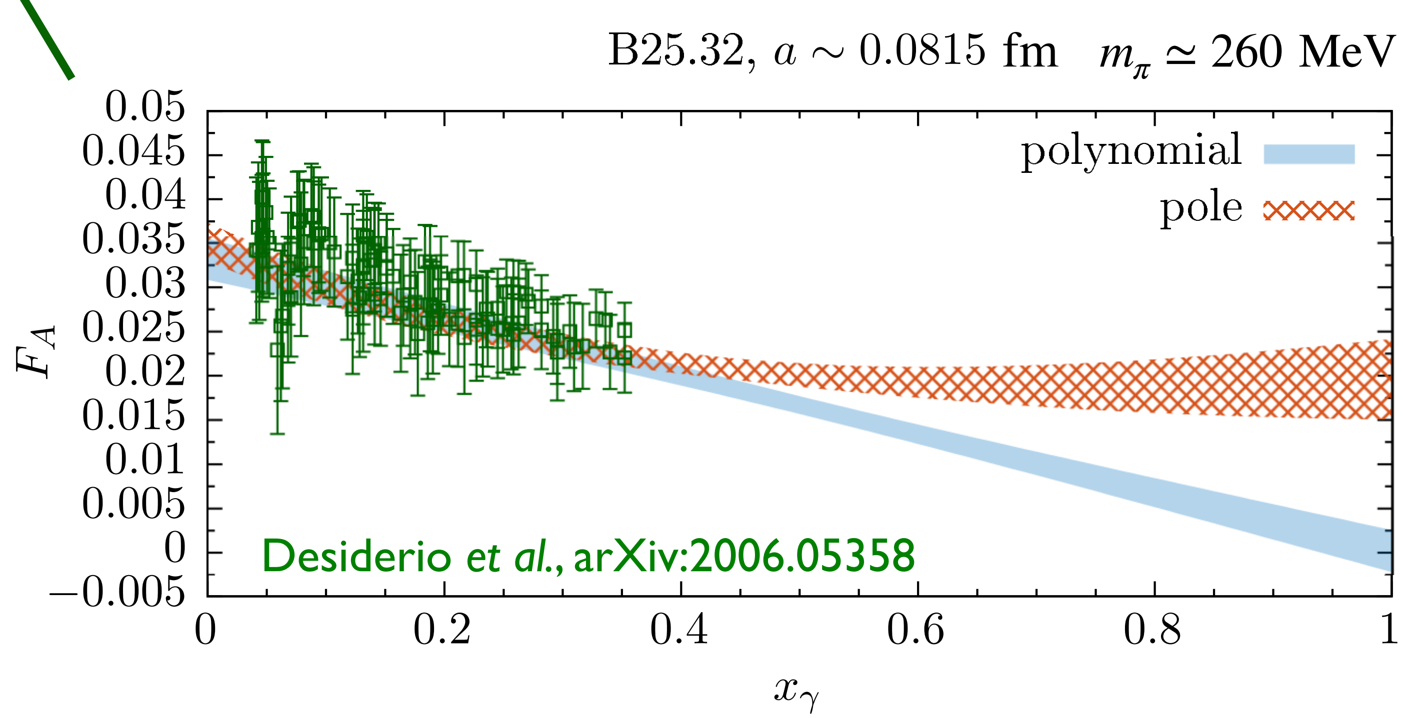


$D_s \to \ell \nu_\ell \gamma$: results (3d method)

