Tropical sampling in momentum space

M.Borinsky: 2008.12310
M.Borinsky, H.Munch, F.Tellander: 2302.08955
M.Borinsky, M.Fraaije: coming soon!

UNIVERSITÄT BERN

Mathijs Fraaije

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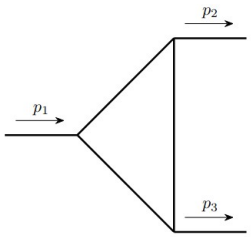

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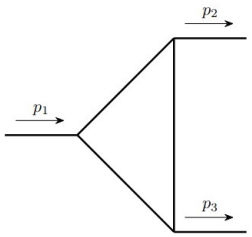

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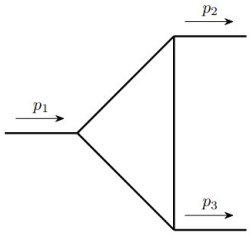

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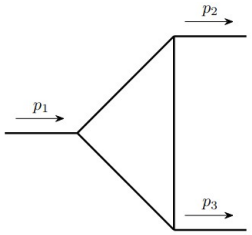

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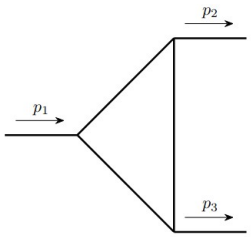

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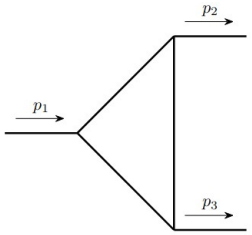

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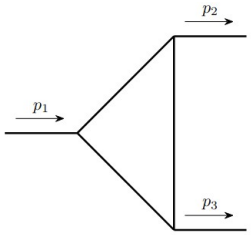

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⬆ These are propagators!

Feynman integrals & measures

Scalar Feynman integral:

$$I_G = \int \prod_{\ell=1}^{L} \frac{\mathrm{d}^D k_\ell}{\prod_{e=1}^{E} D_e(k)^{\nu_e}}$$

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$$\int \mu_G \; f(k) \quad \Rightarrow \quad \int \tilde{\mu}_G \; W \; f(k)$$

W is bounded

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$$k = \sqrt{\frac{\mathcal{V}(x)}{2\lambda}} (Q^T)^{-1}(x) q - (Q(x) Q^T(x))^{-1} u(x)$$

The tropical approximation

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μ_{tr}

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μ_{tr}

W

Sampling from μ_{tr}

Sector decomposition

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The integrand is easy to sample!

Sector importance sampling

But how? There are $E!$ sectors!

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For the case $f(k) = 1$, this method has been used to compute a 17 loop integral!

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Some integrals

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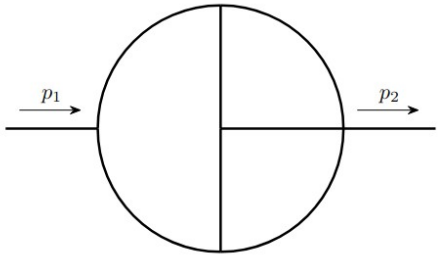

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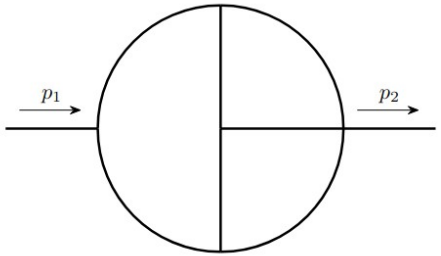

