LUND
UNIVERSITY

New computational strategies for color structure and Lorentz structure

- Lorentz structure with chirality flow

  Thanks to: Joakim Alnefjord, Emil Boman, Andrew Lifson, Christian Reuschle, Simon Plätzer, Adam Warnerbring, Zenny Wettersten

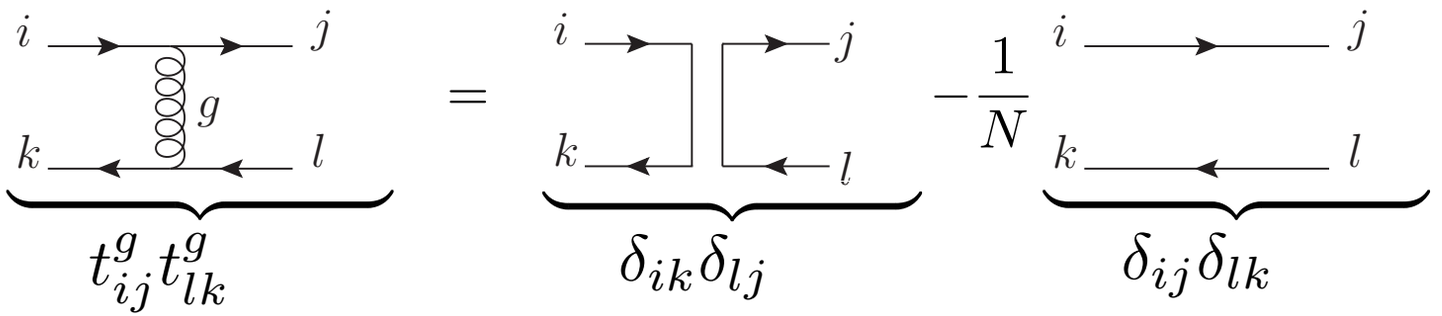
- Color structure in multiplet bases

  Thanks to: Stefan Keppeler, Simon Plätzer, Judith Alcock-Zeilinger

CERN
Feb 2025
Malin Sjödahl

In QCD we translate color to flows

- For each quark-gluon vertex a factor t^a_{ij}, an **su(3)** generator
- Fierz identity



SU($N = 3$) remove gluon indices

$\Rightarrow$ **Color amplitude translated to a flow**

Color flow Feynman rules F. Maltoni, K. Paul, T. Stelzer, S. Willenbrock, hep-ph/0209271 (PRD)

- The Dirac spinor structure transforms under the direct sum representation $\underbrace{(\frac{1}{2},0)}_{\text{left}}\oplus\underbrace{(0,\frac{1}{2})}_{\text{right}}$ in the chiral/Weyl basis

$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \to \begin{pmatrix} e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}+\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} & 0 \\ 0 & e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2}-\bar{\eta}\cdot\frac{\bar{\sigma}}{2}} \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

  i.e. actually two copies of **SL(2,$\mathbb{C}$)**, infinitesimally two **su(2)**
- Can also map a four-vector to a spinor outer product

$$p^{\dot{\alpha}\beta} = (p_\mu \sigma^\mu)^{\dot{\alpha}\beta} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}^{\dot{\alpha}\beta}$$

  We recover the transformation of a four-vector $\Lambda^{\dot{\alpha}}{}_{\dot{\beta}} \Lambda^{\alpha}{}_{\beta} p^{\dot{\beta}\beta}$,
- Lorentz group $\sim$ two copies of su(2), $so(3,1) \cong su(2) \oplus su(2)$
- Can we do something similar for the Lorentz structure?

- Consider spinors

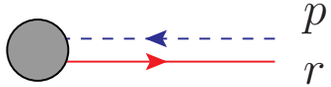
$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) \;\; = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

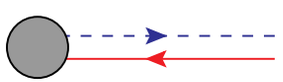
$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p| \,, & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 \,, & \langle p| \end{pmatrix}$$

and translate to left and right flows

$|j\rangle_\alpha =$, $\langle i|^\alpha =$

$|j]^{\dot\beta} =$, $[i|_{\dot\beta} =$

$$\epsilon_L{}^\mu(p,r) \to \frac{|r\rangle[p|}{\langle rp\rangle} = \frac{1}{\langle rp\rangle}$$ 

$$\epsilon_R{}^\mu(p,r) \to \frac{|r][p|}{[pr]} = \frac{1}{[pr]}$$ 

(Polarization vectors originally from Xu, Zang, Chang, Nucl.Phys.B 291 (1987) 392-428, originally 1984)

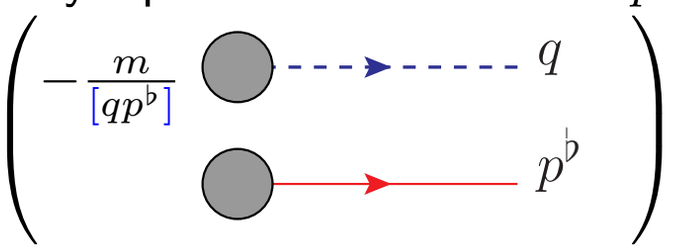
Massive momenta and spinors

- For massless momenta $p^\mu \sigma_\mu \equiv \not{p} = |p]\langle p|$, $p^\mu \bar{\sigma}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$
- Decompose massive momentum p as sum of massless

$$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu = |p^\flat]\langle p^\flat| + \alpha |q]\langle q|$$

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \ , \quad (p^\flat)^2 = q^2 = 0 \ , \quad \alpha = \frac{p^2}{2p\cdot q} = \frac{m^2}{2p\cdot q}$$

- Massive spinors and polarization vectors written in terms of massless Weyl spinors of momentum $p^\flat, q$,

$u^+(p) =$ 

, etc.

- q is arbitrary but physical, as it defines the spin direction s^μ

$$s^\mu = \frac{1}{m}(p^\mu - 2\alpha q^\mu) = \frac{1}{m}(p^\mu - \frac{m^2}{p\cdot q} q^\mu)$$

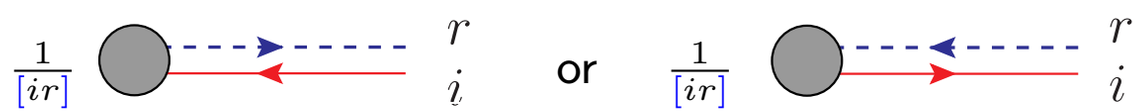
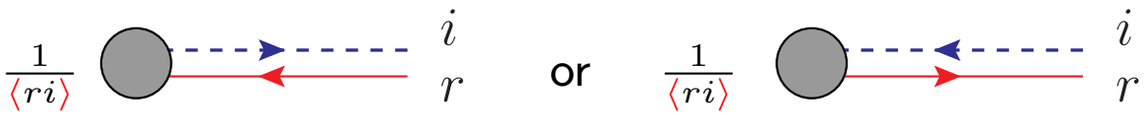
- Amplitudes have to be Lorentz invariant
- Lorentz inner products formed using **the only SL(2,$\mathbb{C}$) invariant object** $\epsilon^{\alpha\beta}$, $\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12}$

$$\underbrace{\epsilon^{\alpha\beta}|i\rangle_\beta}_{\equiv\langle i|^\alpha}|j\rangle_\alpha = \langle i|^\alpha|j\rangle_\alpha = \langle ij\rangle = i \longrightarrow j$$

$$\underbrace{\epsilon_{\dot\alpha\dot\beta}|i]^{\dot\beta}}_{\equiv[i|_{\dot\alpha}}|j]^{\dot\alpha} = [i|_{\dot\alpha}|j]^{\dot\alpha} = [ij] = i \dashrightarrow j \quad ,$$

