Particle Accelerators: An introduction

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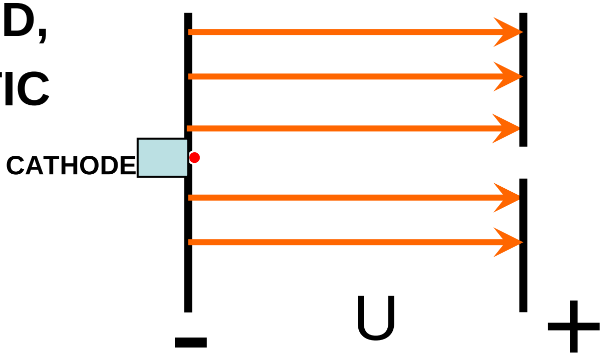
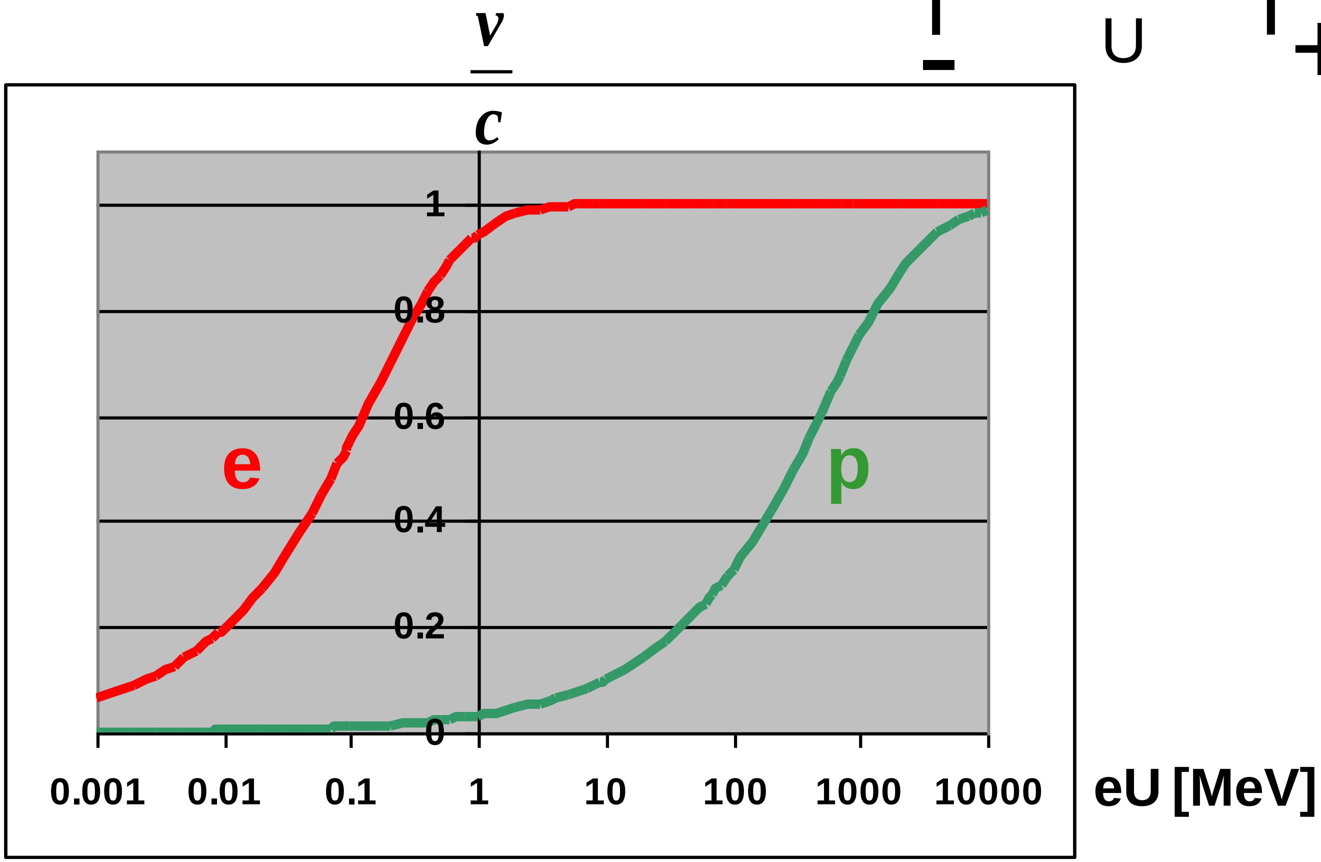
Applied relativity

FOR THE SAME ENERGY EXTRACTED FROM THE FIELD, A PARTICLE WITH LOWER MASS IS MORE RELATIVISTIC



$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$\gamma = 1 + \frac{eU}{E_o}$$



Beams of ultrarelativistic particles: e.g. a race to the Moon

An electron with energy of a few GeV emits a photon…

a race to the Moon!

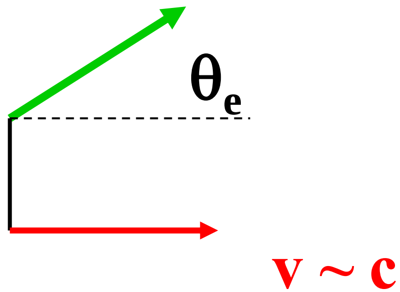
$$\Delta t = \frac{L}{\beta c} - \frac{L}{c} = \frac{L}{\beta c}(1-\beta) \sim \frac{L}{\beta c} \cdot \frac{1}{2\gamma^2}$$

Electron will lose

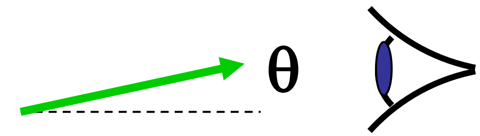
$$\Delta L = L(1-\beta) = \frac{L}{2\gamma^2}$$

- by only 8 meters
- the race will last only 1.3 seconds

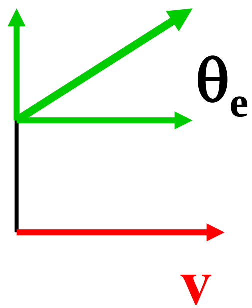
Transformation of angles: collimation

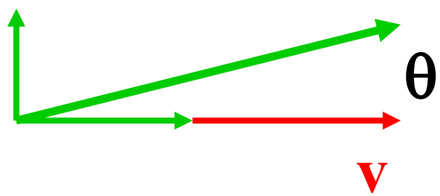


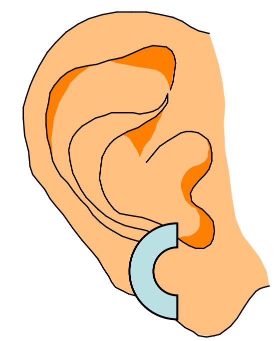
$$\theta = \frac{1}{\gamma} \cdot \theta_e$$



Sound waves (non-relativistic)







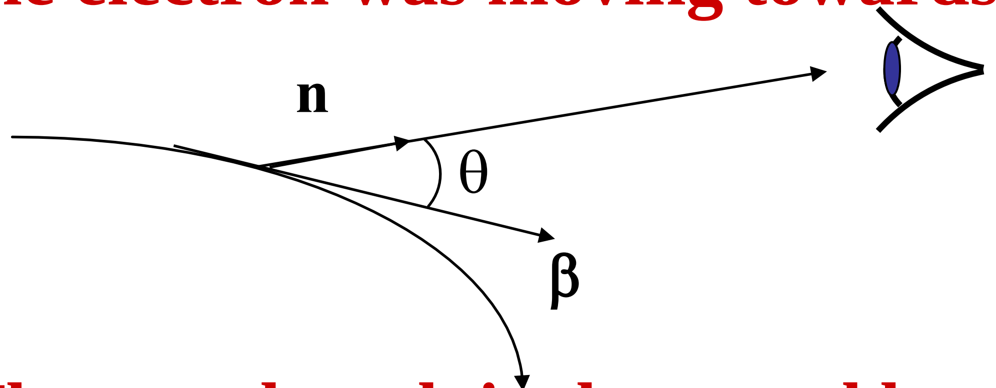
$$\theta = \frac{v_{s\perp}}{v_{s\parallel} + v} = \frac{v_{s\perp}}{v_{s\parallel}} \cdot \frac{1}{1 + \frac{v}{v_s}} \approx \theta_e \cdot \frac{1}{1 + \frac{v}{v_s}}$$

<u>Doppler effect (moving source of sound)</u>

$$\lambda_{heard} = \lambda_{emitted} \left(1 - \frac{v}{v_s} \right)$$

Time compression

Electron with velocity β emits a wave with period T_{emit} while the observer sees a different period T_{obs} because the electron was moving towards the observer



$$T_{obs} = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) T_{emit}$$

The wavelength is shortened by the same factor

$$\lambda_{obs} = (1 - \beta \cos\theta) \lambda_{emit}$$

in ultra-relativistic case, looking along a tangent to the trajectory

$$\lambda_{obs} = \frac{1}{2\gamma^2} \lambda_{emit}$$

since

$$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \cong \frac{1}{2\gamma^2}$$

Electromagnetism

Maxwell equations (poetry)

War es ein Gott, der diese Zeichen schrieb
Die mit geheimnisvoll verborg'nem Trieb
Die Kräfte der Natur um mich enthüllen
Und mir das Herz mit stiller Freude füllen.

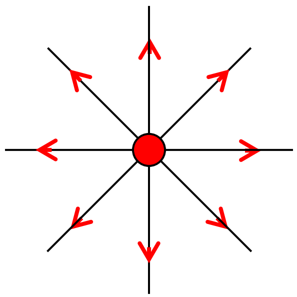
Ludwig Boltzman

Was it a God whose inspiration
Led him to write these fine equations
Nature's fields to me he shows
And so my heart with pleasure glows.