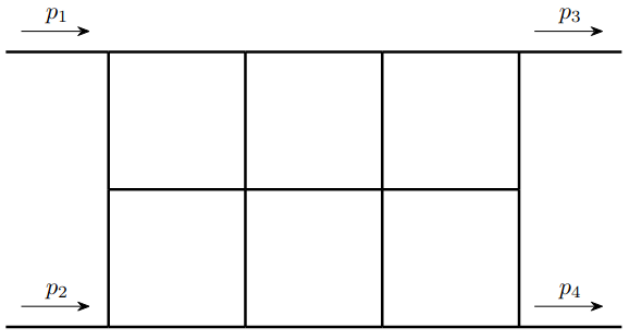

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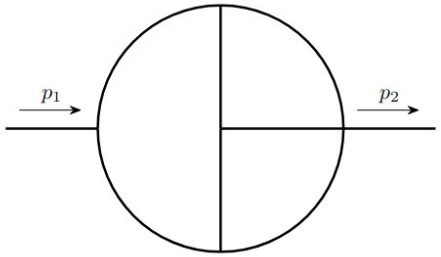

$$N = 10^6$$

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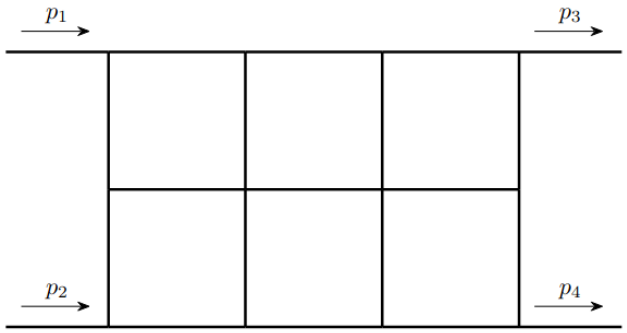

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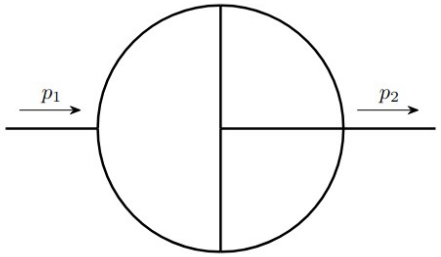

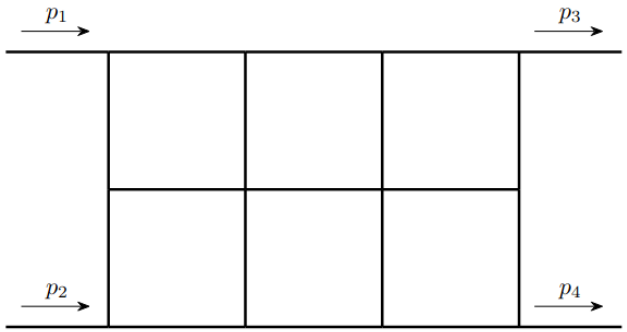

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Multi-channeling	$5.20(14) \times 10^{-6}$	1.44×10^{-7}
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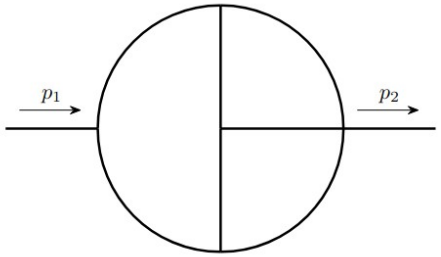

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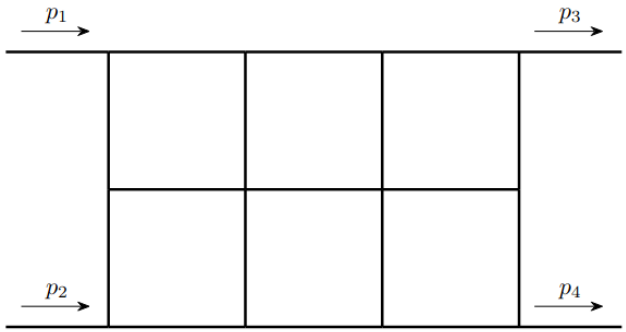

