Cover Your Bases

Asymptotic Distributions of the Profile Likelihood Ratio When Constraining Effective Field Theories in High-Energy Physics

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Introduction

- **Effective Field Theories (EFTs)** are powerful tools in high-energy physics for exploring new physics beyond the Standard Model (SM).
- **Profile Likelihood Ratio (PLR):** A common test statistic used to perform hypothesis tests over EFT parameter space.
- **Wilks' Theorem:** Often assumed to apply, implying the PLR follows a χ^2-distribution.

Motivation

- Wilks' theorem often assumed to hold for PLR tests
  - When it holds, the PLR asymptotically follows a χ^2 distribution with degrees of freedom equal to the number of parameters profiled
- **This assumption is often violated in EFT analyses**
- Consequences:
  - Incorrect p-values
  - Inaccurate confidence intervals
  - Potential for over- or under-coverage

Why Wilks' Theorem Fails

- Wilks' theorem assumes:
  - Best-fit model lies in bulk of function space
  - Not near boundaries of parameter space
- Quadratic EFT parameterization:
  - Cannot fit data fluctuations below SM hypothesis
  - Leads to non-χ^2 behavior

Profile Likelihood Ratio (PLR)

- Gaussian statistical model:

$$p_x(x|c) = \frac{1}{\sqrt{2\pi|\Sigma|}} e^{-\frac{1}{2}\chi^2(x;c)}$$

$$\chi^2(x;c) = (x - \mu(c))^T \Sigma^{-1} (x - \mu(c))$$

- Definition of PLR test statistic:

$$q(x;c) = -2\ln\left[\frac{p_x(x|c)}{\max_{c'} p_x(x|c')}\right] = \chi^2(x;c) - \min_{c'} \chi^2(x;c')$$

EFT Parameterization

- General form of EFT-modulated cross section:

$$\sigma(f;c) \propto \left| \mathcal{M}_{\mathrm{SM}}(f) + \sum_{c_\alpha \in c} c_\alpha \mathcal{M}_{\mathrm{NP}}^{(c_\alpha)}(f) \right|^2$$

- Expanded form:

$$\sigma(f;c) = s(f) + \sum_{\alpha} c_\alpha l_\alpha(f) + \sum_{\alpha,\beta\neq\alpha} c_\alpha c_\beta t_{\alpha,\beta}(f) + \sum_{\alpha} c_\alpha^2 n_\alpha(f)$$

- In vector notation:

$$\mu(c) = s + \sum_{\alpha} c_\alpha l_\alpha + \sum_{\alpha,\beta\neq\alpha} c_\alpha c_\beta t_{\alpha,\beta} + \sum_{\alpha} c_\alpha^2 n_\alpha$$

Consider two simple cases

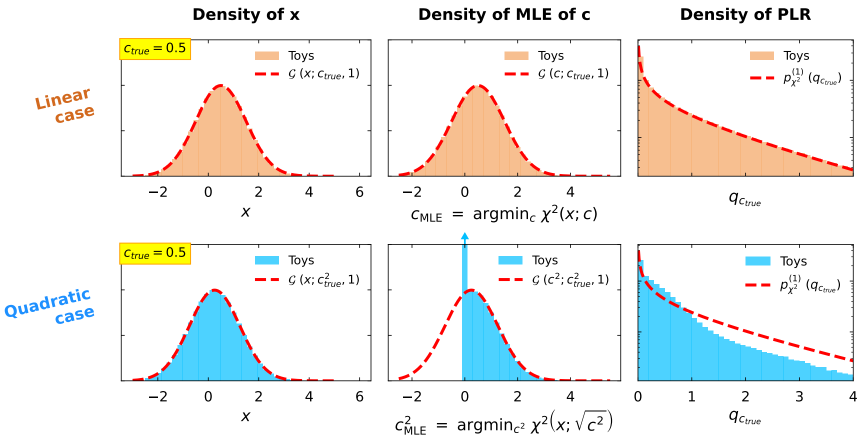
Linear Case

- Model: $\mu(c) = c$
- Represents models where dependence on new physics is dominated by interference with Standard Model

Quadratic Case

- Model: $\mu(c) = c^2$
- Represents models where coupling is purely imaginary or interference is small compared with the pure new physics contribution

Illustration of violation of Wilks' theorem



Solution with One Parameter: Linear Case

- For linear case ($|n| = 0$ and $|l| > 0$):

$$q_{c_{\text{true}}} = \hat{\bar{Z}}_l^2$$

where $\hat{\bar{Z}}_l$ is the projection of $\bar{z}$ onto the unit vector $\hat{\bar{l}}$.