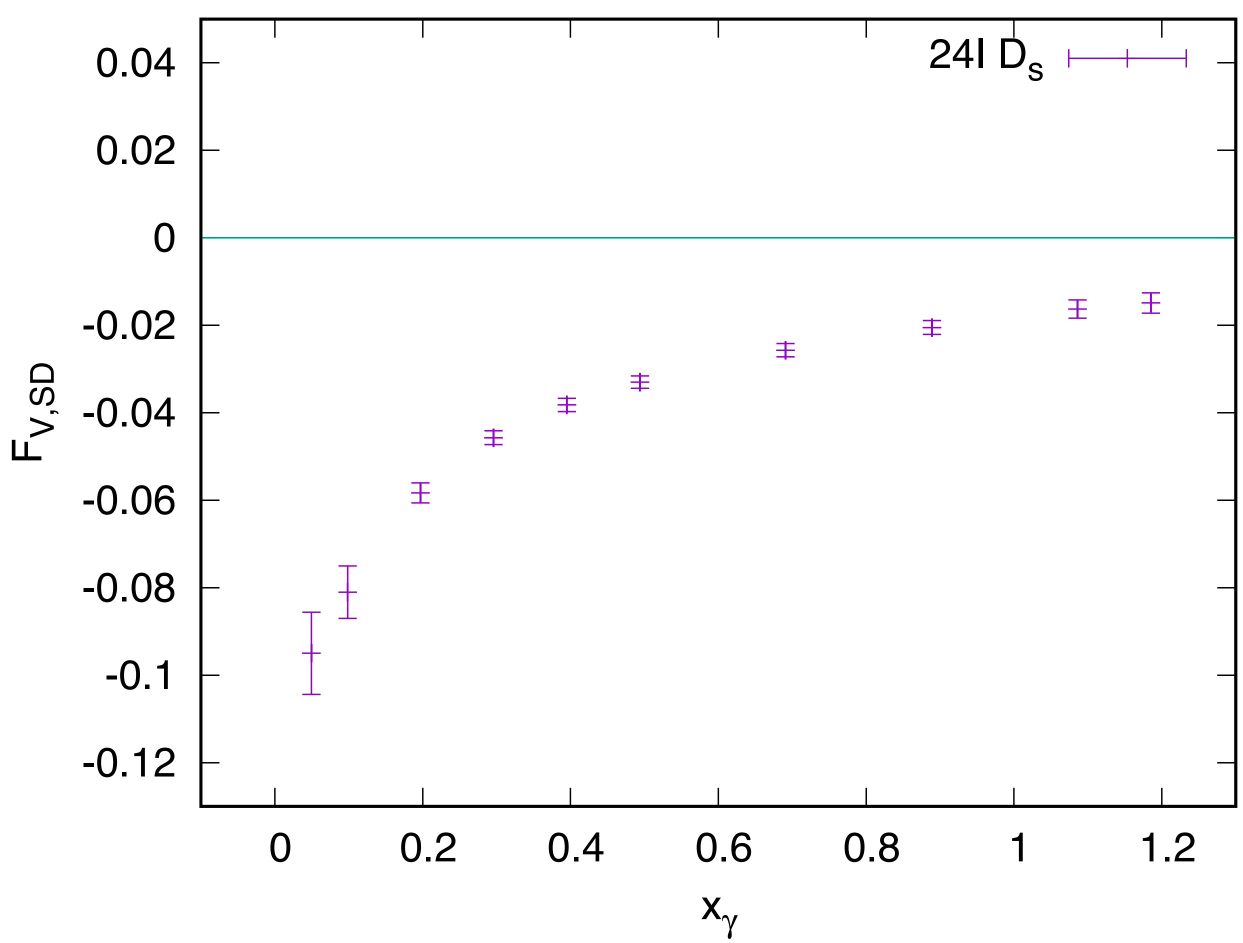


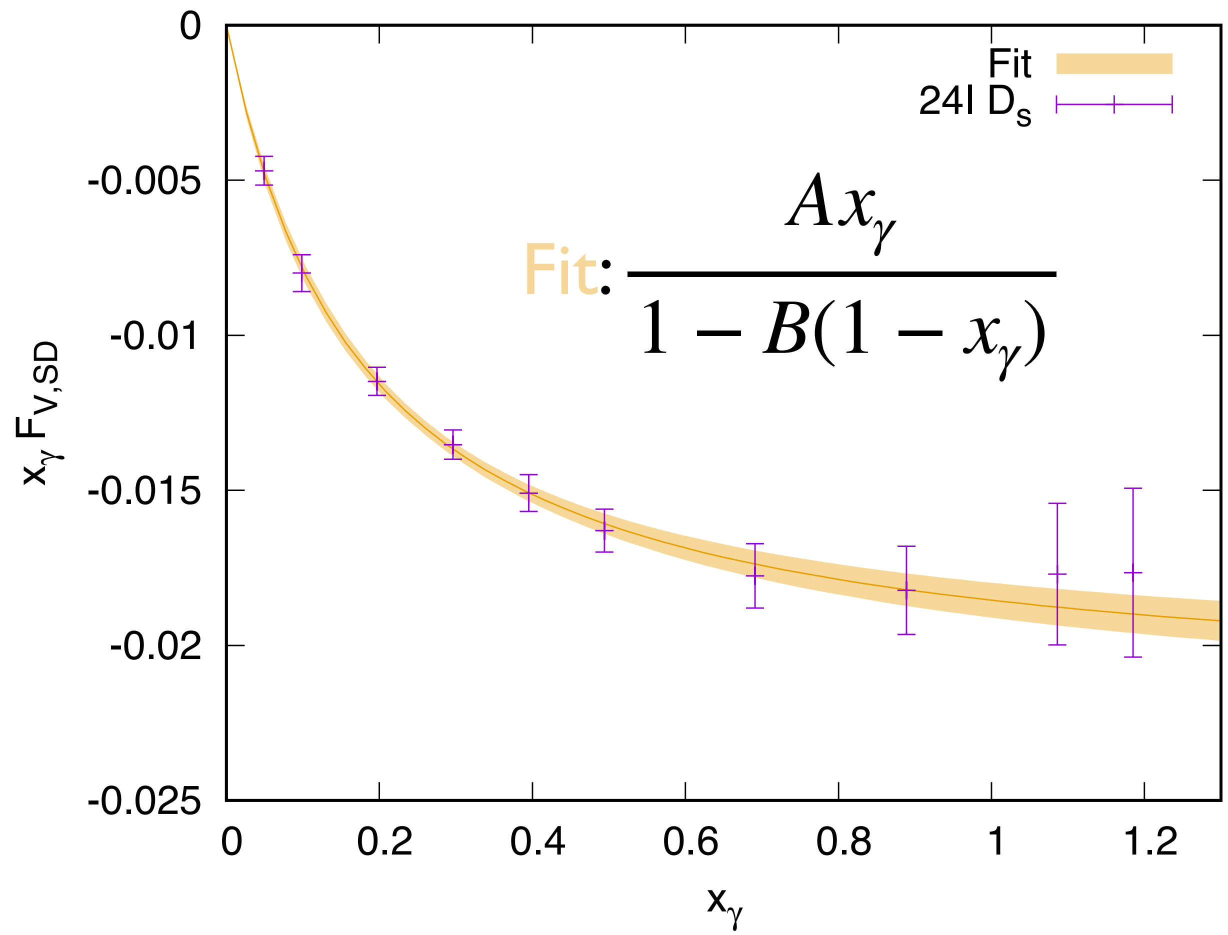


sign: different FFs parameterization



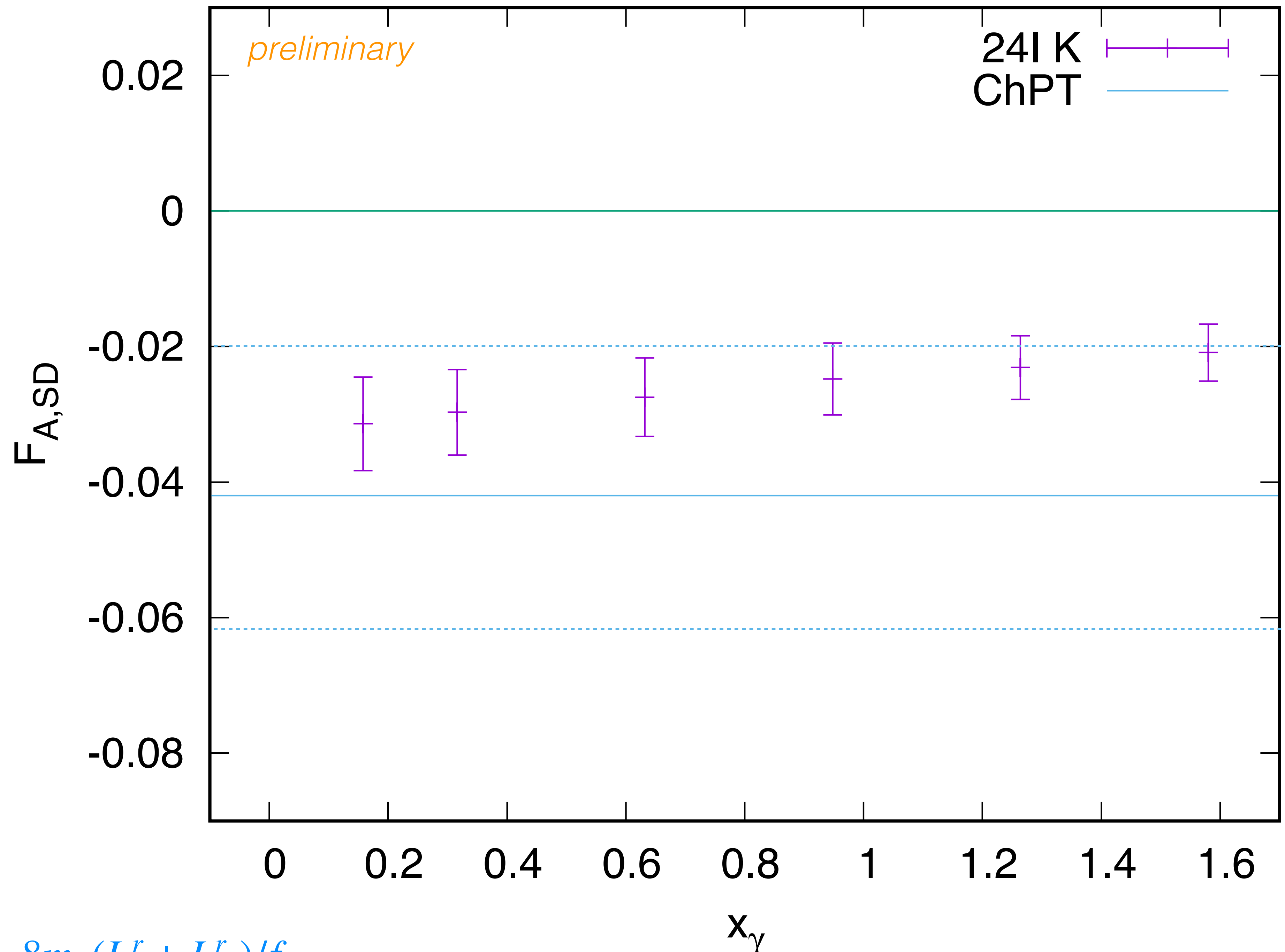
$D_s \to \ell \nu_\ell \gamma$: results (3d method) [2]





- $D_s^+ \to e^+ \nu \gamma$: $\mathcal{B}(E_\gamma > 10\ \mathrm{MeV}) < 1.3 \times 10^{-4}$ SM: $\mathcal{O}(10^{-4})$
  [BESIII Collaboration, arXiv:1902.03351]
- Fit Ansatz inspired by the phenomenological analysis of arXiv:0907.1845

$K \to \ell \nu_\ell \gamma$: results



ChPT : $8m_K(L_9^r + L_{10}^r)/f_K$

J. Bijnens *et al.*, 1993