Multilevel sampling for glueball calculations and more.

FOR 5269

Collaboration

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Based on

[Phys.Rev.D 110 (2024)] SU(3) YM

[Work in progress…] Quenched QCD

DECEMBER 9-11, 2024 CERN

NexTGen

NGT Algorithm Workshop – Lattice QCD at the large scale on exascale computing facilities

FOR 5269

DESY.

Lattice QCD & Signal/Noise problem

Estimator for $C(t) = \langle O(t)\bar{O}(0)\rangle$ affected by statistical error.

Signal

$$C(t) = \langle O(t)\bar{O}(0)^\dagger\rangle$$

$$\sim e^{-Et}$$

Noise/Statistical uncertainty

$$\sigma_C(t) = \sqrt{\langle \left(O(t)\bar{O}(0)\right)^2\rangle - \langle O(t)\bar{O}(0)\rangle^2} \Big/ \sqrt{N}$$

$$\sim \frac{c_0}{\sqrt{N}}, \frac{e^{-\tilde{E}t}}{\sqrt{N}}, \cdots$$

N Monte Carlo (MC) samples

Need sufficiently large distances t in many cases to extract effective masses reliably.

However, Signal in most cases decays faster than Noise,

which in some cases remains constant.

Spectroscopy, Hadron structure, ...

Signal/Noise ratio (S/N) is a ubiquitous problem in Lattice QCD

Multilevel sampling: The idea

If action and observables are local, the two-point correlation function

$$\langle \mathrm{O}(t_1)\bar{\mathrm{O}}(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU] e^{-S[U]}\, \mathrm{O}(U, t_1)\ \bar{\mathrm{O}}(U, t_0)$$

can be factorised into a product of integrals

$$\langle \mathrm{O}(t_1)\mathrm{O}(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU_B] e^{-S_B[U_B]}\ \left[\mathrm{O}^{(2)}(U_B, t_1)\right] \left[\bar{\mathrm{O}}^{(1)}(U_B, t_0)\right]$$

with

$$\left[\mathrm{O}^{(r)}(U_B, t)\right] = \int [dU^{(r)}]\ e^{-S_r[U^{(r)}|U_B]}\ \mathrm{O}(U^{(r)}, t)$$

$$C^{2\mathrm{lvl}}(t_1, t_0) = \langle \left[\mathrm{O}(t_1)\right]\ \left[\bar{\mathrm{O}}(t_0)\right]\rangle = C(t_1, t_0)$$

$$\sigma_C^{2\mathrm{lvl}}(t_1, t_0) \ll \sigma_C^{\mathrm{std}}(t_1, t_0)$$

! Numerically, this is like doing $[N] \times [N]$ sub-measurements with N MC samples.

For 3-pts $[N] \times [N] \times [N]$, etc ...

Multilevel sampling: Pure Gauge Theory

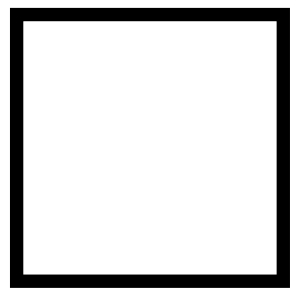
Both pure gauge action and observables are local and multilevel can be adopted straightforwardly

$$\langle \mathrm{O}(t_1)\mathrm{O}(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU_B] e^{-S_B[U_B]} \left[\mathrm{O}^{(2)}(U_B, t_1)\right] \left[\bar{\mathrm{O}}^{(1)}(U_B, t_0)\right]$$

$$S[U] = \frac{\beta}{3} \sum_{x\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\left\{ \mathrm{Tr}\left[1 - U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)\right]\right\}$$

e.g. Wilson plaquette action

$$\mathrm{O}(t) \equiv W(t) = \sum_{\vec{x}} \mathrm{Tr}\left[U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)\right]$$



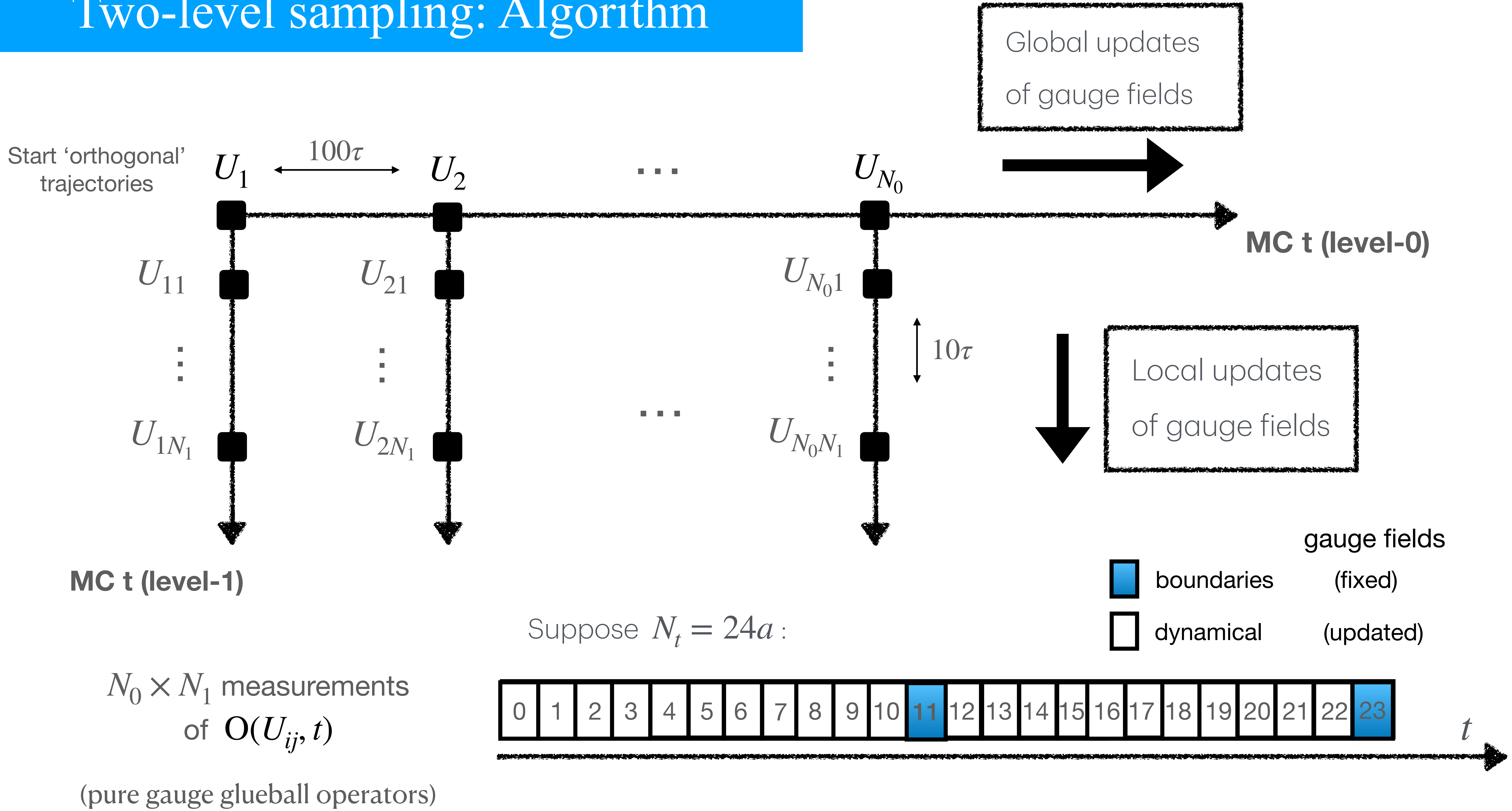
e.g. Wilson plaquette observable

Two-level sampling: Algorithm

In the same way as with the standard algorithm

we generate N_0 gauge configurations (e.g. HMC)

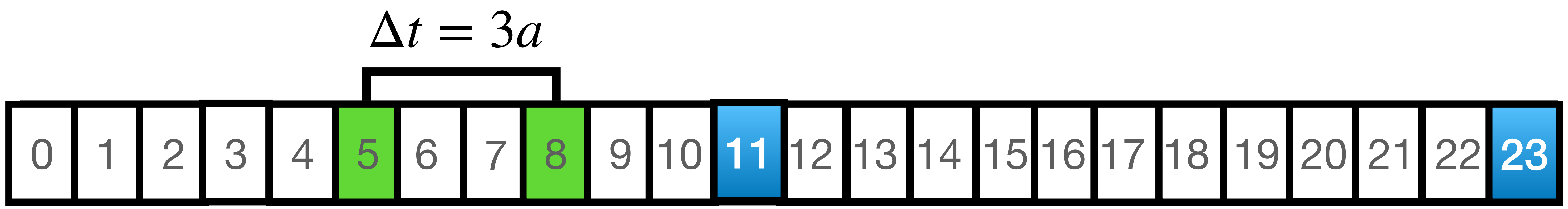
Two-level sampling: Algorithm



Two-level sampling: Algorithm

if t_0 and t_1 belong to the same region, the two-level analysis cannot be adopted

e.g.



Instead, correlate over $N_0 \times N_1$ samples

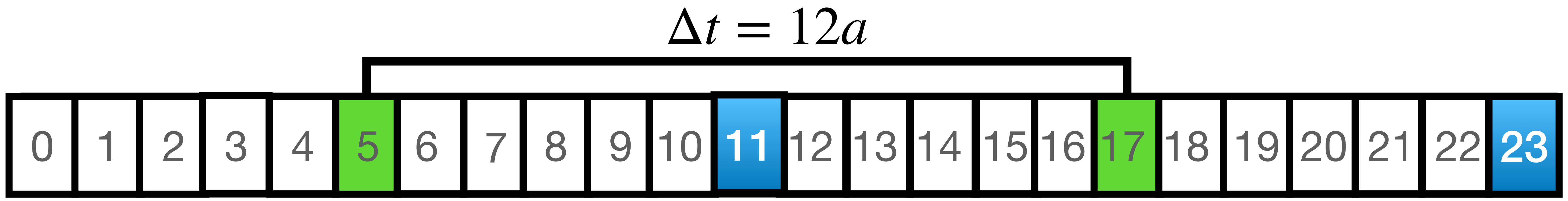
$$C(t_1 - t_0) = \langle \mathrm{O}(U_{ij}, t_1)\ \mathrm{O}(U_{ij}, t_0) \rangle = \frac{1}{N_0 N_1} \sum_{i=1}^{N_0} \sum_{j=1}^{N_1} \mathrm{O}(U_{ij}, t_1) \mathrm{O}(U_{ij}, t_0)$$

Variance scaling $\sigma^2(t_1 - t_0) \propto \frac{1}{N_0 N_1} + \ldots$ **Standard**

Two-level sampling: Algorithm

if t_0 and t_1 belong to different regions

e.g.



Take average over level-1

$$\left[\mathrm{O}(U_i, t)\right] = \frac{1}{N_1} \sum_{j=1}^{N_1} \mathrm{O}(U_{ij}, t)$$

Correlate on level-0

$$C(t_1 - t_0) = \left\langle \left[\mathrm{O}(U_i, t_1)\right] \left[\mathrm{O}(U_i, t_0)\right] \right\rangle = \frac{1}{N_0} \sum_{i=1}^{N_0} \left[\mathrm{O}(U_i, t_1)\right] \left[\mathrm{O}(U_i, t_0)\right]$$

Variance scaling

$$\sigma^2(t_1 - t_0) \propto \frac{1}{N_0 N_1^2} + \ldots$$

Two-level

Numerical Simulations

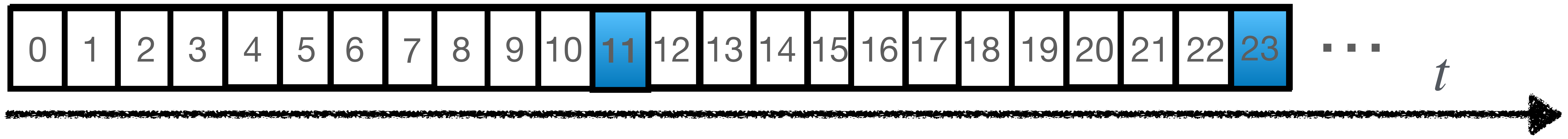
Phys.Rev.D 110 (2024) 5, 054515

***LB**, F. Knechtli, S. Martins, M. Peardon, S. Schaefer, J.A. Urrea-Niño*

In the following:

$$4D\ \mathrm{SU}(3)\ \text{theory}, \quad \beta = 6.2, \quad V = 24^3 \times 48 \qquad N_0 = 101, \ \ N_1 = 1,...,1000$$

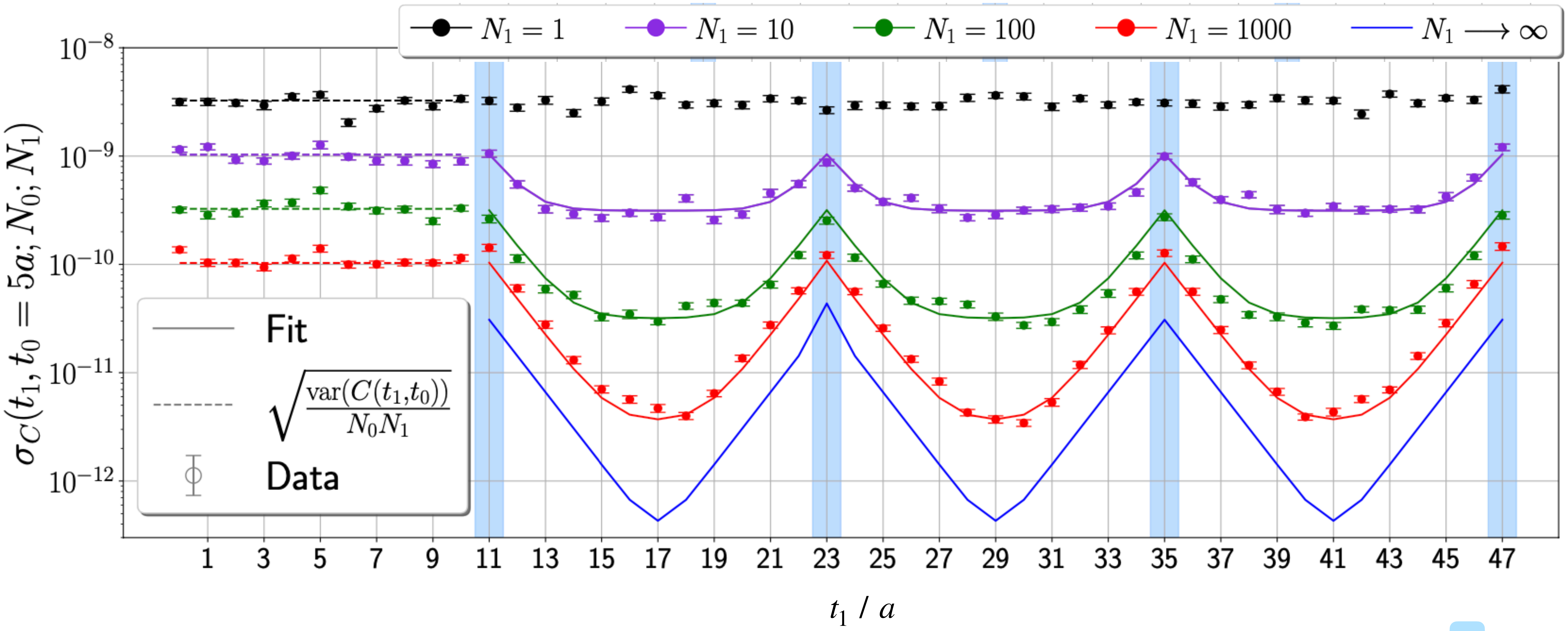
Sub-lattice decomposition on level-1:



$$C(t_1 - t_0) = \langle \mathrm{O}(U_{ij}, t_1)\ \mathrm{O}(U_{ij}, t_0) \rangle$$

Two-level error reduction

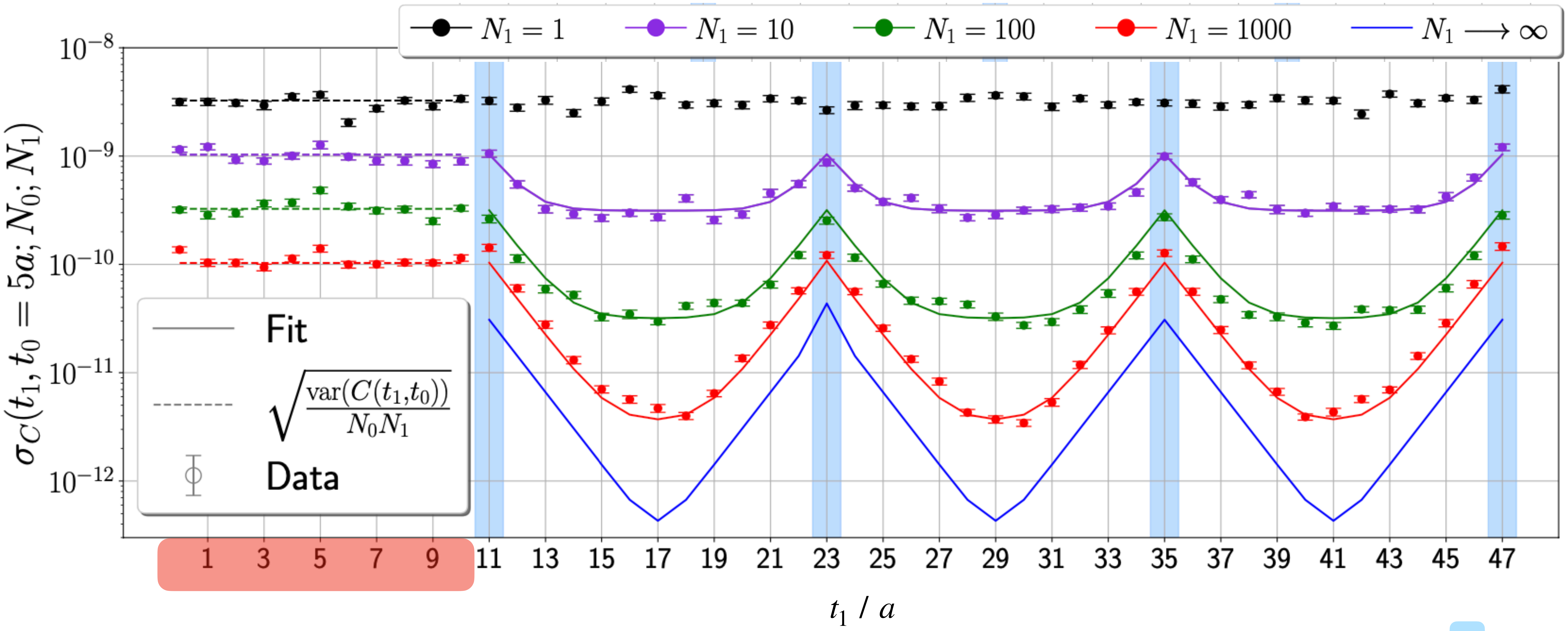
4D SU(3) theory, $\beta = 6.2$, $V/a^4 = 24^3 \times 48$



boundaries $t_B/a = 11,\ 23,\ 35,\ 47.$

Two-level error reduction

4*D* SU(3) theory, $\beta = 6.2$, $V/a^4 = 24^3 \times 48$



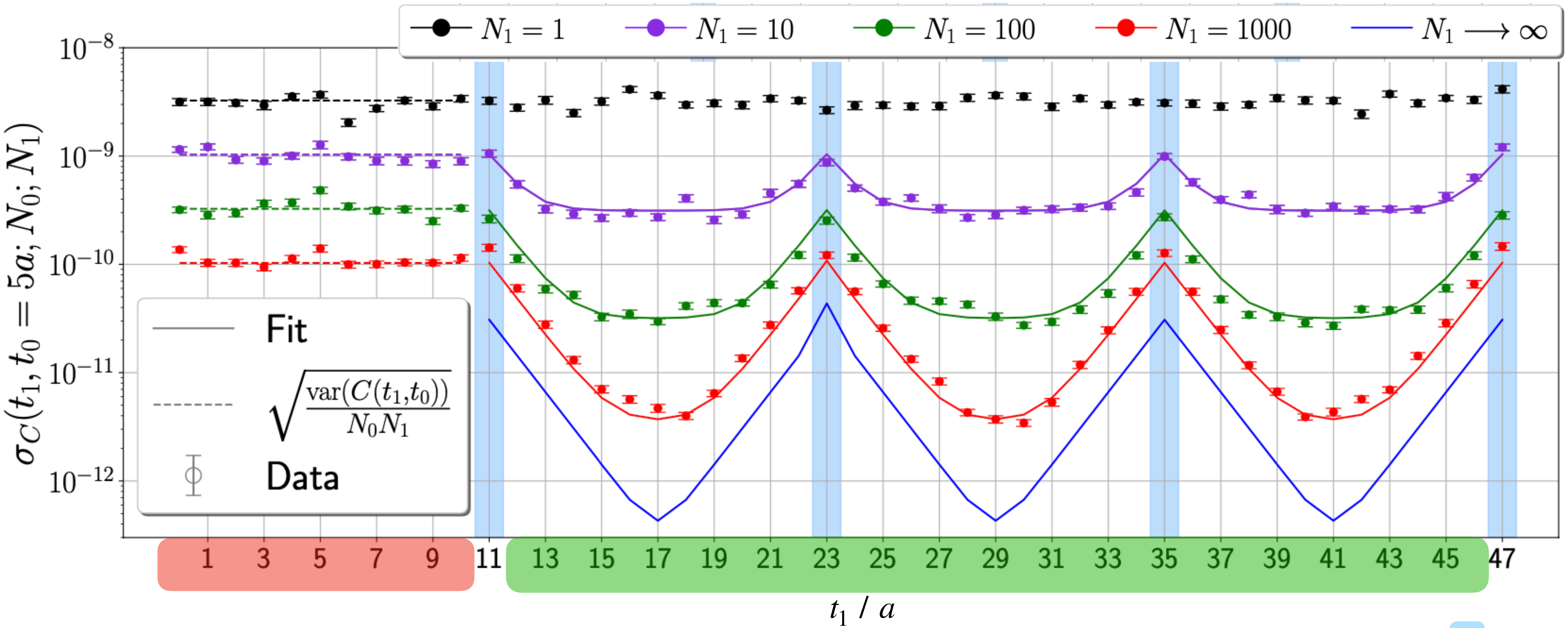
Error scales like $1/\sqrt{N_0 N_1}$

when operators in same region

boundaries $t_B/a = 11, 23, 35, 47.$

Two-level error reduction

4D SU(3) theory, $\beta = 6.2$, $V/a^4 = 24^3 \times 48$



Error scales like $1/\sqrt{N_0 N_1}$

when operators in same region

Error scales exponentially with distance from boundaries

when operators in different regions

$t_B/a = 11, 23, 35, 47.$

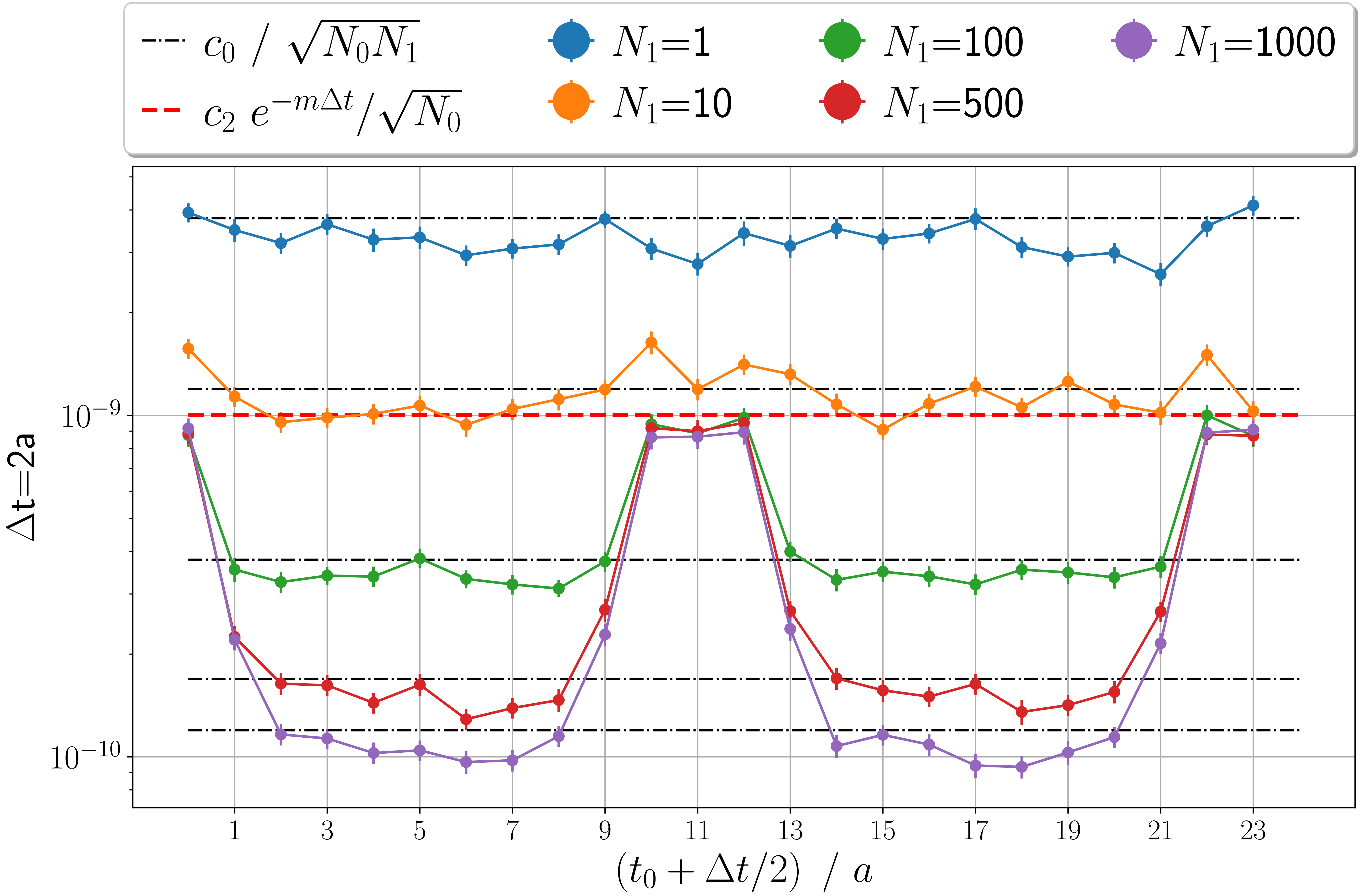
$$\sigma_C^2(t_1, t_0) \approx \frac{c_0^2}{N_0 N_1^2} + \frac{c_1^2}{N_0 N_1}\left[e^{-m\Delta_1} + e^{-m\Delta_0}\right] + \frac{c_2^2}{N_0} e^{-m\Delta_1} e^{-m\Delta_0}$$

$\Delta_{1,0} = |t_{1,0} - t_B|$

Two-level error reduction at fixed Δt

$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



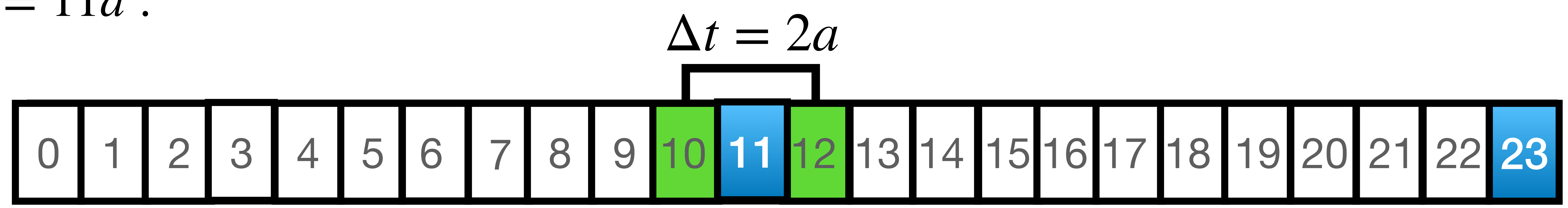
Errors for $C(\Delta t = 2a)$

Distance from boundaries is limiting the 2-lvl error reduction.