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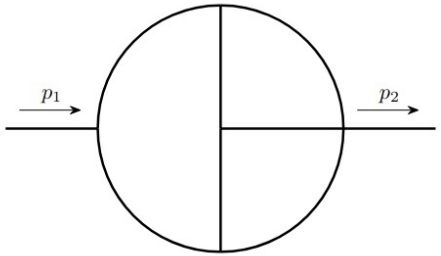

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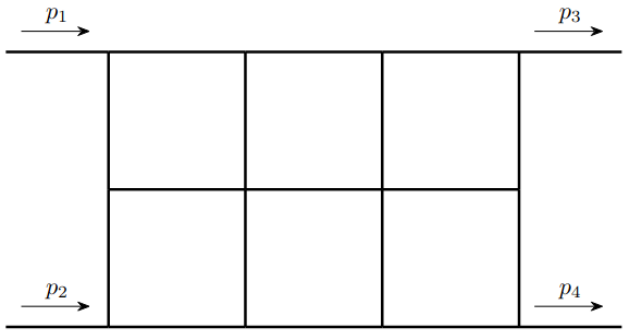

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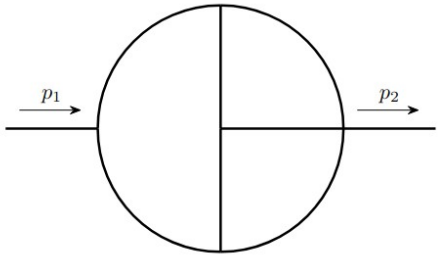

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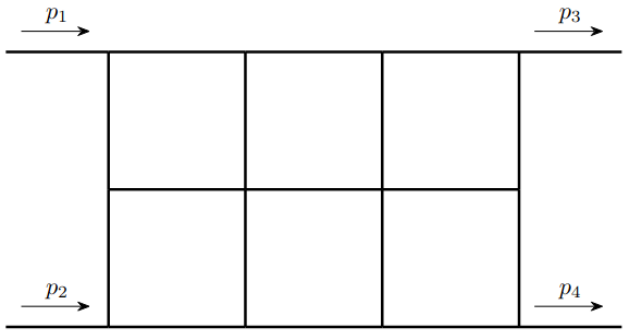

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17 massless propagators!

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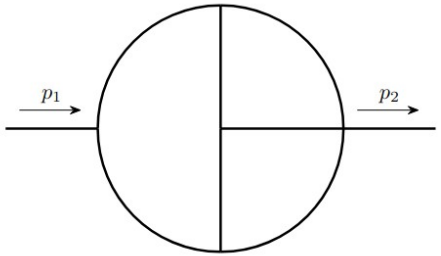

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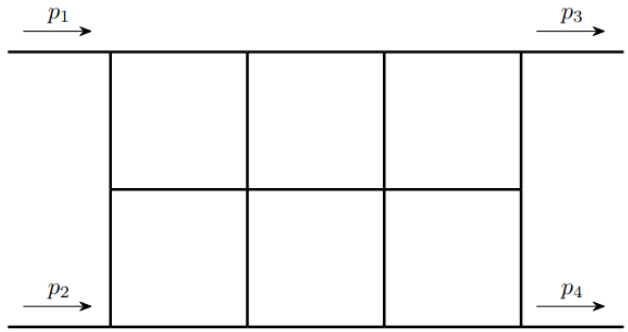

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$$I_{\text{ref}} = 8.4045 \times 10^{-19}$$
$$I = 8.4089(98) \times 10^{-19} \quad \Delta = 0.453\sigma,\ 0.053\%$$

Some visualisations

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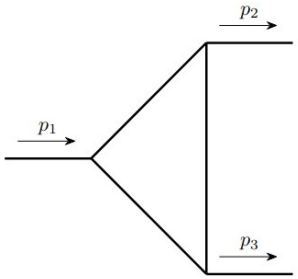