- For purely linear EFT contributions:
  - Wilks' theorem holds
  - PLR follows a χ^2 distribution with one degree of freedom

Solution with One Parameter: Quadratic Case

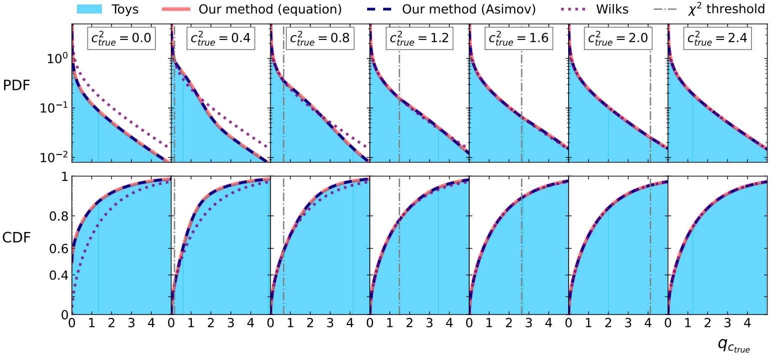
- For quadratic case ($|l| = 0$ and $|n| > 0$):

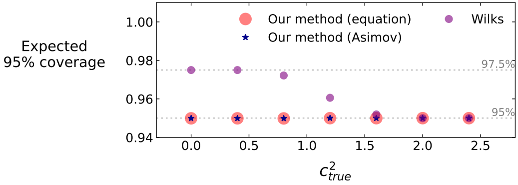
$$p_q(q_{c_{\text{true}}}) = \begin{cases} p^{(1)}_{\chi^2}(q_{c_{\text{true}}}) & q_{c_{\text{true}}} < \bar{N}^2 c^4_{\text{true}} \\ \frac{1}{2} p^{(1)}_{\chi^2}(q_{c_{\text{true}}}) + \mathcal{G}(q_{c_{\text{true}}}; -\bar{N}^2 c^4_{\text{true}}, 2\bar{N} c^2_{\text{true}}) & q_{c_{\text{true}}} \geq \bar{N}^2 c^4_{\text{true}} \end{cases}$$

- For purely quadratic EFT contributions:
  - PLR follows a mixture of χ^2 and Gaussian distributions
  - Distribution depends on the true parameter value

Quadratic Case (1D) - Example

Quadratic case (1D scan)





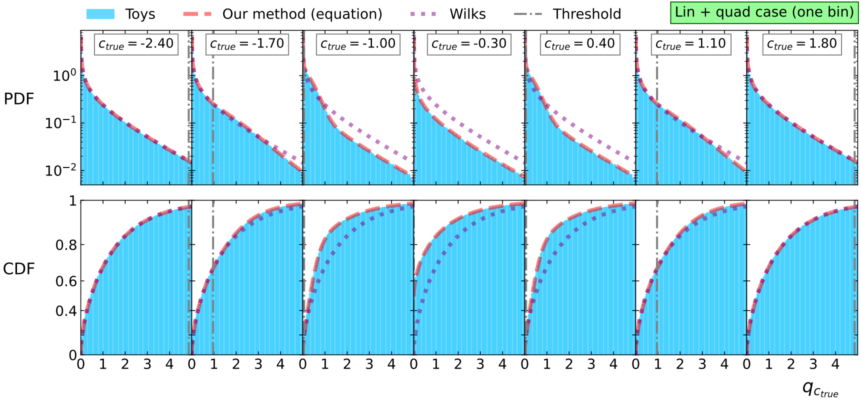
Quadratic Case (1D) - Example results

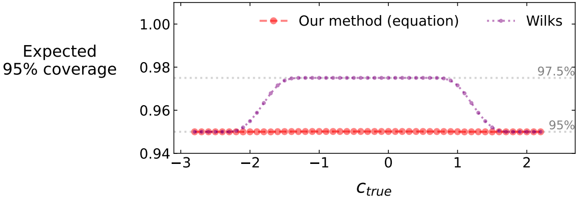
- Key observations:
  - When $c_{\text{true}}^2 = 0$, PDF is $\frac{1}{2}\delta(q_{c_{\text{true}}}) + \frac{1}{2}p_{\chi^2}^{(1)}(q_{c_{\text{true}}})$
  - As c_{true}^2 increases, χ^2-validity threshold moves upwards
  - For large c_{true}^2, distribution follows chi^2 distribution
  - Our method agrees with pseudo-experiments for all c_{true}^2 values
- Coverage analysis:
  - Wilks' theorem fails to exclude c_{true}^2 in half of expected cases when c_{true}^2 close to zero
  - Wilks' theorem leads to 97.5% coverage for hypotheses close to Standard Model
  - Our method provides correct 95% coverage for all c_{true}^2 values