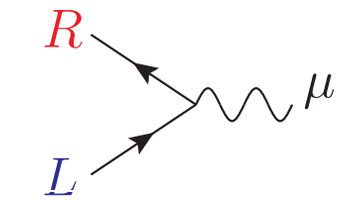
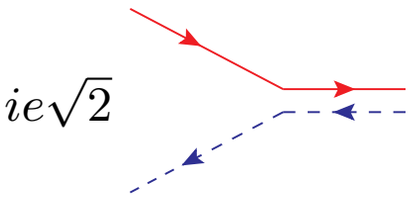
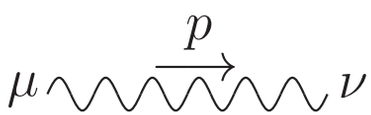
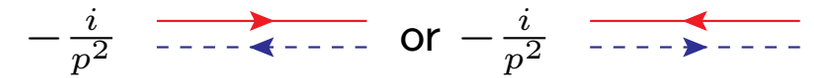
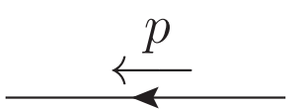
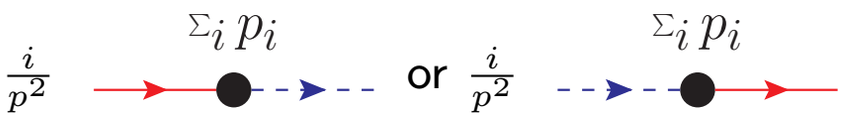
- $\Longrightarrow$ Amplitudes are built up of contractions of form $\langle ij\rangle, [ij] \sim \sqrt{s_{ij}}$
- If we manage to create a flow picture, the "flow" must contract left and right indices separately

The QED flow rules: massless particles

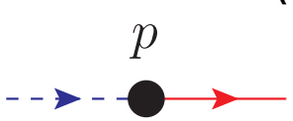
Species	Feynman	Flow
$\bar{u}^R(p_i)$		
$v^R(p_j)$		
$v^L(p_j)$		
$\bar{u}^L(p_i)$		
$\epsilon_R{}^\mu(p_i, r)$		
$\epsilon_L{}^\mu(p_i, r)$		

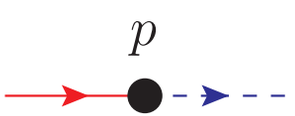
Use L and R chiral spinors $\underbrace{su(2)}_{\text{dotted}}$, $\underbrace{su(2)}_{\text{undotted}}$ for incoming $-/+$ or outgoing $+/-$ helicity (A. Lifson, C. Reuschle and MS, 2003.05877 (EPJC))

Vertices and propagators

Feynman	Flow
	
	
	

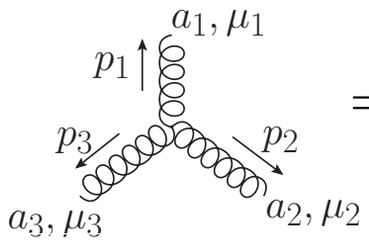
Fermion, split $\not{p}_{4d} \equiv p_\mu \gamma^\mu = p_\mu \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ into two terms

$\not{p} \equiv p^\mu \sigma_\mu^{\dot{\alpha}\beta} =$ 

$\bar{\not{p}} \equiv p_\mu \bar{\sigma}^\mu_{\alpha\dot{\beta}} =$ 

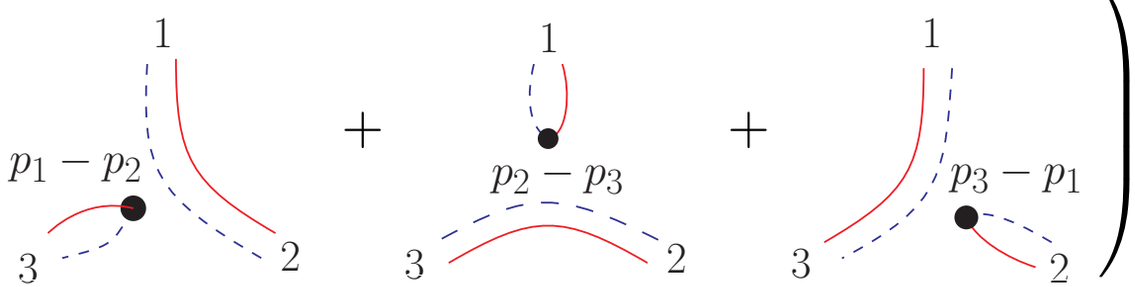
Stitch together such that arrow directions match

The non-abelian QCD vertices

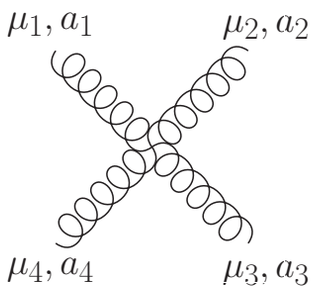


$$= i\frac{g_s}{\sqrt{2}} i f^{a_1 a_2 a_3} \left(g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2 - p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3 - p_1)^{\mu_2} \right)$$

$$\rightarrow i\frac{g_s}{\sqrt{2}} i f^{a_1 a_2 a_3} \frac{1}{\sqrt{2}} \left(\right. \quad + \quad + \quad \left. \right) ,$$

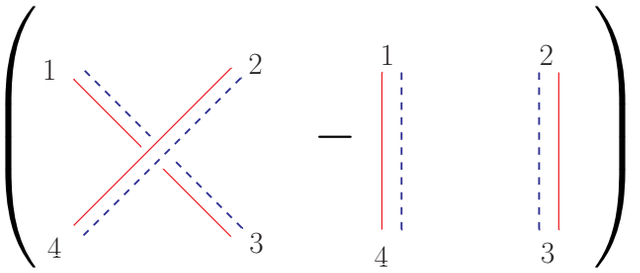


Here f^{abc} is the color factor in the vertex, the SU(3) structure constants (A.Lifson, C.Reuschle and M. Sjodahl 2011.10075 (EPJC))



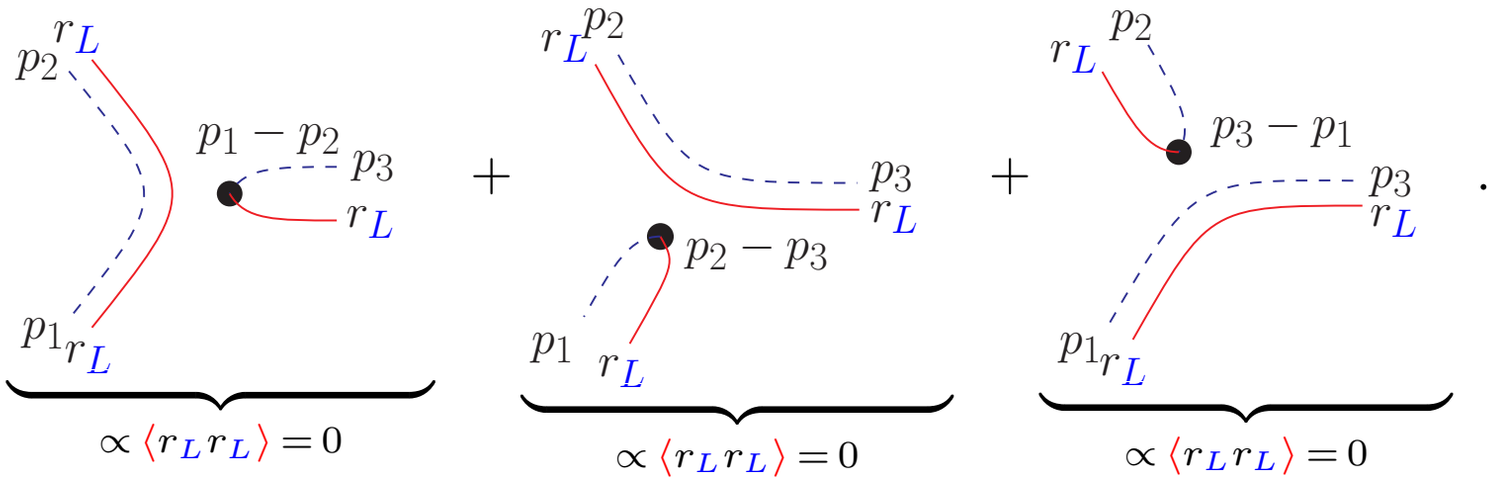
$$= \quad i\left(\frac{g_s}{\sqrt{2}}\right)^2 \sum_{Z(2,3,4)} if^{a_1 a_2 b} if^{b a_3 a_4} \left(g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}\right)$$

$$\rightarrow \quad i\left(\frac{g_s}{\sqrt{2}}\right)^2 \sum_{Z(2,3,4)} if^{a_1 a_2 b} if^{b a_3 a_4}$$