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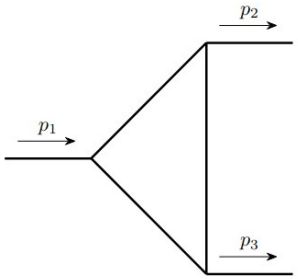

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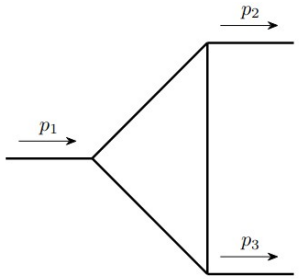

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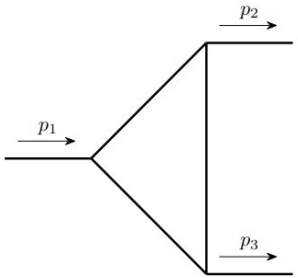
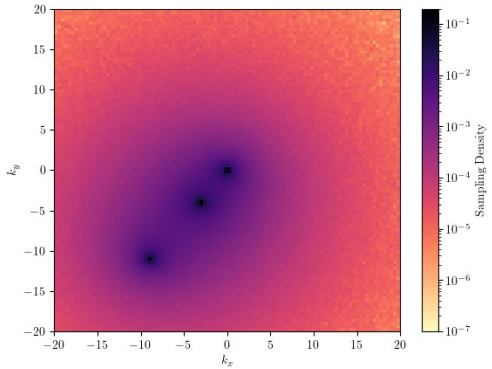
$$\vec{k} = 0 \ , \ \vec{k} = (-3, -4) \ , \ \vec{k} = (-9, -11)$$

Some visualisations

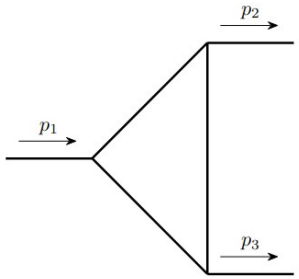


$$\vec{p}_1 = (3, 4),$$
$$\vec{p}_2 = (-6, -7),$$
$$\vec{p}_3 = \vec{p}_1 - \vec{p}_2.$$

$$\vec{k} = 0 \ , \ \vec{k} = (-3, -4) \ , \ \vec{k} = (-9, -11)$$



Some visualisations



$$\vec{p}_1 = (3, 4),$$
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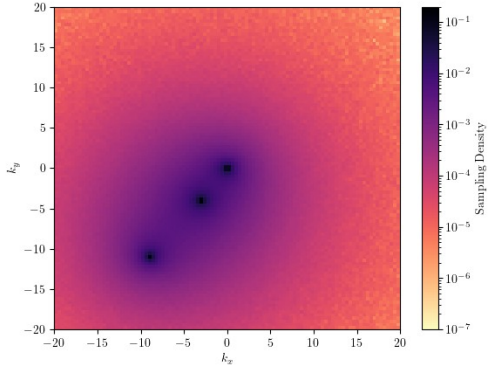
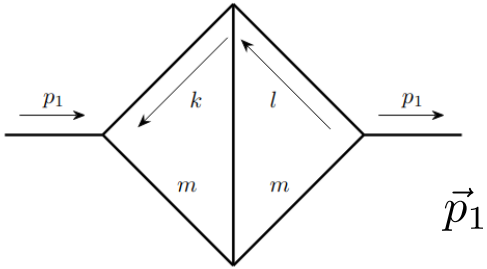
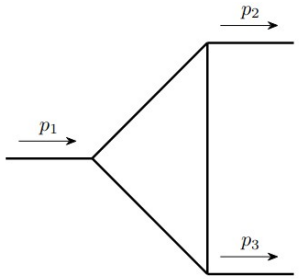
$$\vec{k} = 0 \ , \ \vec{k} = (-3, -4) \ , \ \vec{k} = (-9, -11)$$