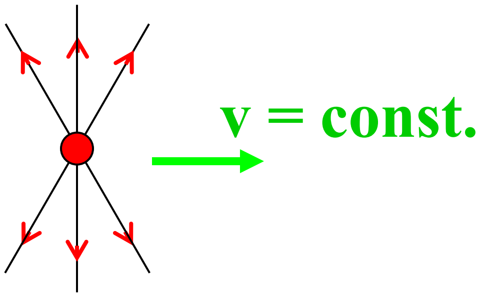
translated by John P. Blewett

Field of a charge

At rest: Coulomb field



Moving with constant velocity



Charge in an external electromagnetic field

$$F = e(E + [v \times B])$$

Lorentz force

Particle motion in electromagnetic fields

Lorentz force $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Same force for:

- Magnetic field B = 1 Tesla (typical for magnets)
- Electric field E = $3 \cdot 10^8$ V/m (presently out of reach)

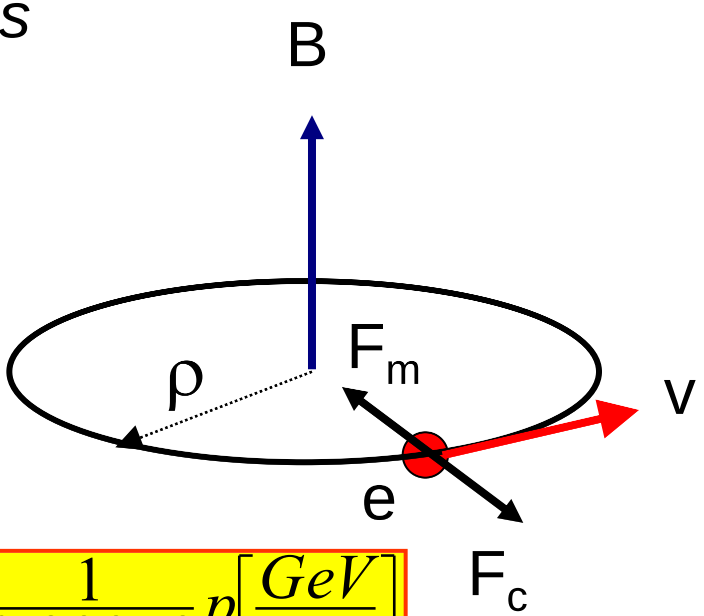
Magnetic fields are used exclusively to bend and focus ultra-relativistic particles

- Constant magnetic field

- Magnetic rigidity $B\rho = \frac{p}{e}$

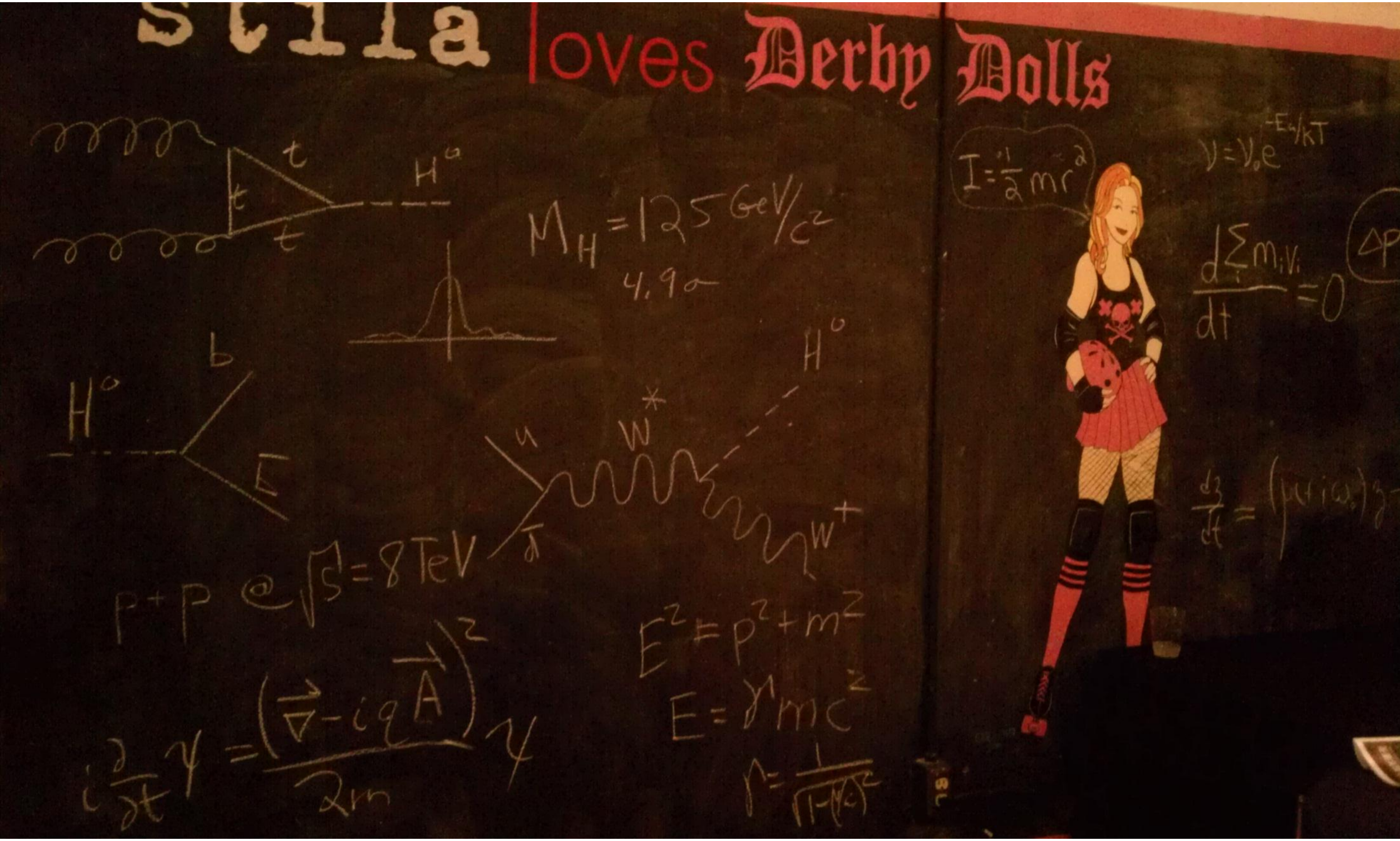
or, in practical units

$$[T \cdot m]\ B\rho = \frac{1}{0.29979} p \left[\frac{GeV}{c}\right]$$



Roller derby in Los Angeles on 7 July 2012

Under the sign of the Higgs on 7 July 2012

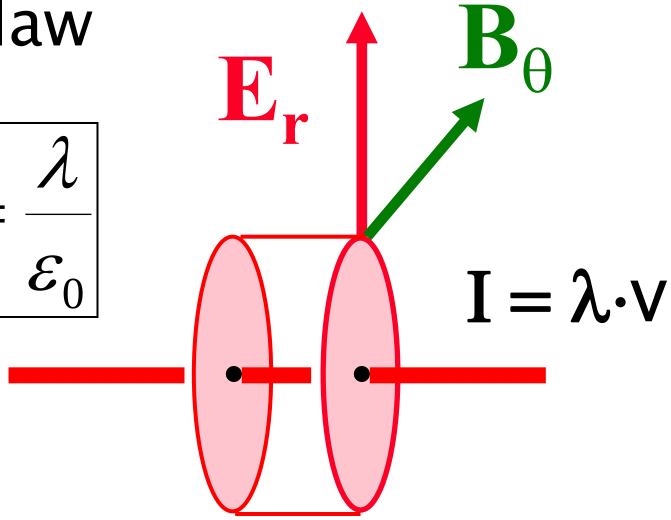


Fields of a long bunch (linear charge density λ)

Transverse electric field: from Gauss law

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$$

$$2\pi r \cdot E_r = \frac{\lambda}{\varepsilon_0}$$



Transverse magnetic field: from Ampere law

$$B_\theta = \frac{\mu_0 \lambda}{2\pi r} \mathrm{v} = \frac{\lambda}{2\pi\varepsilon_0 r} \cdot \frac{\mathrm{v}}{c^2}$$

$$2\pi r \cdot B_\theta = \mu_0 I$$

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$$B_\theta[T] = \frac{1}{c} E_r \left[\frac{V}{m}\right]$$

$$\mu_0 = 4\pi \cdot 10^{-7}\ {}^{V\cdot s}\!/_{A\cdot m}$$

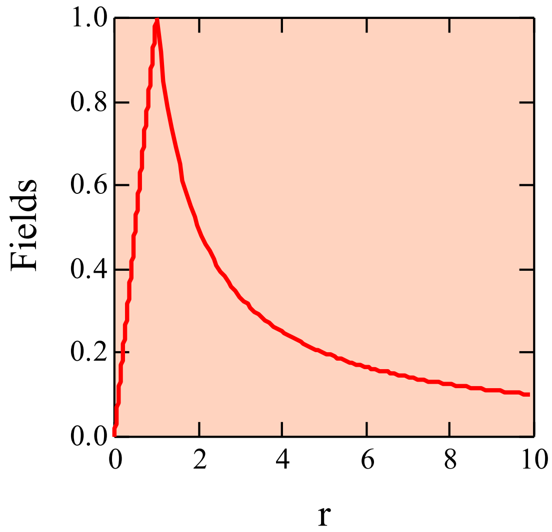
$$\varepsilon_0 = 8.85 \cdot 10^{-12}\ {}^{C}\!/_{V\cdot m}$$

