Non-perturbative determination of the Sphaleron Rate in Quantum Chromo-Dynamics

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Based on:

- "*Sphaleron Rate of $N_f = 2 + 1$ QCD*"
  CB, F. D'Angelo, M. D'Elia, L. Maio, M. Naviglio
  Phys. Rev. Lett. **132** (2024) 5, 051903 [2308.01287]

- "*Sphaleron rate from a modified Backus–Gilbert inversion method*"
  CB, F. D'Angelo, M. D'Elia, L. Maio, M. Naviglio
  Phys. Rev. D **108** (2023) 7, 074515 [2305.17120]

Gauge topology and QCD phenomenology

$$\mathcal{S}_{\mathrm{QCD}}(\theta) = \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{Tr}\left[G_{\mu\nu}(x)G^{\mu\nu}(x)\right] + \int d^4x \sum_{\mathrm{f}=1}^{N_{\mathrm{f}}} \overline{\psi}_{\mathrm{f}}(x)(\not{D} + m_{\mathrm{f}})\psi_{\mathrm{f}}(x) + \theta Q$$

$$Q = \frac{g^2}{32\pi^2}\int \mathrm{d}^4x\, \varepsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[G^{\mu\nu}(x)G^{\rho\sigma}(x)\right] \in \mathbb{Z}$$

The non-trivial topological structure of $\mathrm{SU}(N_{\mathrm{c}})$ has several intriguing theoretical and phenomenological implications:

- Topological charge breaks $\mathrm{U}(1)_{\mathrm{A}}$ via chiral anomaly $\implies$ Witten–Veneziano mechanism for η' mass
- The θ-term coupling the **topological charge** to the standard QCD action is a source of CP-violation $\implies$ strong-CP problem
- Intense theoretical/experimental research activities around strong-CP: axions, neutron electric dipole moment

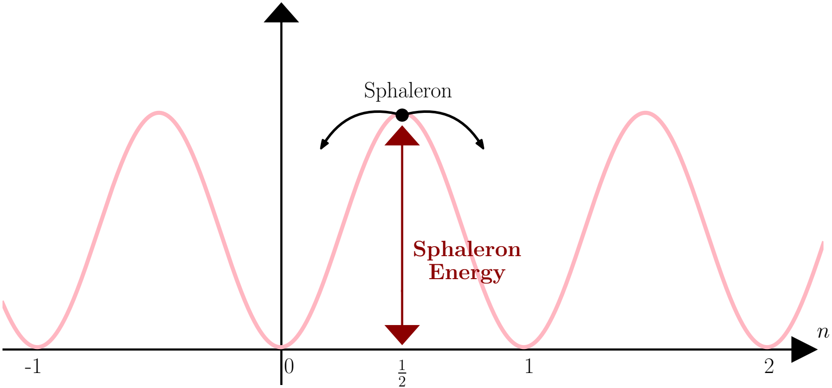
This seminar is about a somewhat underdeveloped topic in QCD topology:
QCD topological sphaleron thermal transitions

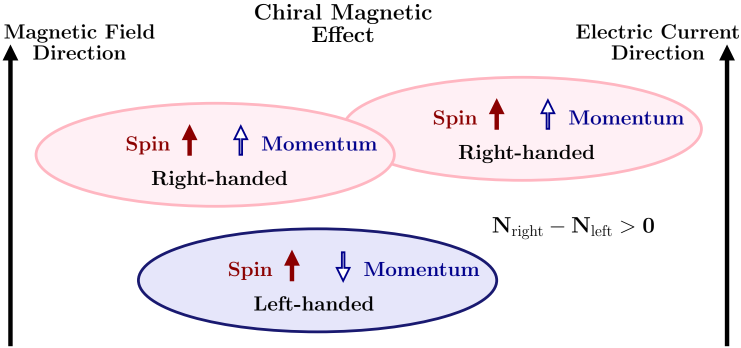
The strong sphaleron rate

Sphaleron rate = rate of real time thermal transitions above sphaleron barriers separating topologically-inequivalent QCD vacua