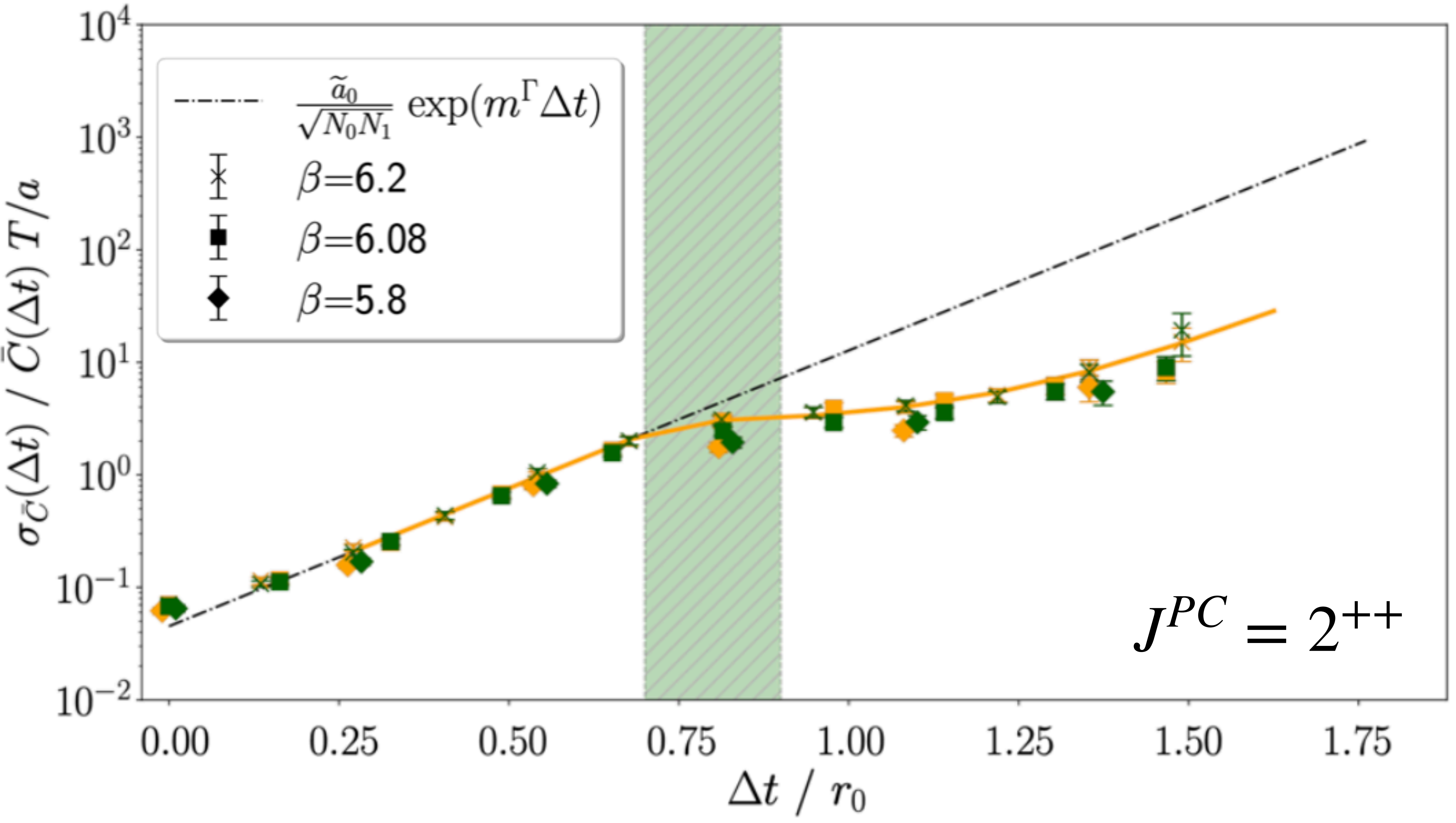
Multilevel is not efficient at short distance!

Doing $N_1 > 10$ is just a wast of resources with the multilevel analysis

$t_0 + \Delta t/2 = 11a$:



Noise/Signal towards continuum limit



- Short distance scaling: $$\frac{\sigma^2_{\overline{C}}(\Delta t)}{\overline{C}^2(\Delta t)} \approx \tilde{a}_0^2 \frac{e^{2m^\Gamma \Delta t}}{N_0 N_1}$$
- Long distance scaling: $$\frac{\sigma^2_{\overline{C}}(\Delta t)}{\overline{C}^2(\Delta t)} \approx \frac{\tilde{c}_0^2}{N_0 N_1^2} e^{2m^\Gamma \Delta t} + \frac{2\tilde{c}_1^2}{N_0 N_1} e^{2m^\Gamma \Delta t/2} + \frac{\tilde{c}_2^2}{N_0}$$

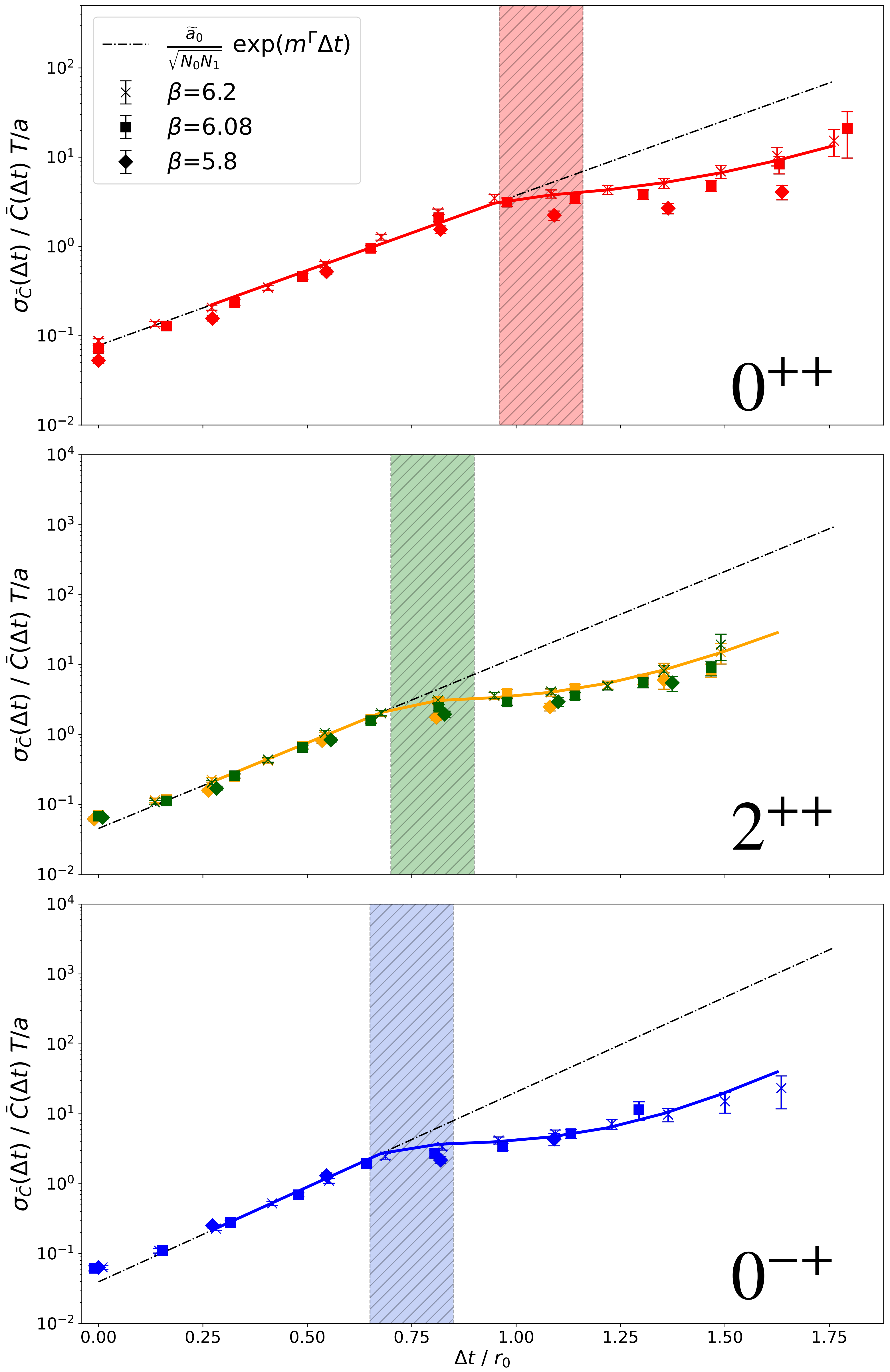
Weighted average

$$\overline{C}(\Delta t) = \frac{\sum_{t_0} w(t_0) C(\Delta t = t_1 - t_0)}{\sum_{t_0} w(t_0)}$$

$$w(t_0) = 1 \;/\; \sigma^2_C(t_0)$$

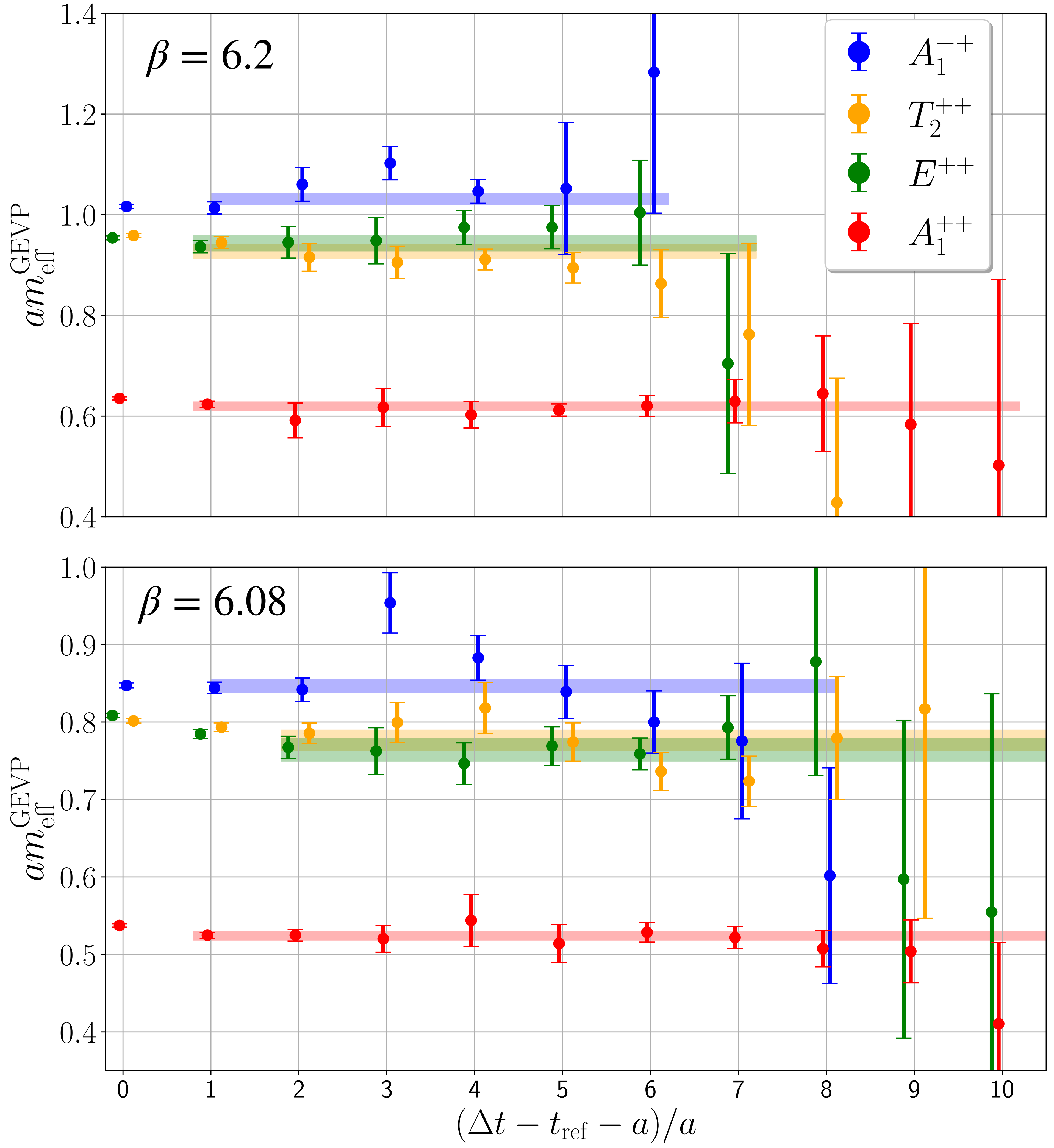
Coloured band highlights transition between different scalings

Noise/Signal towards continuum limit



Consistency across all glueball channels

GEVP effective masses



Results agree with state-of-the-art calculations that use $\mathcal{O}(10^5)$ configs

[A. Athenodorous, M. Teper, 2020]

Multilevel algorithm: Quenched QCD

Action is still pure gauge, thus local, but observables can be traces of fermionic propagators (non-local)

$$\langle \mathrm{O}(t_1)\bar{\mathrm{O}}(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU] e^{-S[U]}\, \mathrm{O}(t_1)\ \bar{\mathrm{O}}(t_0)$$

$$S[U] = \frac{\beta}{3} \sum_{x\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\left\{ \mathrm{Tr}\left[1 - U_\mu(x) U_\nu(x+\mu) U^\dagger_\mu(x+\nu) U^\dagger_\nu(x) \right] \right\}$$

e.g. Wilson plaquette action

$$\mathrm{O}(t_1)\bar{\mathrm{O}}(t_0) = \mathrm{Tr}\left[D^{-1}(t_1, t_0) D^{-1}(t_0, t_1) \right]$$

e.g. scalar meson two-point functions

$D^{-1}(t_1, t_0)$ depends on the values of the gauge fields over the full lattice

Need factorisation of $D^{-1}(t_1, t_0)$ into local regions.

Phys.Rev.D 93 (2016) 9, 094507 [M. Ce', S. Schaefer, L. Giusti]

Fermionic observables investigated

Combine multilevel + distillation to compute correlations of

$$\mathrm{O}(t) \in \left\{ \mathrm{O}_{\pi\pi}^{\mathrm{I}=0}(t),\ \mathrm{O}_{\Gamma}^{\mathrm{I}=0} = \bar{q}\Gamma q \right\} \text{ with } \Gamma = \gamma_5, \gamma_4\gamma_5, \gamma_i, \gamma_5\gamma_i, \gamma_i\gamma_j, I\,.$$

The 2-pts of these observables have severe signal/noise

due to the variance being constant for disconnected contributions.

$$\langle \mathrm{O}_{\gamma_5}(t)\ \bar{\mathrm{O}}_{\gamma_5}(t_0)\rangle = -\langle \mathrm{Tr}\left[\gamma_5 D^{-1}(t,t_0)\gamma_5 D^{-1}(t,t_0)^\dagger\right]\rangle + \langle \mathrm{Tr}\left[\gamma_5 D^{-1}(t,t)\right] \mathrm{Tr}\left[\gamma_5 D^{-1}(t_0,t_0)\right]\rangle$$

Connected Disconnected

We want to apply a two-level integration to the disconnected piece:

$$\langle \mathrm{O}_\Gamma(t)\bar{\mathrm{O}}_\Gamma(t_0)\rangle_{\text{disc.}} = \frac{1}{\mathcal{Z}} \int [dU] e^{-S[U]}\ \mathrm{Tr}\left[\mathrm{D}^{-1}(\mathrm{t},\mathrm{t})\Gamma\right]\ \mathrm{Tr}\left[\mathrm{D}^{-1}(\mathrm{t}_0,\mathrm{t}_0)\Gamma\right]$$

$$\longrightarrow \frac{1}{\mathcal{Z}_B} \int [dU_B] e^{-S[U_B]} \left[\mathrm{Tr}\left[\mathrm{D}^{-1}(\mathrm{t},\mathrm{t})\Gamma\right]\right]\left[\ \mathrm{Tr}\left[\mathrm{D}^{-1}(\mathrm{t}_0,\mathrm{t}_0)\Gamma\right]\right]$$

Ultimate Physics goal (Full QCD)

Study $f_0(1500), f_0(1760) \to \pi\pi, \ldots$

and decay of other glueball candidates

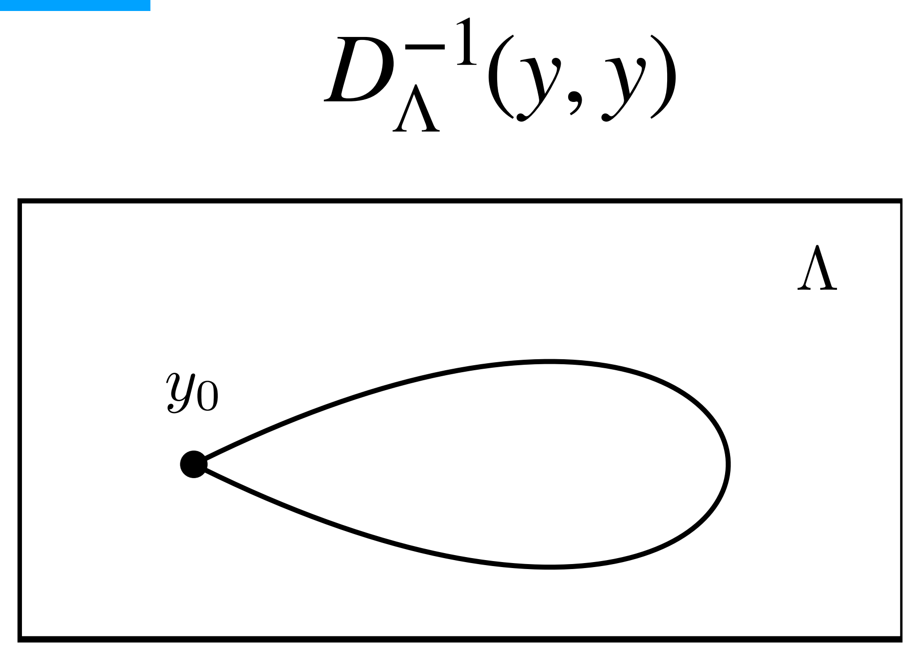
e.g. X(2370) announced by BESIII @CERN

See J. Urrea-Niño's talk @Lattice2024

Factorisation of Fermion Propagator for quark loops

Challenge: Propagators depend on gauge fields over all space-time Λ

$$\mathrm{Tr}\left[\mathrm{D}^{-1}(t,t)\Gamma\right] \qquad \text{non-local}$$

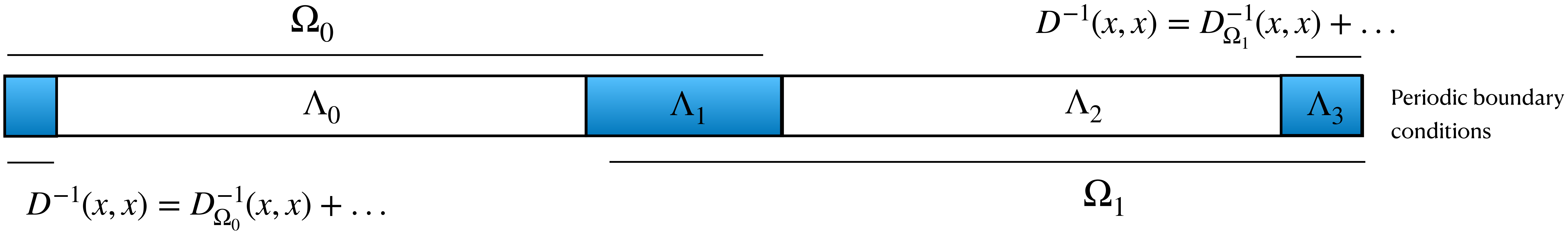


Solution:

- Factorise propagators in different regions

M. Ce`, L. Giusti, S. Schaefer *[PRD.93.094507]* *[PRD.95.034503]*

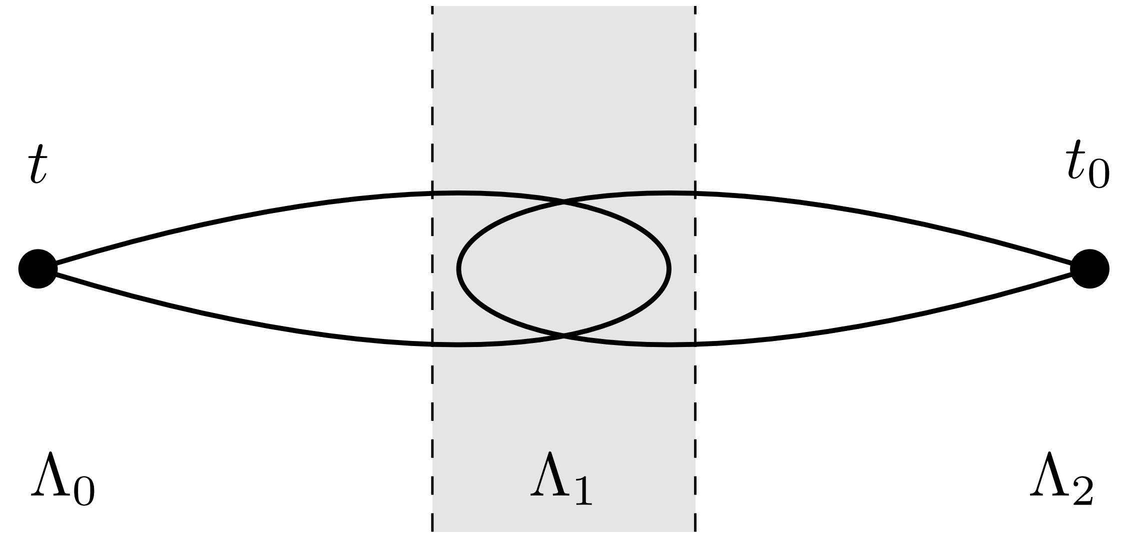
When solving Dirac equations, consider only ~ half temporal extent and approximate propagators:



Domain decomposition used also for vector 2-pts in HPV *[Phys.Lett.B 816 (2021)]*

T. Harris, L. Giusti, et al.

Two-level sampling for disconnected diagrams



This approximation allows to factorise the disconnected diagram

$$\langle \mathrm{O}_{\gamma_5}(t)\bar{\mathrm{O}}_{\gamma_5}(t_0)\rangle_{\mathrm{disc.}} = \frac{1}{\mathcal{Z}} \int [dU] e^{-S[U]}\ \mathrm{Tr}\left[\gamma_5 \mathrm{D}^{-1}_{\Omega_1}(\mathrm{t},\mathrm{t})\right]\ \mathrm{Tr}\left[\gamma_5 \mathrm{D}^{-1}_{\Omega_0}(\mathrm{t}_0,\mathrm{t}_0)\right] \quad + \ \text{correction}$$

which now allows 2-level integration:

$$= \frac{1}{\mathcal{Z}} \int [dU_B] e^{-S_B[U_B]}\ \left[\mathrm{Tr}\left[\gamma_5 D^{-1}_{\Omega_1}(t,t)\right]\right]_{\Omega_1} \left[\gamma_5 D^{-1}_{\Omega_0}(t_0,t_0)\right]_{\Omega_0} \quad + \ \text{correction}$$

$$\left[\mathrm{Tr}\left[\gamma_5 D^{-1}_{\Omega_r}(t,t)\right]\right]_{\Omega_r} = \int [dU^{(r)}]\ e^{-S_r[U^{(r)},U_B]}\ \mathrm{Tr}\left[\gamma_5 \mathrm{D}^{-1}_{\Omega_r}(t,t)\right]$$

Multilevel estimators and correction

- Use full propagators to compute estimator in the standard way:

$$\mathcal{O}_{ij}(t,t) = \ \mathrm{Tr}\left[\gamma_5 \mathrm{D}^{-1}(\mathrm{U_{ij}}, \mathrm{t}, \mathrm{t})\right]$$

$$C^{\mathrm{std}}(t,t_0) = \frac{1}{N_0 N_1}\sum_{i,j} \mathcal{O}_{ij}(t,t)\ \mathcal{O}_{ij}(t_0,t_0)$$

$$\sigma^2_{C^{\mathrm{std}}}(t,t_0) \approx \frac{c_0}{N_0 N_1}$$

Multilevel estimators and correction

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$$C^{\mathrm{std}}(t,t_0) = \frac{1}{N_0 N_1}\sum_{i,j} \mathcal{O}_{ij}(t,t)\ \mathcal{O}_{ij}(t_0,t_0) \qquad \sigma^2_{C^{\mathrm{std}}}(t,t_0) \approx \frac{c_0}{N_0 N_1}$$

- Use block propagators to compute estimator in the standard way and w/ two-level: $\mathcal{O}^{\Omega_r}_{ij}(t,t) = \mathrm{Tr}\left[\gamma_5 \mathrm{D}^{-1}_{\Omega_\mathrm{r}}(\mathrm{U_{ij}}, \mathrm{t}, \mathrm{t})\right]$

$$C^{\mathrm{approx,1lvl}}(t,t_0) = \frac{1}{N_0 N_1}\sum_{i,j} \mathcal{O}^{\Omega_1}_{ij}(t,t)\mathcal{O}^{\Omega_0}_{ij}(t_0,t_0) \qquad \sigma^2_{C^{\mathrm{approx,1lvl}}}(t,t_0) \approx \frac{c_0}{N_0 N_1}$$

$$C^{\mathrm{approx,2lvl}}(t,t_0) = \frac{1}{N_0}\sum_{i}^{N_0}\left[\frac{1}{N_1}\sum_{j=1}^{N_1}\mathcal{O}^{\Omega_1}_{ij}(t,t)\right]\left[\frac{1}{N_1}\sum_{j=1}^{N_1}\mathcal{O}^{\Omega_0}_{ij}(t_0,t_0)\right] \qquad \sigma^2_{C^{\mathrm{approx,2lvl}}}(t,t_0) \approx \frac{c_0}{N_0 N_1^2} + \dots$$

Multilevel estimators and correction

- Use full propagators to compute estimator in the standard way: $\mathcal{O}_{ij}(t,t) = \mathrm{Tr}\left[\gamma_5 \mathrm{D}^{-1}(\mathrm{U_{ij}}, \mathrm{t}, \mathrm{t})\right]$

$$C^{\mathrm{std}}(t,t_0) = \frac{1}{N_0 N_1}\sum_{i,j} \mathcal{O}_{ij}(t,t)\ \mathcal{O}_{ij}(t_0,t_0) \qquad \sigma^2_{C^{\mathrm{std}}}(t,t_0) \approx \frac{c_0}{N_0 N_1}$$

- Use block propagators to compute estimator in the standard way and w/ two-level: $\mathcal{O}^{\Omega_r}_{ij}(t,t) = \mathrm{Tr}\left[\gamma_5 \mathrm{D}^{-1}_{\Omega_\mathrm{r}}(\mathrm{U_{ij}}, \mathrm{t}, \mathrm{t})\right]$

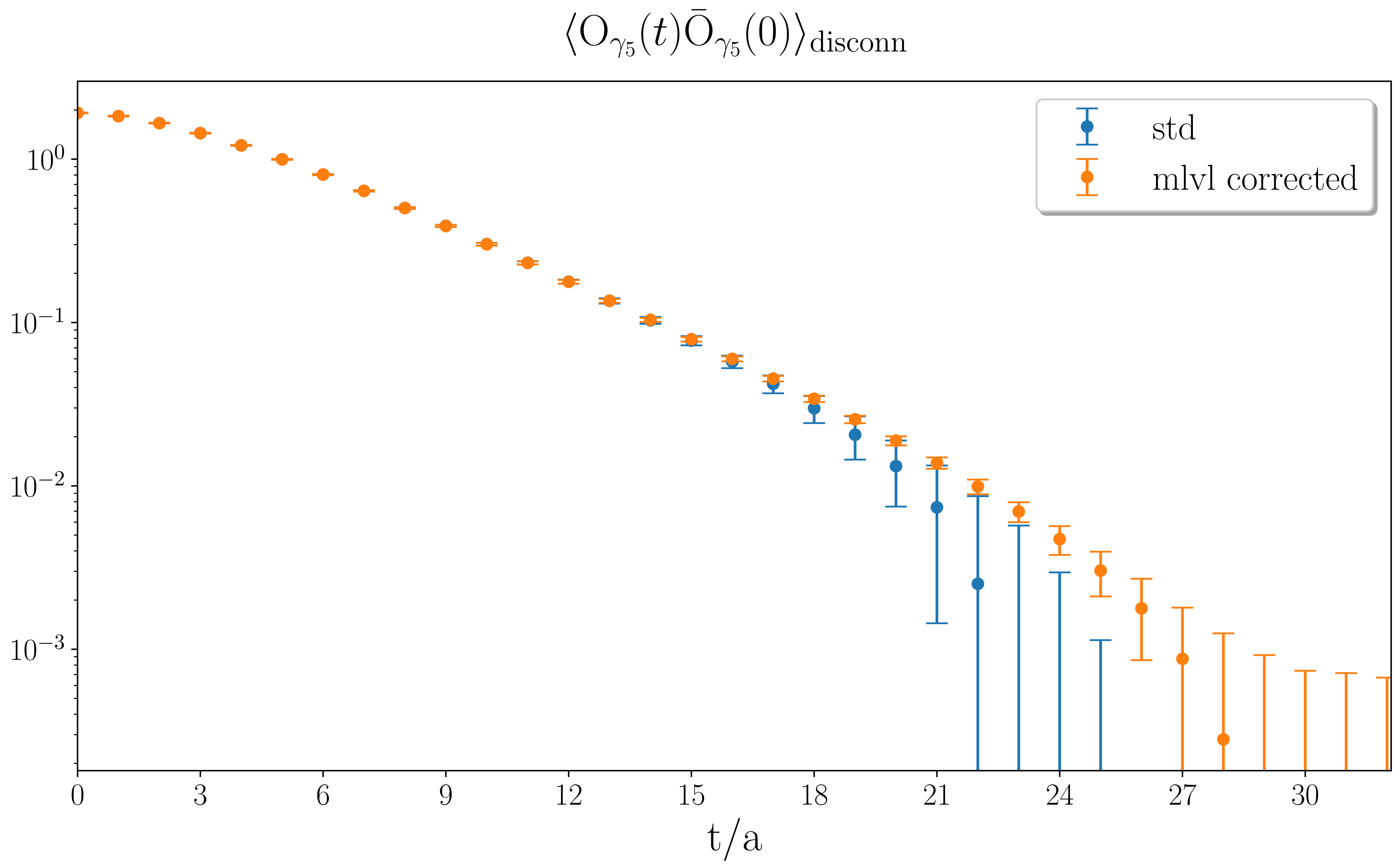
$$C^{\mathrm{approx,1lvl}}(t,t_0) = \frac{1}{N_0 N_1}\sum_{i,j} \mathcal{O}^{\Omega_1}_{ij}(t,t)\mathcal{O}^{\Omega_0}_{ij}(t_0,t_0) \qquad \sigma^2_{C^{\mathrm{approx,1lvl}}}(t,t_0) \approx \frac{c_0}{N_0 N_1}$$

$$C^{\mathrm{approx,2lvl}}(t,t_0) = \frac{1}{N_0}\sum_{i}^{N_0}\left[\frac{1}{N_1}\sum_{j=1}^{N_1}\mathcal{O}^{\Omega_1}_{ij}(t,t)\right]\left[\frac{1}{N_1}\sum_{j=1}^{N_1}\mathcal{O}^{\Omega_0}_{ij}(t_0,t_0)\right] \qquad \sigma^2_{C^{\mathrm{approx,2lvl}}}(t,t_0) \approx \frac{c_0}{N_0 N_1^2} + \dots$$