Fields in the bunch

- Round uniform distribution

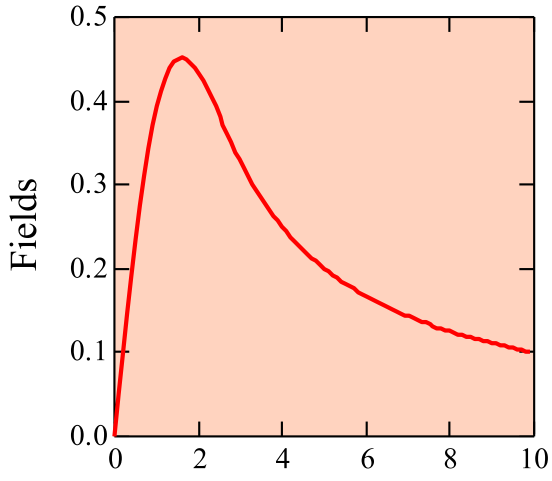
$$E_r = \frac{eN}{2\pi\varepsilon_0 l}\cdot\frac{1}{r} \qquad r > a$$

$$E_r = \frac{eN}{2\pi\varepsilon_0 l}\cdot\frac{r}{a^2} \qquad r < a$$

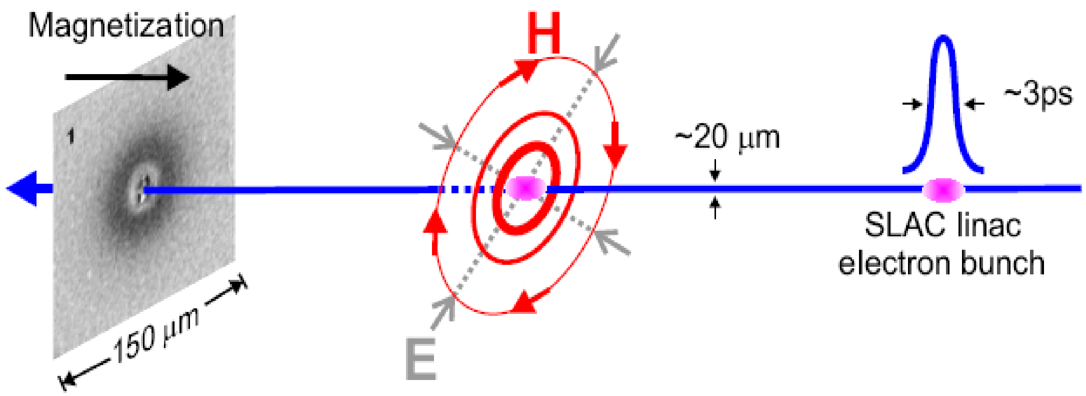


- Round Gaussian distribution

$$E_r = \frac{eN}{2\pi\varepsilon_0 l\sigma}\left[\frac{1-e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}}{\frac{r}{\sigma}}\right]$$



Using large magnetic fields of electron beam



$$E_r\left[\frac{GV}{m}\right] \approx \frac{eNr}{4\pi\varepsilon_0 l\sigma^2} \approx 14.4\,\frac{N\left[10^{10}\right]r[\mu m]}{l[mm]\cdot\sigma^2[\mu m]}$$

$$B_\theta[T] \approx \frac{E_r}{c} \approx 50\,\frac{N\left[10^{10}\right]r[\mu m]}{l[mm]\cdot\sigma^2[\mu m]}$$

Force seen by a test particle

- "Fellow–traveler": E and B nearly cancel

$$\text{Force} = eE_r - e\text{v}B_\theta = eE_r\left(1-\frac{\text{v}^2}{c^2}\right) = \frac{1}{\gamma^2}eE_r$$

- Particle travelling in the opposite direction: contributions from E and B add

$$\text{Force} = eE_r + e\text{v}B_\theta \approx 2eE_r$$

  - For round Gaussian distribution

$$F = \frac{e^2N}{\pi\varepsilon_0 l}\cdot\frac{1}{r}\cdot\left(1-e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2}\right)$$

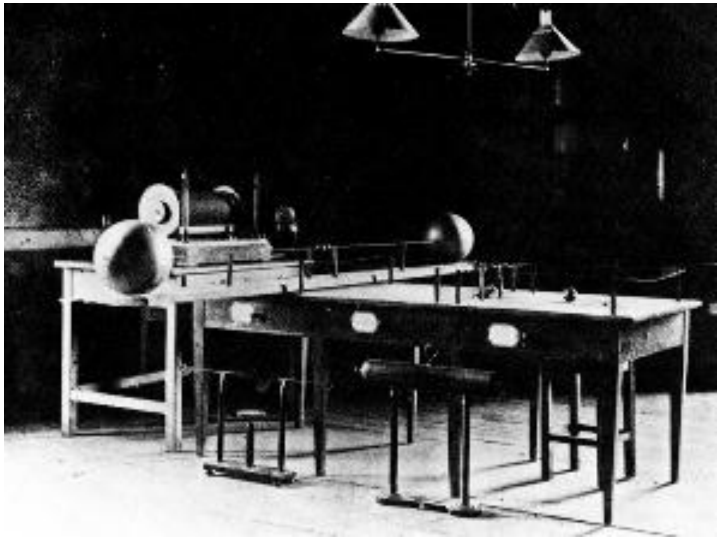
Synchrotron radiation

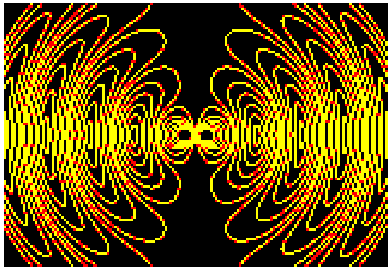
THEORETICAL UNDERSTANDING →

1873 Maxwell's equations

→ made evident that changing charge densities would result in electric fields that would radiate outward

1887 Heinrich Hertz demonstrated such waves:





….. this is of no use whatsoever !

Accelerated charges radiate EM waves

An electron of energy **E** in a magnetic field **B**

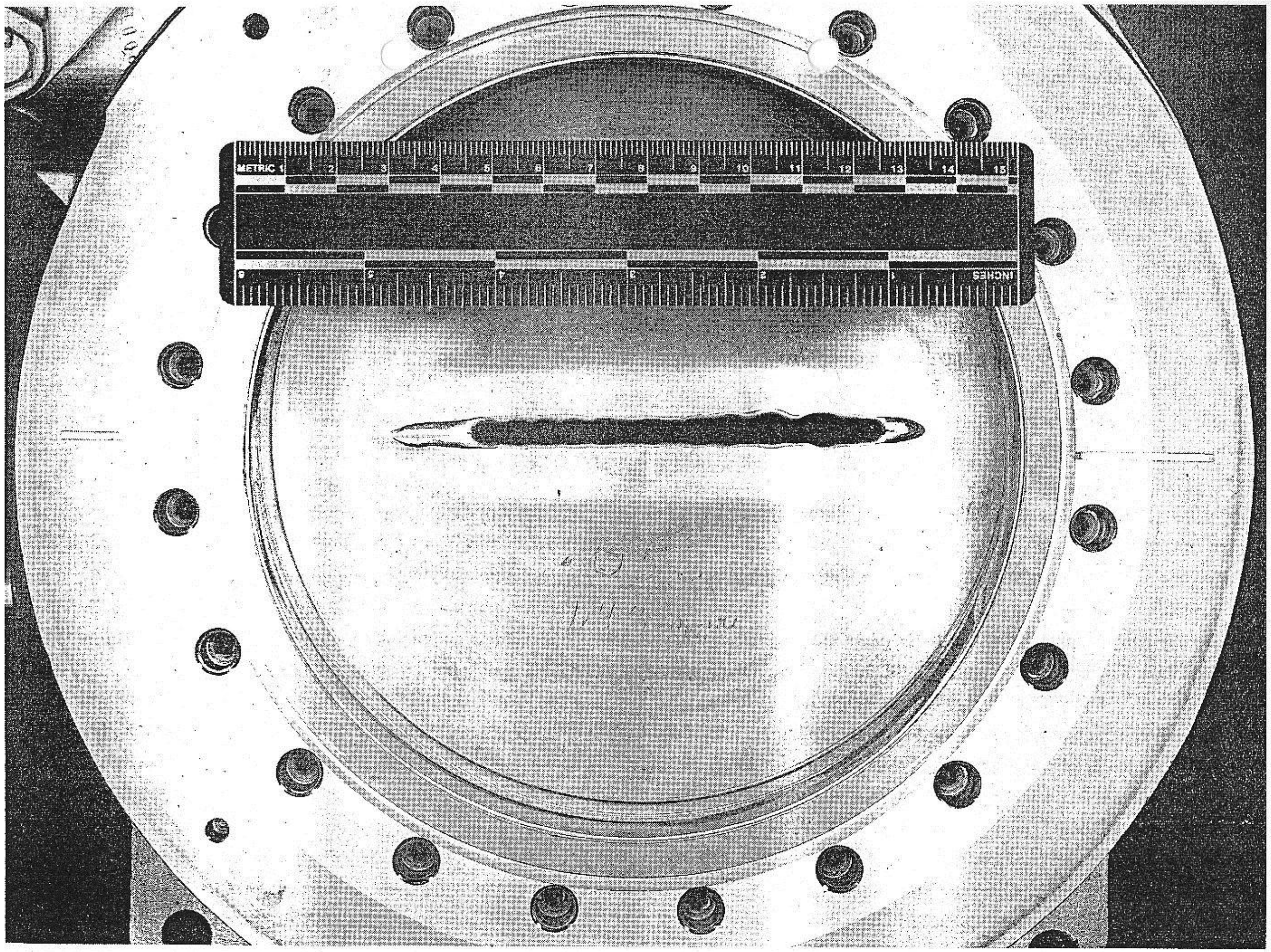
Power emitted is proportional to:

$$P \propto E^2 B^2$$

$$P_\gamma = \frac{c C_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2}$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{\left(m_e c^2\right)^3} = 8.858 \cdot 10^{-5} \left[\frac{\mathrm{m}}{\mathrm{GeV}^3}\right]$$

The power is all too real!