$$\Gamma_{\text{Sphal}} \equiv \lim_{\substack{V\to\infty \\ t_{\mathrm{M}}\to\infty}} \frac{1}{V t_{\mathrm{M}}} \left\langle \left[\int_0^{t_{\mathrm{M}}} \mathrm{d}t'_{\mathrm{M}} \int_V \mathrm{d}^3x \, q(t'_{\mathrm{M}}, \vec{x}) \right]^2 \right\rangle = \int \mathrm{d}t_{\mathrm{M}} \mathrm{d}^3x \, \langle q(t_{\mathrm{M}}, \vec{x}) q(0, \vec{0}) \rangle \, ,$$

$$q(x) = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\{G^{\mu\nu}(x) G^{\rho\sigma}(x)\} \qquad Q = \int \mathrm{d}^4x \, q(x) = \Delta n$$





A sphaleron transition creates a local imbalance in the number of chiral quark species (Index theorem): $Q = n_{\rm Left} - n_{\rm Right} \equiv n_5$

Even if strong interactions preserve CP, $\langle n_{\rm Left} - n_{\rm Right}\rangle = 0$ only at equilibrium $\implies$ $\Gamma_{\rm Sphal}$ drives the relaxation of $n_{\rm Left} - n_{\rm Right}$ and can be interpreted in terms of chiral imbalance creation/annihilation rate: $\frac{\rm d}{{\rm d}t}\frac{n_5}{V} \simeq -\frac{n_5}{V}\Gamma_{\rm Sphal}$

When sphaleron transitions create chiral imbalances in the presence of strong magnetic fields (e.g., fireball of heavy-ion collisions)

$\longrightarrow$ **Chiral Magnetic Effect** Fukushima et al., Phys. Rev. D 78 (2008) 074033 [arXiv:0808.3382]

Sphalerons and axions

Recently, a new source of thermal axion production has been recognized

Notari et al., Phys. Rev. Lett. 131 (2023) 1, 011004 [2211.03799]

Axions could have been produced from pion-pion scattering in the hot and dense cosmological medium permeating the primordial Universe

In this context, Γ_{Sphal} enters the Boltzmann equation governing the axion number cosmological evolution, and can be interpreted in terms of rate of axion creation/annihilation

$$\frac{\mathrm{d}f_{\vec{p}}}{\mathrm{d}t} = (1 + f_{\vec{p}})\Gamma^{<} - f_{\vec{p}}\Gamma^{>} \qquad (f_{\vec{p}} = \text{axion distribution function})$$

$$\Gamma^{>} = \mathrm{e}^{E/T}\Gamma^{<} = \frac{\Gamma_{\mathrm{Sphal}}}{2Ef_a^2}$$

$$\Gamma^{>}/\Gamma^{<} = \text{rate of axion annihilation/creation}$$

The sphaleron rate as an inverse problem

Sphaleron transitions stem from purely strongly-coupled **non-perturbative** dynamics. The approach of Lattice QCD is a natural tool to address its determination.

Γ_{Sphal} is hard to compute on the lattice. Only few attempts limited to quenched theory

Kotov (2018) Altenkort et al. (2021) [arXiv:2012.08279] Mancha & Moore (2022) [arXiv:2210.05507]

Main obstacle: $\Gamma_{\text{Sphal}} \to$ *real time correlation function*

Lattice QCD $\to$ Euclidean correlators.

In the context of transport properties of a thermal medium, there is a way to make contact with Euclidean field theory Meyer, Eur. Phys. J. A 47 (2011) 86 [arXiv:1104.3708]

$$\Gamma_{\text{Sphal}} = 2T \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega} \qquad \text{(Kubo equation)}$$

Here $\rho(\omega)$ is the spectral density of the Euclidean time correlator of $q(t, \vec{x})$

$$G_{\text{E}}(t) = -\int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \left[\frac{\cosh\left(\frac{\omega}{2T} - \omega t\right)}{\sinh\left(\frac{\omega}{2T}\right)} \right] \qquad G_{\text{E}}(t) \equiv \int \mathrm{d}^3 x \, \langle q(t, \vec{x}) q(0, \vec{0}) \rangle$$

Determining Γ_{Sphal} from lattice QCD is equivalent to the problem of determining $G_{\text{E}}(t)$ and inverting the integral relation between $G_{\text{E}}(t)$ and $\rho(\omega) \longrightarrow$ Inverse Problem!