Compute correction: $\epsilon(t,t_0) = C^{\mathrm{std}}(t,t_0) - C^{\mathrm{approx,1lvl}}(t,t_0)$

Compute corrected 2-level: $C^{\mathrm{2lvl}}(t,t_0) = C^{\mathrm{approx,2lvl}}(t,t_0) + \epsilon(t,t_0)$

Comparison of Final results: Pseudoscalar quark loops ($\eta - \eta'$ channel)



Numerical Simulations

$\beta = 6.0\,;\ (a = 0.0876\ \text{fm})\,;\ m_\pi = 760\ \text{MeV}$

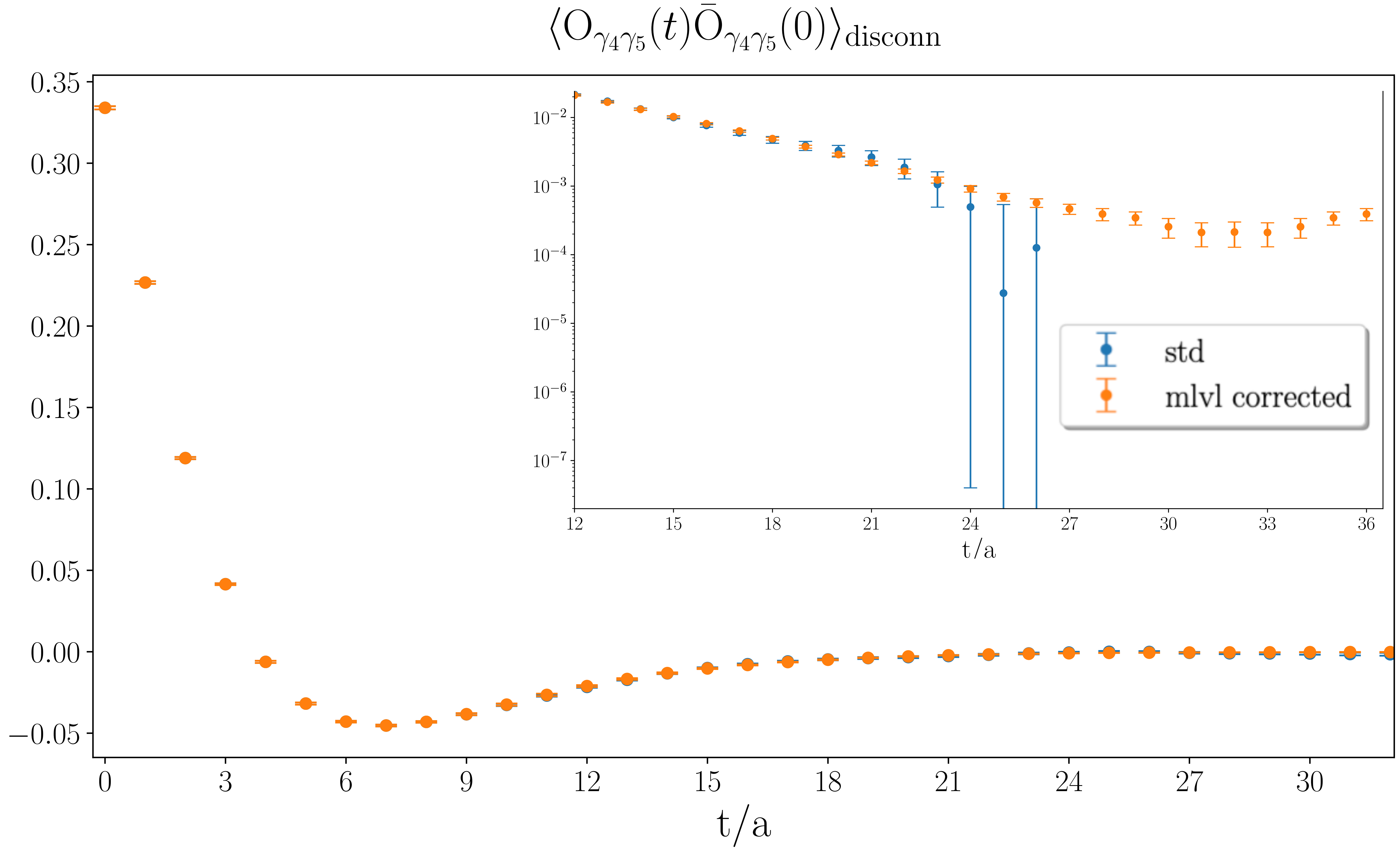
$V/a^4 = 16^3 \times 64$

$N_0 = 101\,;\ N_1 = 200$

Signal reached for much longer distances

Substantial improvement of two-level sampling over standard method.

Comparison of Final results: Pseudoscalar quark loops ($\eta - \eta'$ channel)



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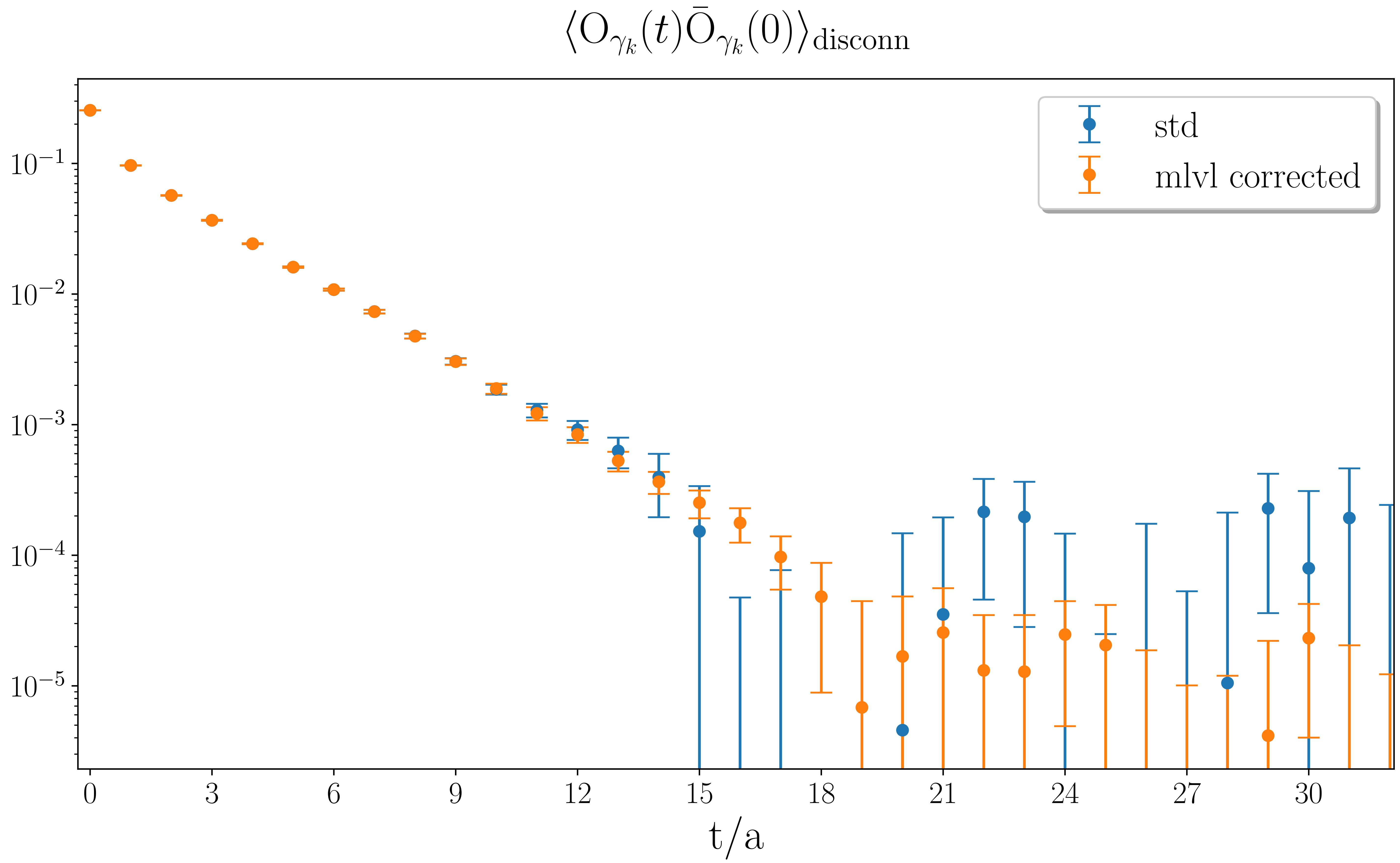
$V/a^4 = 16^3 \times 64$

$N_0 = 101\,;\ N_1 = 200$

Substantial improvement of two-level sampling over standard method.

2-level enables to reconstruct signal for the full temporal extent.

Comparison of Final results: Vector quark loops (ω channel)



Numerical Simulations

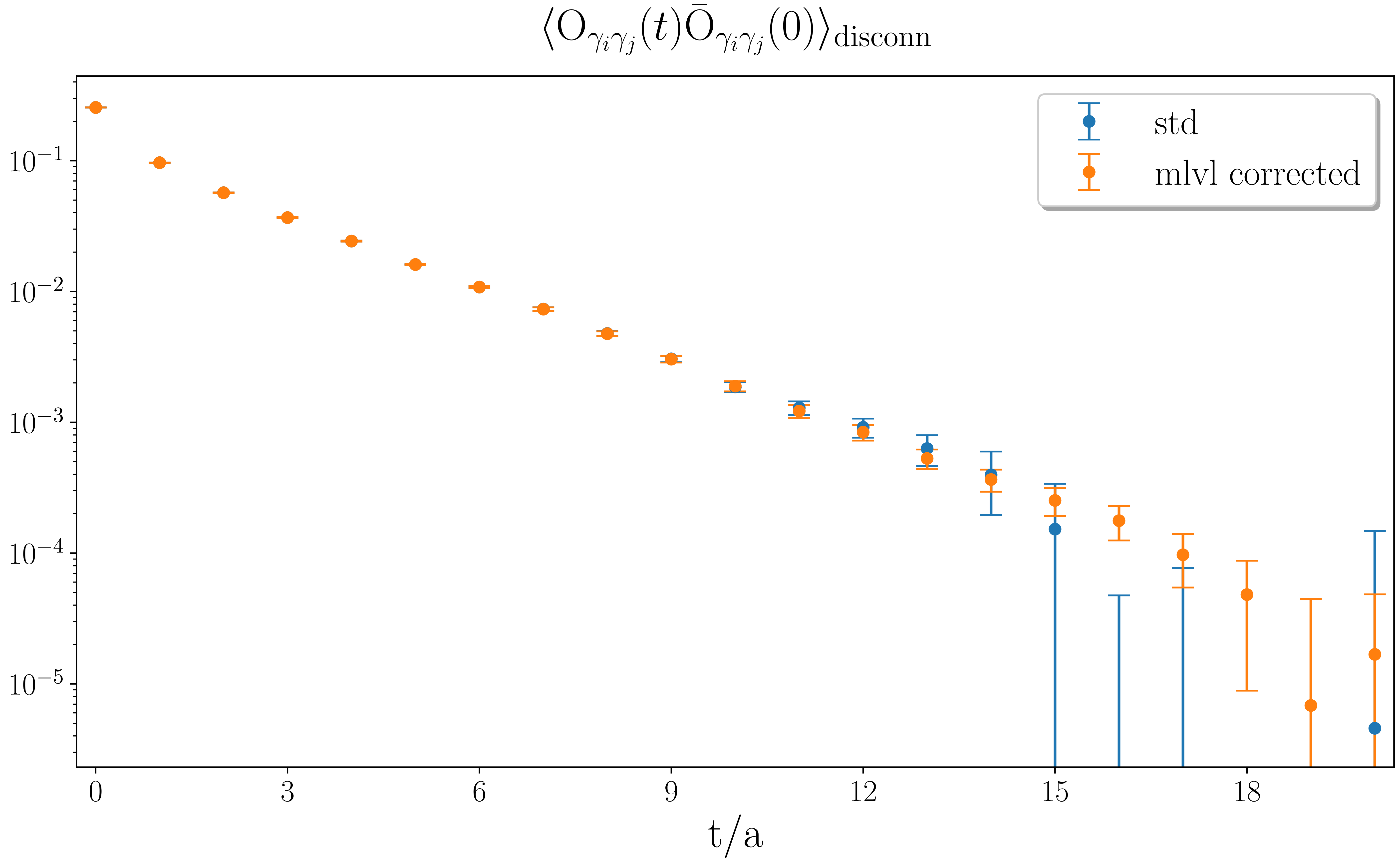
$\beta = 6.0\,;\,(a = 0.0876\text{ fm})\,;\; m_\pi = 760\text{ MeV}$

$V/a^4 = 16^3 \times 64$

$N_0 = 101\,;\, N_1 = 200$

Signal reached for longer distances.

Comparison of Final results: Tensor quark loops (f_2 channel)



Numerical Simulations

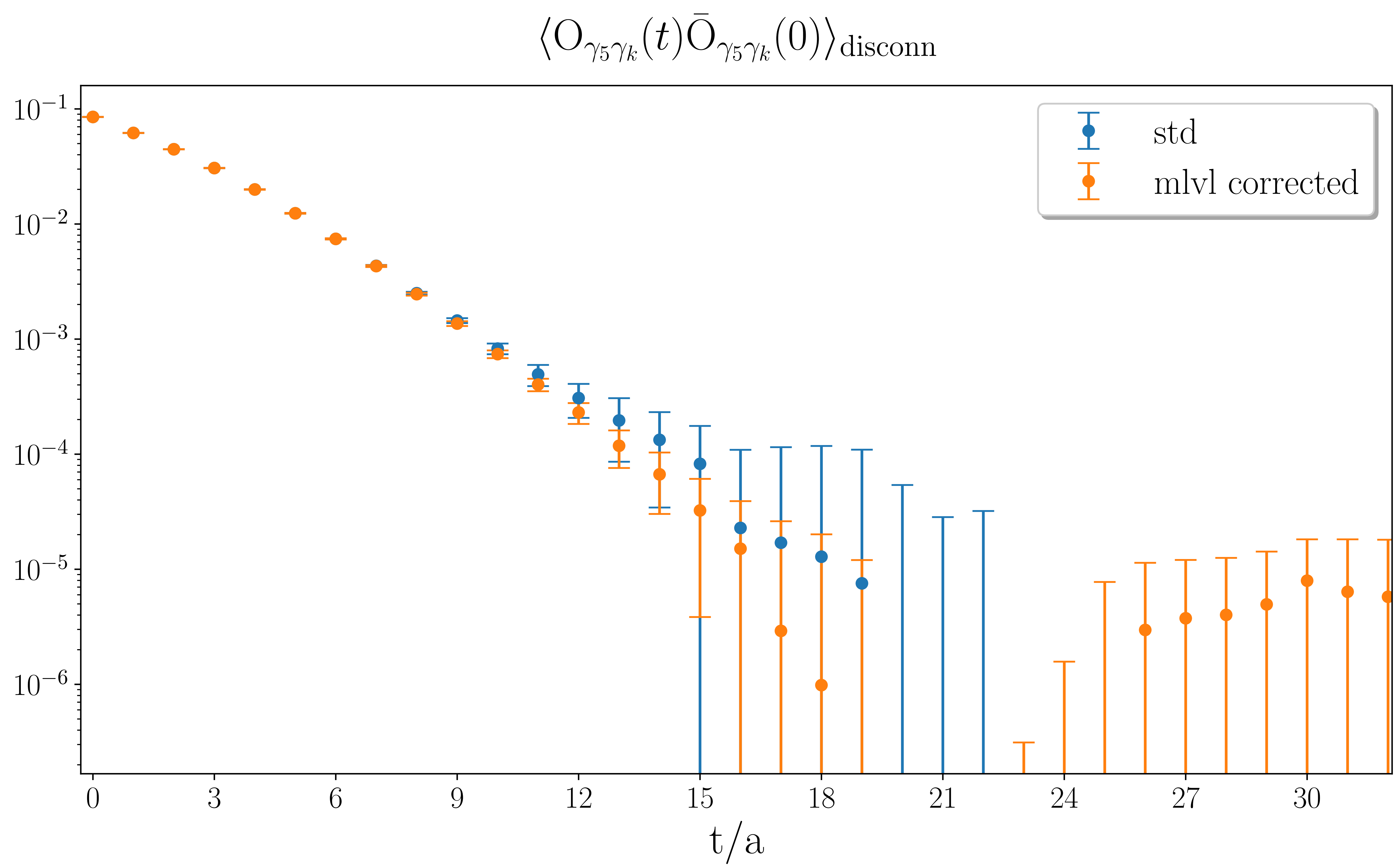
$\beta = 6.0\,;\,(a = 0.0876\ \mathrm{fm})\,;\ m_\pi = 760\ \mathrm{MeV}$

$V/a^4 = 16^3 \times 64$

$N_0 = 101\,;\,N_1 = 200$

Signal reached for longer distances.

Comparison of Final results: Axial-Vector quark loops (a_1 channel)



Numerical Simulations

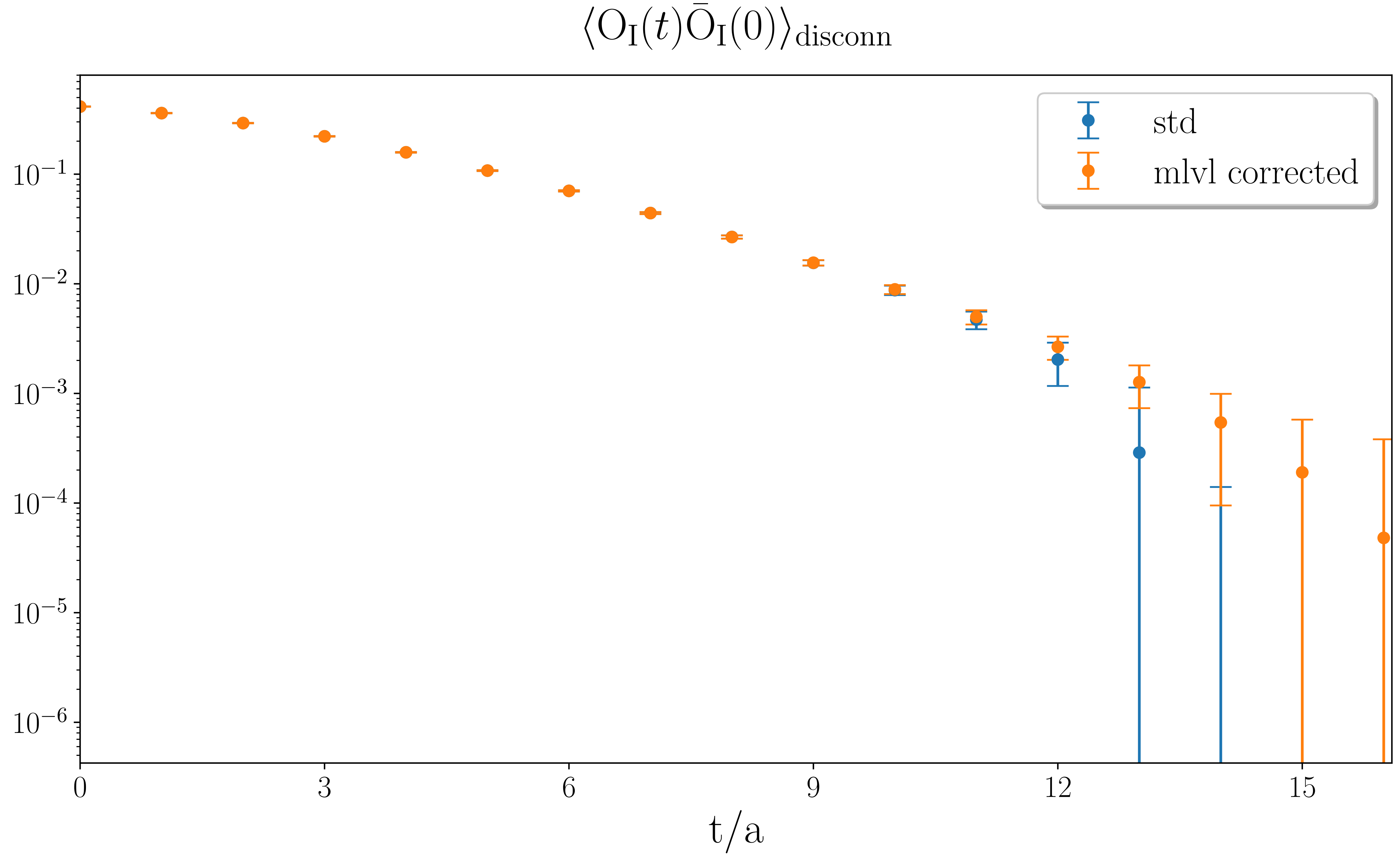
$\beta = 6.0\,;\,(a = 0.0876\ \text{fm})\,;\ m_\pi = 760\ \text{MeV}$

$V/a^4 = 16^3 \times 64$

$N_0 = 101\,;\,N_1 = 200$

Signal reached for slightly longer distances.

Comparison of Final results: Scalar quark loops (f_0/σ channel)



Numerical Simulations

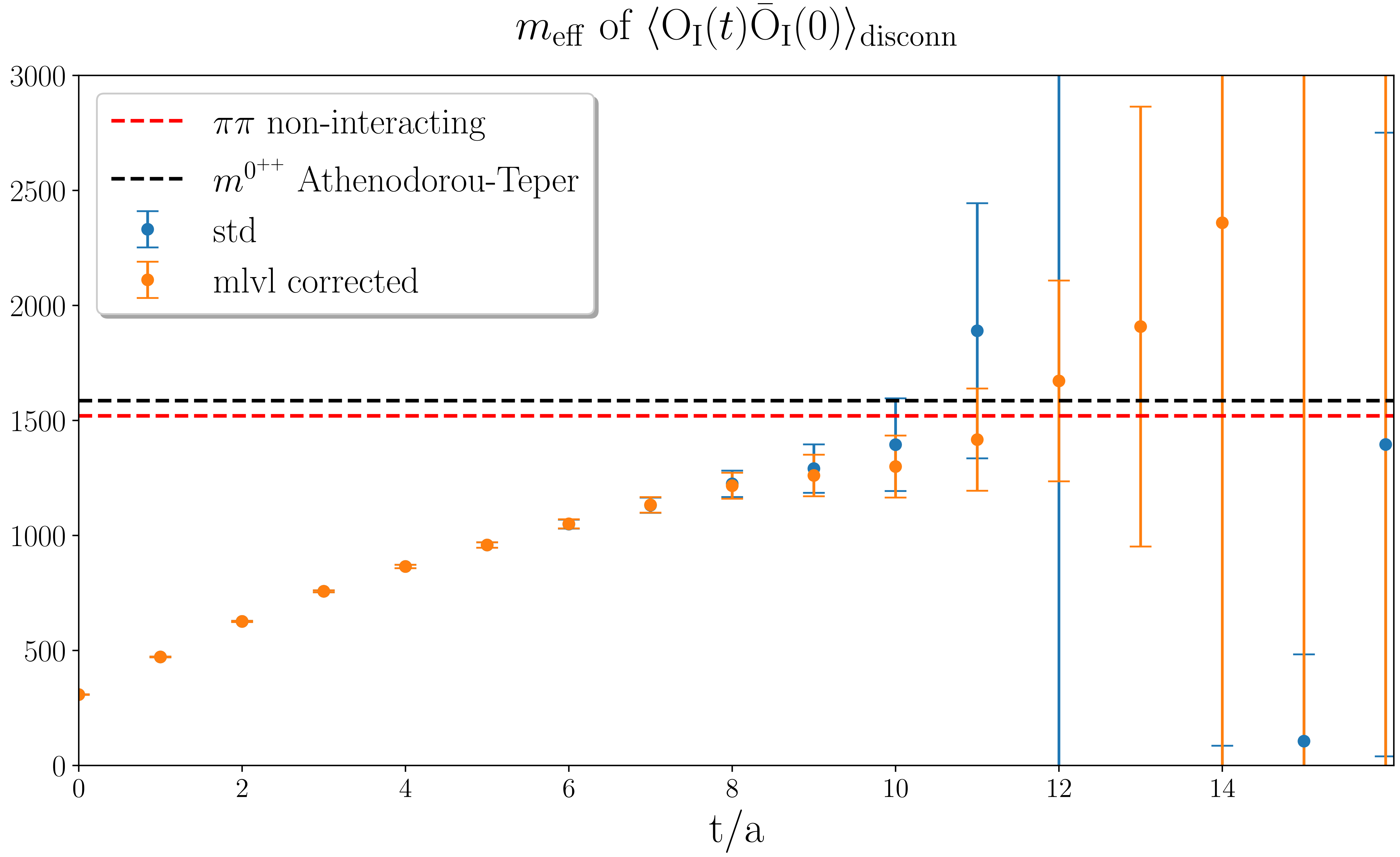
$\beta = 6.0\,;\,(a = 0.0876\ \text{fm})\,;\ m_\pi = 760\ \text{MeV}$

$V/a^4 = 16^3 \times 64$

$N_0 = 101\,;\,N_1 = 200$

Signal improves a bit (decays quite fast)

Comparison of Final results: Scalar quark loops (f_0/σ channel)