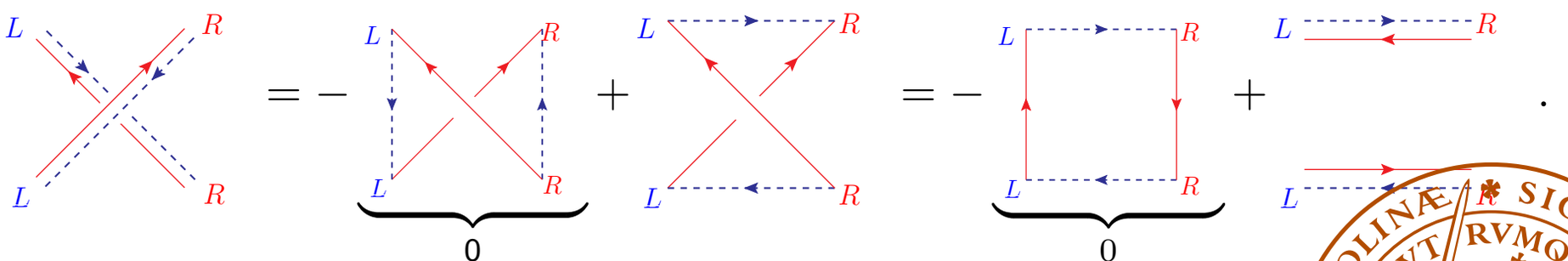


where $Z(2,3,4)$ denotes the set of cyclic permutations of the integers $2, 3,$ and 4,

The gluon vertices can be simplified due to a nice gauge choice, for example, we can not have three left gluons

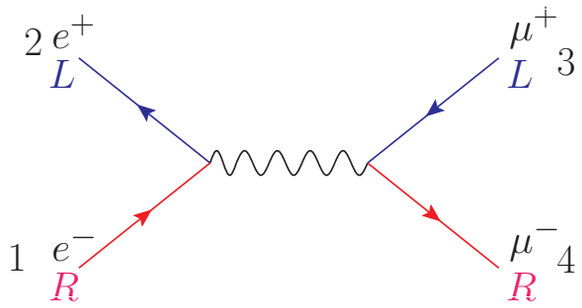


The four-gluon vertex simplifies further due to the Schouten identity

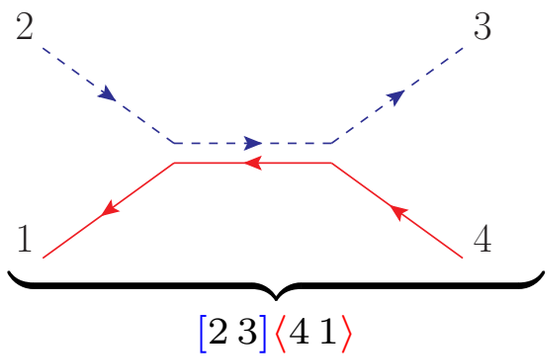


Simplest QED example

- Chirality flow, super easy and intuitive

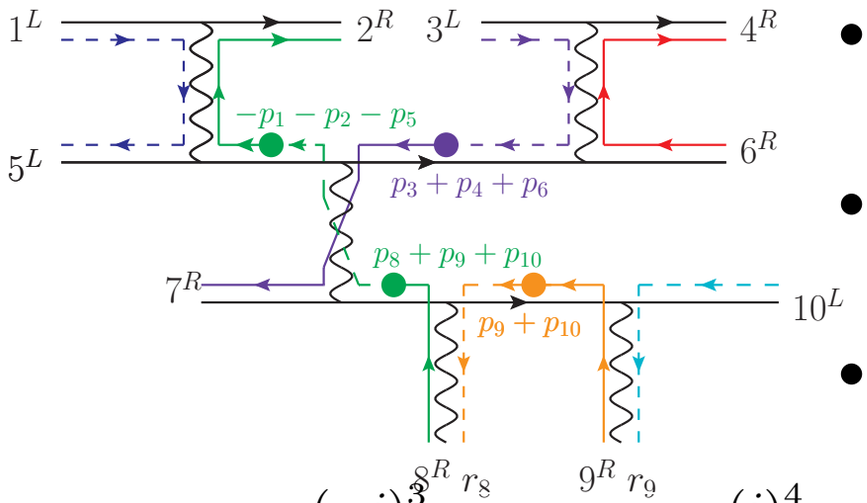


$$= \frac{2ie^2}{s_{e^+e^-}}$$



$$[2\,3]\langle 4\,1\rangle$$

A complicated QED example



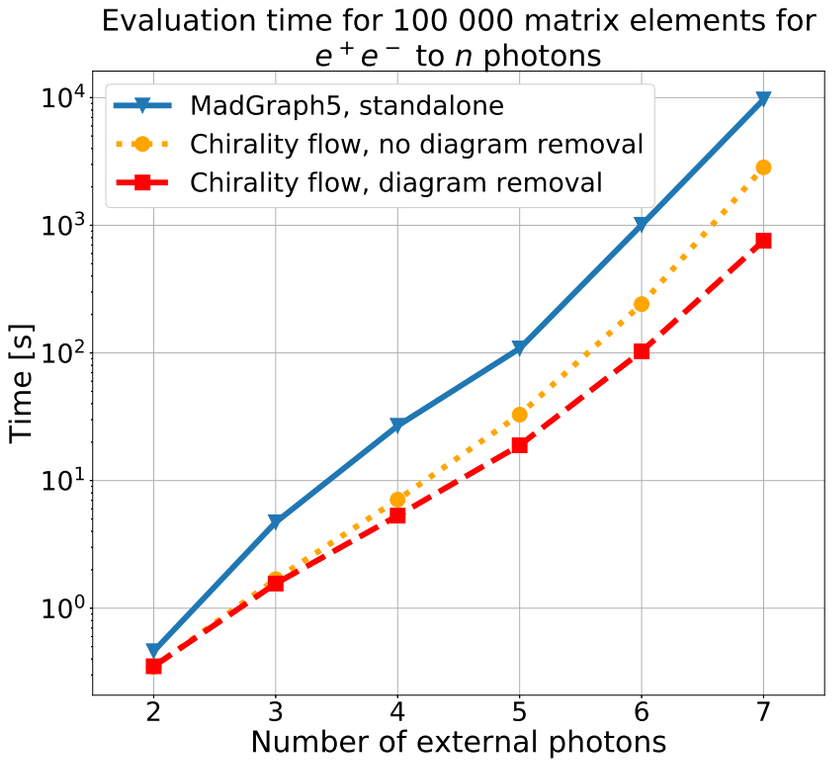
- Pick any consistent arrow direction
- Compare to standard QFT: 12 γ^μ matrices
- Here all particles crossed to outgoing