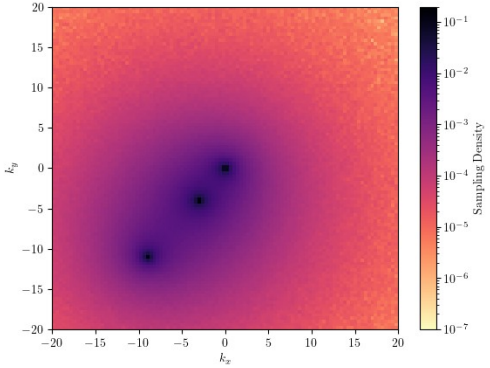
$$\vec{p}_1 = (1)$$

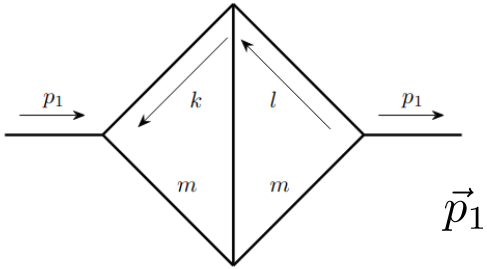
Some visualisations



$$\vec{p}_1 = (3, 4),$$
$$\vec{p}_2 = (-6, -7),$$
$$\vec{p}_3 = \vec{p}_1 - \vec{p}_2.$$

$$\vec{k} = 0 \ , \ \vec{k} = (-3, -4) \ , \ \vec{k} = (-9, -11)$$

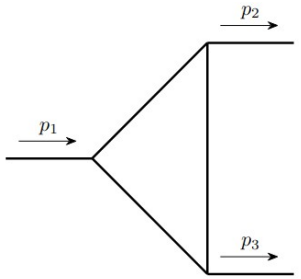




$$\vec{p}_1 = (1)$$

$$\vec{k} = 0 \ , \ \vec{l} = 0 \ , \ \vec{k} = \vec{l}$$

Some visualisations



$$\vec{p}_1 = (3, 4),$$
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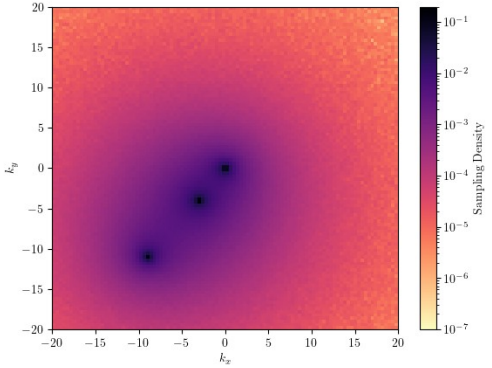
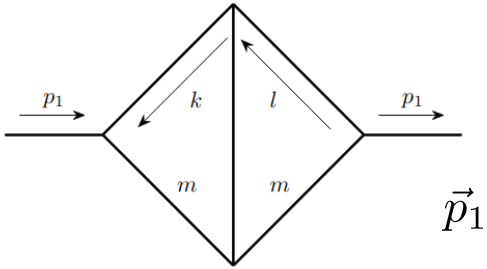
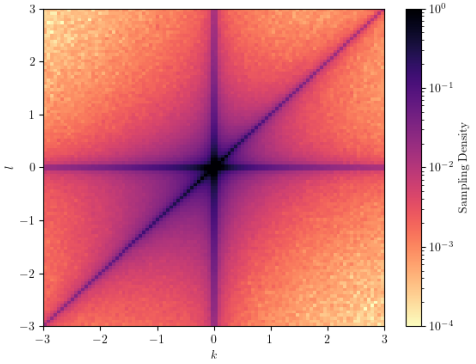




$$\vec{p}_1 = (1)$$

$$\vec{k} = 0 \ , \ \vec{l} = 0 \ , \ \vec{k} = \vec{l}$$



The Future

- Use in $\gamma\mathrm{loop}$ for cross sections
- Combine with learning methods (MadNIS)
- Recast other integration problems into this form: $\int \mu_G \, f(k)$