Numerical Simulations

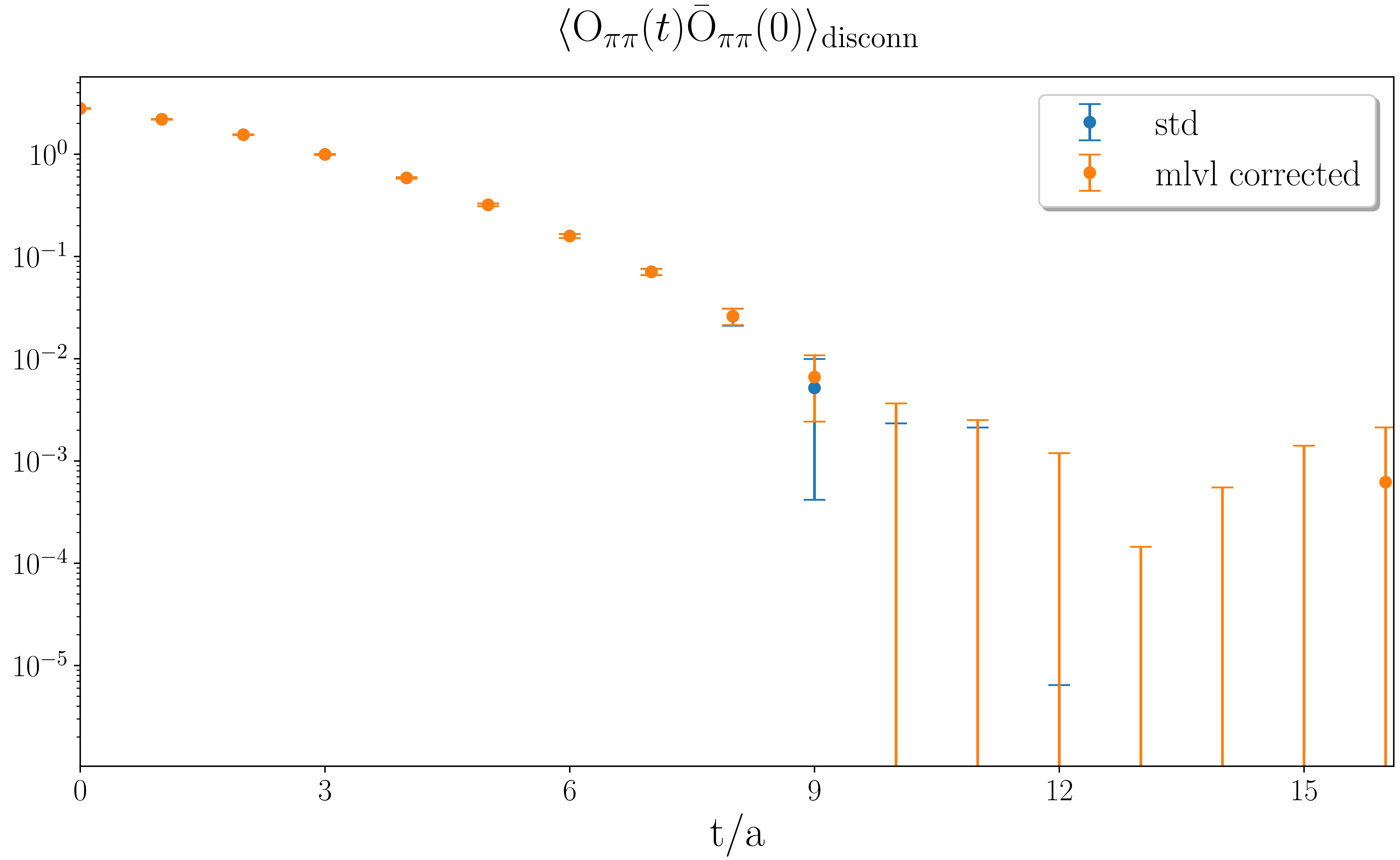
$\beta = 6.0\,;\,(a = 0.0876\ \text{fm})\,;\ m_\pi = 760\ \text{MeV}$

$V/a^4 = 16^3 \times 64$

$N_0 = 101\,;\,N_1 = 200$

- Need to further reduce error for reliable estimate
- Need to include connected piece

Comparison of Final results: $\pi\pi$ quark loops (I = 0 channel)



Numerical Simulations

$\beta = 6.0\,;\,(a = 0.0876\ \mathrm{fm})\,;\ m_\pi = 760\ \mathrm{MeV}$

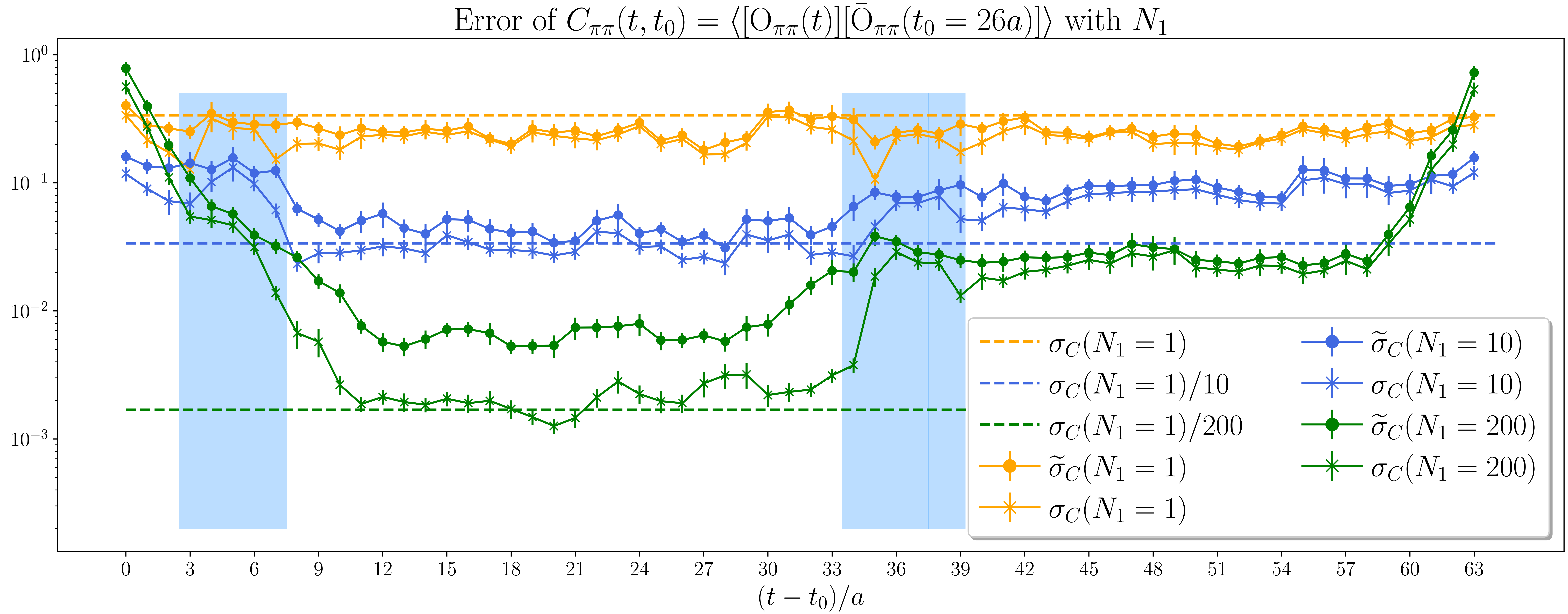
$V/a^4 = 16^3 \times 64$

$N_0 = 92\,;\,N_1 = 198$

Many exceptional configs ($\mathcal{O}(50)/20{,}000$)

2-level Error Reduction

$\sigma_C(t, t_0; N_1) =$ 2lvl Error of $\langle[\mathrm{O}_{\pi\pi}(t)]\ [\bar{\mathrm{O}}_{\pi\pi}(t_0)]\rangle$ $\quad$ $\tilde{\sigma}_C(t, t_0; N_1)$ Error of corrected observable



Error increases by a factor 2 – 4 after correction

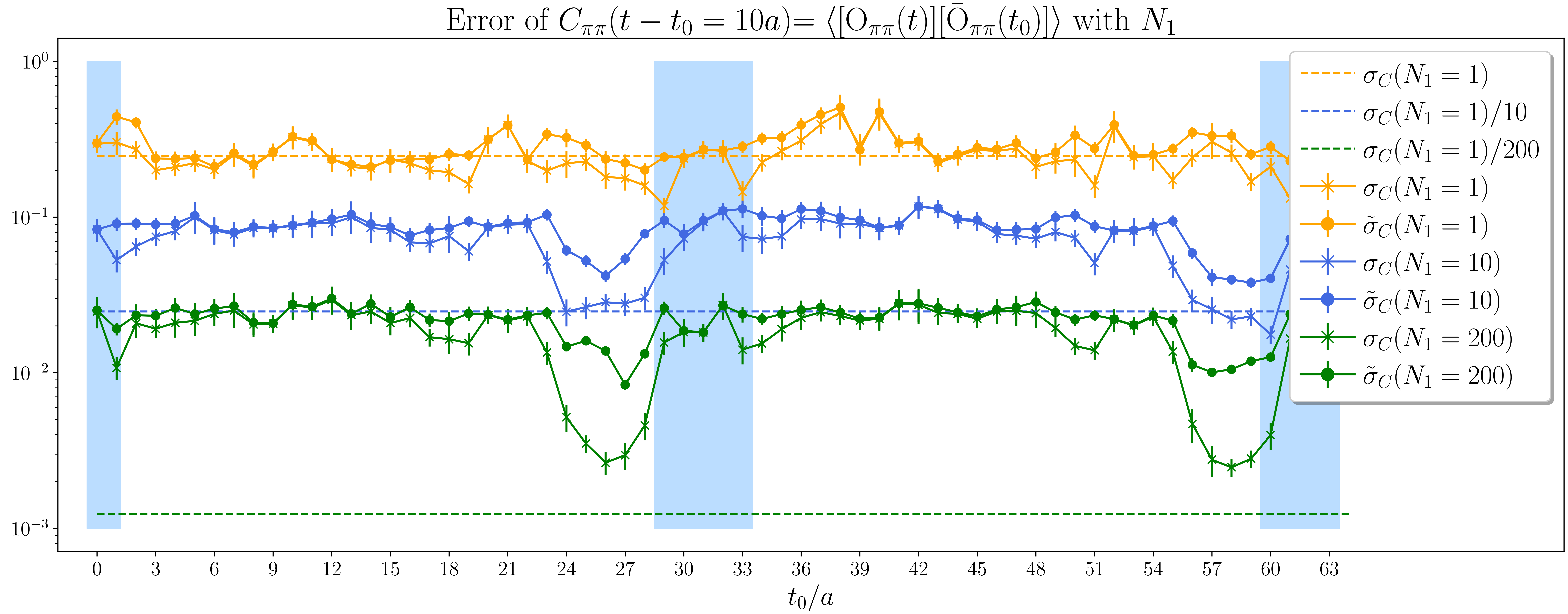
e.g. $\tilde{\sigma}_C(t - t_0 = 15a; N_1 = 200) \approx 2 * 10^{-3}$

$\sigma_C(t - t_0 = 15a; N_1 = 200) \approx 7 * 10^{-3}$

Multilevel Error Reduction at fixed distance

$\sigma_C(t, t_0; N_1) =$ Error of $\langle [\mathrm{O}_{\pi\pi}(t)]\ [\bar{\mathrm{O}}_{\pi\pi}(t_0)] \rangle$

$\tilde{\sigma}_C(t, t_0; N_1)$ Error of corrected observable

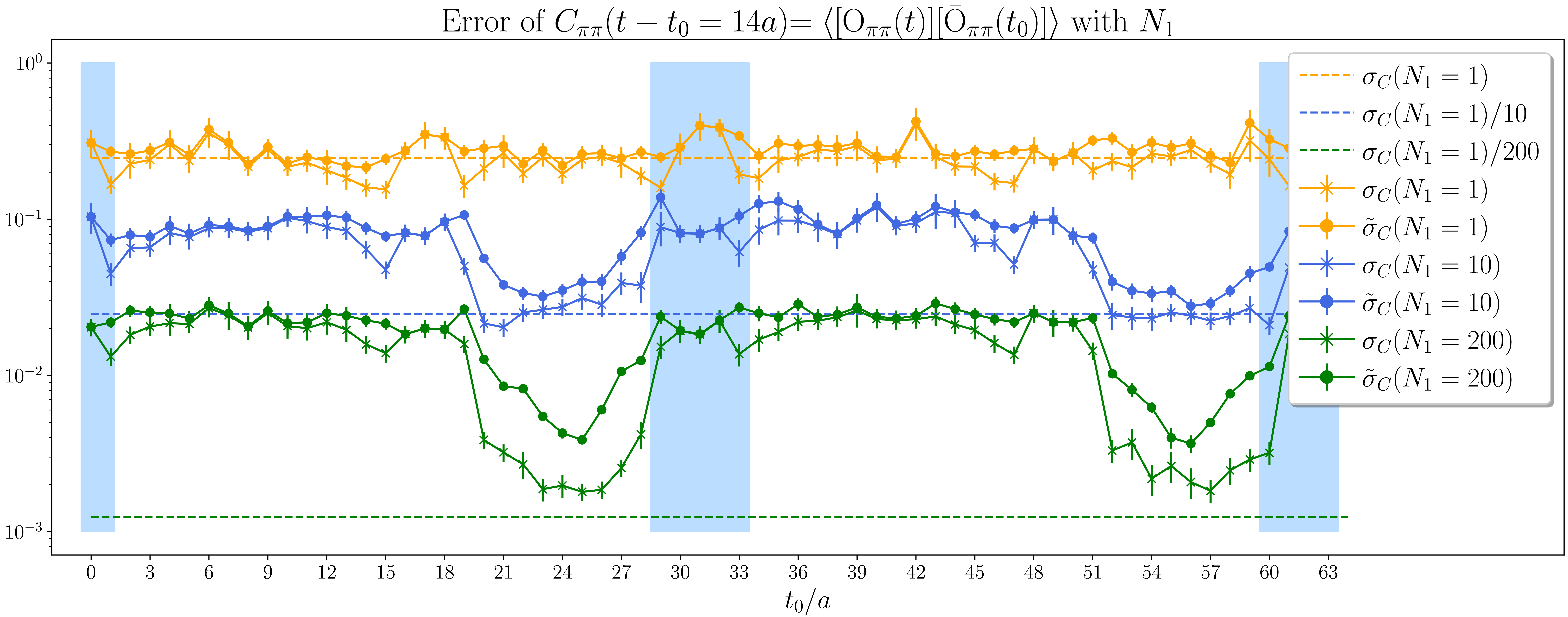


This motivates us to compute the first correction with 2-level integration.

Multilevel Error Reduction at fixed distance

$\sigma_C(t, t_0; N_1) =$ Error of $\langle [\mathrm{O}_{\pi\pi}(t)]\ [\bar{\mathrm{O}}_{\pi\pi}(t_0)] \rangle$

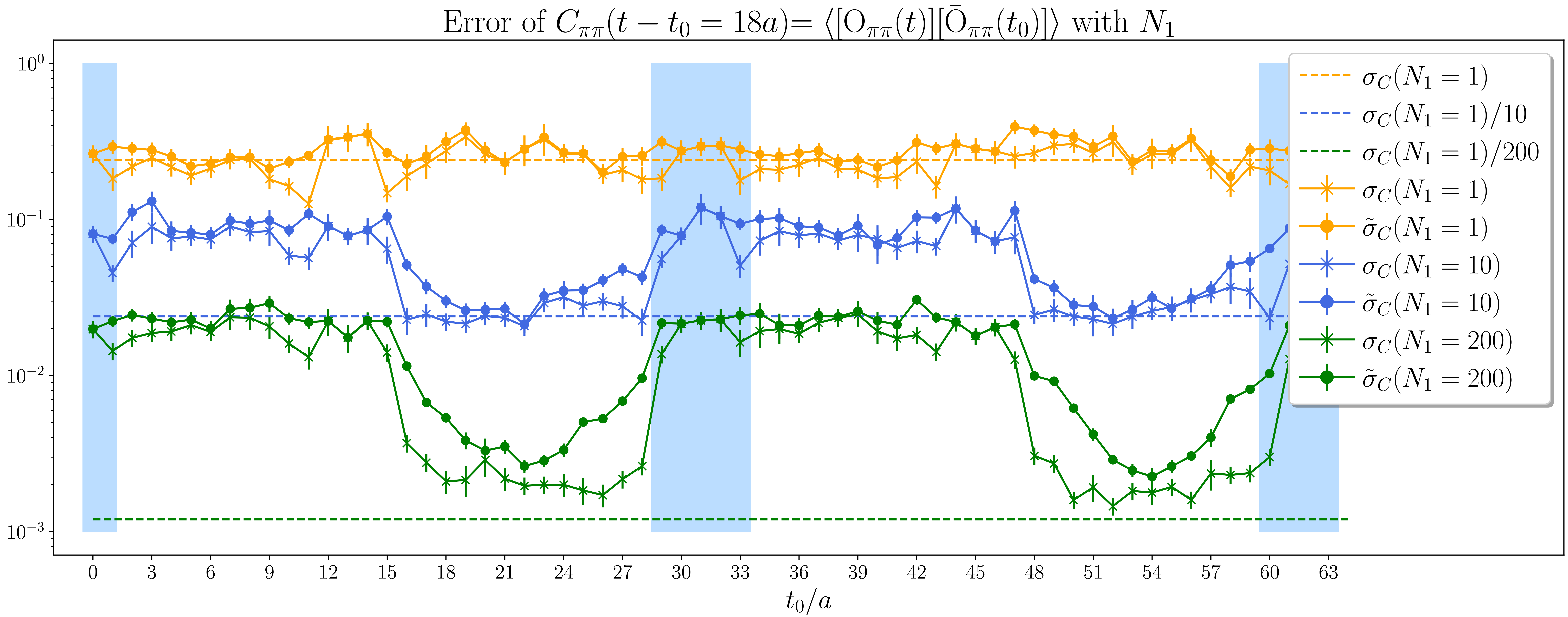
$\tilde{\sigma}_C(t, t_0; N_1)$ Error of corrected observable



Multilevel Error Reduction at fixed distance

$\sigma_C(t, t_0; N_1) =$ Error of $\langle [\mathrm{O}_{\pi\pi}(t)]\ [\bar{\mathrm{O}}_{\pi\pi}(t_0)] \rangle$

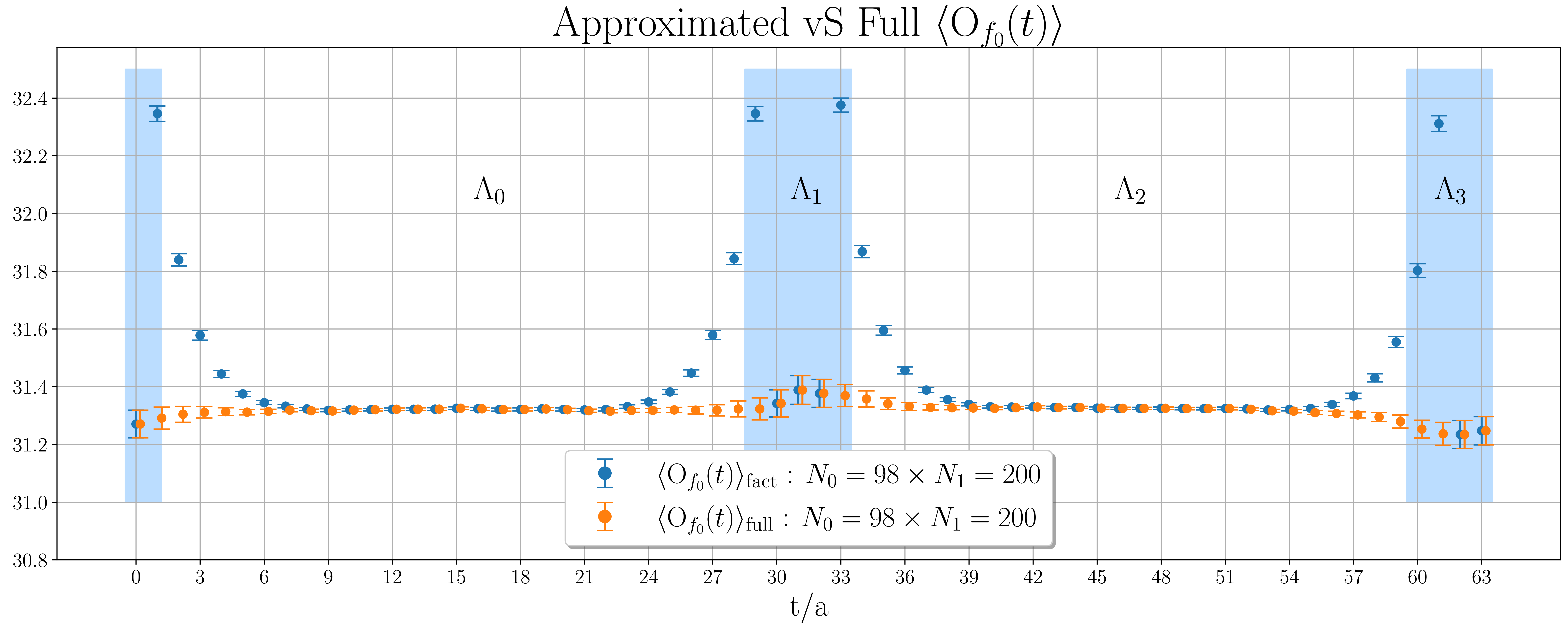
$\tilde{\sigma}_C(t, t_0; N_1)$ Error of corrected observable



Similarities with systematic study in pure gauge.

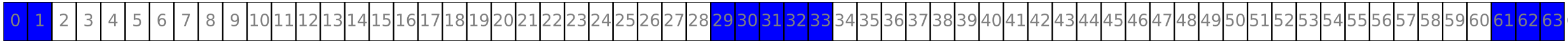
How good is the approximation? 1-pts

$$O_{f_0}(t) = \sum_{\vec{x}} \bar{q}(\vec{x}, t)\ \mathrm{I}\ q(\vec{x}, t)$$



Longer the distance from the other region, better the approximation

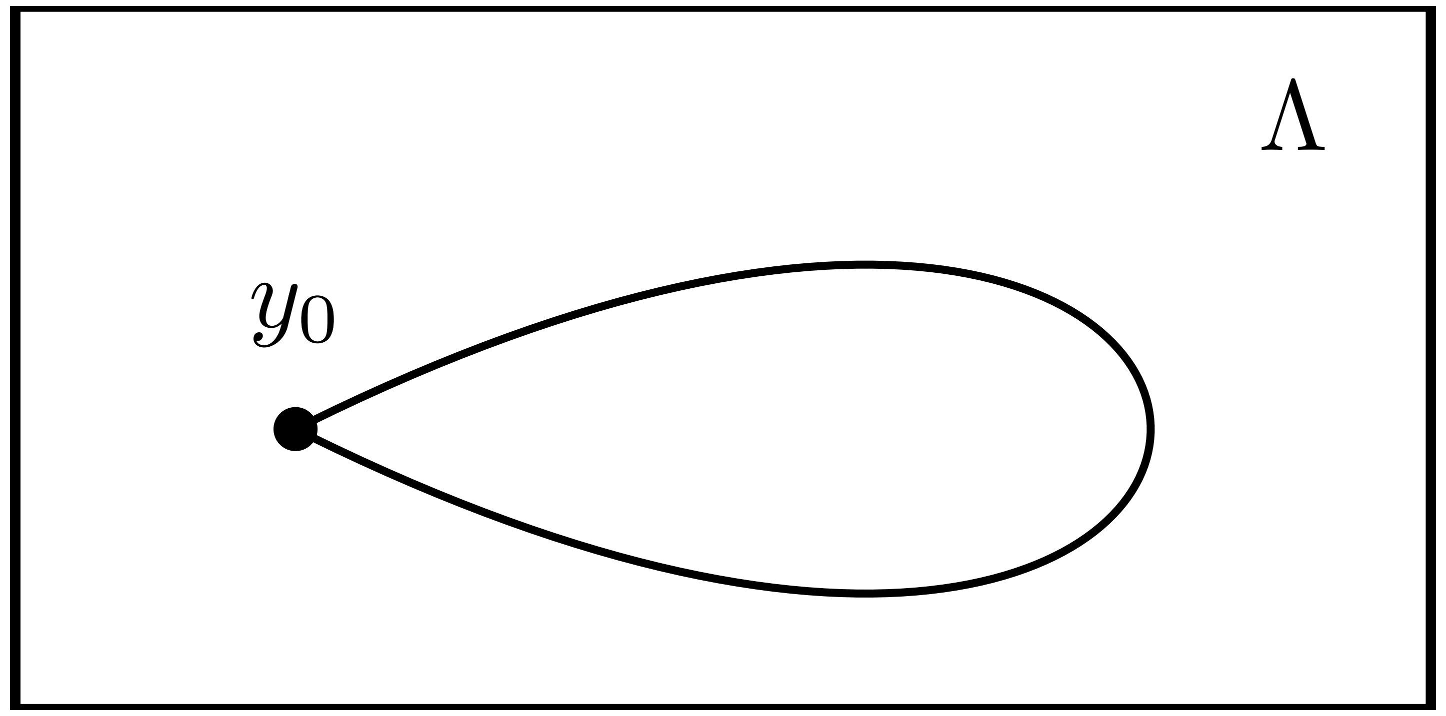
current sub-lattice decomposition



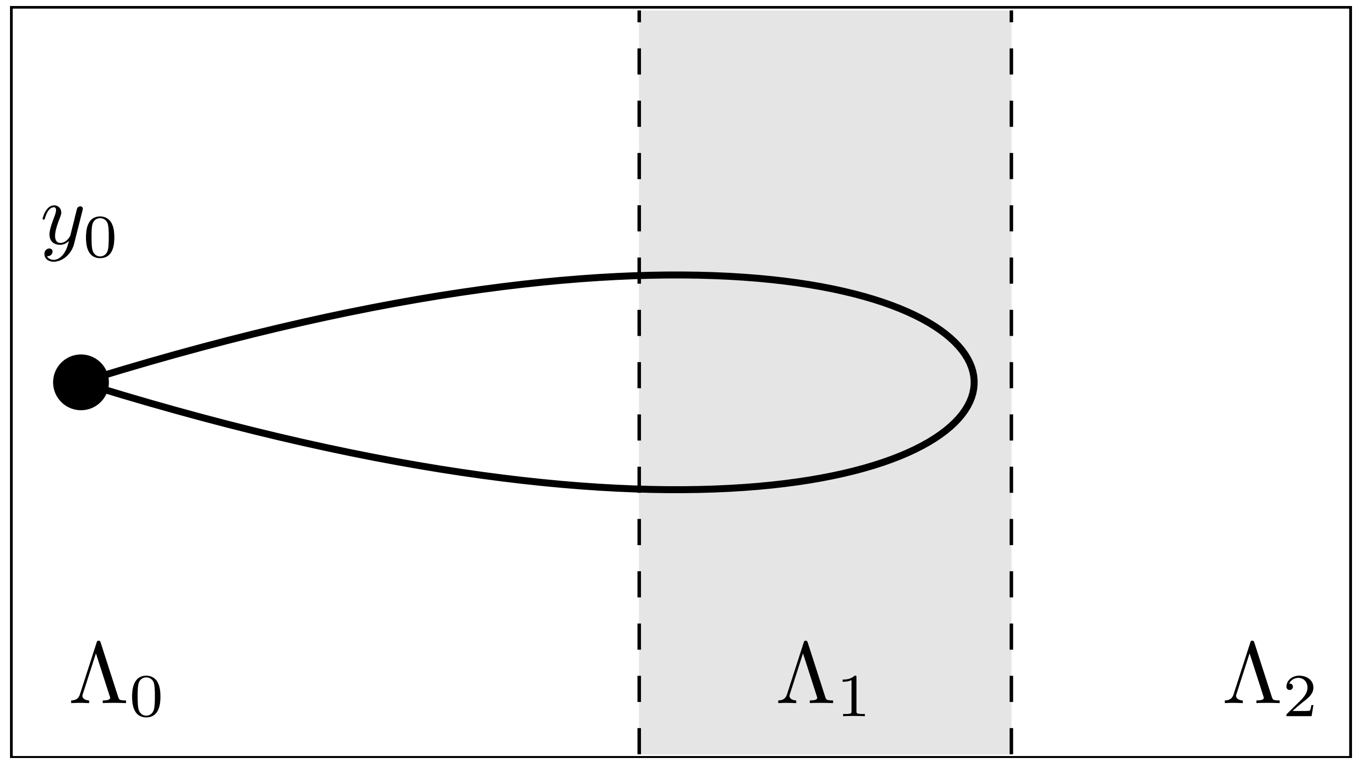
Factorisation of Fermion Propagator for quark loops