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\ s_{3\,4}\ s_{8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\ s_{3\,4\,6}\ s_{8\,9\,10}\ s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8][9r_9]}}_{\text{polarization vectors}} [15]\langle 64\rangle[10\ r_9]$$

$$\times \left(\underbrace{\langle 9\,9\rangle}_{0}[9r_8] + \langle 910\rangle[10r_8] \right) \left(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle \right)$$

$$\times \left(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\ 1]\langle 12\rangle - \langle 8\,10\rangle[10\ 5]\langle 52\rangle \right)$$

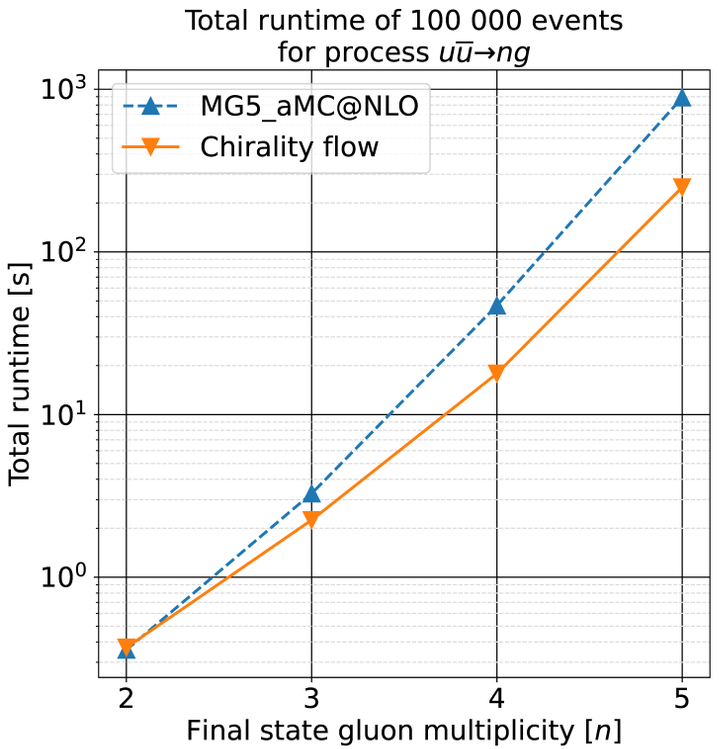
Implementation, massless QED



Evaluation time in `MadGraph5_aMC@NLO`, $e^+e^- \to n$ photons

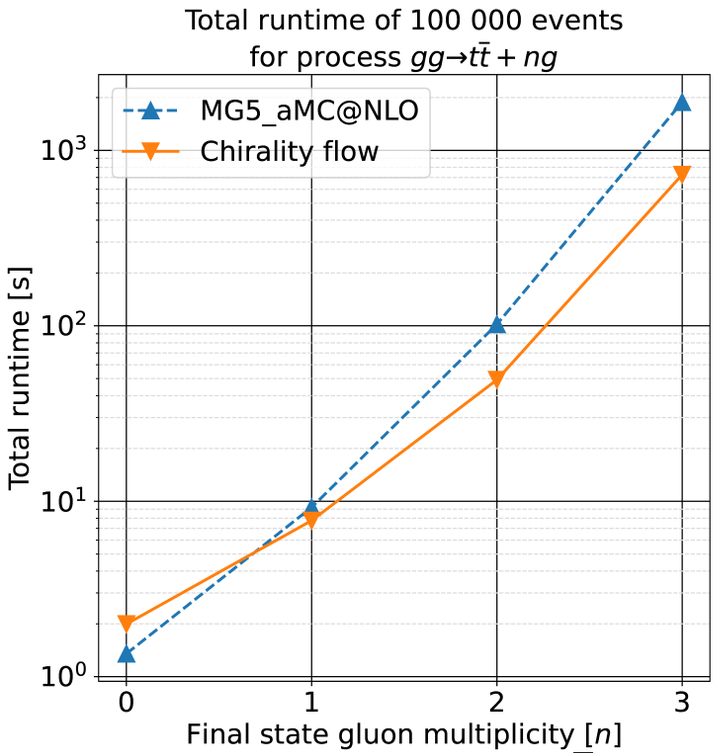
(A. Lifson, M. Sjodahl and Z. Wettersten 2203.13618 (EPJC))

Implementation, massless QCD



Runtime for the Lorentz structure of $u\bar{u} \to ng$ for the standalone version of `MadGraph5_aMC@NLO` and for the chirality-flow branch. (E. Boman, A. Lifson, M. Sjodahl, A. Warnerbring, Z. Wettersten, 2312.07447 (JHEP))

Implementation, massive QCD



Runtime for the Lorentz structure of $gg \to t\bar{t} + ng$, for the standalone version of `MadGraph5_aMC@NLO` and for the chirality-flow branch.

(E. Boman, A. Lifson, M. Sjodahl, A. Warnerbring, Z. Wettersten, 2312.07447 (JHEP))

Conclusion and outlook

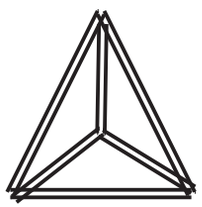
- Splitting Lorentz structure into su(2), su(2), we are able to recast all standard model Feynman rules to chirality-flow rules
- Speedup from more compact Lorentz structure for (massless) fermions
- Speedup from setting reference vector for massless gauge bosons
- ... and q-vector, i.e., spin direction for massive particles
- To simplify calculations for massive particles, we should *not* use helicity, but we should also not use the same spin direction. Instead we should use the same reference vector

$$s^\mu = \frac{1}{m}(p^\mu - 2\alpha q^\mu) = \frac{1}{m}(p^\mu - \frac{m^2}{p\cdot q}q^\mu)$$

- Expect even larger effects from this for electroweak physics due to $W^\pm$ chiral coupling

LUND
UNIVERSITY

Treating color structure with Wigner 6j coefficients

- Orthogonal multiplet bases
- All you need is 
- Color structure treatment using group invariant Wigner 3j and 6j symbols

Thanks to my collaborators: Judith Alcock-Zeilinger, Stefan Keppeler, Simon Plätzer, Johan Thorén

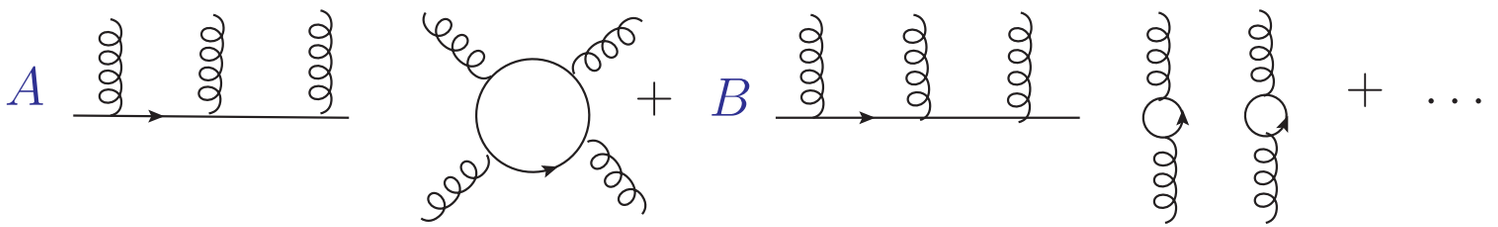
CERN
Feb 2025
Malin Sjödahl

Dealing with color space

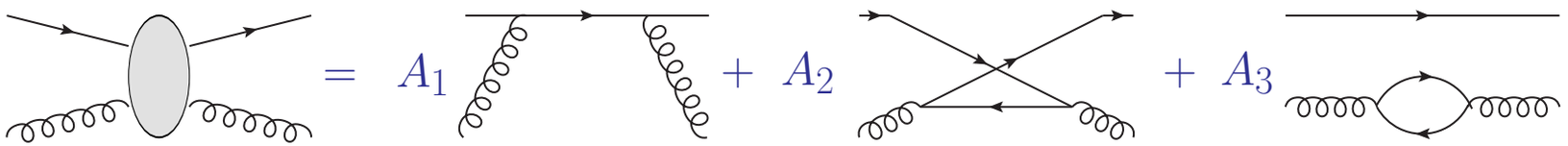
Due to confinement we never observe individual colors