'ig. 12. Damaged X-ray ring front end gate valve. The power incident on the valve was approximately 1 kW for a duration estimated to 2–10 min and drilled a hole through the valve plate.

Synchrotron radiation power

Power emitted is proportional to:

$$P \propto E^2 B^2$$

$$P_{\mathrm{SR}} = \frac{c C_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2}$$

$$P_{\mathrm{SR}} = \frac{2}{3} \alpha \hbar c^2 \frac{\gamma^4}{\rho^2}$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{\left(m_e c^2\right)^3} = 8.858 \cdot 10^{-5} \left[\frac{\mathrm{m}}{\mathrm{GeV}^3}\right]$$

$$\alpha = \frac{1}{137}$$

$$\hbar c = 197 \ \mathrm{Mev} \cdot \mathrm{fm}$$

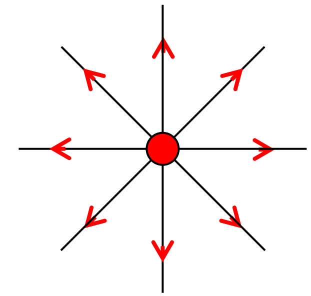
Energy loss per turn:

$$U_0 = C_\gamma \cdot \frac{E^4}{\rho}$$

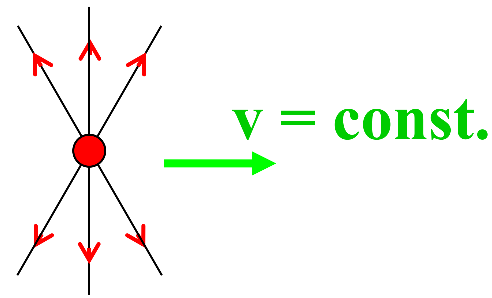
$$U_0 = \frac{4\pi}{3} \alpha \hbar c \frac{\gamma^4}{\rho}$$

Why do they radiate?

Charge at rest: Coulomb field, no radiation

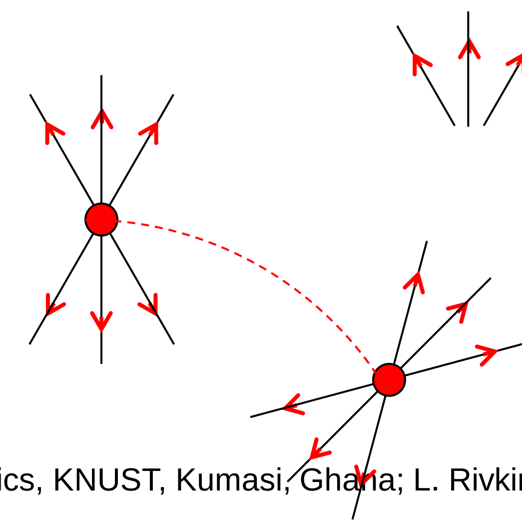


Uniformly moving charge
does not radiate

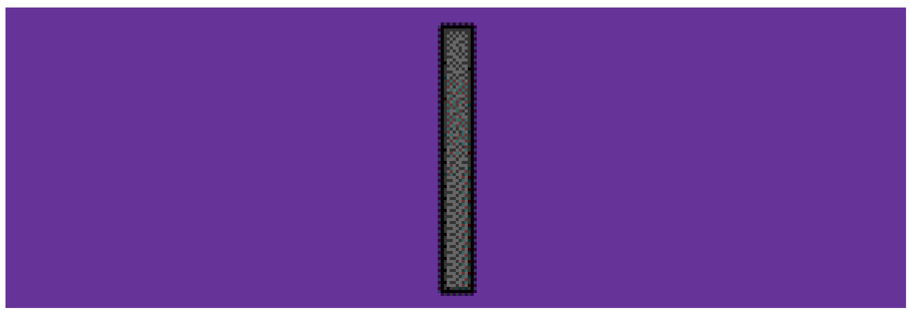


But! Cerenkov!

Accelerated charge:
fields separate from the charge



Bremsstrahlung
or
breaking radiation



Liénard–Wiechert potentials

$$\varphi(t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left[r\left(1-\vec{\mathbf{n}}\cdot\vec{\boldsymbol{\beta}}\right)\right]_{ret}} \qquad \vec{\mathbf{A}}(t) = \frac{q}{4\pi\varepsilon_0 c^2}\left[\frac{\vec{\mathbf{v}}}{r\left(1-\vec{\mathbf{n}}\cdot\vec{\boldsymbol{\beta}}\right)}\right]_{ret}$$

and the electromagnetic fields:

$$\nabla\cdot\vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$

(Lorentz gauge)

$$\vec{\mathbf{B}} = \nabla\times\vec{\mathbf{A}}$$

$$\vec{\mathbf{E}} = -\nabla\varphi - \frac{\partial\vec{\mathbf{A}}}{\partial t}$$

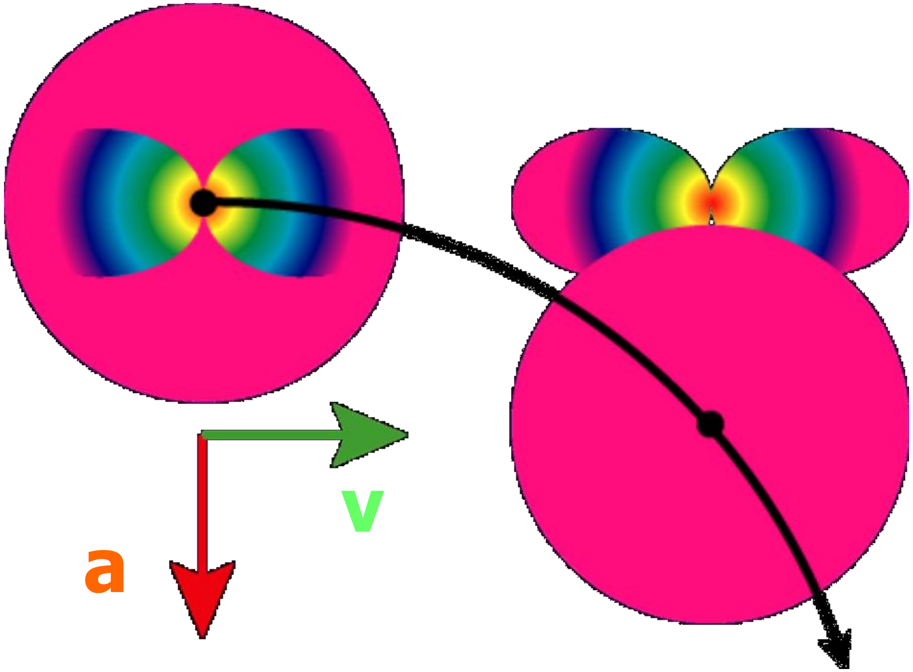
Fields of a moving charge

$$\vec{\mathbf{E}}(t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{\mathbf{n}} - \vec{\boldsymbol{\beta}}}{\left(1 - \vec{\mathbf{n}} \cdot \vec{\boldsymbol{\beta}}\right)^3 \gamma^2} \cdot \boxed{\frac{1}{\mathbf{r}^2}}\right]_{ret} +$$

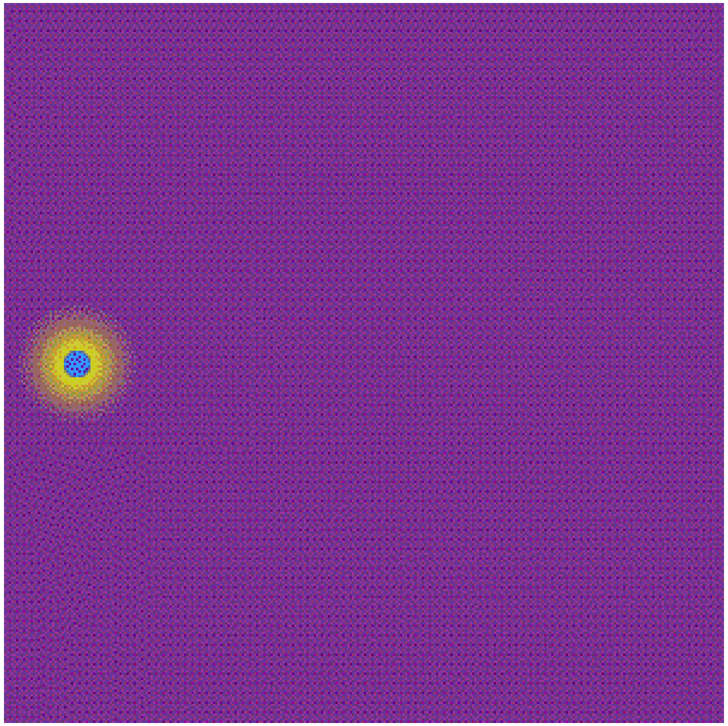
$$\frac{q}{4\pi\varepsilon_0 c}\left[\frac{\vec{\mathbf{n}} \times \left[\left(\vec{\mathbf{n}} - \vec{\boldsymbol{\beta}}\right) \times \boxed{\dot{\vec{\boldsymbol{\beta}}}}\right]}{\left(1 - \vec{\mathbf{n}} \cdot \vec{\boldsymbol{\beta}}\right)^3 \gamma^2} \cdot \boxed{\frac{1}{\mathbf{r}}}\right]_{ret}$$

$$\vec{\mathbf{B}}(t) = \frac{1}{c}\left[\vec{\mathbf{n}} \times \vec{\mathbf{E}}\right]$$