Inverse problems belong te a very general broad class of mathematical problems which is ubiquitous in Physics (and beyond), vast literature of different possible strategies to found numerical approximate solutions.

Recent Lattice review: Rothkopf EPJ Web Conf. 274, 01004 (2022) [arXiv:2211.10680]

Our strategy:

CB et al., Phys. Rev. D 108 (2023) 7, 074515 [arXiv:2305.17120]
CB et al., Phys. Rev. Lett. 132 (2024) 5, 051903 [arXiv:2308.01287]

Recent Hansen–Lupo–Tantalo modification (HLT) of the model-independent **Backus–Gilbert** method

Hansen, Lupo, Tantalo, Phys. Rev. D **99** (2019) 094508 [arXiv:1903.06476]

HLT-modified Backus–Gilbert method

We look for a solution for $\rho(\omega)/\omega$ of this form:

$$\frac{\bar{\rho}(\bar{\omega})}{\bar{\omega}} = -\pi \int \mathrm{d}t\, g(\bar{\omega}, t) G_{\mathrm{E}}(t)$$

$$\frac{\Gamma_{\mathrm{Sphal}}}{2T} = \left[\frac{\bar{\rho}(\bar{\omega})}{\bar{\omega}}\right]_{\bar{\omega}=0} = -\pi \int \mathrm{d}t\, g(0, t) G_{\mathrm{E}}(t)$$

Using the relation between $G_{\mathrm{E}}(t)$ and $\rho(\omega)/\omega$

$$G_{\mathrm{E}}(t) = -\int_0^\infty \frac{\mathrm{d}\omega}{\pi} \left[\frac{\rho(\omega)}{\omega}\right] \left[\frac{\omega \cosh\left(\frac{\omega}{2T} - \omega t\right)}{\sinh\left(\frac{\omega}{2T}\right)}\right] \equiv -\int_0^\infty \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega} K(\omega, t),$$

we obtain the following consistency relation:

$$\frac{\bar{\rho}(\bar{\omega})}{\bar{\omega}} = \int_0^\infty \mathrm{d}\omega\, \Delta(\omega, \bar{\omega}) \frac{\rho(\omega)}{\omega}, \qquad \text{where} \quad \Delta(\omega, \bar{\omega}) = \int \mathrm{d}t\, g(\bar{\omega}, t) K(\omega, t).$$

The more $\Delta(\omega, \bar{\omega})$ is **peaked** around $\bar{\omega}$ as a function of ω, the better $\bar{\rho}(\bar{\omega})$ will approximate $\rho(\omega)$. Extreme case: $\Delta(\omega, \bar{\omega}) = \delta(\omega - \bar{\omega}) \implies \bar{\rho}(\bar{\omega}) = \rho(\bar{\omega})$.

Determination of g coefficients

We determine the unknown g coefficients requiring that $\Delta(\omega, 0)$ is as close as possible to a given target function, chosen on the basis of physical reasoning.

Inspired by the relation between $G_{\rm E}(t)$ and $\rho(\omega)$, we choose:

$$\delta_\sigma(\omega,0) = \left(\frac{2}{\sigma\pi}\right)^2 \frac{\omega}{\sinh(\omega/\sigma)}, \qquad \delta_\sigma(\omega,0) \underset{\sigma\to 0}{\longrightarrow} \delta(\omega),$$

This is a smeared Dirac delta function.

The optimal coefficients are determined by minimizing:

$$F[g] = (1-\lambda)A[g] + \lambda\, B[g], \quad \lambda \in [0,1).$$

$$A[g] = \int_0^\infty \mathrm{d}\omega\, [\Delta(\omega,0) - \delta_\sigma(\omega,0)]^2\, \mathrm{e}^{2\omega}, \quad B[g] = \frac{1}{G_{\rm E}^2(0)} \sum_{t,t'=0}^{1/T} \mathrm{Cov}(t,t')\, g(0,t) g(0,t').$$

λ is a free parameter to be discussed later.