Open boundary conditions

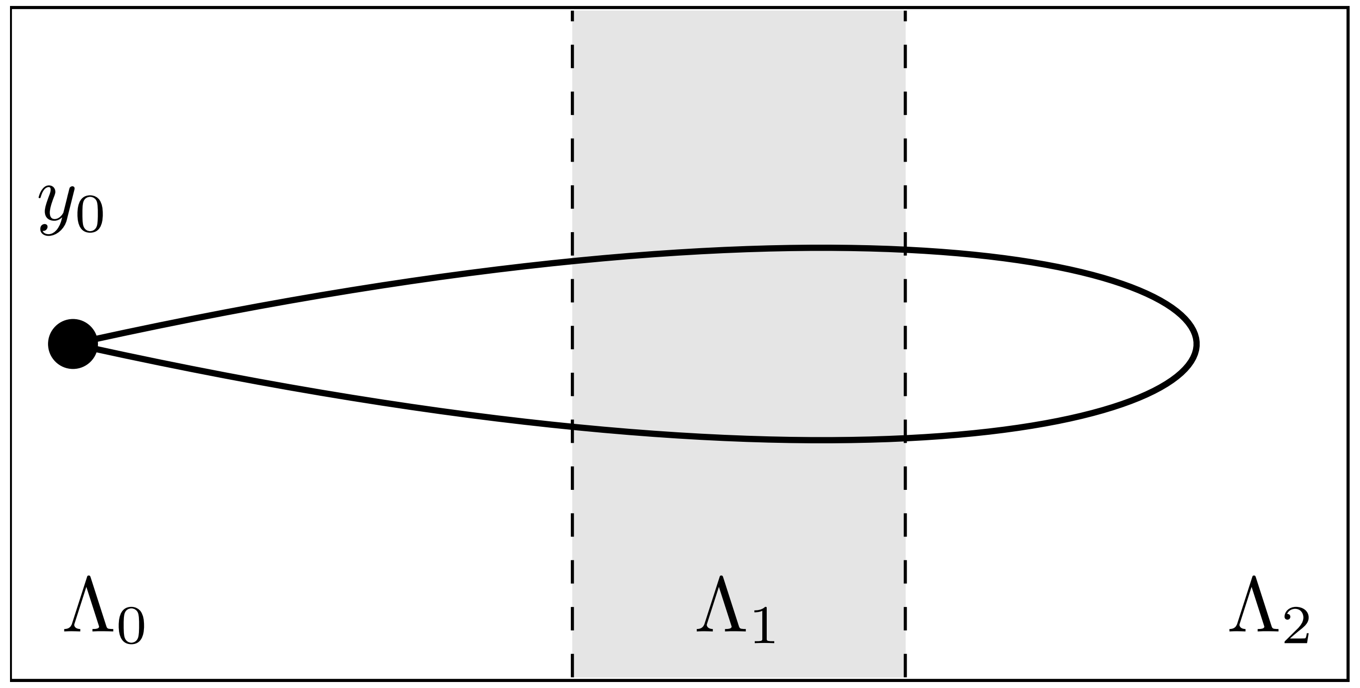
$$D_{\Lambda}^{-1}(y,y) = D^{(0)}(y,y) = D^{-1}_{\Lambda_0\cup\Lambda_1}(y,y) + D^{(1)}(y,y) + \dots$$



=

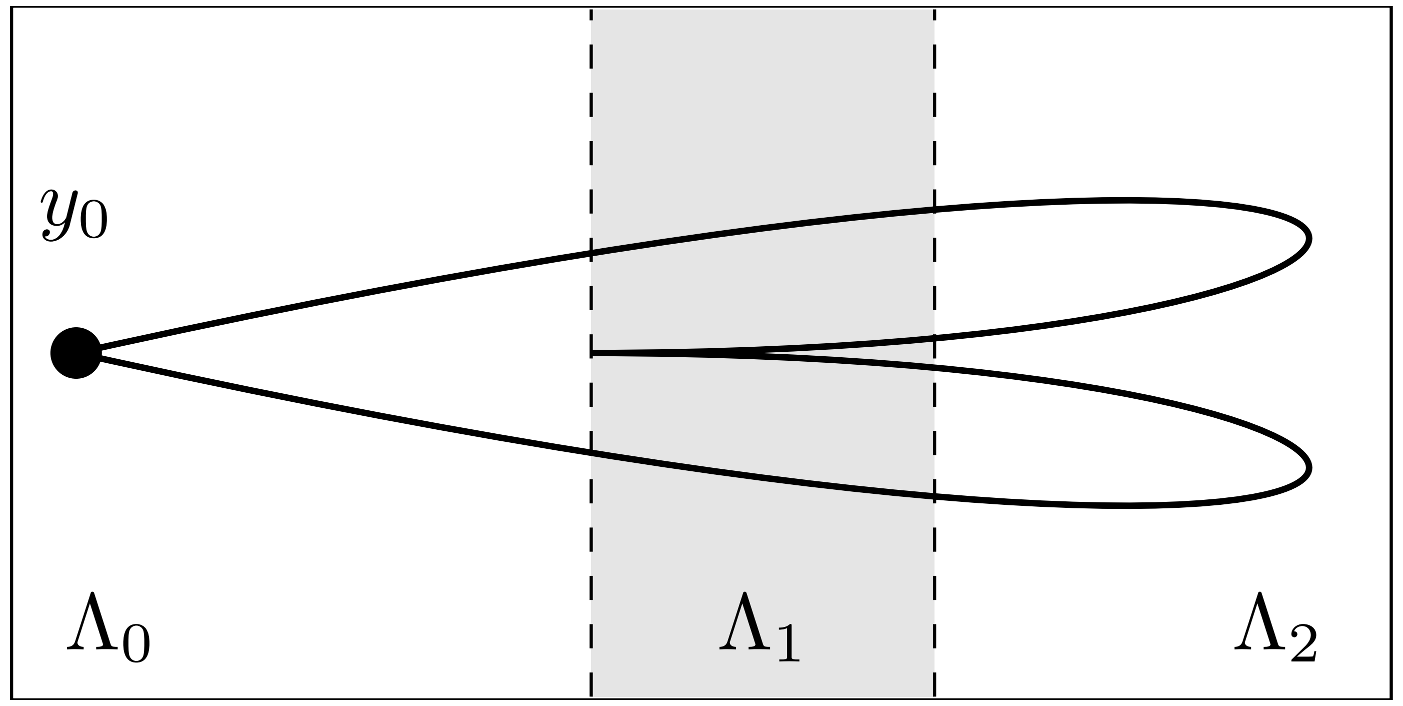


+



$D^{(1)}(y,y)$

+



$+\dots$

$$||D^{(2)}(y,y)|| \propto ||D^{-1}_{\Lambda_0\cup\Lambda_1}(y,y)||\, e^{-m||\Lambda_1||}$$

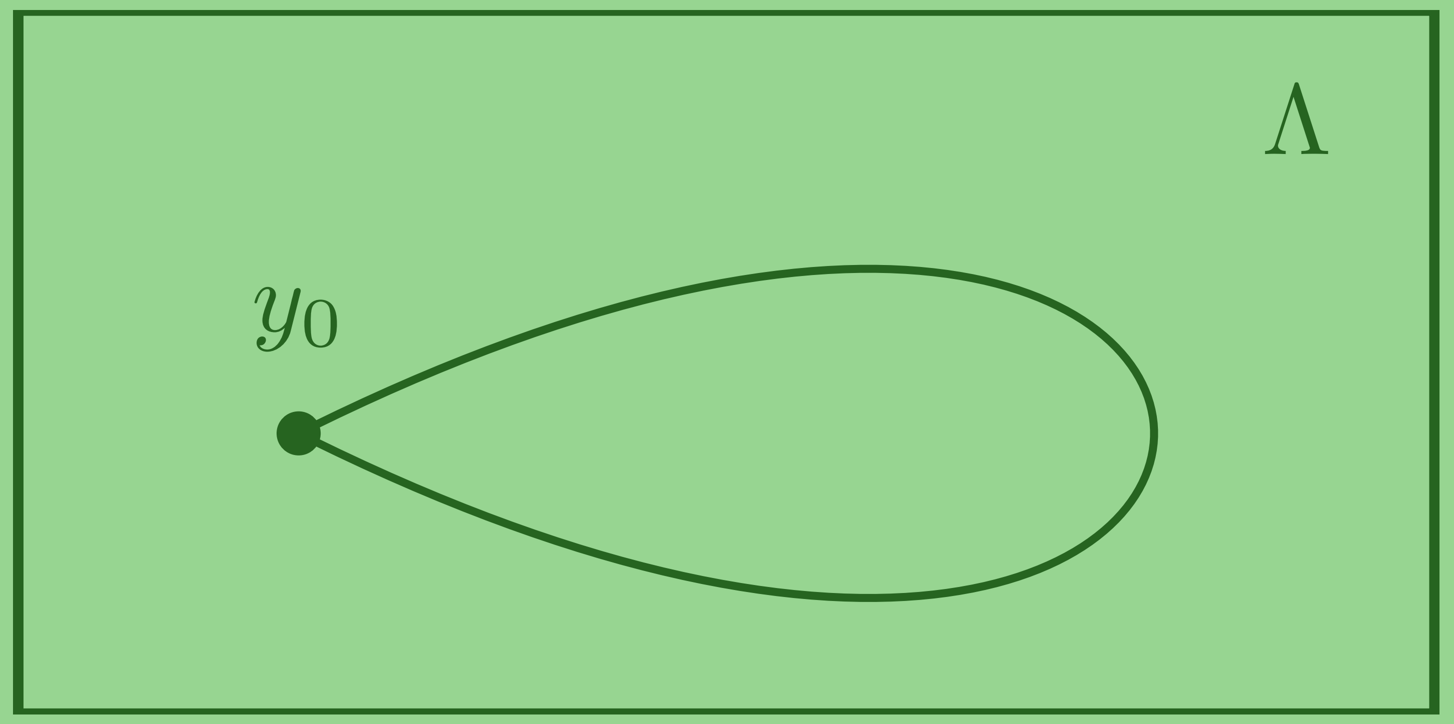
Ce`-Giusti-Schaefer

Lepage-Parisi

Factorisation of Fermion Propagator for quark loops

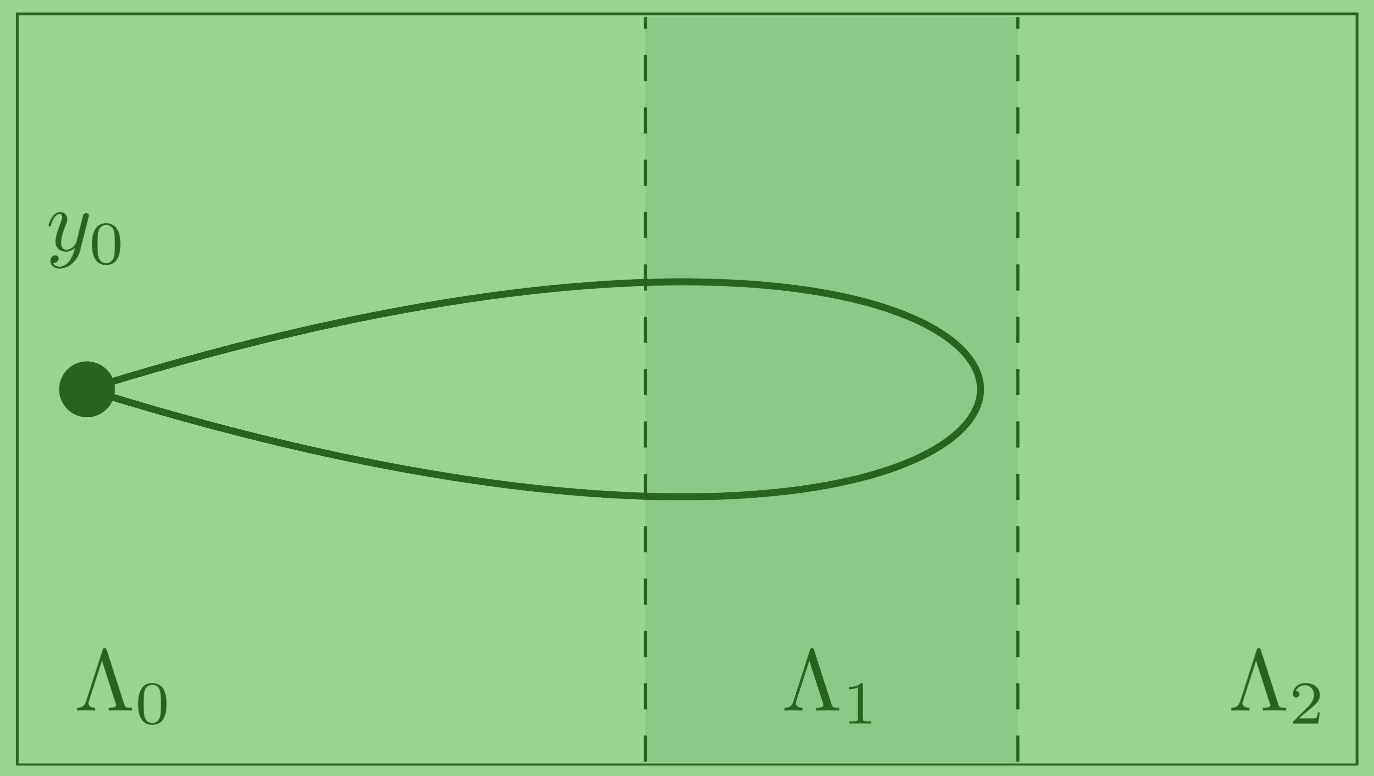
Open boundary conditions

$D_{\Lambda}^{-1}(y,y)$

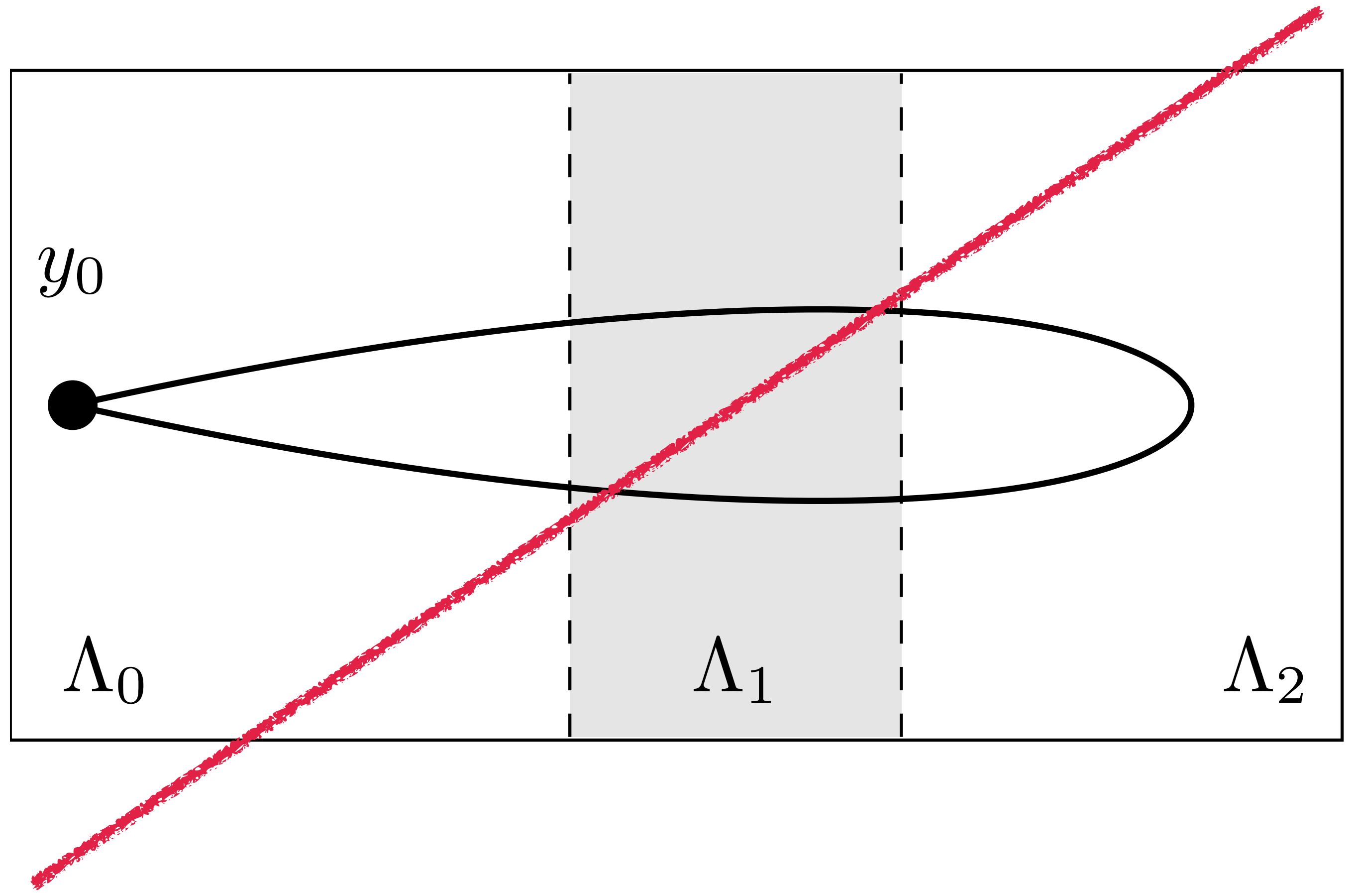


$=$

$D^{(0)}(y,y) = D_{\Lambda_0 \cup \Lambda_1}^{-1}(y,y)$

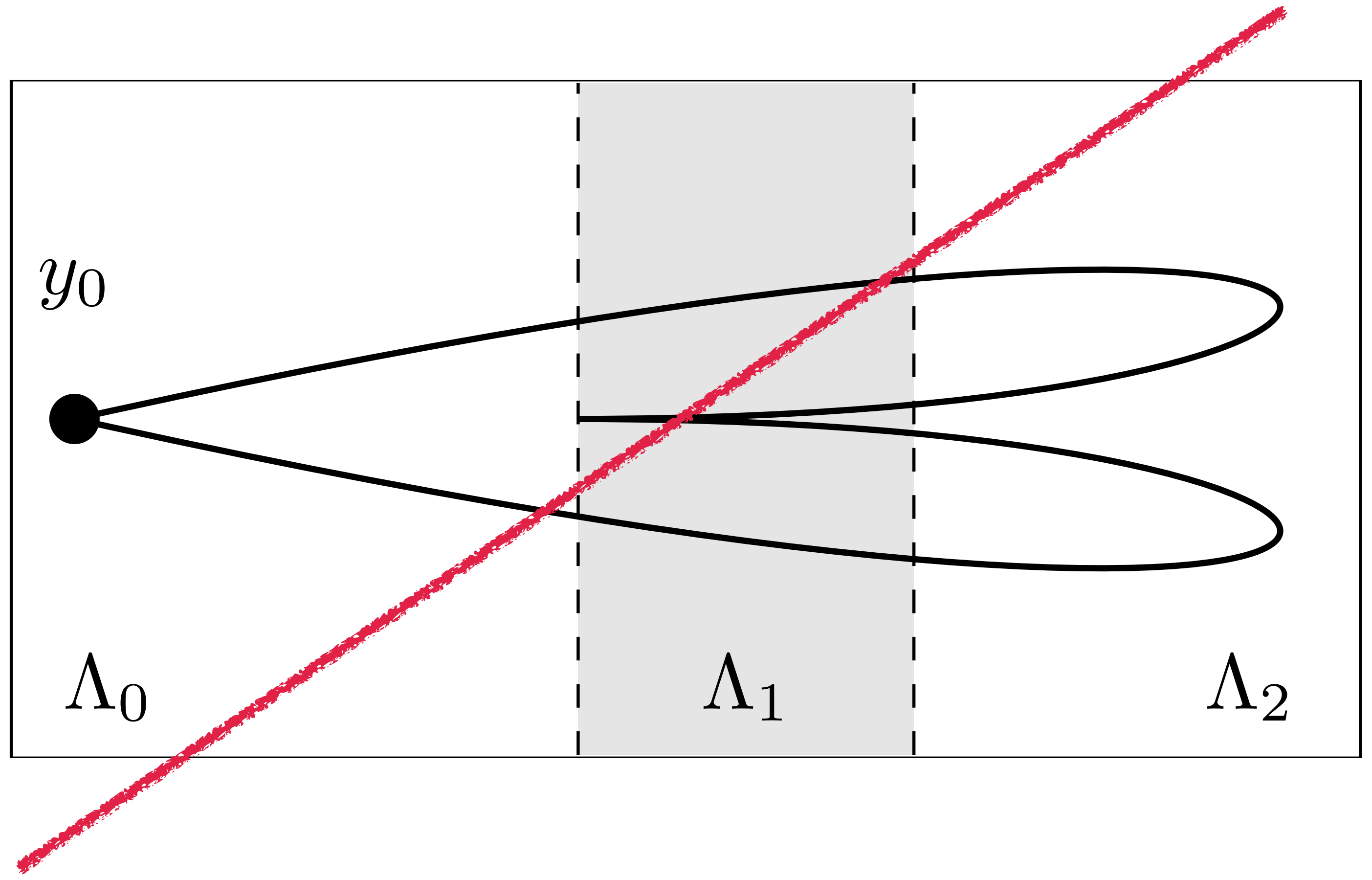


$+$



$D^{(1)}(y,y)$

$+$



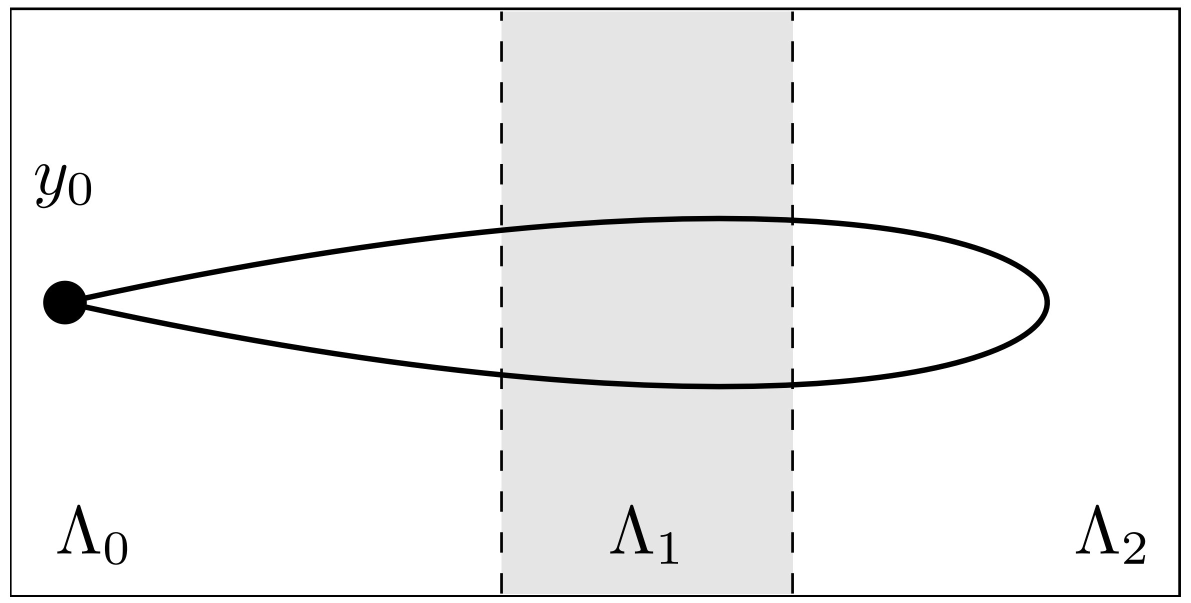
$||D^{(2)}(y,y)|| \propto ||D_{\Lambda_0 \cup \Lambda_1}^{-1}(y,y)|| \, e^{-m||\Lambda_1||}$