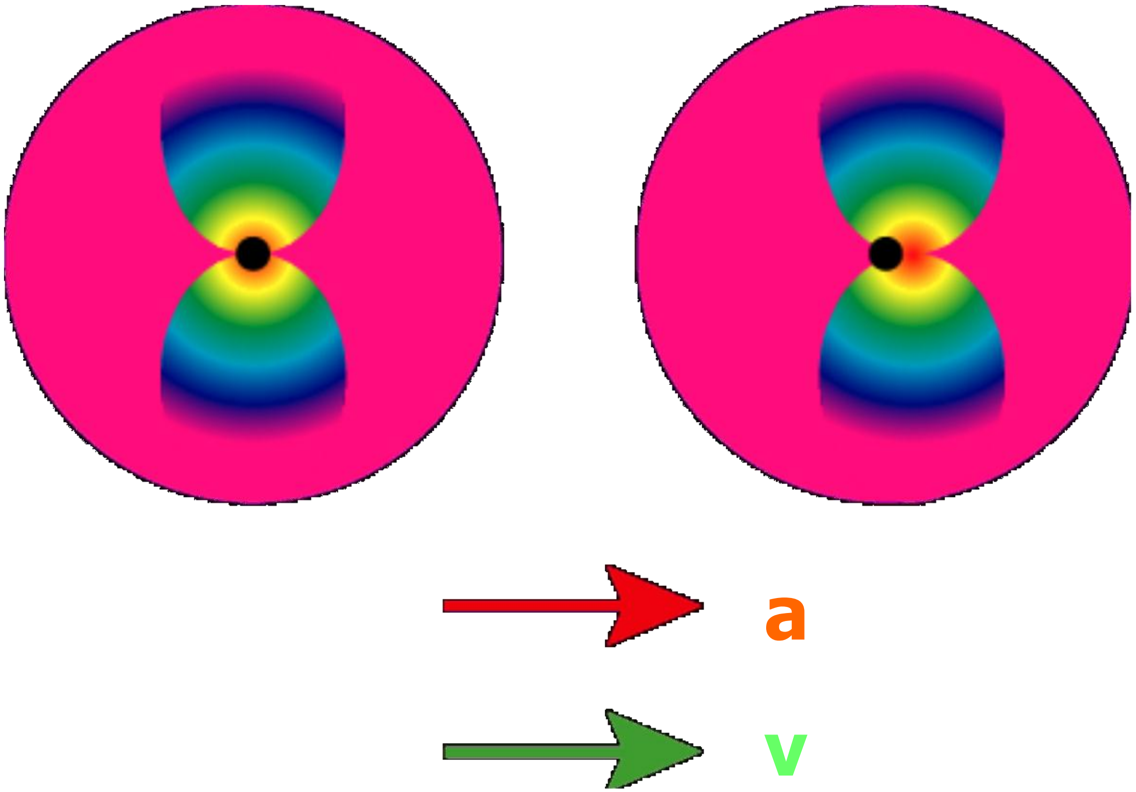
Transverse acceleration



Radiation field quickly separates itself from the Coulomb field



Longitudinal acceleration

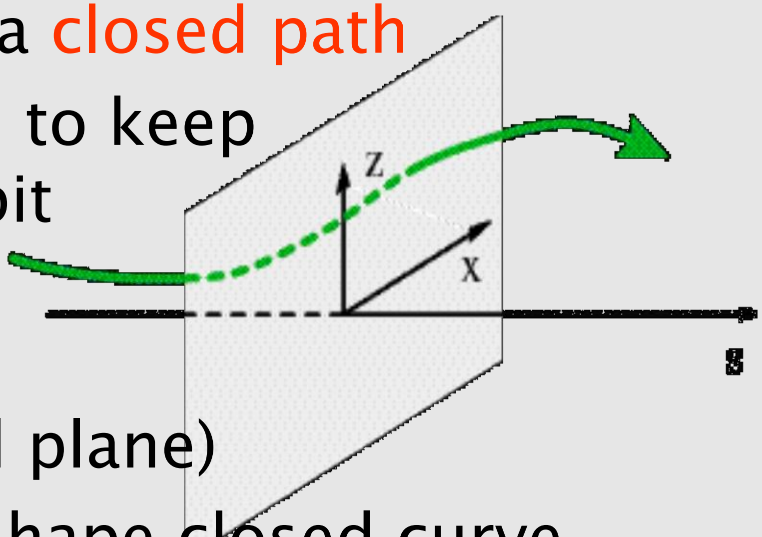


Radiation field cannot separate itself from the Coulomb field

High Energy Storage Ring

To store relativistic particles ($v \approx c$) in a ring for ~ 10h they travel a distance of diameter of Pluto's orbit

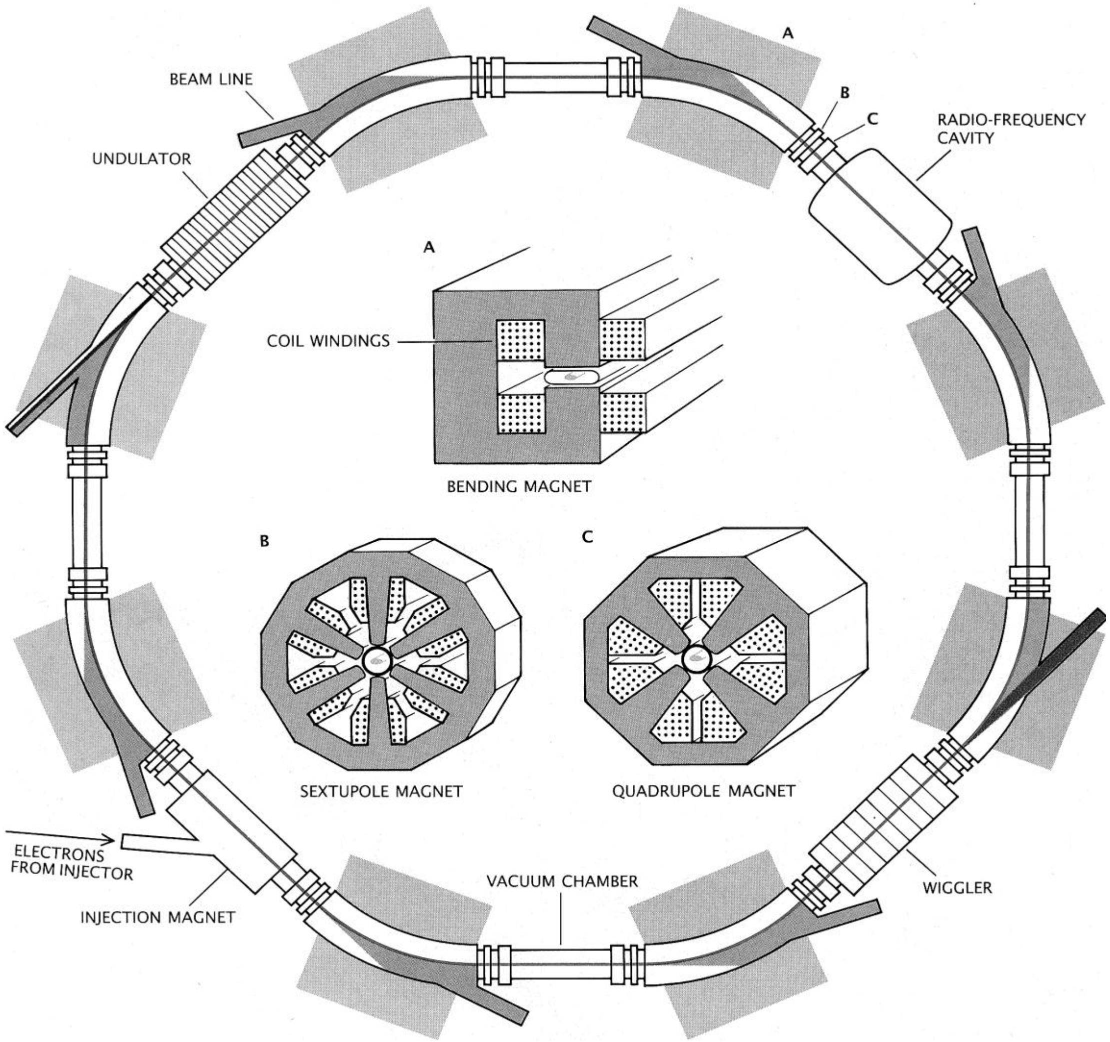
- Trajectories are bent into a closed path
- Beams need to be focused to keep particles close to ideal orbit (stability questions)



Ideal orbit (usually in horizontal plane)

- Smooth, roughly circular shape closed curve, consisting of arcs and straight sections
- Magnets are placed along the ideal orbit, design fields adjusted, so that particles of nominal energy follow the ideal orbit for ever and ever and ever ...