- We average over incoming colors
- We sum over outgoing colors
- $\rightarrow$ we sum over the colors of all external partons
- As always in quantum mechanics we also sum over all degrees of freedom that can interfere with each other $\rightarrow$ we sum over the colors of all internal particles
- $\rightarrow$ We sum over all colors of all particles

- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair



(an incoming quark is the same as an outgoing anti-quark)

- The above type of color structure can be used as a spanning set, a "trace basis"

Example: Number of spanning vectors for N_g gluons (without imposing charge conjugation invariance). These numbers are representative also for N_g gluons plus $q\bar{q}$-pairs.

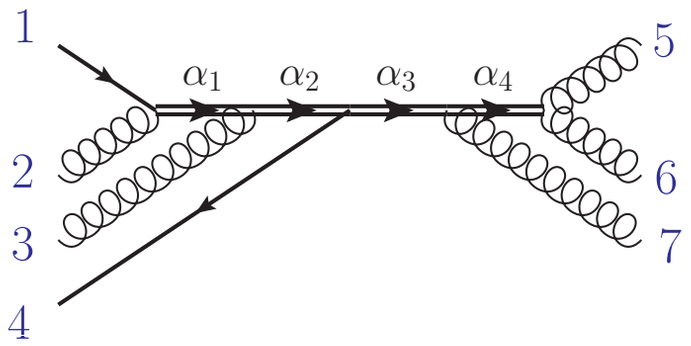
N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$	LO Vectors $N_c \to \infty$
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880
11	614 000	14 684 570	3 628 800
12	3 609 760	176 214 841	39 916 800

(Y. Du, MS & J. Thorén, 1503.00530 (JHEP))

Multiplet bases

- QCD is based on SU(3) $\rightarrow$ the color space may be decomposed into irreducible representations
- Orthogonal basis vectors corresponding to irreducible representations may be constructed
- The construction of the corresponding basis vectors is non-trivial, and a general strategy was presented relatively recently (S. Keppeler & MS, 1207.0609 (JHEP); MS & J. Thoren 1809.05002 (JHEP))
- With general, I mean general: general number of quarks and gluons, general order in α_s and general N_c

- In multiplet bases partons are grouped into representations



  and can thus be characterized by a chain of representations $\alpha_1, \alpha_2, ...$ (In principle we have to differentiate between different vertices as well)

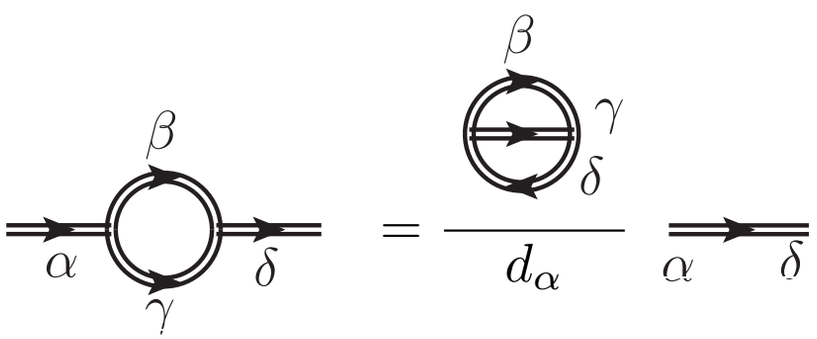
- These vectors are orthogonal ($\rightarrow$ minimal) by construction

Decomposing color structure in multiplet bases

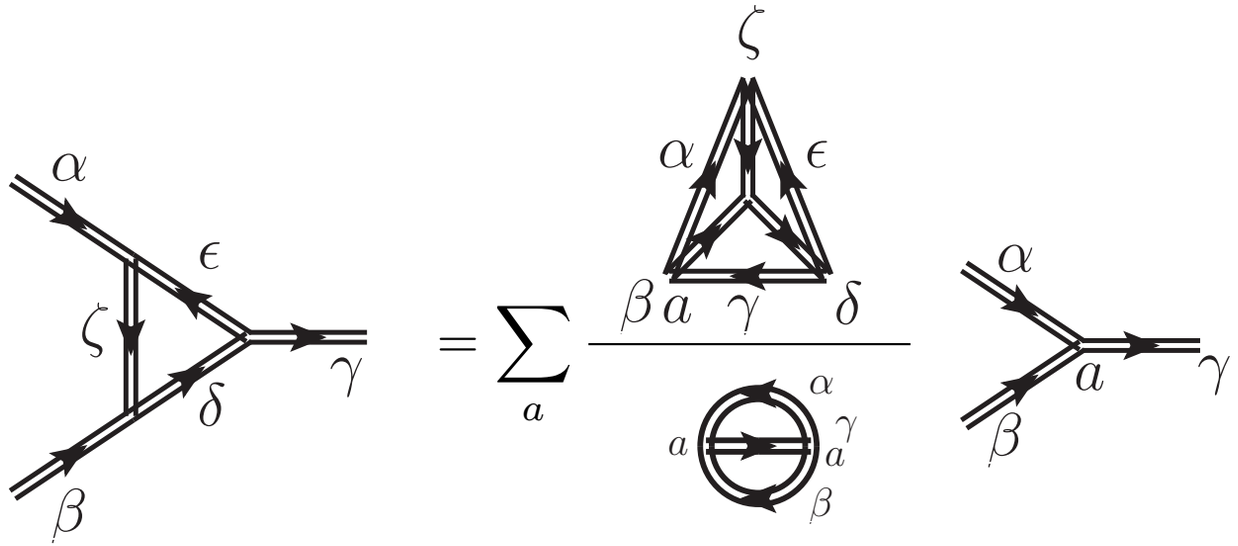
To simplify the color structure we need a few rules:

- Dimension relation

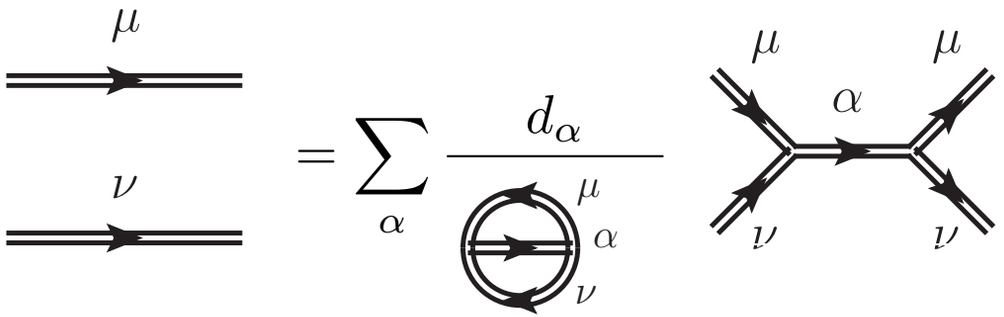
- Two-vertex loops give just a constant

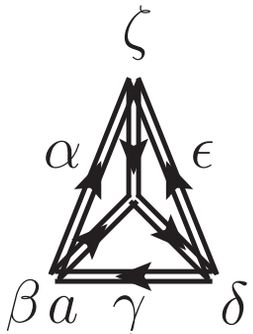
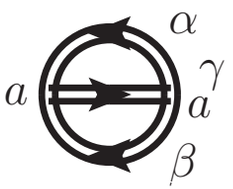