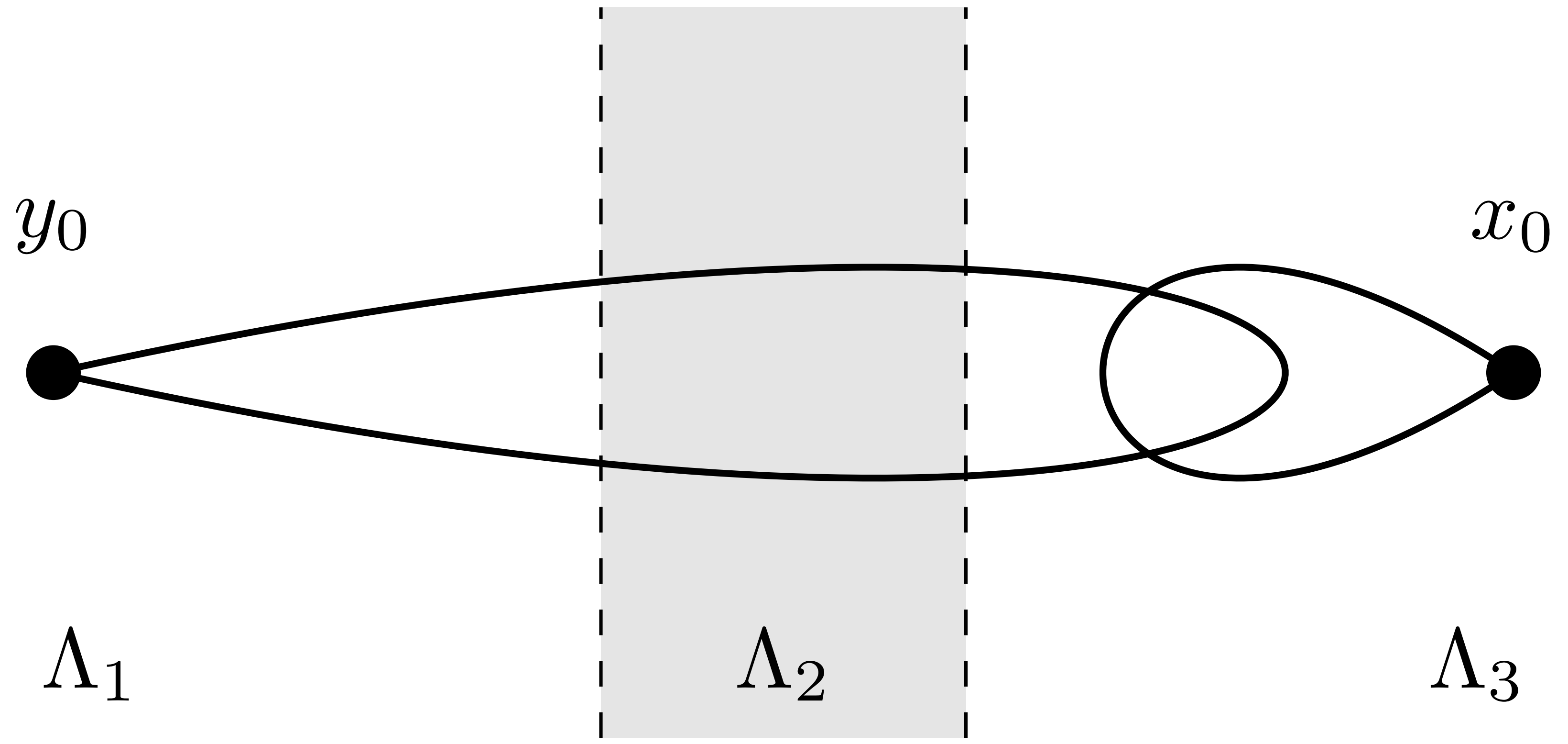
$+ \ldots$

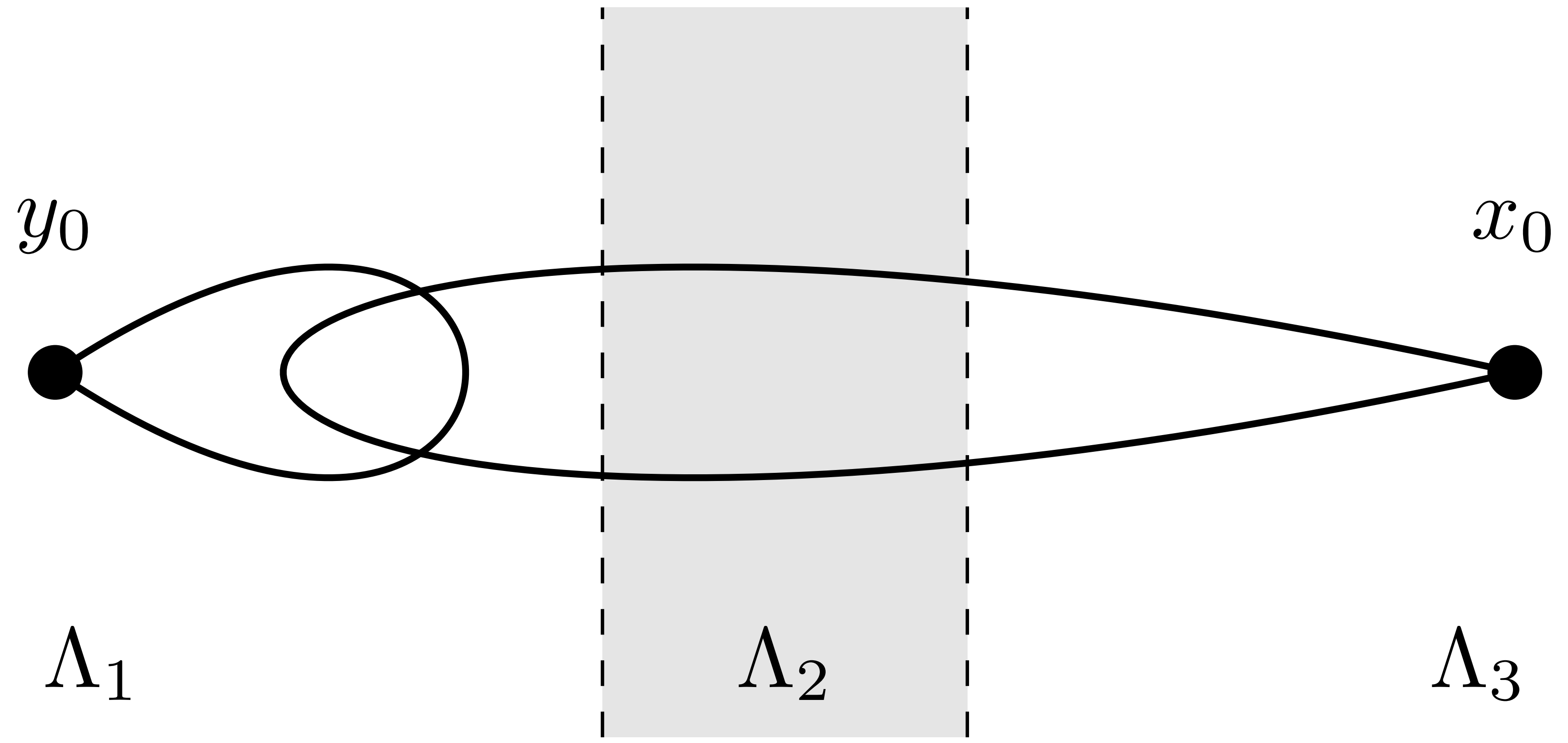
Ce`-Giusti-Schaefer

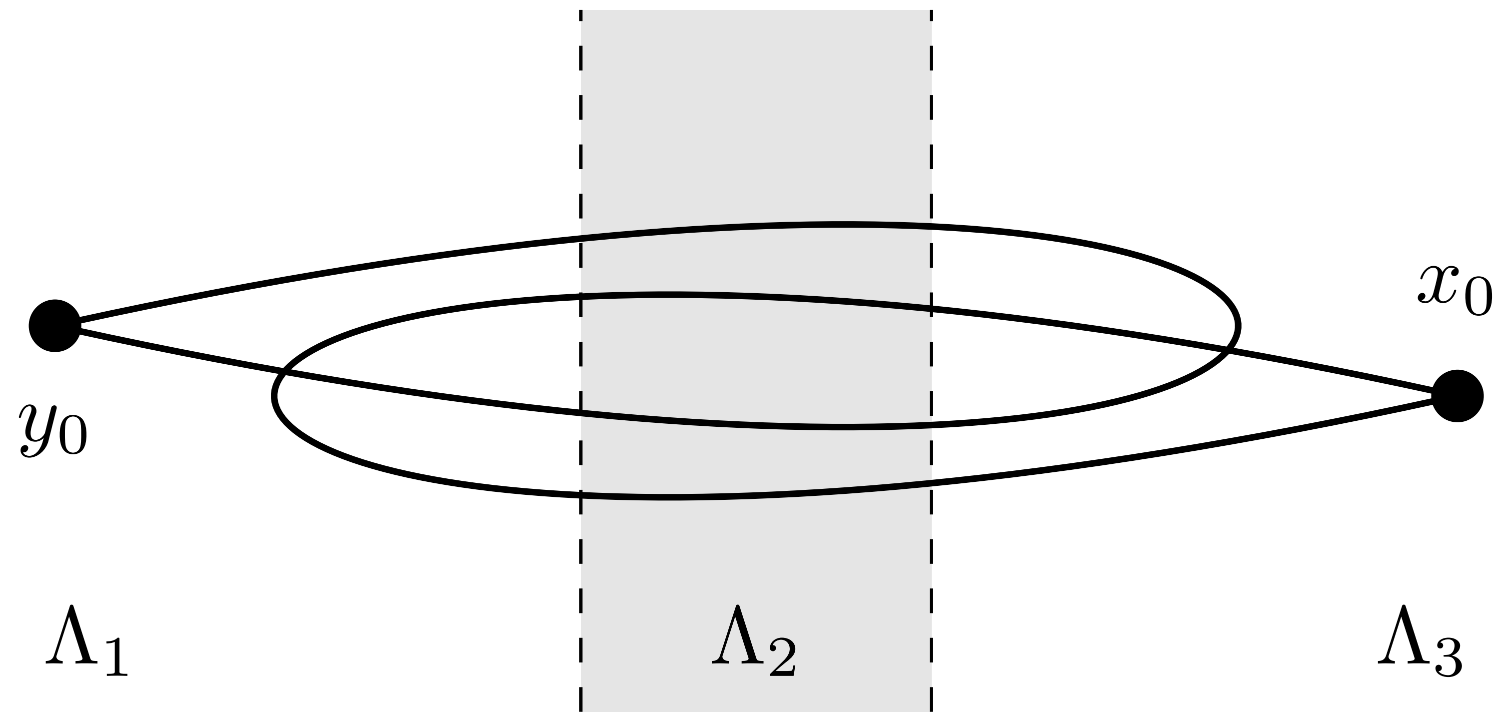
Lepage-Parisi

First corrections to quark loops with 2-level









Conclusions

- ☑ Two-level sampling for pure gauge glueballs very effective and well understood
  - ☑ Fit ansatz describes short and long distance scalings, as well as transition point
  - ☑ GEVP effective masses agree with literature

[LB et al. *Phys.Rev.D* 110 (2024) 5, 054515]

- Two-level sampling for glueballs in quenched QCD + distillation
  - ☆ Approximating Dirac inversions to local domains enables multilevel measurements
  - ☑ Results show multilevel error reduction for different observables $O_{2\pi}, O_{f_0}, O_{\gamma_5}, O_{\gamma_4\gamma_5}, \ldots$
  - ☆ Significant improvement for correlators whose signal drops w/ $E \lesssim 1.2$ GeV

    However, not significant improvement for heavier states $O_{2\pi}, O_{f_0}$
  - ☐ Compute first corrections with 2-level, which will improve the error

[Expected preprint on arXiv in the next months]

- Full QCD is work in progress...

Acknowledgements

Gauge generation and block solvers based on modified version of OpenQCD

Distillation measurements on QCDlib

Computing resources: PAX in Zeuthen and Juwels in Jülich

FOR 5269

DESY.

Thank you !

Backup

Multilevel sampling: Background

Based on an old idea by Nobel laureate G. Parisi et al. [*Phys.Lett.B* 128 (1983) 418-420]

Further developed by M. Lüscher, P. Weisz ~20 years later: [*JHEP* 09 (2001) 010]

Many other applications in pure gauge theory: *JHEP* 07 (2002) 049, *JHEP* 01 (2003) 048, *Nucl.Phys.B* 664 (2003) 213-232, *Nucl.Phys.B* 671 (2003) 103-132, *Nucl.Phys.B* 677 (2004) 273-288, *Nucl.Phys.B* 692 (2004) 209-231, *JHEP* 01 (2004) 030, …, *Phys.Rev.D* 110 (2024) 5, 054515 **[LB]**

One test application in quenched QCD: *Phys.Rev.D* 93 (2016) 9, 094507 [M. Ce', S. Schaefer, L. Giusti]

and two in full QCD afaik: *Phys.Rev.D* 95 (2017) 3, 034503 [M. Ce', S. Schaefer, L. Giusti]
Phys.Lett.B 816 (2021) 136191 [L. Giusti, T.Harris, et al.]

Multilevel algorithm: Full QCD (e.g. $N_f = 2$)

Both the action and the observables depend on fermionic propagators (non-local)

$$\langle \mathrm{O}(t_1)\mathrm{O}^\dagger(t_0)\rangle = \frac{1}{\mathcal{Z}} \int [dU] e^{-S[U]} \det(D)^2 \; \mathrm{O}(U, t_1)\; \mathrm{O}^\dagger(U, t_0)$$

$$\det(D)^2 e^{-S[U]}$$

Probability distribution depends on $\det(D)$ (non-local)

$$\mathrm{O}(t_1)\mathrm{O}(t_0) = \mathrm{Tr}\left[D^{-1}(t_1, t_0) D^{-1}(t_1, t_0)\right]$$

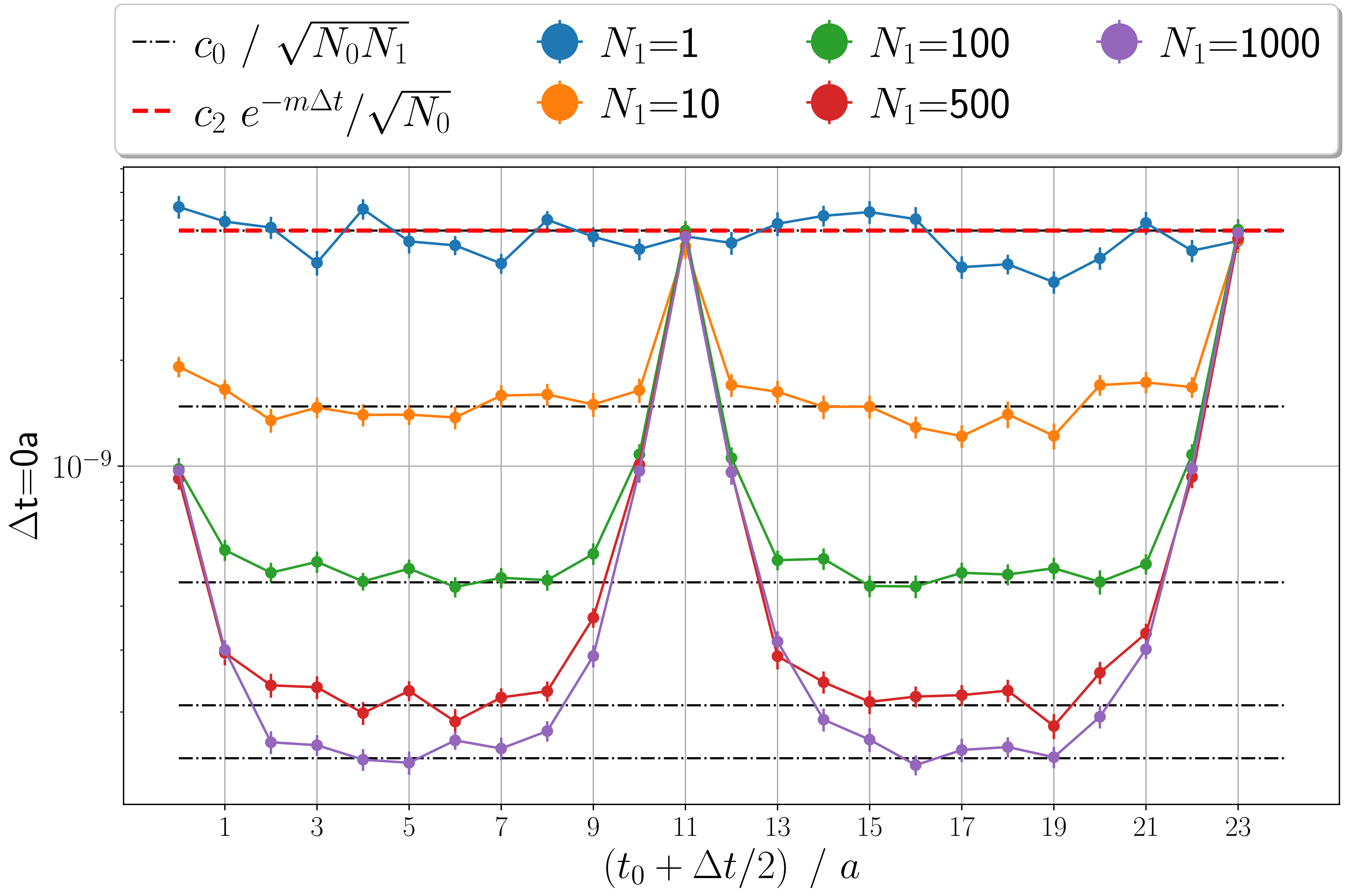
$D^{-1}(t_1, t_0)$ depends on the values of the gauge fields on the full lattice

Need both factorisation of $D^{-1}(t_1, t_0)$ and $\det(D)$ to use multi-level.

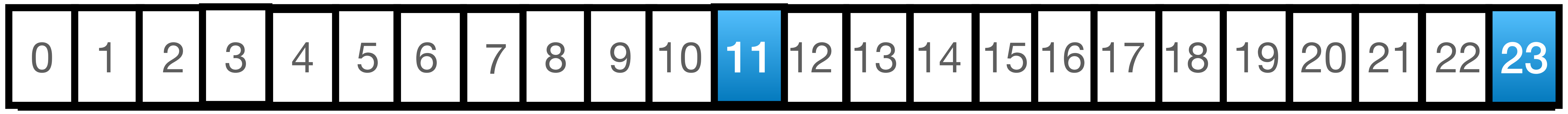
Multilevel error reduction at fixed Δt

$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 0a)$

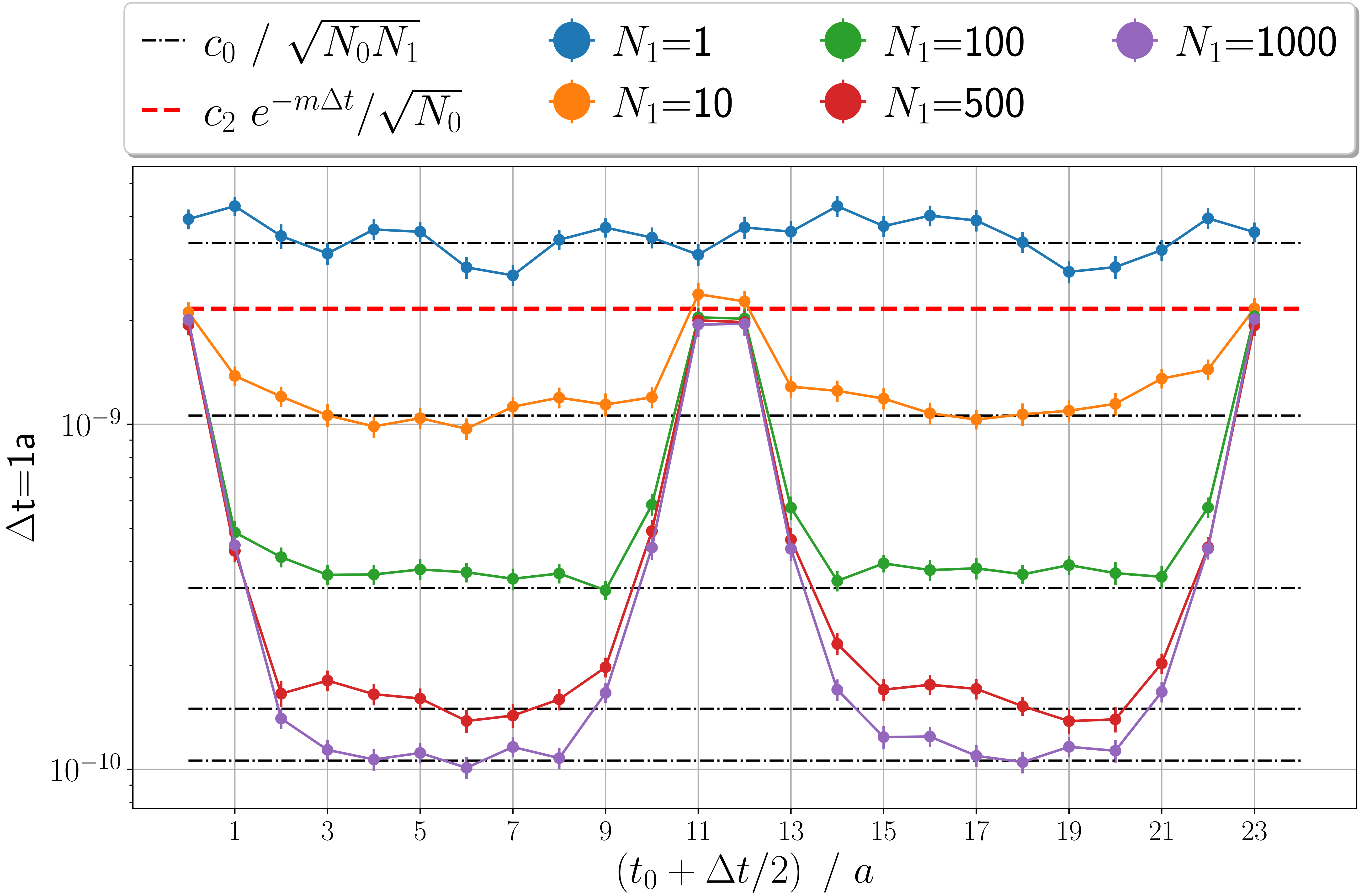


Multilevel error reduction at fixed Δt

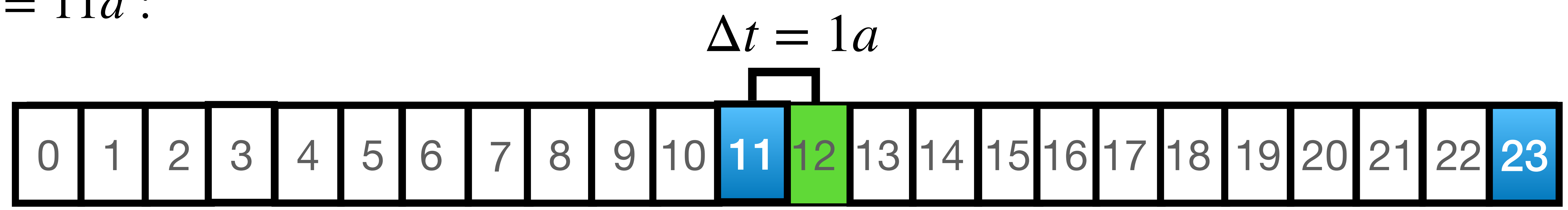
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$

Errors for $C(\Delta t = 1a)$



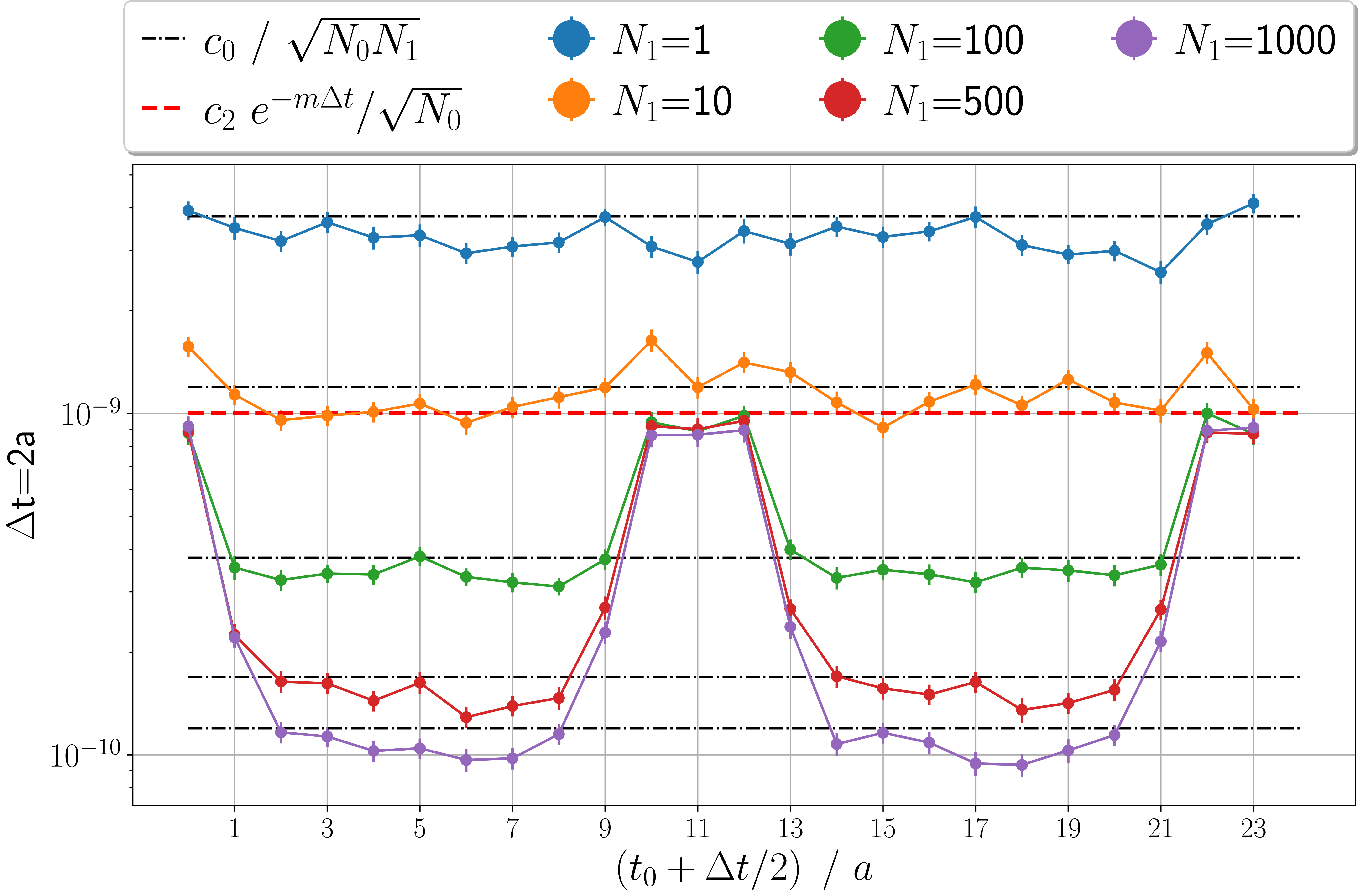
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$

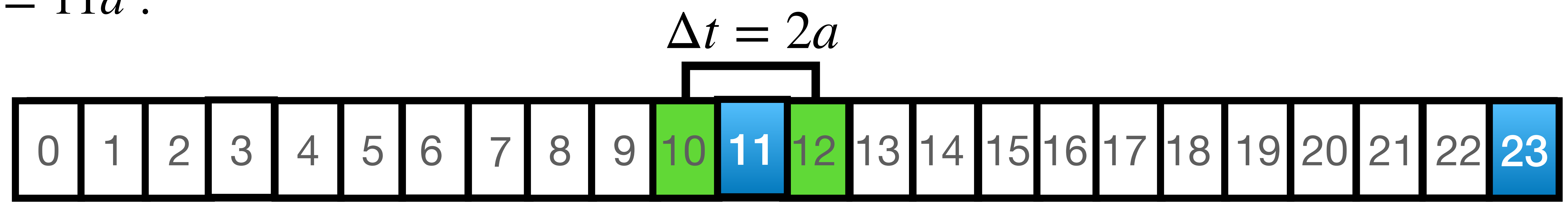


Errors for $C(\Delta t = 2a)$

Multilevel is not efficient at short distance!

Doing $N_1 > 100$ is just a wast of resources for multilevel measurements

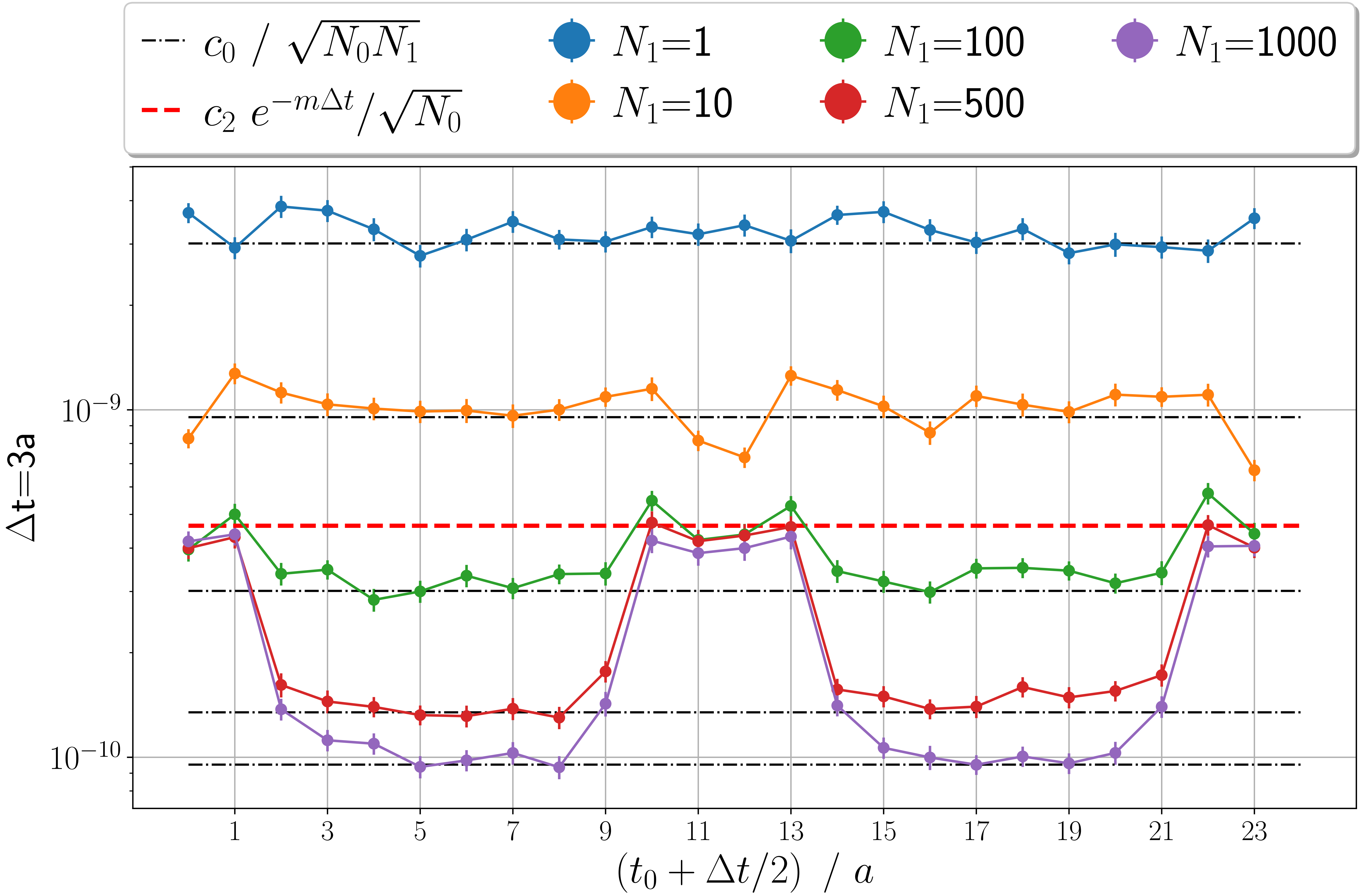
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

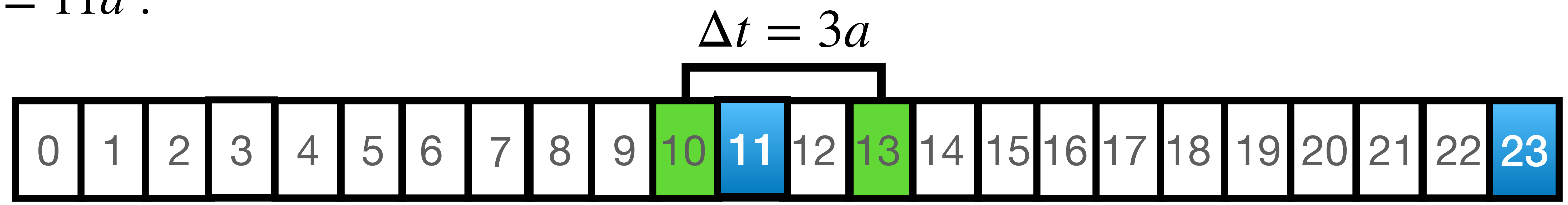
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 3a)$