Storage ring layout



Particle motion in electromagnetic fields

Lorentz force $\mathbf{F} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$

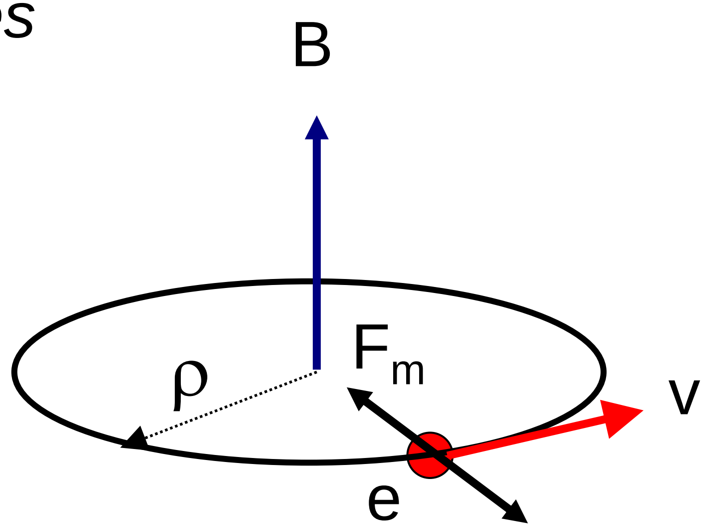
Same force for:

- Magnetic field B = 1 Tesla (typical for magnets)
- Electric field E = $3 \cdot 10^8$ V/m (presently out of reach)

Magnetic fields are used exclusively to bend and focus ultra-relativistic particles

- Constant magnetic field
- Magnetic rigidity $B\rho = \frac{p}{e}$

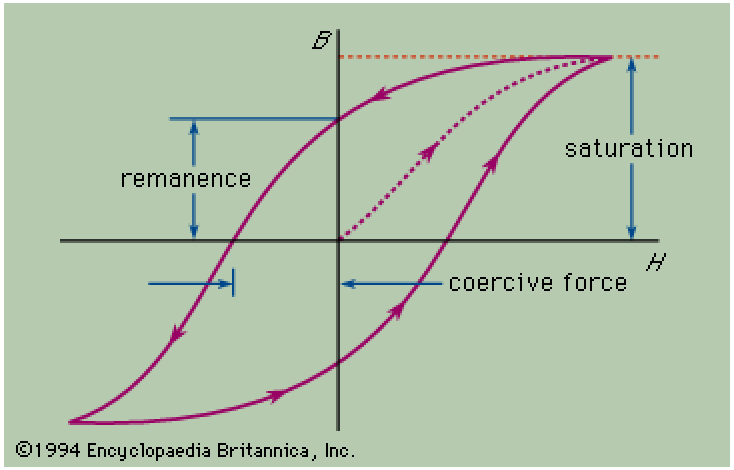
or, in practical units

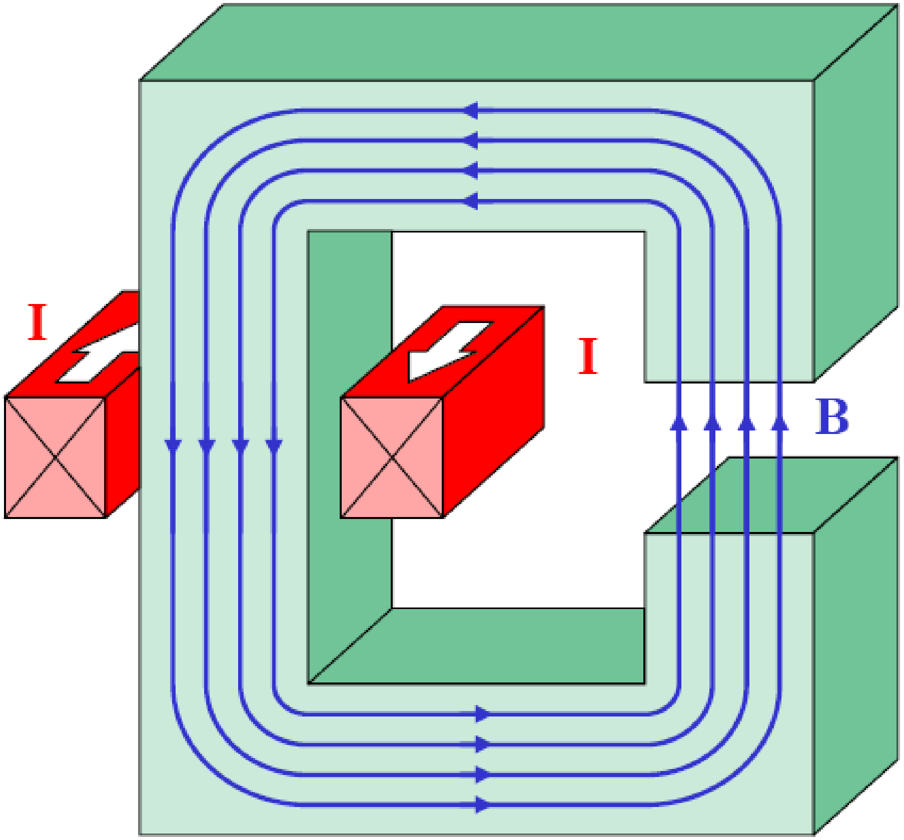


$$[T \cdot m]\ B\rho = \frac{1}{0.29979} p \left[\frac{GeV}{c}\right]$$

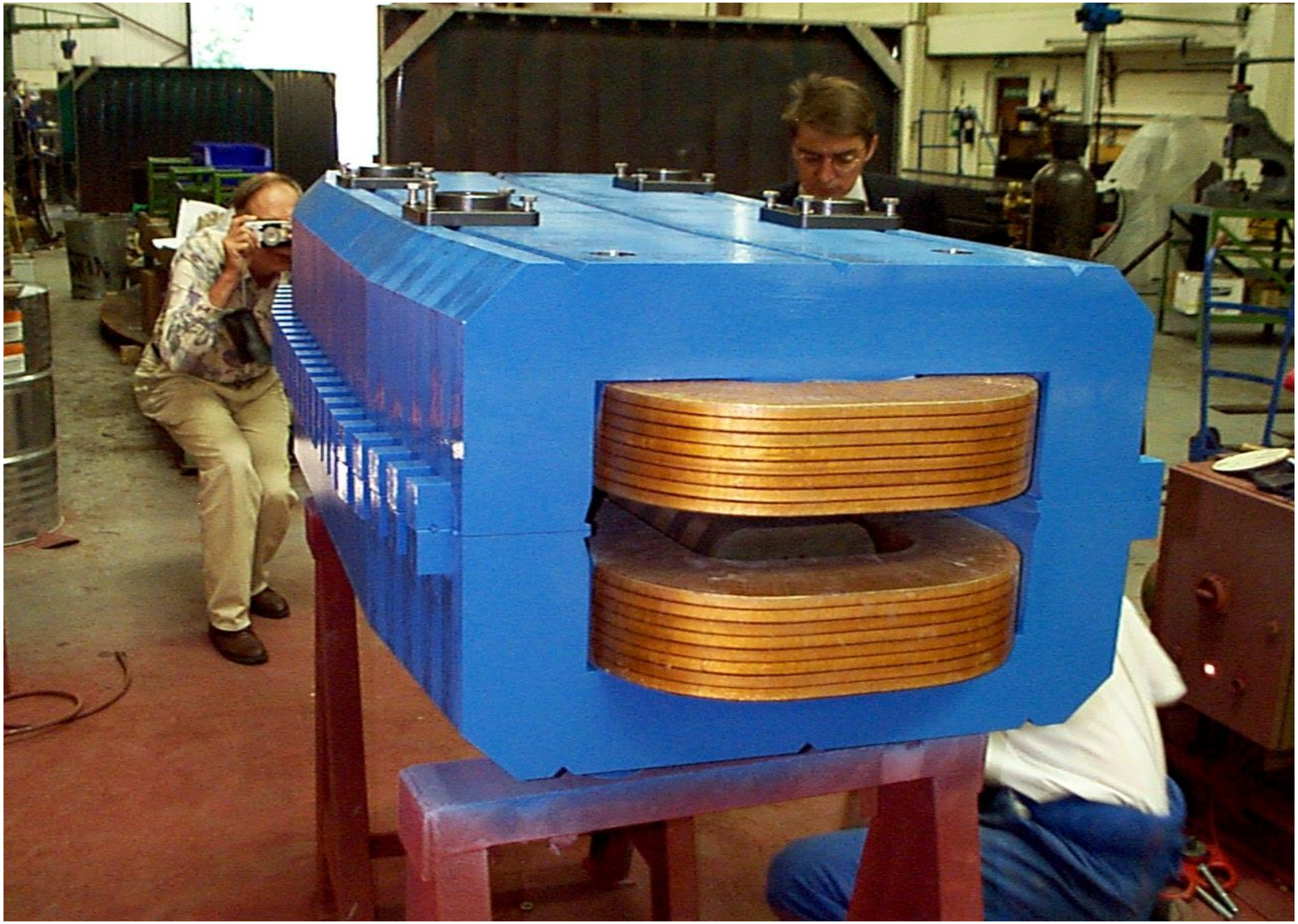
Bending magnets (iron dominated)

Iron saturates at 2 T





SLS dipole

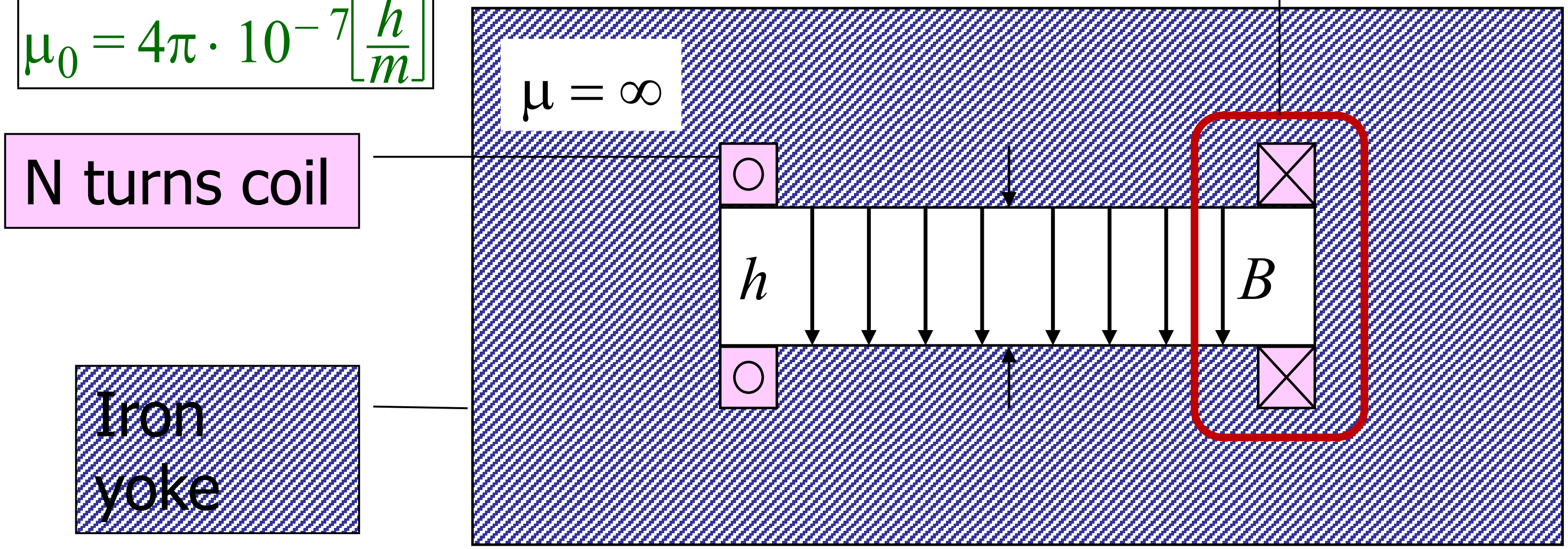


Bending magnets (dipoles)