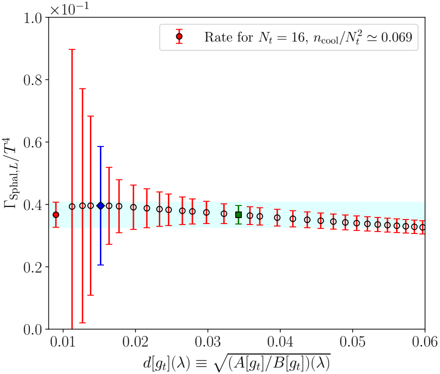
Crucial point of HLT: smearing width σ is an **input of the method**
Can be changed to check systematic effects on reconstructed spectral density

Role of the parameter λ

The functional $F[g]$ to be minimized is the sum of two pieces:

$$A[g] = \int_0^\infty \mathrm{d}\omega \, [\Delta(\omega,0) - \delta_\sigma(\omega,0)]^2 \, \mathrm{e}^{2\omega}, \qquad B[g] = \frac{1}{G_{\mathrm{E}}^2(0)} \sum_{t,t'=0}^{1/T} \mathrm{Cov}(t,t') \, g(0,t) g(0,t'),$$

$A[g] \to$ distance between Δ and δ_σ $\qquad$ $B[g] \to$ statistical errors on $G_{\mathrm{E}}(t)$.



- $\lambda \to 0$: $B[g]$ is neglected $\implies$ error on Γ_{Sphal} dominated by violent statistical fluctuations
- $\lambda \to 1$: $A[g]$ is neglected $\implies$ error on Γ_{Sphal} dominated by systematic effects, being $\Delta(\omega,0)$ unconstrained $\implies$ **we have no control on its shape and width**
- We look for an intermediate regime where statistical error dominates over systematic

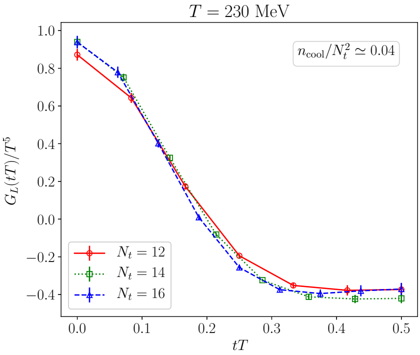
Error estimate: statistically-dominated regime when $d^2 \equiv (A/B) \ll 1$.
Final error keeps into account observed variations of Γ_{Sphal}.

Lattice determination of the top. charge dens. correlator

$$\frac{G_{\rm L}(tT)}{T^5} = \frac{N_t^5}{N_s^3} \sum_{n_{\vec{x}}, n_{\vec{y}}} \langle q_{\rm L}(n_t/N_t, n_{\vec{x}}) q_{\rm L}(0, n_{\vec{y}}) \rangle$$

$q_{\rm L}$ = clover discr. computed after smoothing due to UV noise.

Smoothing = kills small-scale fluctuations below smoothing radius $r_s \propto \sqrt{\text{smoothing}}$.



In our work we compute $G_{\rm L}(t)$ after $n_{\rm cool}$ cooling steps.

Smoothing radius given by $r_s/a \simeq \sqrt{8n_{\rm cool}/3} \implies n_{\rm cool}/N_t^2 \propto (r_s T)^2$.

$t > r_s$: correlator is negative because of *reflection positivity* (q is CP-odd)

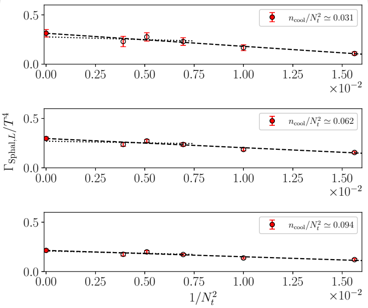
$t < r_s$: correlator is unphysical and positive (overlapping sources)

$L_s = aN_s \qquad T = 1/(aN_t) \qquad a \to 0$ with L_s, T fixed

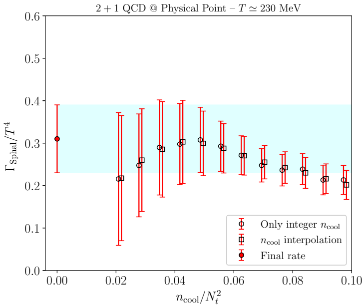
$\implies N_s/N_t$ fixed $\qquad a \leftrightarrow 1/N_t$.