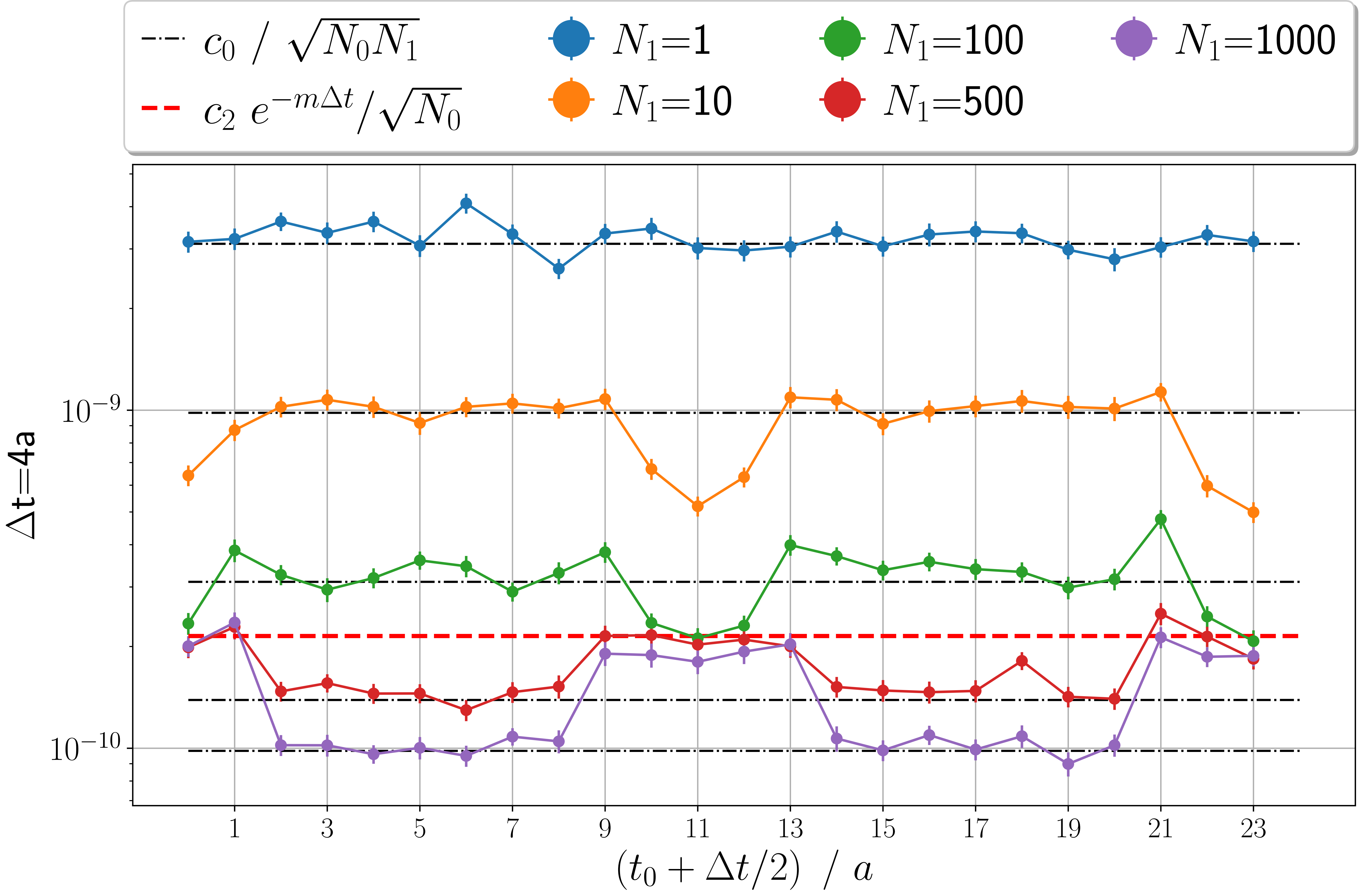
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

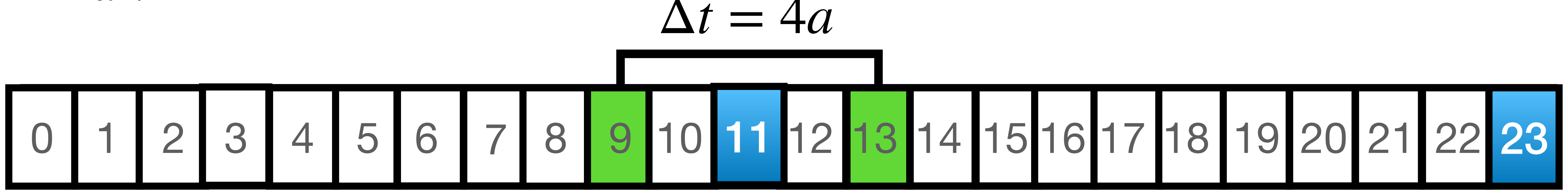
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 4a)$

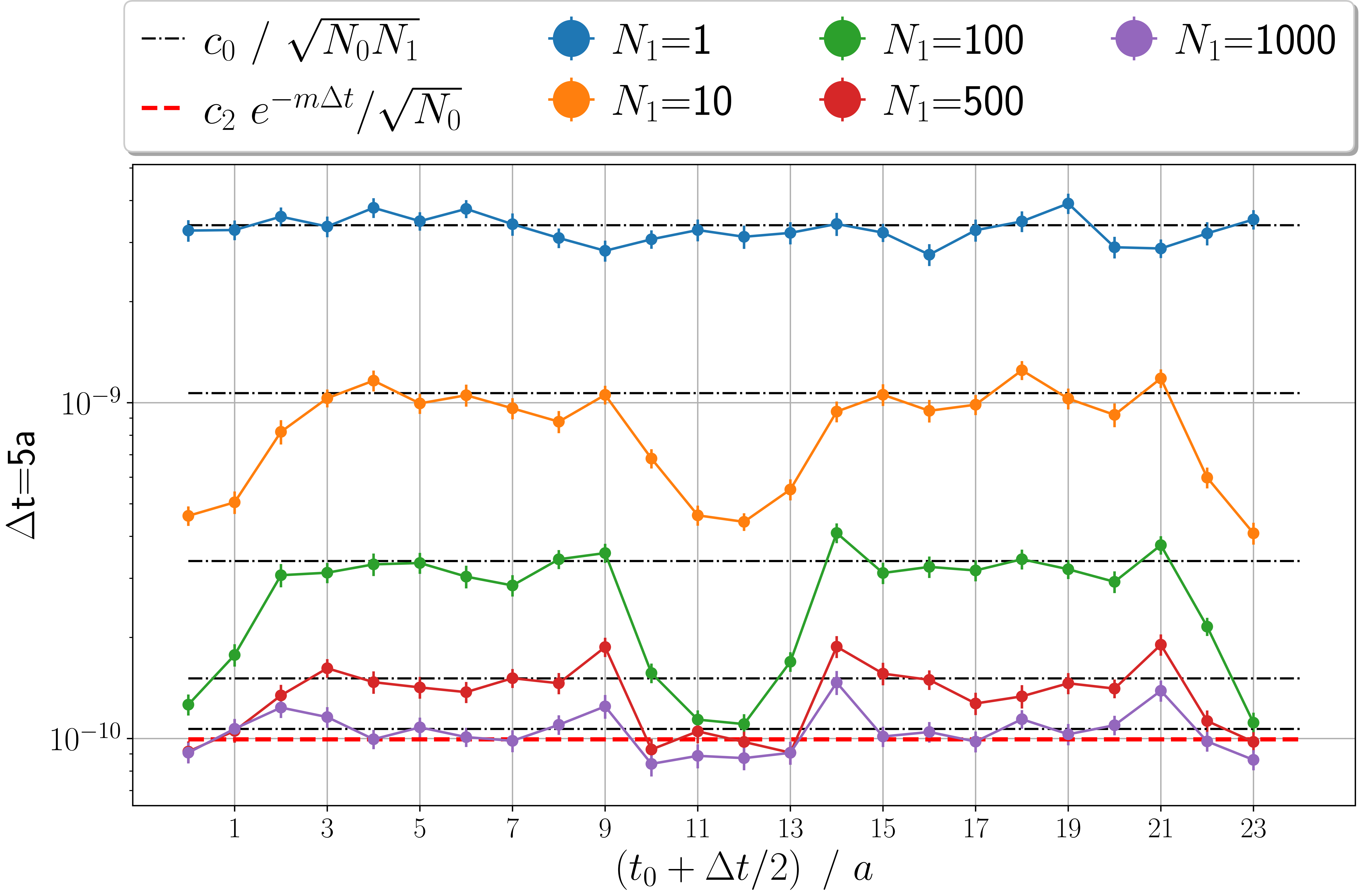
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

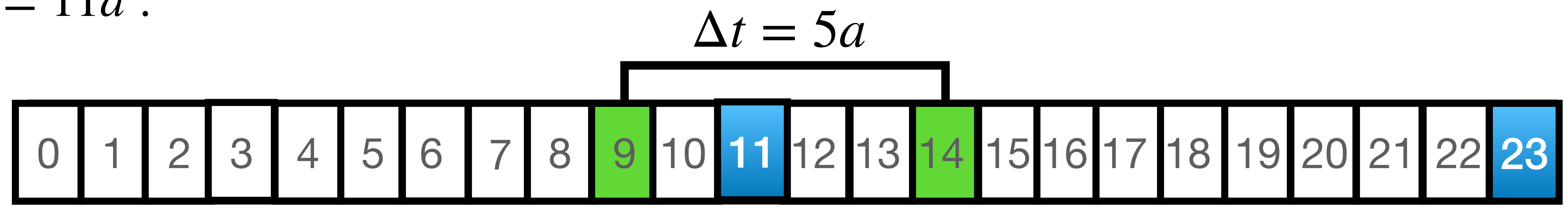
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 5a)$

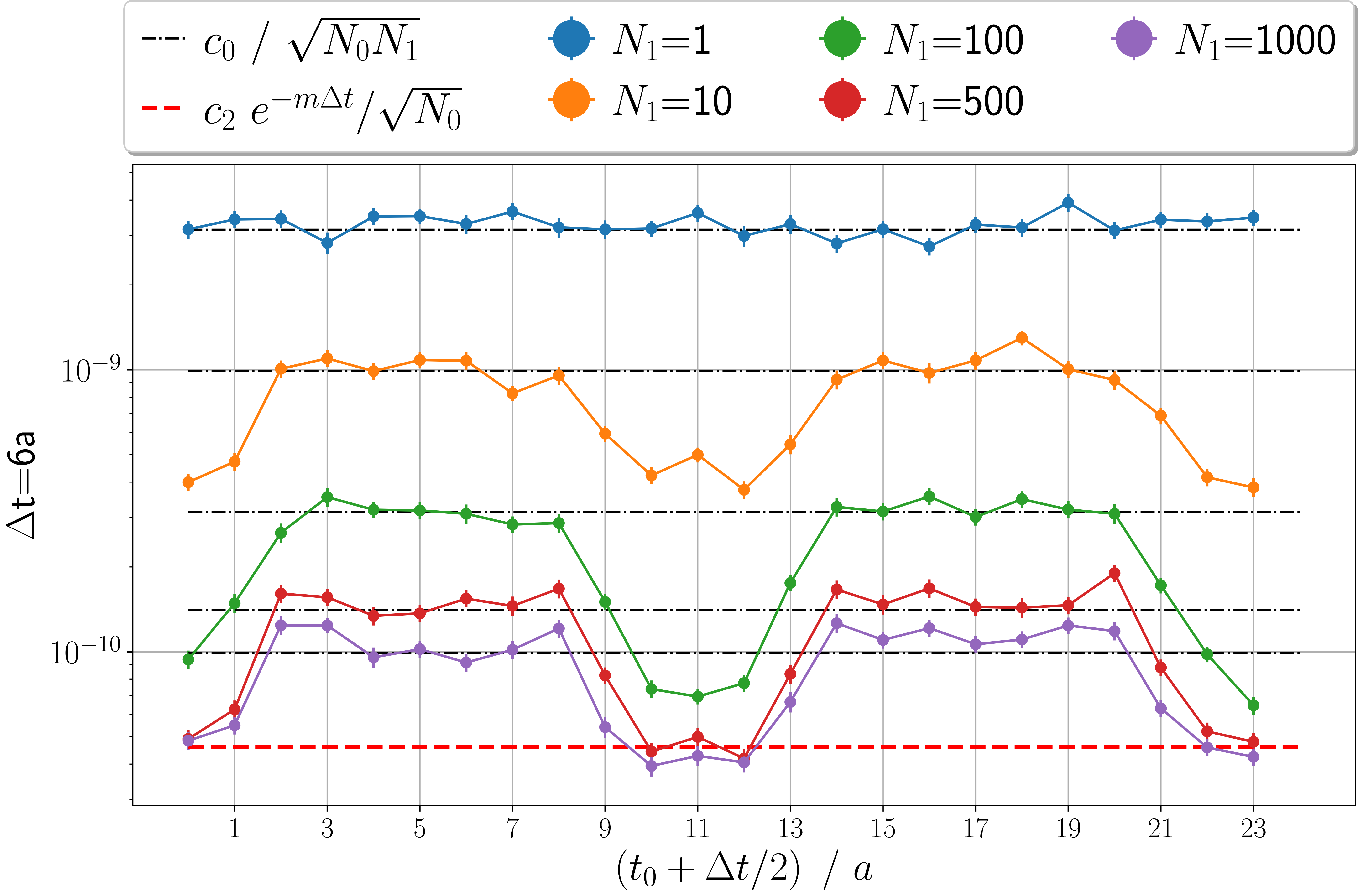
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

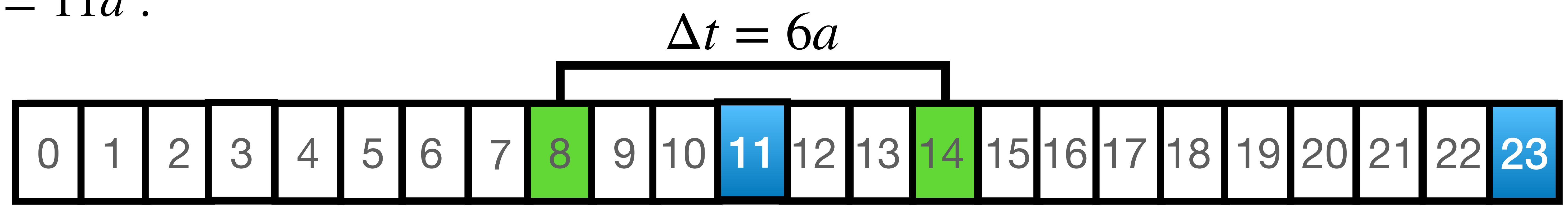
$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 6a)$

Multilevel starts to outperform standard scaling

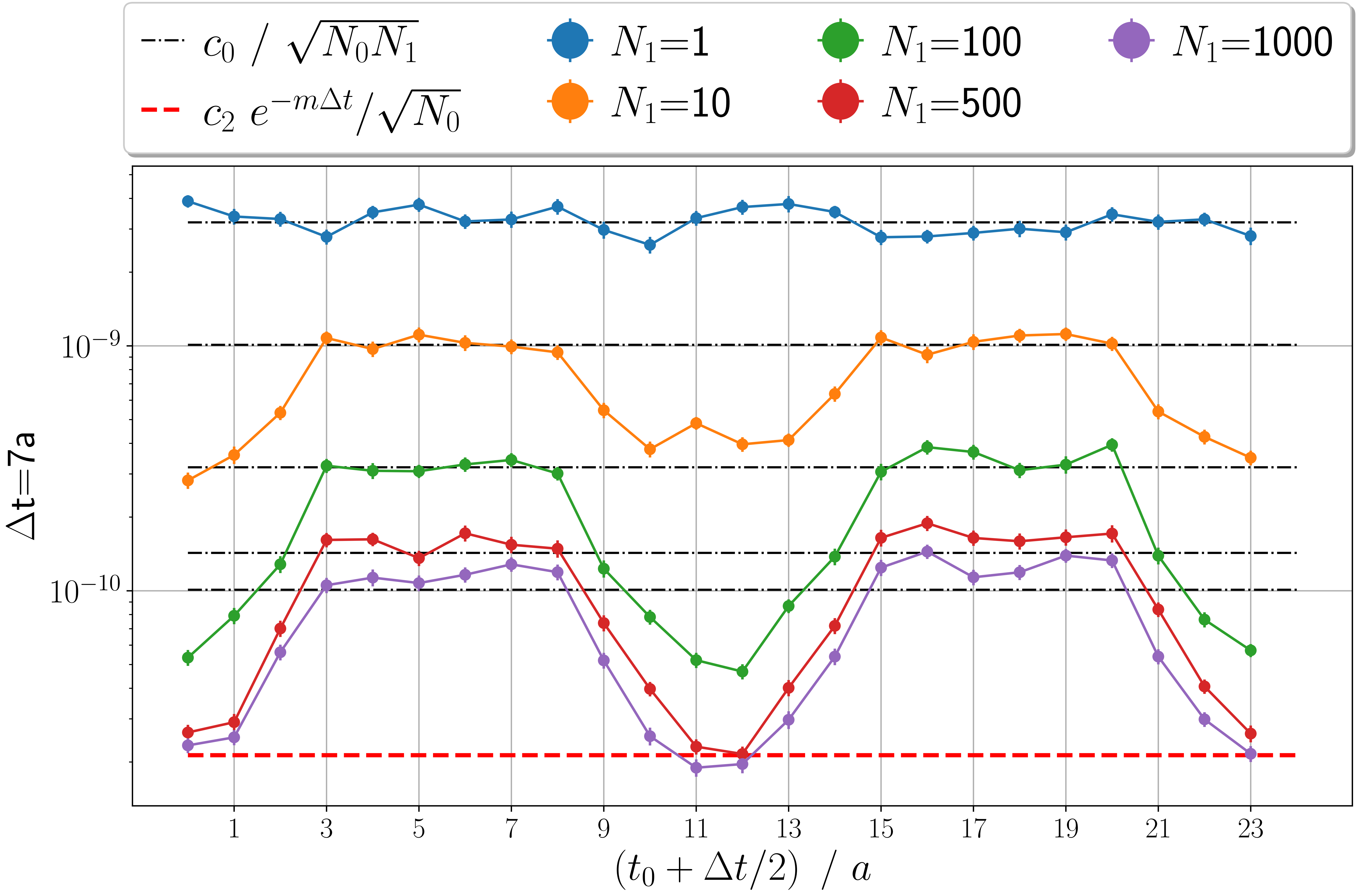
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

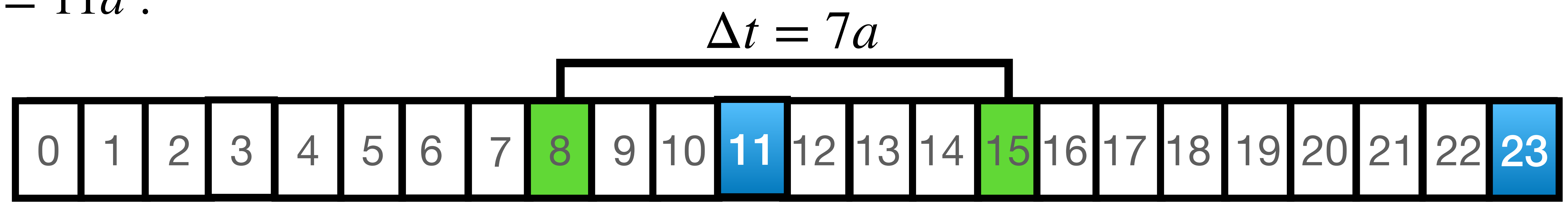
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 7a)$

$t_0 + \Delta t/2 = 11a$:

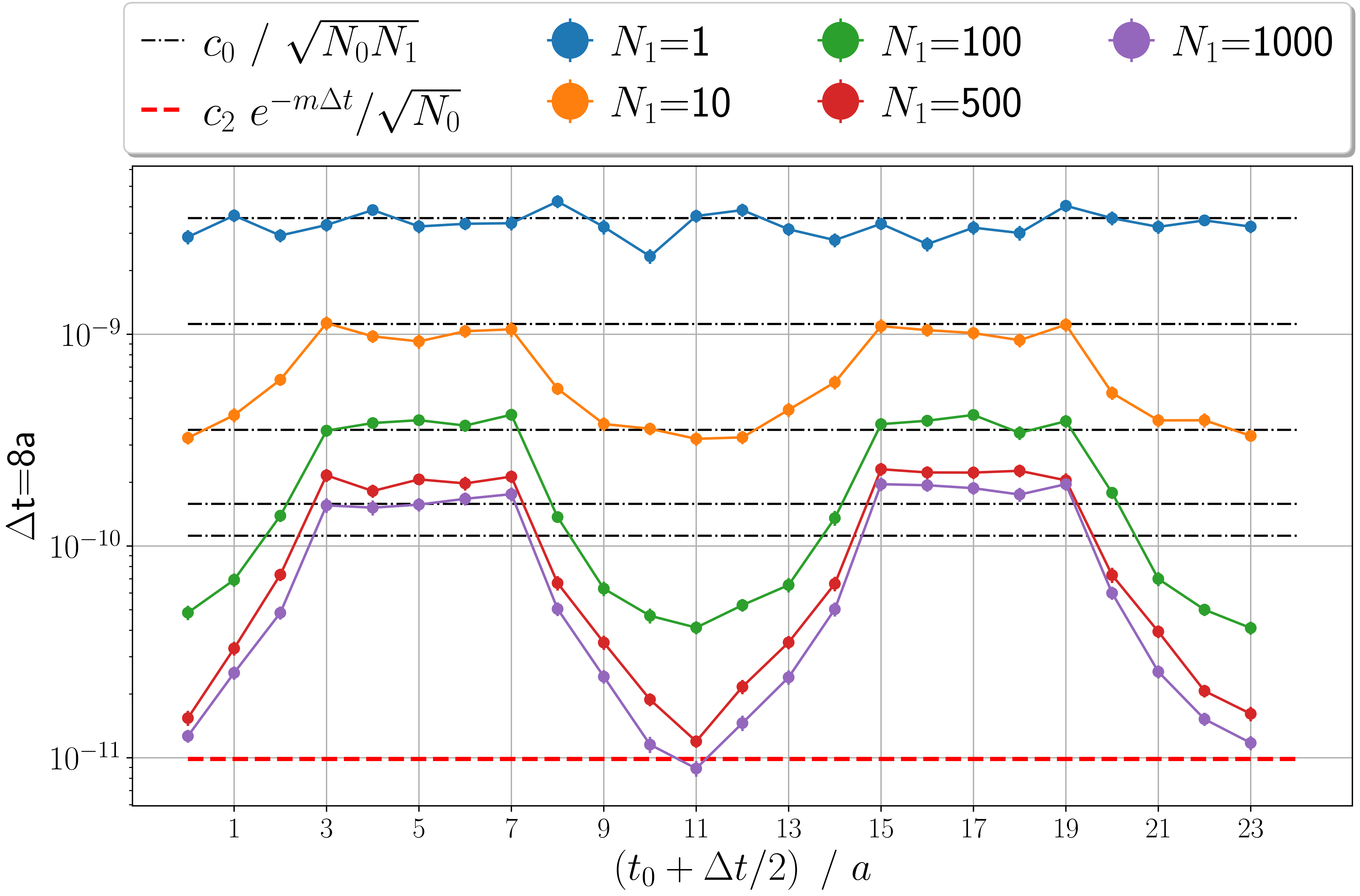


Multilevel error reduction at fixed Δt

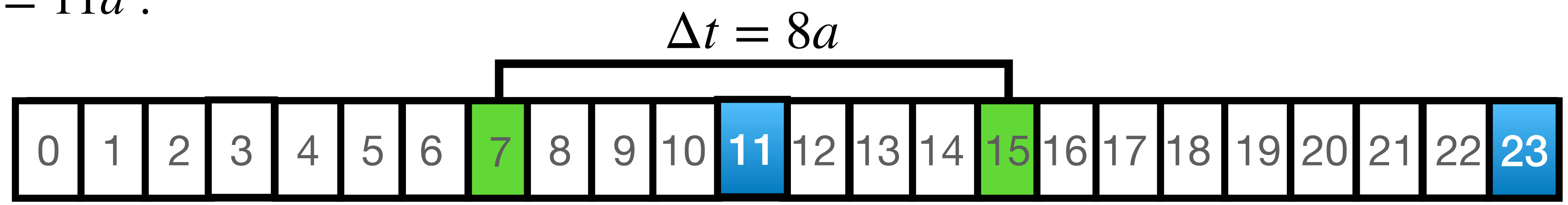
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$

Errors for $C(\Delta t = 8a)$



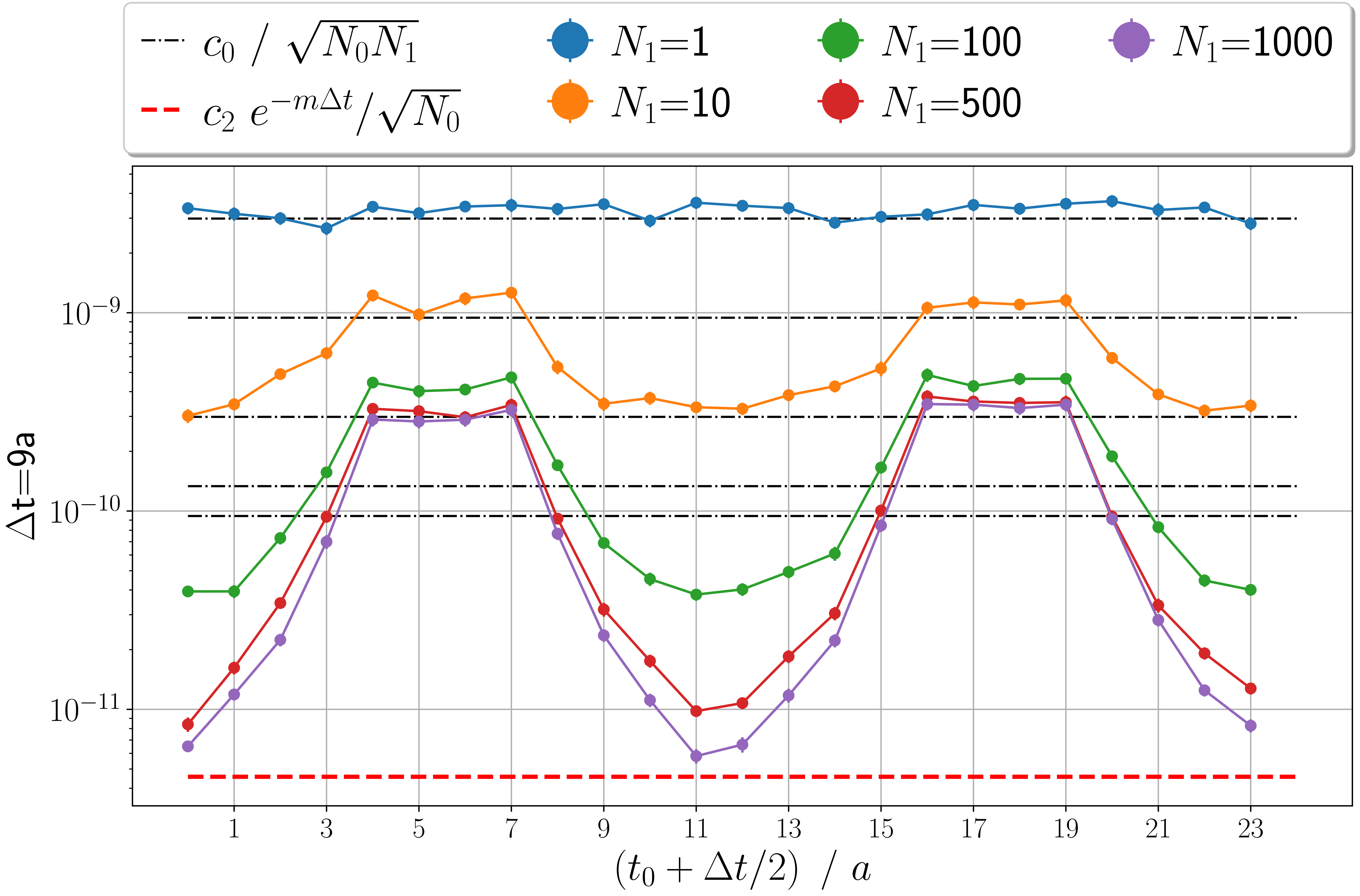
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

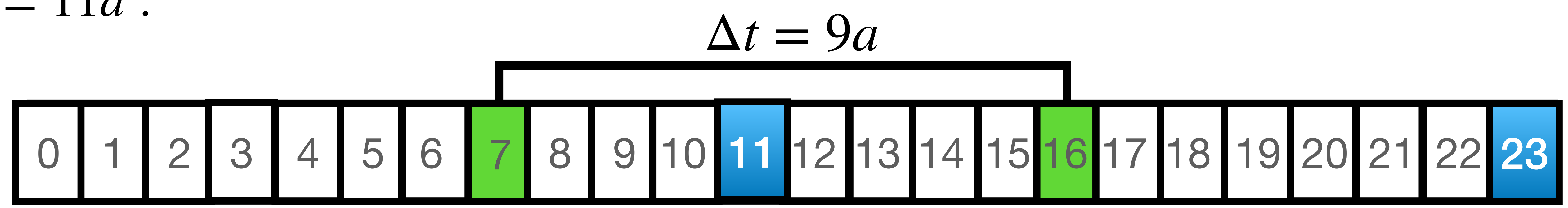
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 9a)$