Iron dominated magnets (B < 2 Tesla)

$$\oint \frac{B}{\mu} dl = 2NI$$

$$\mu_0 = 4\pi \cdot 10^{-7} \left[\frac{h}{m} \right]$$



$$B[T] = \frac{2\mu_0 NI [Amp \cdot turns]}{h[m]}$$

$e.g.\ for\ h = \pi\ [cm]$

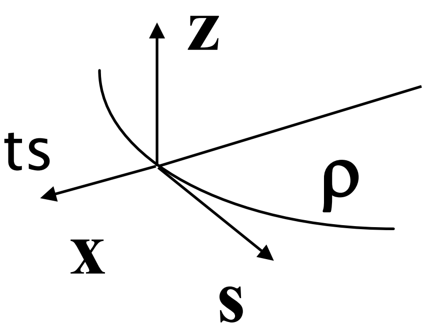
$N \cdot I = 20'000\ [Amp \cdot turns]$

$B = 1.6\ Tesla$

System of coordinates

Design orbit in horizontal plane

- consists of arcs and straight segments



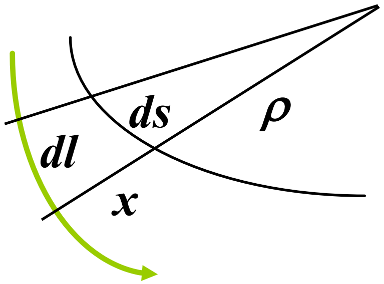
Local curvilinear coordinates:

- x, z transverse displacements from design orbit
- s measured along the design orbit
- ρ(s) local radius of curvature (depends on field)

- Length element

$$dl = ds \cdot \frac{\rho + x}{\rho}$$

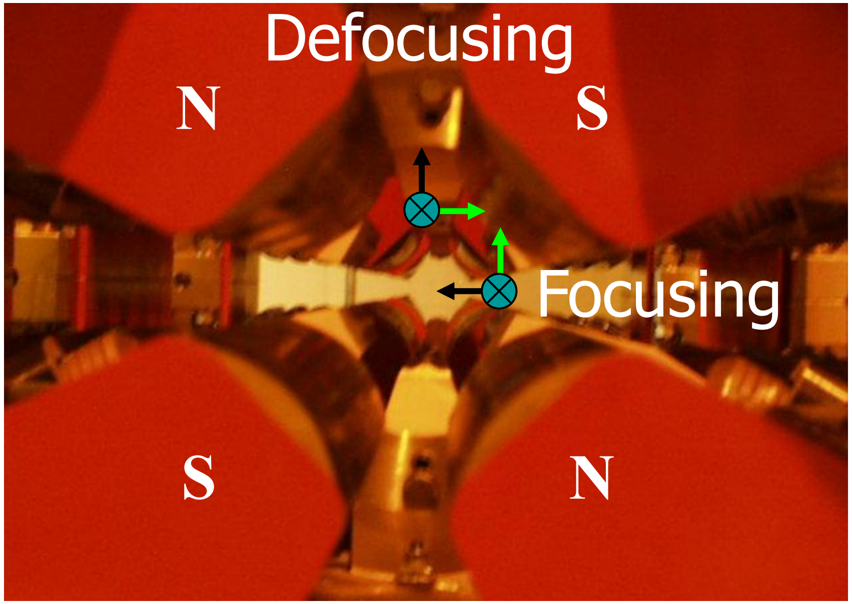
$$dl^2 = dx^2 + dz^2 + \left(1 + \frac{x}{\rho}\right)^2 ds^2$$