Summary of Further Sections

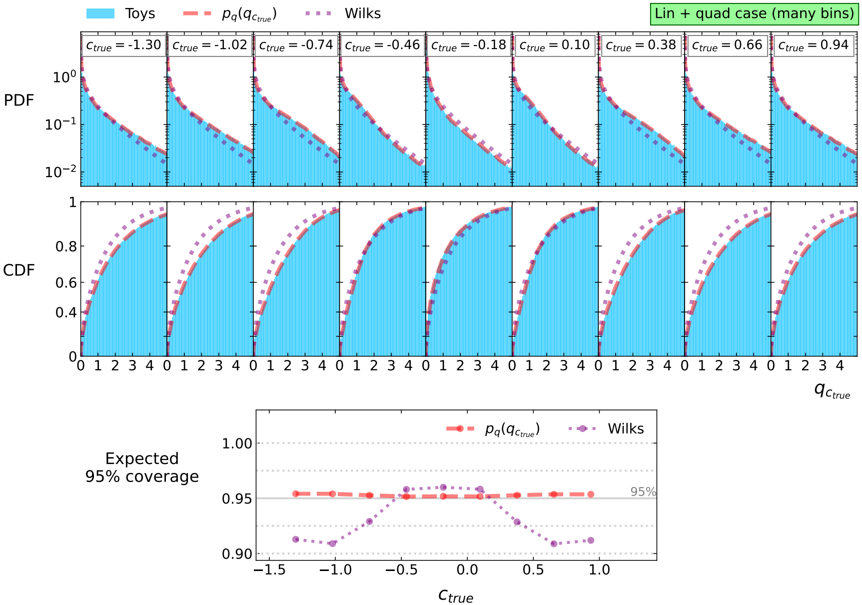
- Section 4: Two-parameter cases
  - Linear x Linear
  - Linear x Quadratic
  - Quadratic x Quadratic
- Section 5: One-parameter case with both linear and quadratic contributions
  - Single-bin measurements
  - Multi-bin measurements

Linear plus quadratic example (one bin)





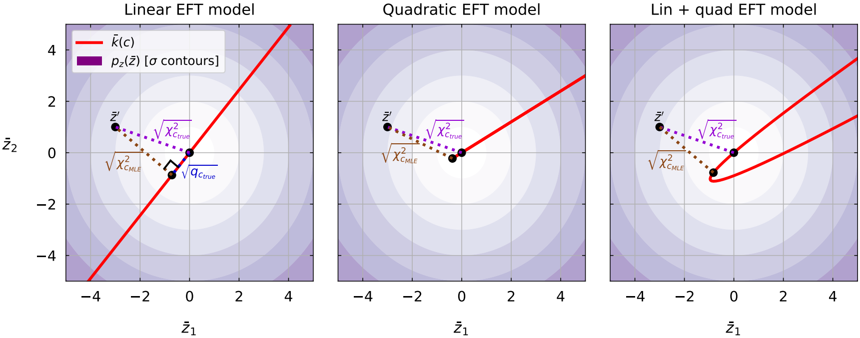
Linear plus quadratic example (multiple bins)



Future work

- Extend to more (or arbitrary number of) parameters
- Include interference terms
- Consider non-Gaussian nuisance parameters
- Improve the “linear plus quadratic” case
- Study “quadratic plus quartic” cases

Geometric interpretation



Geometric interpretation (explanation)

- Prediction (red line): straight for linear/quadratic, curved for lin+quad
- Prediction shape influences decomposition of $p(q)$ into χ^2 and Gaussian
- Problem: maximize likelihood (minimize χ^2)
- χ^2 = Euclidean distance in uncorrelated normalized basis
- Linear case: prediction spans full space, χ^2 for all observations
- Quadratic case: prediction turns at $c = 0$, resulting in χ^2 + Gaussian
- Lin+quad: prediction curve complicates mode splitting
- Lin+quad requires solving complex integrals for $p(q)$, often via numerical methods
- Quadratic curve appears linear due to mathematical structure
- Lin+quad curve is curvy, blending linear and quadratic terms