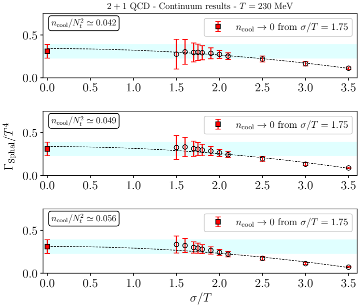
Our strategy: continuum limit $a \to 0$ of $\Gamma_{\rm Sphal}$ at fixed $r_s T$ and σ/T

Results in 2+1 QCD – $T = 230$ MeV $\simeq 1.48\,T_c$



- Continuum limit at fixed $r_s T$ is smooth (lattice artifacts small already for the correlator)
- $\Gamma_{\rm Sphal}$ exhibits a plateau for small enough values of r_s
  r_s kills short-scale fluctuations, small r_s do not affect large-scale fluctuations ($\Gamma_{\rm Sphal} \propto \lim_{\omega\to 0} \rho(\omega)/\omega$)
- Also possible systematics due to finite smearing width σ are under control

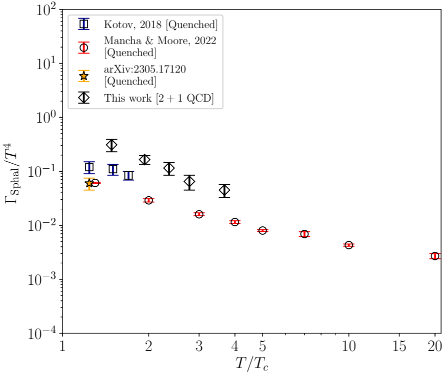


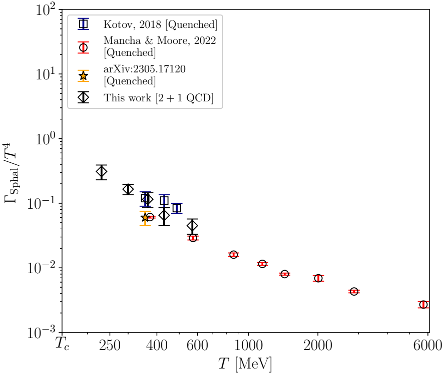


First determination of $\Gamma_{\text{Sphal}}(T)$ in 2+1 QCD

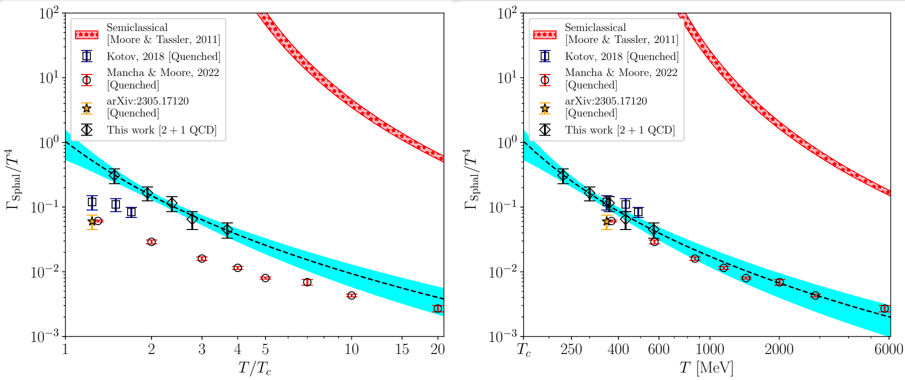
Sphaleron rate of 2+1 QCD at the physical point for 200 MeV $\lesssim T \lesssim$ 600 MeV

CB et al., Phys. Rev. Lett. 132 (2024) 5, 051903 [arXiv:2308.01287]





Full QCD rate: slightly larger than quenched results, but same ballpark



Semiclassical estimate Moore & Tassler JHEP 02 (2011) 105 [arXiv:1011.1167]

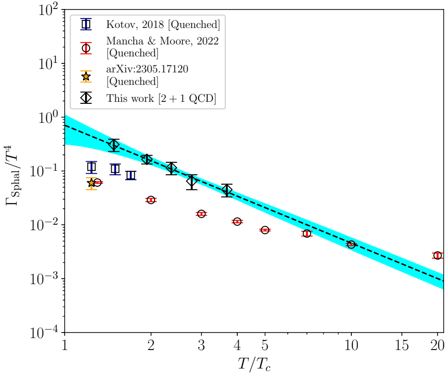
$$\Gamma_{\text{Sphal}}(T) \propto \alpha^5_{\text{strong}}(T) \qquad \alpha^{(1-\text{loop})}_{\text{strong}}(T^2) = \frac{4\pi}{9}\frac{1}{\log\left(T^2/\Lambda^2_{\text{QCD}}\right)}$$