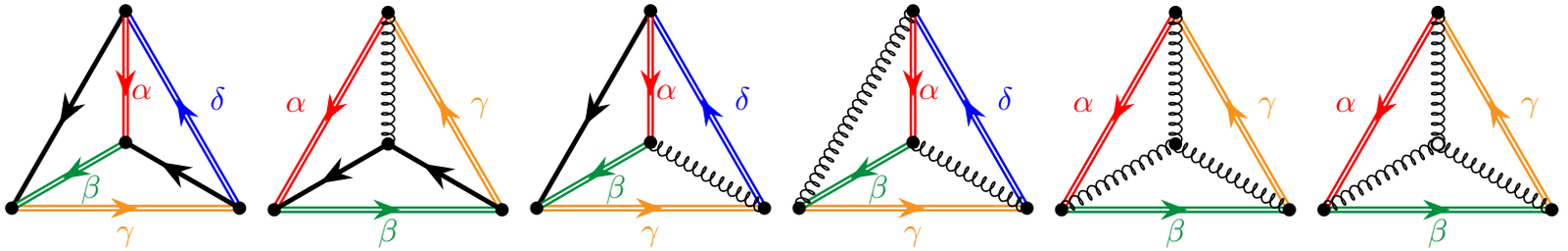
- Vertex correction relation



- For longer loops we need the completeness relation (which the Fierz identity is a special case of)

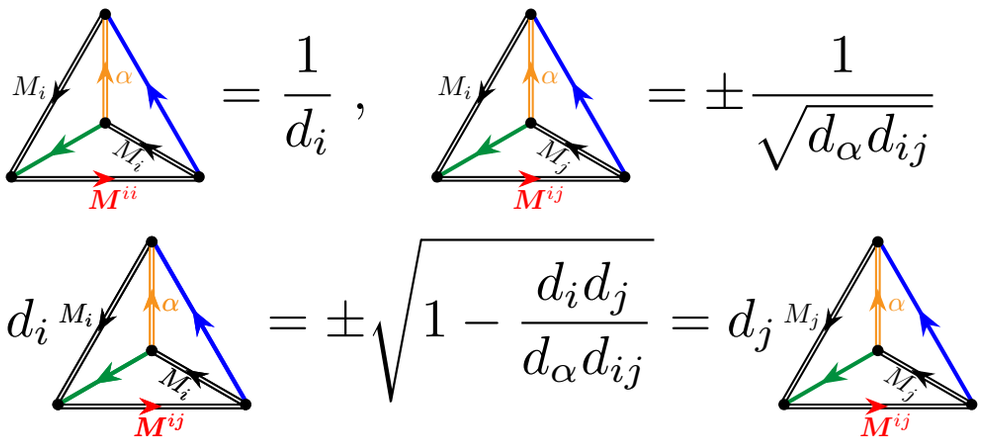


- The symbols  and  are Wigner 6j and 3j coefficients and their values can be calculated once and for all
- Knowing the 3j and 6j Wigner coefficients we can calculate with color without explicitly writing down color bases
- For QCD, where every representation is 8, 3 or $\overline{3}$, it turns out that we only need 6js of form



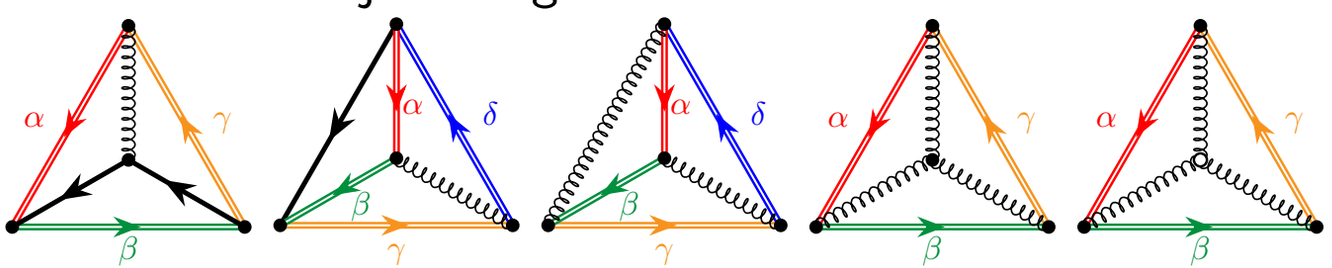
- Wigner 6j and 3j coefficients and their values can be calculated once and for all (Some in M.S. & J. Thorén, 1507.03814 (JHEP), 1809.05002 (JHEP)) ... but this still builds on constructing bases on symmetrizers and anti-symmetrizers
- ... but if everything can be calculated in terms of 6js, perhaps we can calculate 6js in terms of 6js?
- Algorithms recently presented (6js with quarks: J. Alcock-Zeilinger, S. Keppeler, S. Plätzer and MS, 2209.15013 (J. Math. Phy.), 6js with gluons: S. Keppeler, S.Plätzer and M. Sjodahl 2312.16688 (JHEP))
- Presently implementing 6js (Stefan Keppeler, Pascal Lutz and Malin Sjödahl)

- For quarks, solve equation system

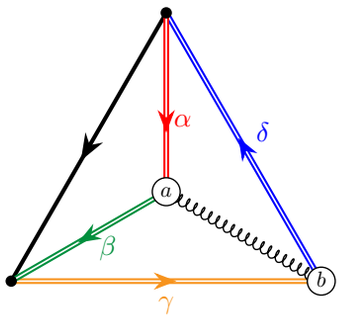


(Judith Alcock-Zeilinger, Stefan Keppeler, Simon Plätzer and MS, 2209.15013 (J. Math. Phy.))

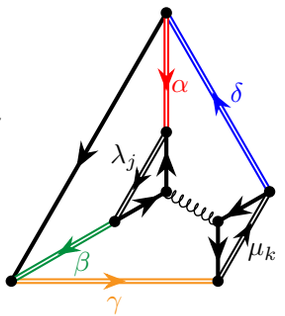
- Also need the 6js with gluons



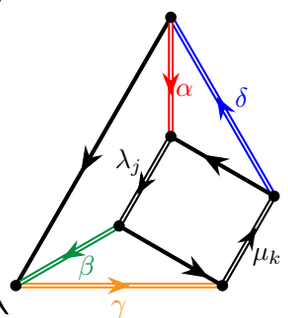
- A gluon example:



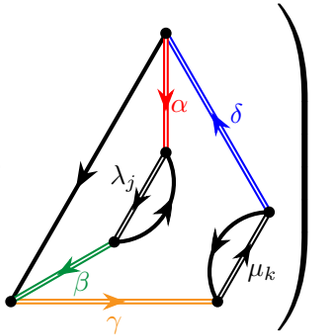
$$= \sum_{j=1}^{a} \sum_{k=1}^{b} C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}$$



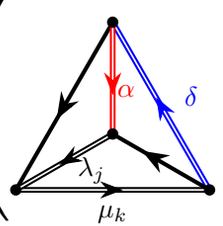
$$= \sum_{j=1}^{a} \sum_{k=1}^{b} \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2-1}$$

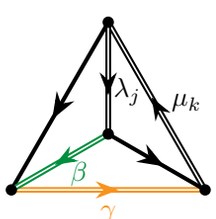


$$- \frac{1}{N}$$



$$= \sum_{j=1}^{a} \sum_{k=1}^{b} \frac{C_{aj}^{\beta\alpha} C_{bk}^{\delta\gamma}}{N^2-1}$$





$$- \frac{\delta_{\alpha\beta}\,\delta_{\gamma\delta}}{N d_\alpha\, d_\gamma}$$

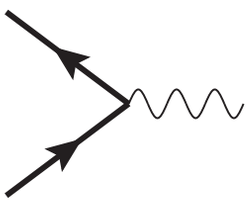
Summary

- We can calculate in orthogonal multiplet bases *without* explicitly constructing the corresponding bases
- Instead only the Wigner 6j coefficients are needed
- We can calculate them in a way which scales only as the square of the number of quarks
- An implementation on its way
- Opens up for orthogonal treatment of color structure
- Currently looking into Berends-Giele recursion with Rikkert Frederix and a master student, Oskar Bolinder

Thank you for your attention!