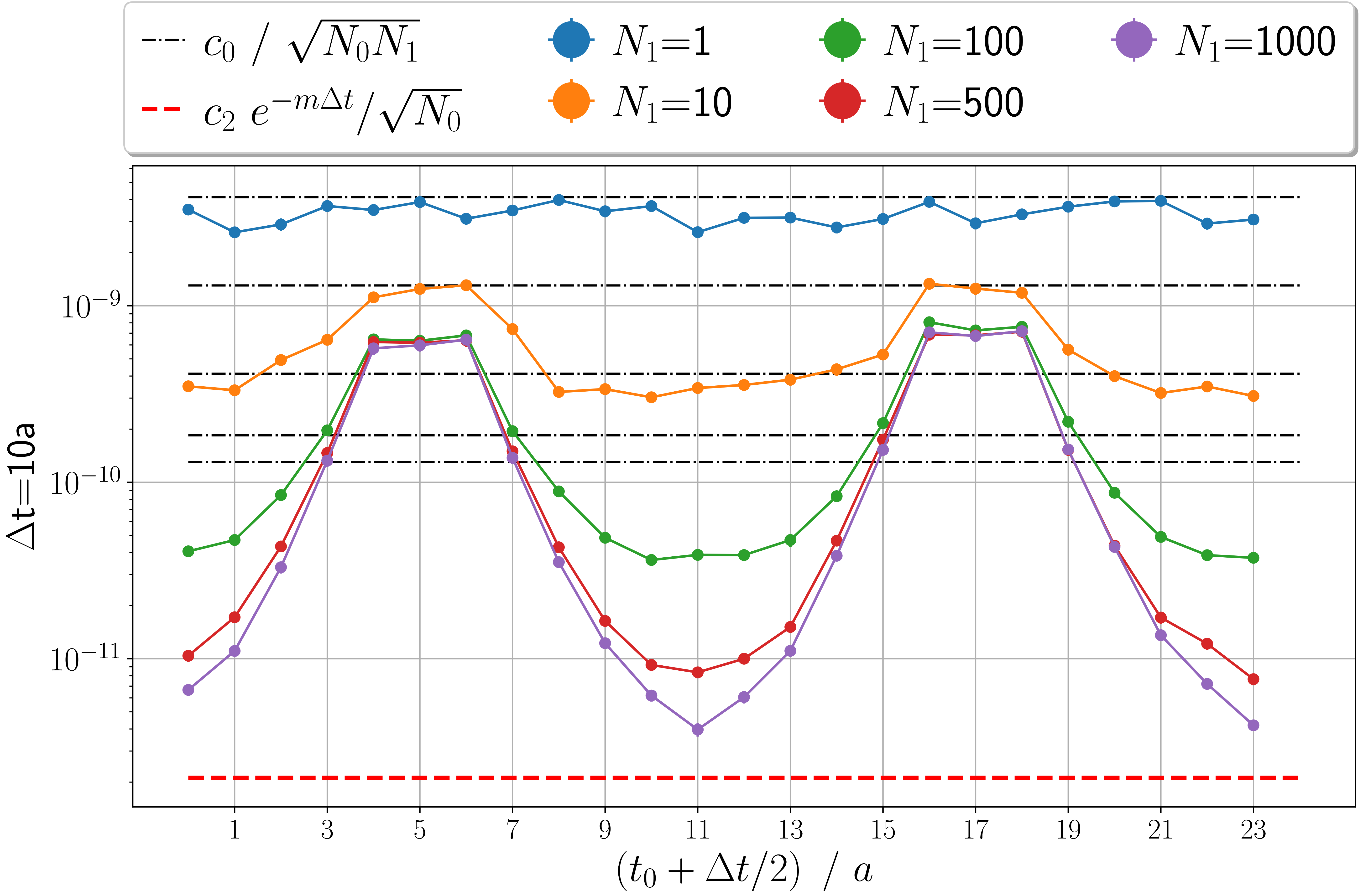
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

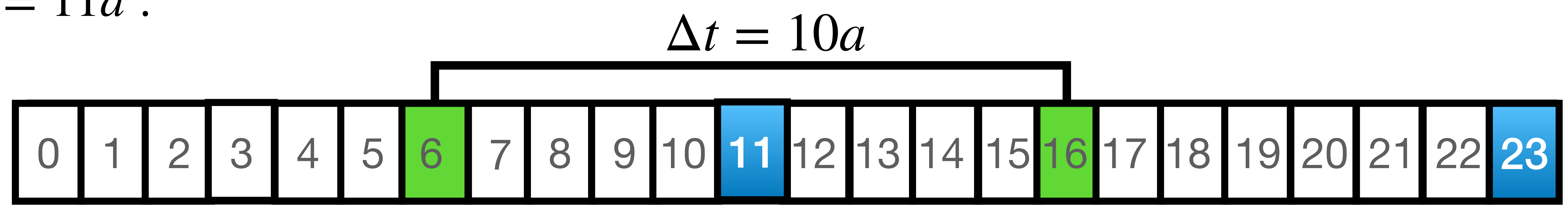
$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 10a)$

Multilevel error is reduced by 1000 with $N_1 = 1000$ for $t_0 + \Delta t/2 = 11a$

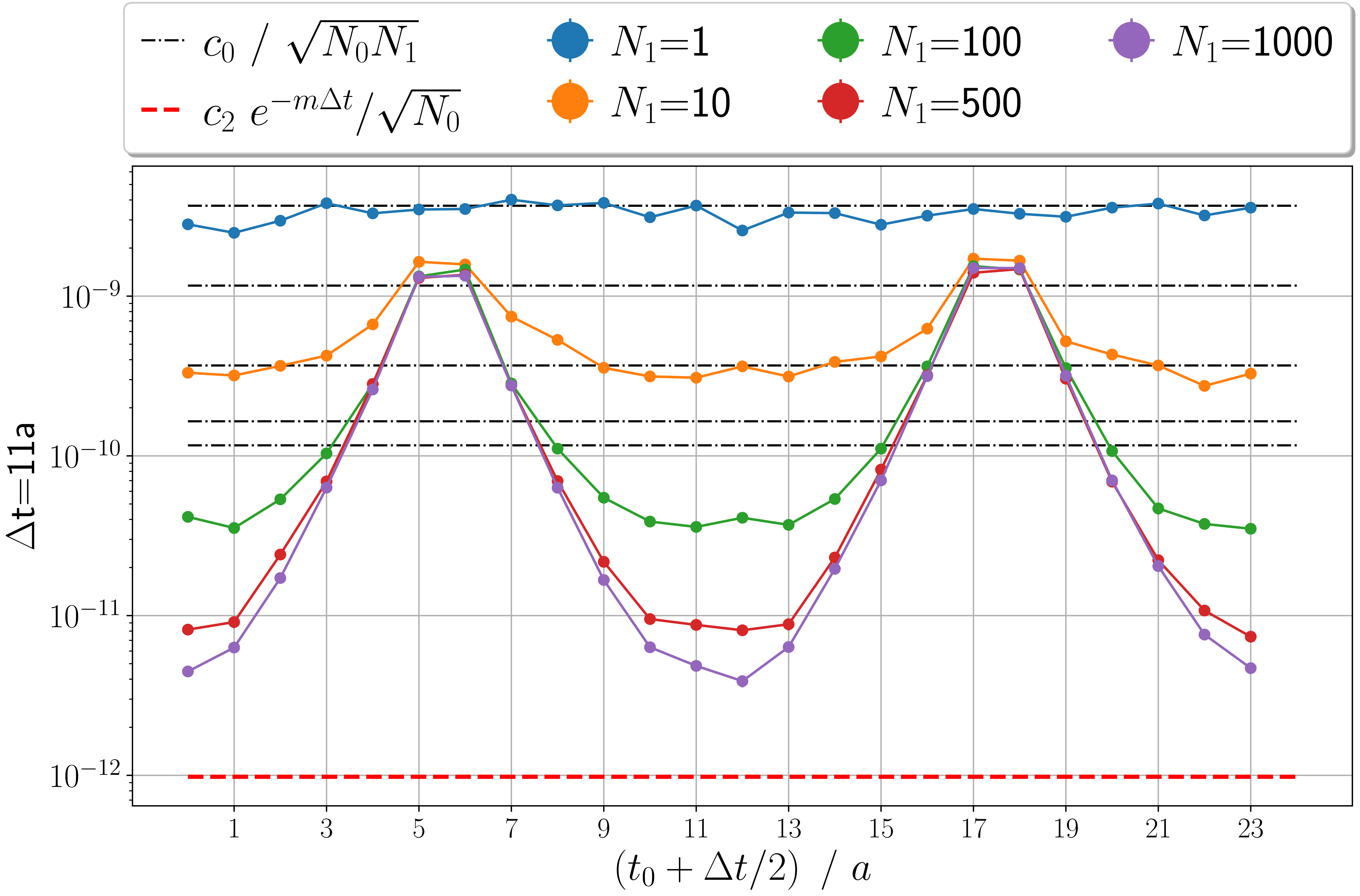
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

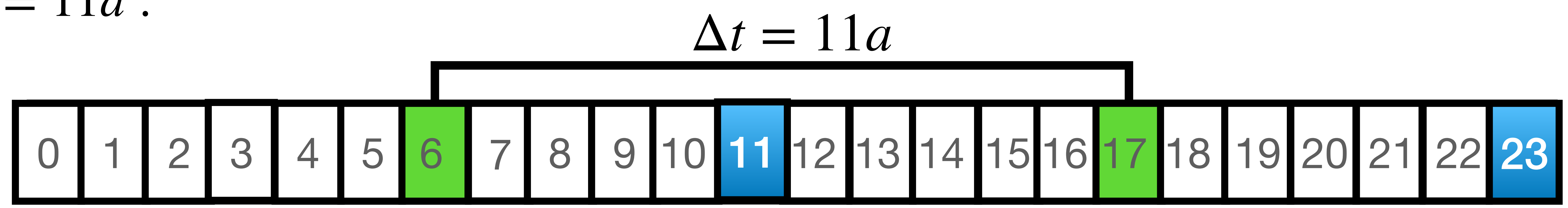
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 11a)$

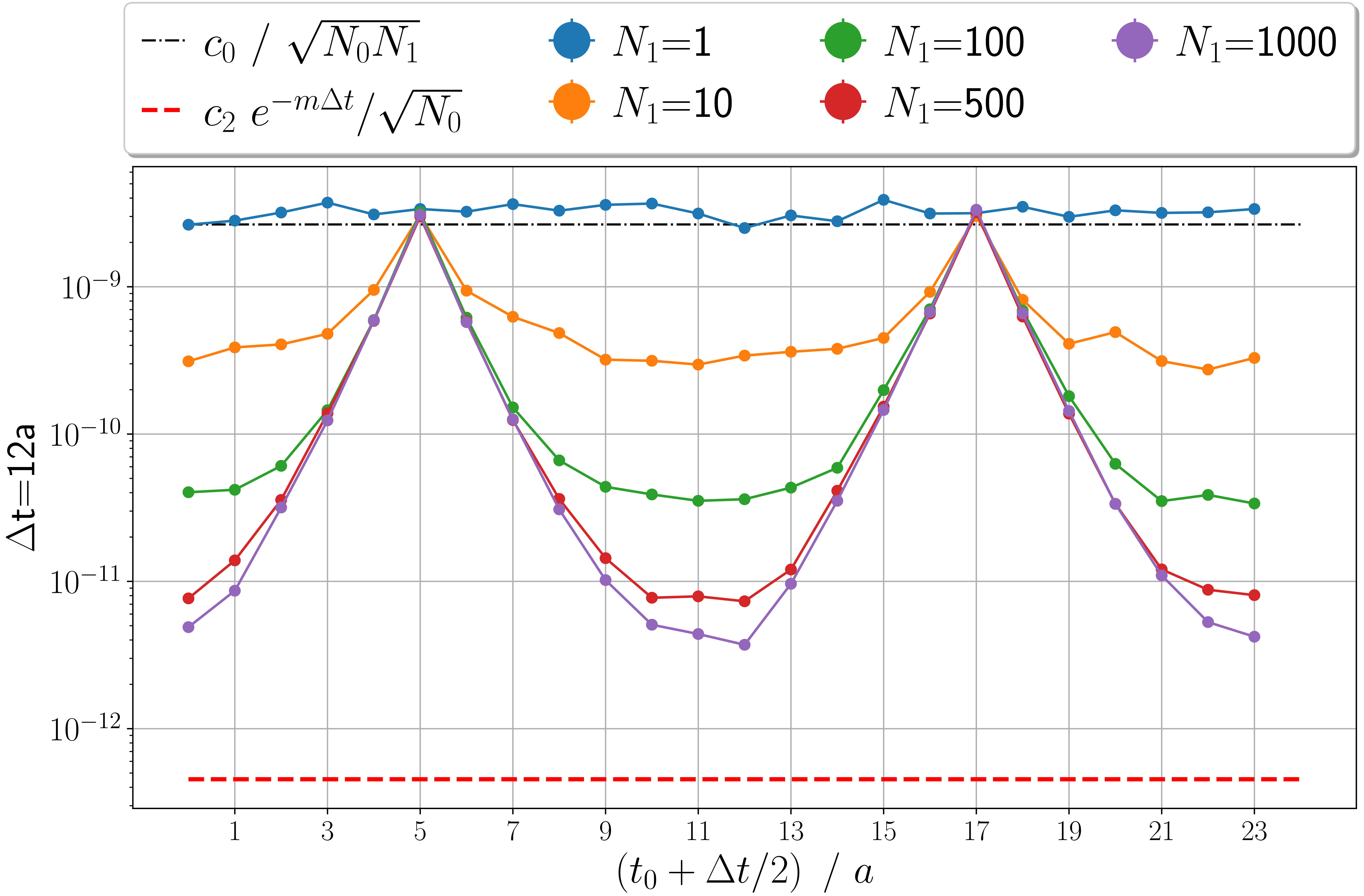
$t_0 + \Delta t/2 = 11a$:



Multilevel error reduction at fixed Δt

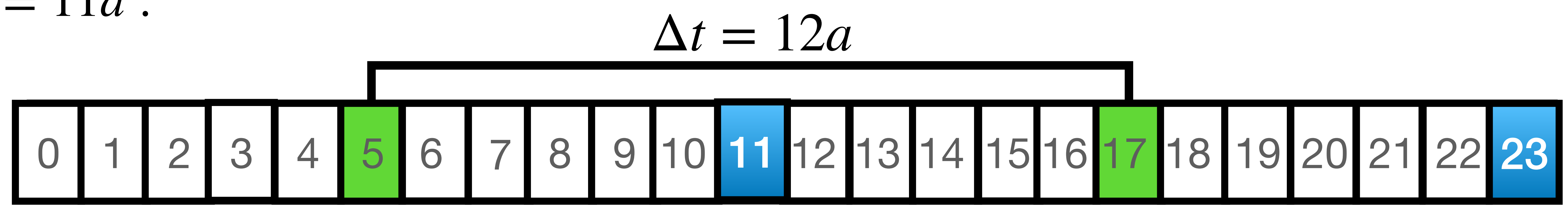
$N_0 = 101 \quad V/a^4 = 48 \times 24^3$

$\beta = 6.2,\ R^{PC} = E^{++}$



Errors for $C(\Delta t = 12a)$

$t_0 + \Delta t/2 = 11a$:



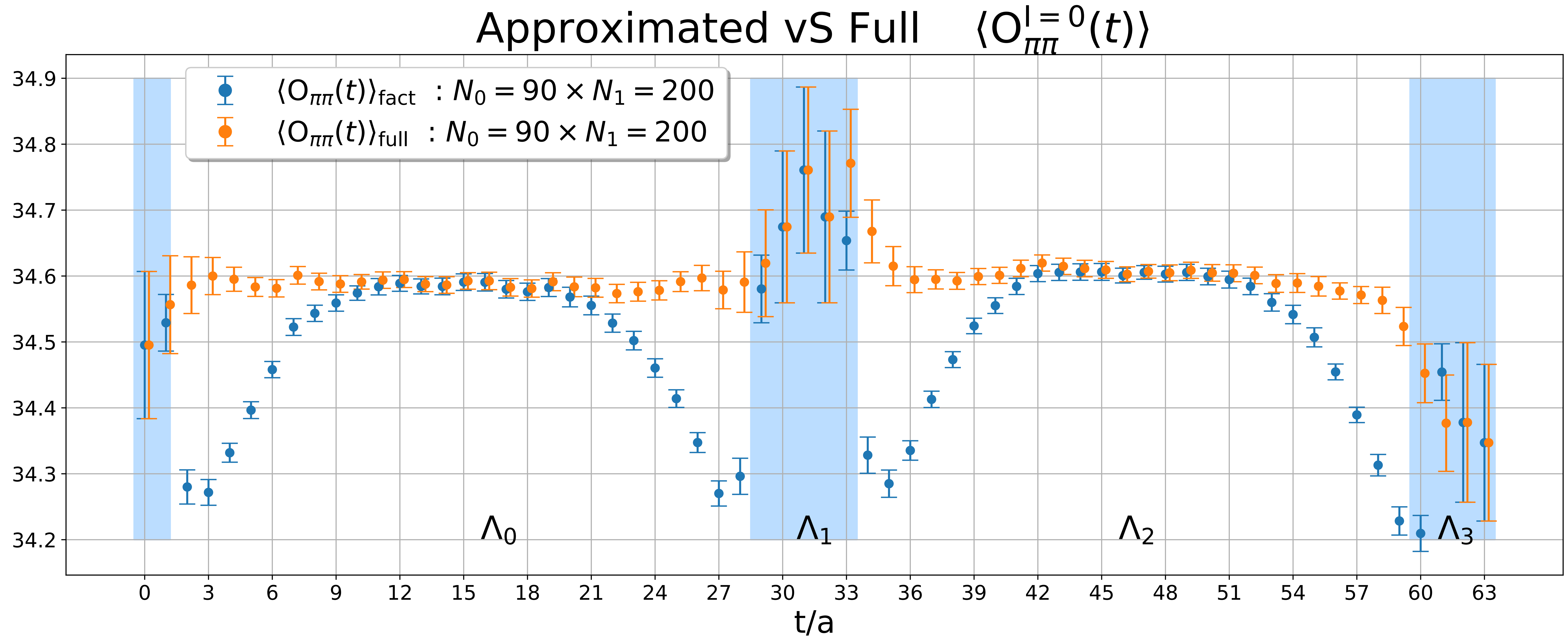
Notice

width δt of active regions

is better

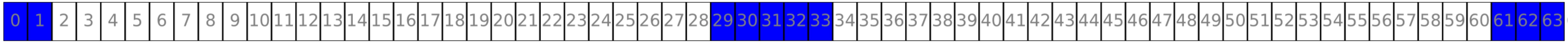
to reduce error at $\Delta t = \delta t - a$

How good is the approximation? 1-pts



Longer the distance from the other region, better the approximation

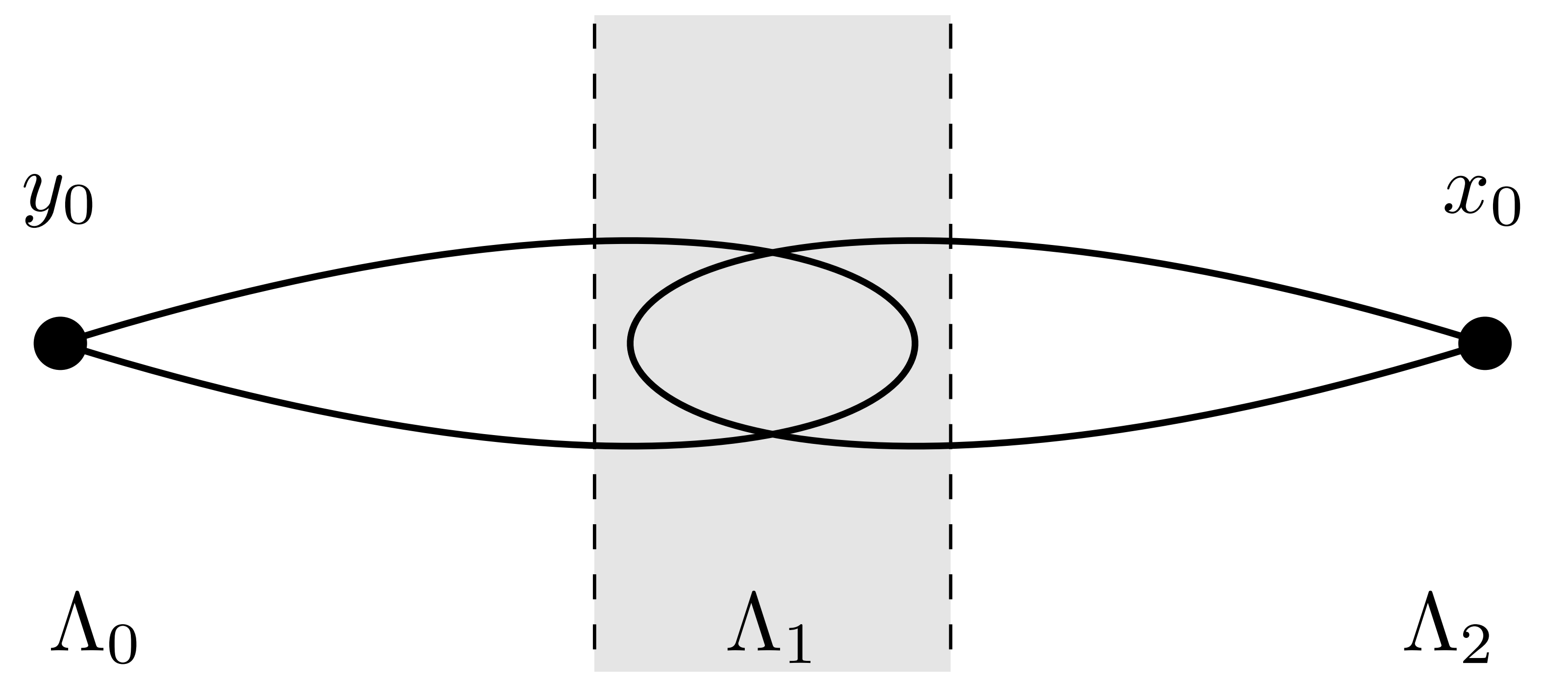
current sub-lattice decomposition



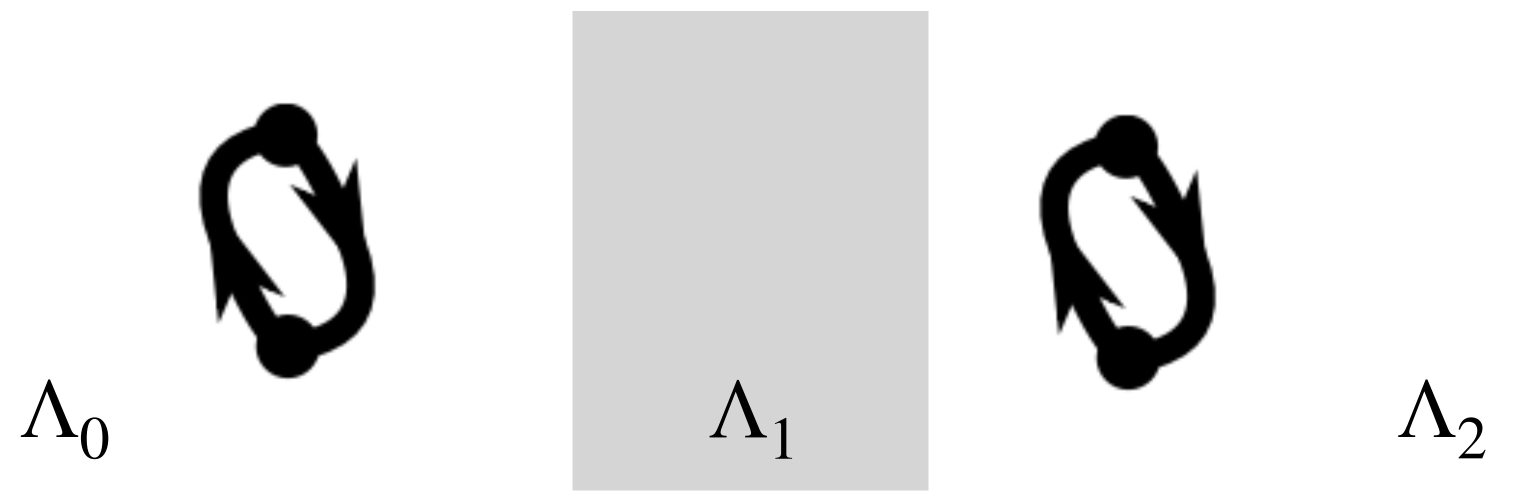
Disconnected diagrams with 2-level

$$C_{f_0}^{(0)}(y_0 - x_0) = \langle \sum_{\vec{y},\vec{x}} \left[\mathrm{Tr}\left(\mathrm{O}(y) \right) \right] \left[\mathrm{Tr}\left(\bar{\mathrm{O}}(x) \right) \right] \rangle$$

$$\mathrm{O}(y) = \mathrm{O}_{f_0}(y)$$



$$\mathrm{O}(y) = \mathrm{O}_{\pi\pi}(y)$$

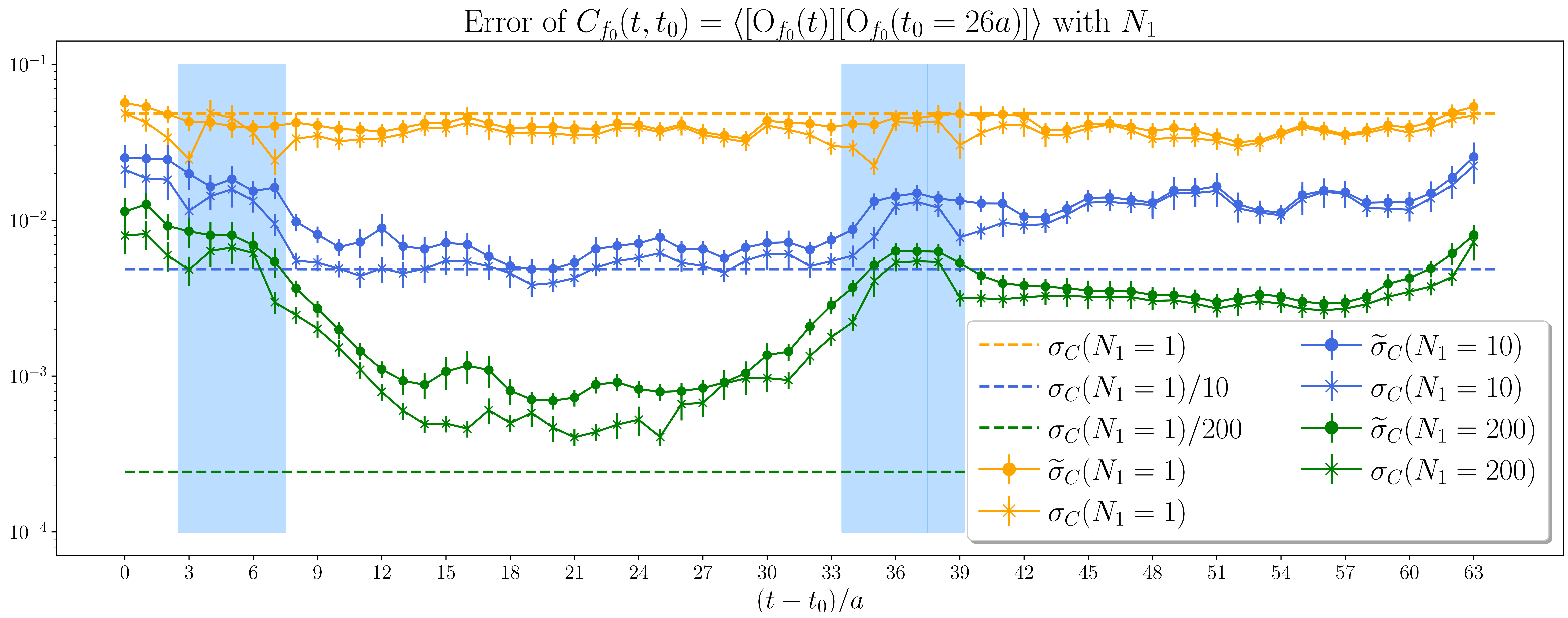


$\mathscr{V}$-diagram

Multilevel Error Reduction

$\sigma_C(t, t_0; N_1) =$ Error of $\langle [\mathrm{O}_{f_0}(t)]\ [\bar{\mathrm{O}}_{f_0}(t_0)] \rangle$

$\tilde{\sigma}_C(t, t_0; N_1)$ Error of corrected observable



Promiscuous topics on multilevel

@Lattice 2024 (Liverpool)

- Study of glueball scattering with a 3-level integration in pure gauge theory — M. Bruno, M. Hansen, A. Rago

- Compute glueball gravitational form factors with a standard integration — D. Hackett, F. Romero-Lopez, D. Pefkou, P. Shanahan

from glueball 3-pts $\langle \mathrm{G(t)}\ \mathrm{T}_{\mu\nu}(\tau)\ \mathrm{G(0)} \rangle$

Use a 3-level integration instead →

$$\langle [\mathrm{G(t)}]\ [\mathrm{T}_{\mu\nu}(\tau)]\ [\mathrm{G(0)}] \rangle$$

... [**LB**, S.Martins]

$$\sigma^2_{C_{3pt}}(t, \tau, t_0) \sim \frac{1}{N_0 N_1^3}$$

- Test multi-dimensional multilevel for pure gauge and quenched observables

$$C(x, y) = \langle [[[[\mathrm{O}(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \mathrm{y}_4)]]]]\ \ [[[[\mathrm{O}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4)]]]] \rangle$$

$$\sigma^2_C(t, t_0) \sim \frac{1}{N_0 (N_1^2)^4}\ ?$$