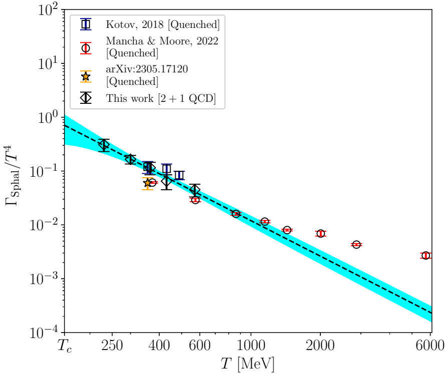
Fit to semiclassical-inspired ansatz:

$$\frac{\Gamma_{\text{Sphal}}}{T^4} = \left[\frac{A_1}{\log(T^2/T_c^2) + A_2}\right]^C \qquad \text{(Fit Ansatz)}$$

Perfectly works for $C = 5$. If C left as free parameter: 5 with 100% error

$\Longrightarrow$ larger temperatures are needed to determine the exponent





Other fit functions are possible: $\Gamma_{\rm Sphal}/T^4 = A(T/T_c)^{-B}$ gives excellent description with $B = 2.19(38) \implies \Gamma_{\rm Sphal} \sim T^2$

Recent studies of electric conductivity of Quark-Gluon Plasma suggest that $\Gamma_{\rm Sphal} \sim BT^2$ for intense magnetic fields B.

Almirante et al. (2024) [arXiv:2406.18504]

More investigations are needed to clarify the actual temperature dependence of the sphaleron rate.

Future outlooks

- Reach larger temperatures at GeV scale.

Not easy because $T \sim 1$ GeV on $N_t \sim 12$–16 requires $a \sim 0.02$–0.01 fm

$\implies$ Need better algorithms to mitigate **topological freezing**

Parallel Tempering on Boundary Conditions
CB et al., JHEP **08** (2024) 236 [arXiv:2404.14151]

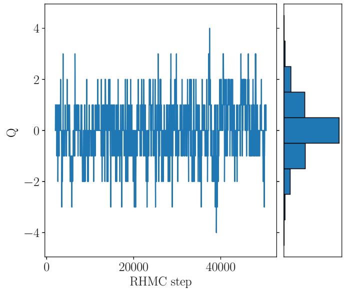
[NGT Algorithm Workshop @ CERN 9–11 Dec. 2024]

- Momentum-dependence of the sphaleron rate.

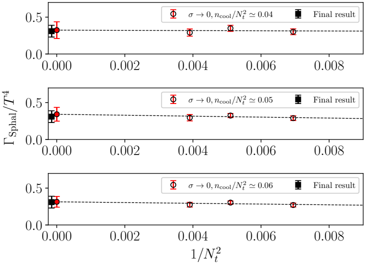
Phenomenologically relevant for axion cosmology.

$$\Gamma_{\mathrm{Sphal}}(p) \equiv \int \mathrm{d}^4 x \, \mathrm{e}^{\mathrm{i} p_\mu x^\mu} \, \langle q(x) q(0) \rangle$$