Fermion vertices

- Fermion-vector vertex

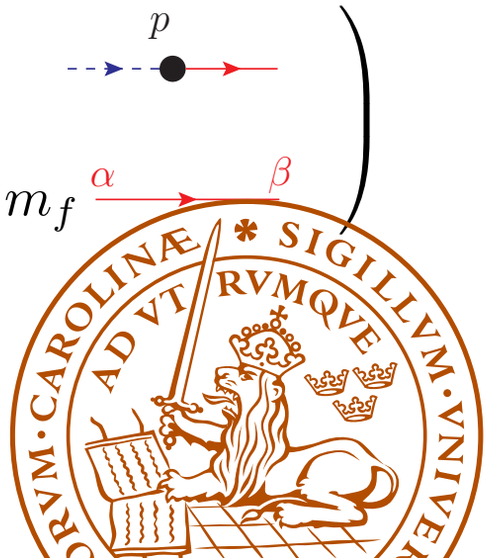


$$= ie(P_L C_L + P_R C_R)\gamma^\mu = ie\sqrt{2}\begin{pmatrix} 0 & C_R \\ C_L & 0 \end{pmatrix}$$

Left and right chiral couplings may differ, in particular $C_R = 0$ for $W^\pm \Rightarrow$ **electroweak sector nice** in chirality flow

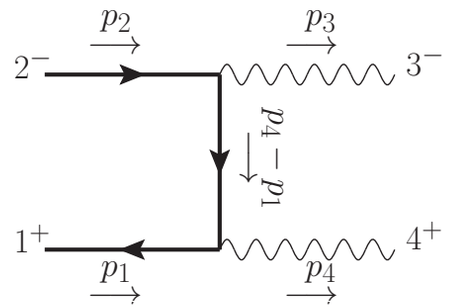
- Fermion propagator

$$\frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f \delta^{\dot\alpha}{}_{\dot\beta} & \sqrt{2}p^{\dot\alpha\beta} \\ \sqrt{2}\bar p_{\alpha\dot\beta} & m_f \delta_\alpha{}^\beta \end{pmatrix} = \frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f\ \dot\alpha \to \dot\beta & p \to \\ p \to & m_f\ \alpha \to \beta \end{pmatrix}$$



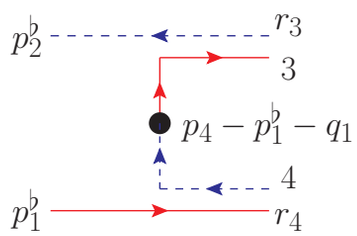
A massive example

Consider the diagram of $e^+_{1+}e^-_{2-} \to \gamma_3^{-=R}\gamma_4^{+=L}$ and include the mass m_e

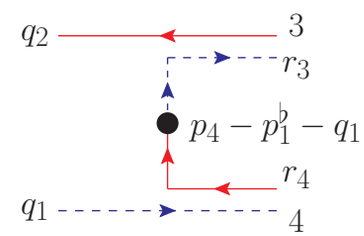


- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4

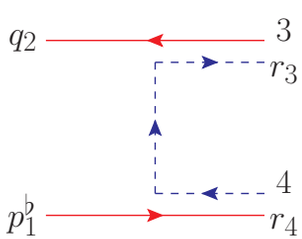
$$= \frac{-2ie^2}{((p_4-p_1)^2 - m_e^2)[3r_3]\langle 4r_4\rangle} \left\{ \right.$$



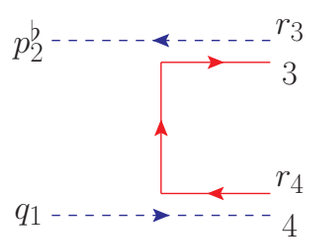
$$- \frac{m_e}{[q_1 p_1^\flat]} \frac{m_e}{\langle p_2^\flat q_2\rangle}$$



$$+ m_e \left(\frac{m_e}{\langle p_2^\flat q_2\rangle}$$



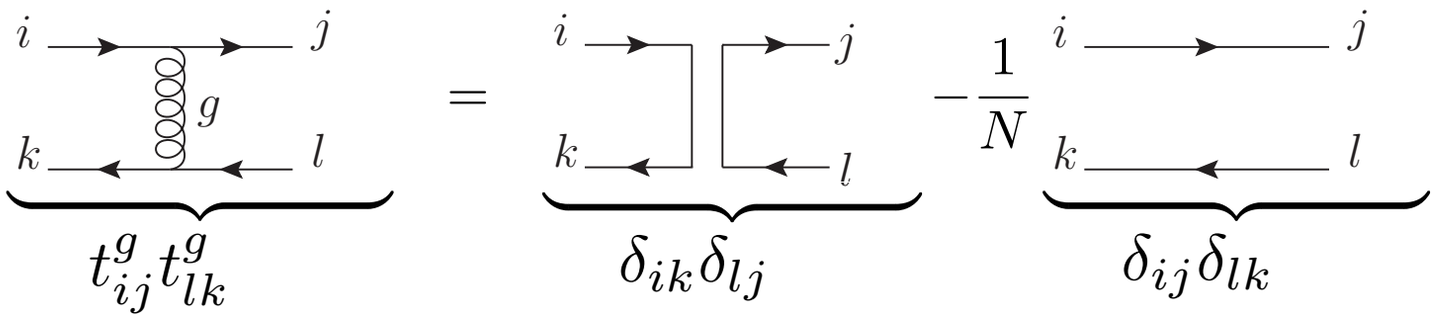
$$- \frac{m_e}{[q_1 p_1^\flat]}$$



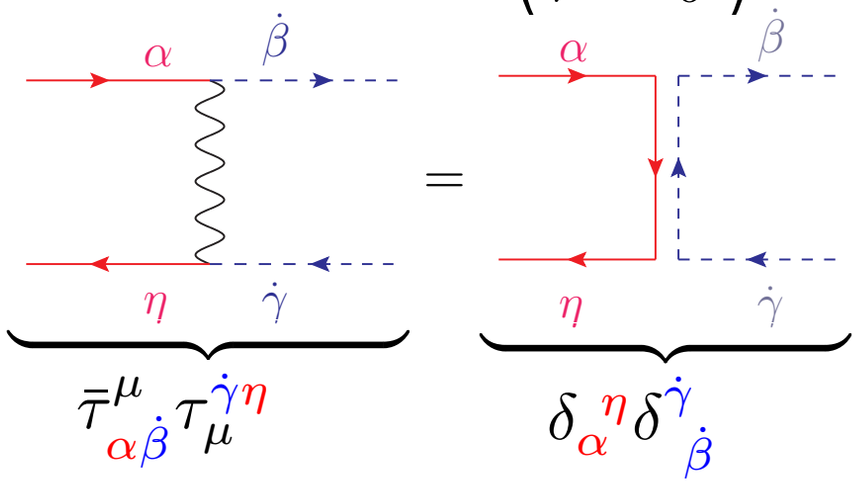
$$\left. \right) \right\}$$

Towards chirality flow: Photon exchange

- Recall color, a single $SU(N)$: generators $t^a \to \delta$'s

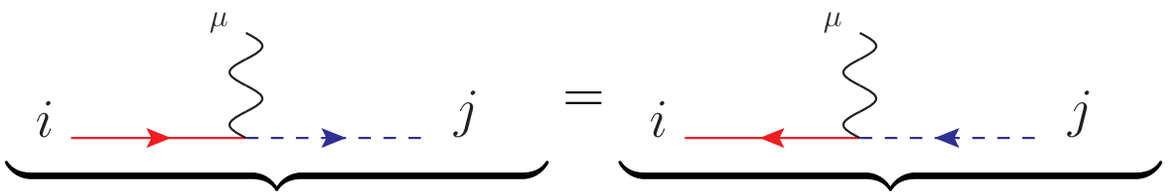


- For the Lorentz structure $\gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & \tau^\mu \\ \bar{\tau}^\mu & 0 \end{pmatrix}$ in vertices



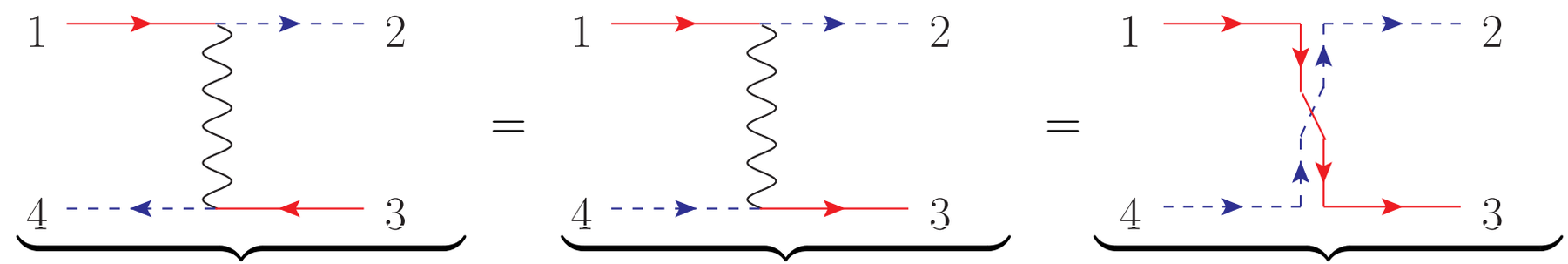
Non-matching arrows?

Arrows opposed? Flip them (when contracted between external spinors), first use charge conjugation



$\langle i|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|j]^{\dot{\beta}}$ $\qquad$ $[j|_{\dot{\alpha}}\tau^{\mu,\dot{\alpha}\beta}|i\rangle_{\beta}$

Then use Fierz identity $\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tau_{\mu}^{\dot{\gamma}\eta} = \delta_{\alpha}{}^{\eta}\delta^{\dot{\gamma}}{}_{\dot{\beta}}$ to remove vector index



$(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}})(\langle 3|^{\gamma}\bar{\tau}_{\mu,\gamma\dot{\eta}}|4]^{\dot{\eta}})$ $\qquad$ $(\langle 1|^{\alpha}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}|2]^{\dot{\beta}})\underbrace{([4|_{\dot{\eta}}\tau_{\mu}^{\dot{\eta}\gamma}|3\rangle_{\gamma})}_{\text{flipped}}$ $\qquad$ $\langle 13\rangle[42]$

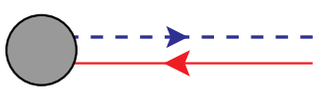
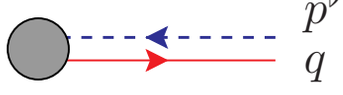
Backup: Massive polarization vectors

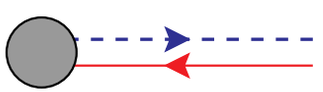
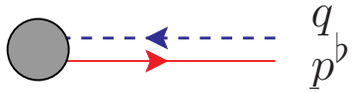
External gauge bosons, for helicity states $p^\flat \to p_f$, $\alpha \to 1$, $q \to p_b$

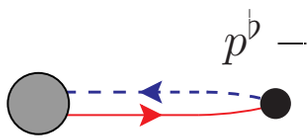
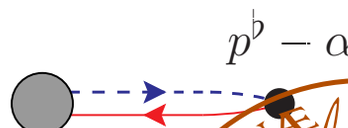
$$\epsilon_L{}^\mu(p) \to \frac{|p^\flat]\langle q|}{\langle qp^\flat\rangle} \quad \text{or} \quad \frac{|q\rangle[p^\flat|}{\langle qp^\flat\rangle} \qquad \epsilon_R{}^\mu(p) \to \frac{|q]\langle p^\flat|}{[p^\flat q]} \quad \text{or} \quad \frac{|p^\flat\rangle[q|}{[p^\flat q]}$$

$$\epsilon_0^\mu(p) = s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu)$$

for incoming (outgoing) bosons, use L/R for $-/+$ $(-/+)$ spin along s^μ-axis. Translate to chirality flow

$\epsilon_L{}^\mu(p) \to \dfrac{1}{\langle qp^\flat\rangle}$  , or $\epsilon_L{}^\mu(p) \to \dfrac{1}{\langle qp^\flat\rangle}$ 

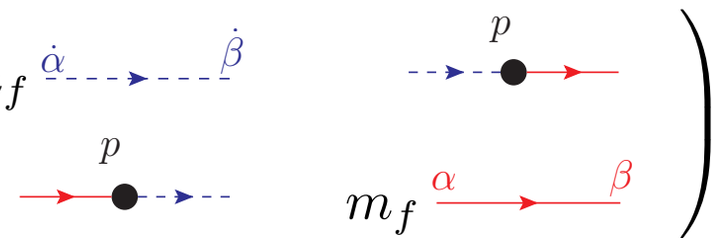
$\epsilon_R{}^\mu(p) \to \dfrac{1}{[p^\flat q]}$  , or $\epsilon_R{}^\mu(p) \to \dfrac{1}{[p^\flat q]}$ 

$\epsilon_0^\mu(p) \to \dfrac{1}{m\sqrt{2}}$  , or $\epsilon_0^\mu(p) \to \dfrac{1}{m\sqrt{2}}$ 

Backup: Massive Fermions

- Fermion propagator

$$\frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f \delta^{\dot{\alpha}}{}_{\dot{\beta}} & \sqrt{2} p^{\dot{\alpha}\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot{\beta}} & m_f \delta_{\alpha}{}^{\beta} \end{pmatrix} = \frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f & \\ & \end{pmatrix}$$



- Propagators and vertices don't always contribute factor $\tau/\bar{\tau}$
  $\Rightarrow$ may have even number of $\tau/\bar{\tau}$-matrices
- Have to update arrow swap procedure to include even number of $\tau/\bar{\tau}$. Arrow flips may induce minus signs! Care must be taken

$$\langle i|\bar{\tau}^{\mu_1}\tau^{\mu_2}\ldots\bar{\tau}^{\mu_{2n+1}}|j] = [j|\tau^{\mu_{2n+1}}\bar{\tau}^{\mu_{2n}}\ldots\tau^{\mu_1}|i\rangle$$

$$\langle i|\bar{\tau}^{\mu_1}\tau^{\mu_2}\ldots\tau^{\mu_{2n}}|j\rangle = -\langle j|\bar{\tau}^{\mu_{2n}}\bar{\tau}^{\mu_{2n-1}}\ldots\tau^{\mu_1}|i\rangle$$

$$[i|\tau^{\mu_1}\bar{\tau}^{\mu_2}\ldots\bar{\tau}^{\mu_{2n}}|j] = -[j|\tau^{\mu_{2n}}\bar{\tau}^{\mu_{2n-1}}\ldots\bar{\tau}^{\mu_1}|i]$$