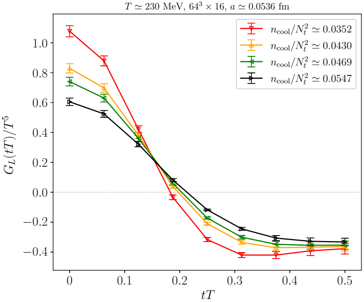
Back-up slides



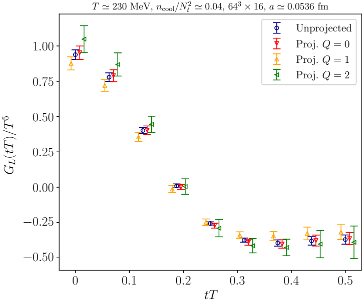
Q
RHMC step
0
20000
40000
4
2
0
−2
−4



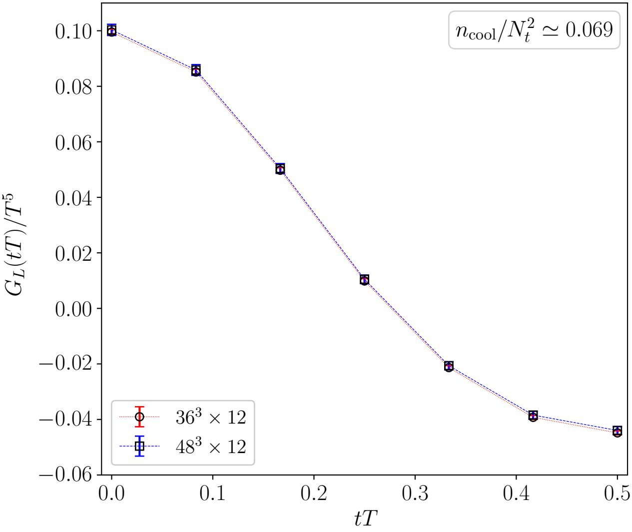
$\Gamma_{\rm Sphal}/T^4$
$\sigma \to 0, n_{\rm cool}/N_t^2 \simeq 0.04$
$\sigma \to 0, n_{\rm cool}/N_t^2 \simeq 0.05$
$\sigma \to 0, n_{\rm cool}/N_t^2 \simeq 0.06$
Final result
$1/N_t^2$
0.000
0.002
0.004
0.006
0.008
0.0
0.5



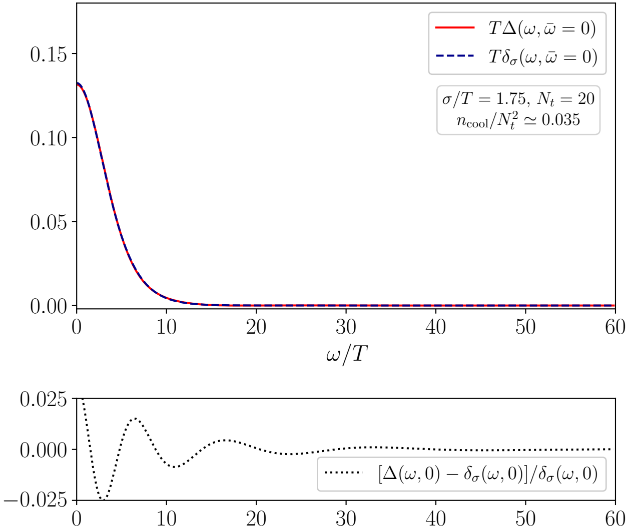
$T \simeq 230$ MeV, $64^3 \times 16$, $a \simeq 0.0536$ fm
$n_{\rm cool}/N_t^2 \simeq 0.0352$
$n_{\rm cool}/N_t^2 \simeq 0.0430$
$n_{\rm cool}/N_t^2 \simeq 0.0469$
$n_{\rm cool}/N_t^2 \simeq 0.0547$
$G_L(tT)/T^5$
tT



$T \simeq 230$ MeV, $n_{\rm cool}/N_t^2 \simeq 0.04$, $64^3 \times 16$, $a \simeq 0.0536$ fm
Unprojected
Proj. $Q = 0$
Proj. $Q = 1$
Proj. $Q = 2$
$G_L(tT)/T^5$
tT



$n_{\rm cool}/N_t^2 \simeq 0.069$
$G_L(tT)/T^5$
tT
$36^3 \times 12$
$48^3 \times 12$
0.10
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
0.0
0.1
0.2
0.3
0.4
0.5


$T\Delta(\omega, \bar{\omega} = 0)$
$T\delta_\sigma(\omega, \bar{\omega} = 0)$
$\sigma/T = 1.75,\ N_t = 20$
$n_{\rm cool}/N_t^2 \simeq 0.035$
0.15
0.10
0.05
0.00
0
10
20
30
40
50
60
ω/T
0.025
0.000
−0.025
$[\Delta(\omega, 0) - \delta_\sigma(\omega, 0)]/\delta_\sigma(\omega, 0)$