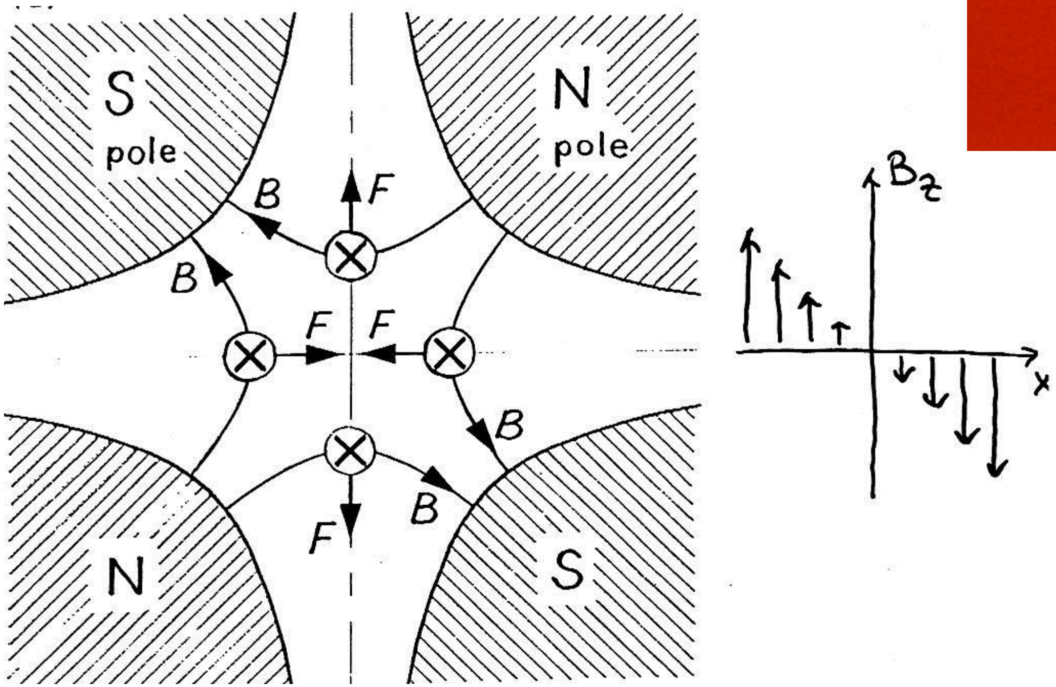


Quadrupole lens

Focusing in one plane

Defocusing in the other plane

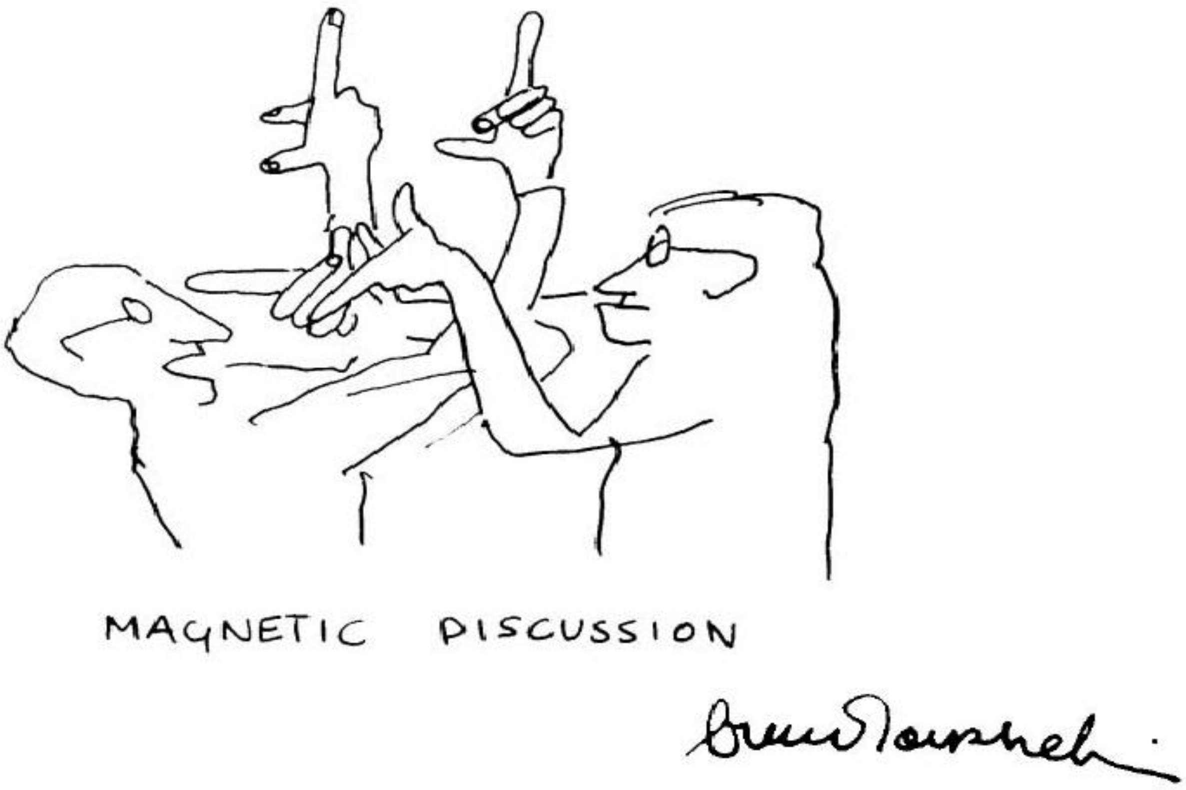




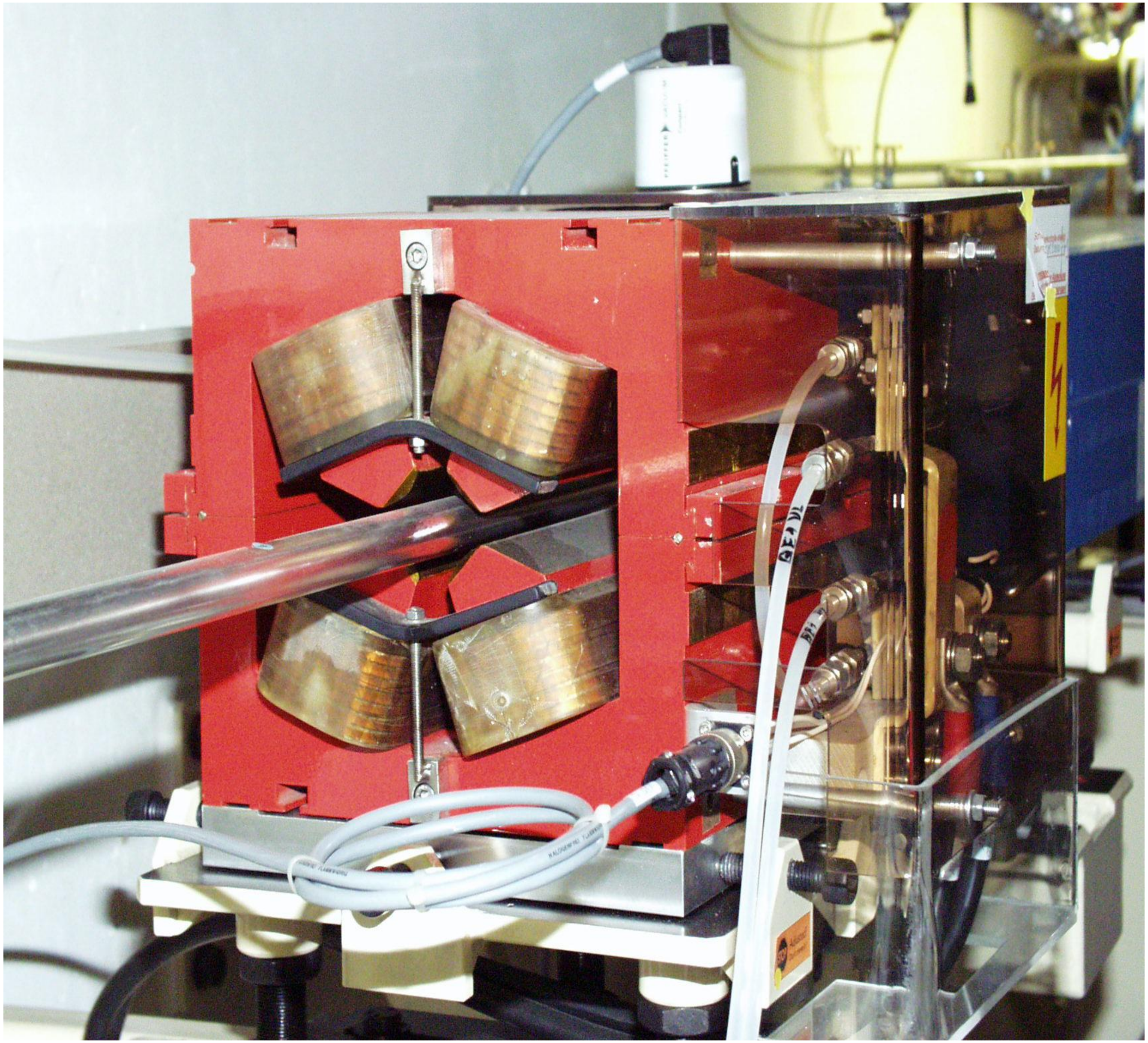
$$\nabla \times B = 0 \quad \Rightarrow \quad \frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}$$

Linear restoring force

Theoretical magnetism (after Bruno Touschek)



Focusing elements

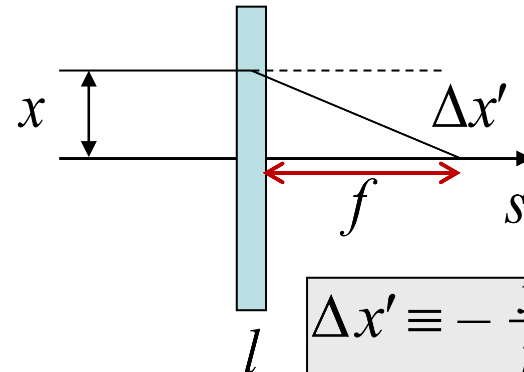


Focal length of a thin quad

Particle off-axis in a quad

- sees constant magnetic field (thin lens!) and is bent by it

- the slope $x' \equiv \frac{dx}{ds}$ changes by



$$\Delta x' = -\frac{l}{\rho} = -l\left(\frac{eB_z}{p}\right)$$

$$\Delta x' \equiv -\frac{x}{f}$$

- Defining the focal length

1 Dioptre = 1 m^{-1}

$$\frac{1}{f} = \frac{e}{p} g \cdot l \quad \textit{with gradient} \quad g \equiv \frac{\partial B_z}{\partial x}$$

$$\frac{1}{f} = \frac{g \cdot l}{(B\rho)} \quad [m^{-1}]$$

Hamiltonian dynamics (brief reminder)

A dynamical system is described by a Hamiltonian

$$H(q, p, t)$$

q	coordinate
p	canonical momentum
t	independent variable (time)

The equations of motion: Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

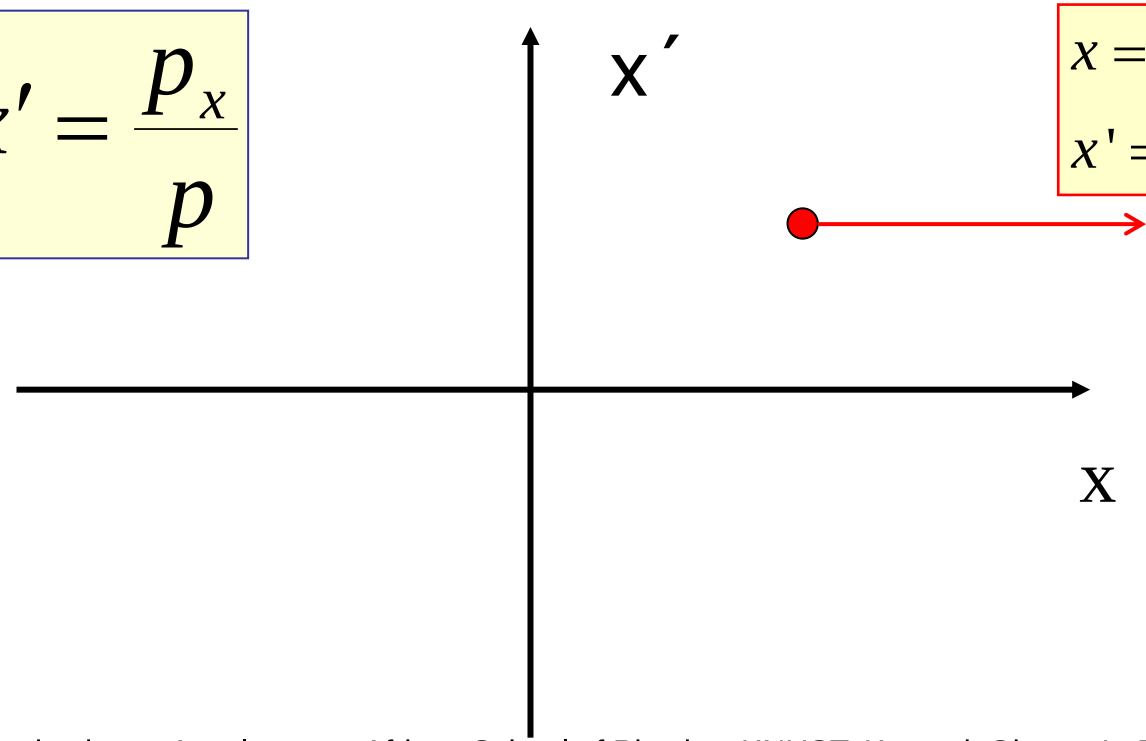
Phase space

$\left(x, p_x\right)$ – canonical variables

$\left(x, x'\right)$ – a point in 2-d phase space

$$x' = \frac{p_x}{p}$$

$$x = x_o + s \cdot x_o'$$
$$x' = x_o'$$



Conservative Hamiltonian systems

$$H = \frac{p^2}{2m} + V(q,t)$$

The equations of motion are:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q} \equiv F(q,t)$$

The Hamiltonian is conserved, ist value – energy

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0$$

$$\frac{dH}{dt} = 0$$

Phase space

$$(q, p)$$

– a point in 2-d phase space

$$\mathrm{v} = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial p} \right)$$

– velocity vector in 2-d phase space

$$\nabla H(q, p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial p} \right)$$

– the gradient of the Hamiltonian, orthogonal to velocity

The motion is along the curves of *H = const*

DRIFT SECTION

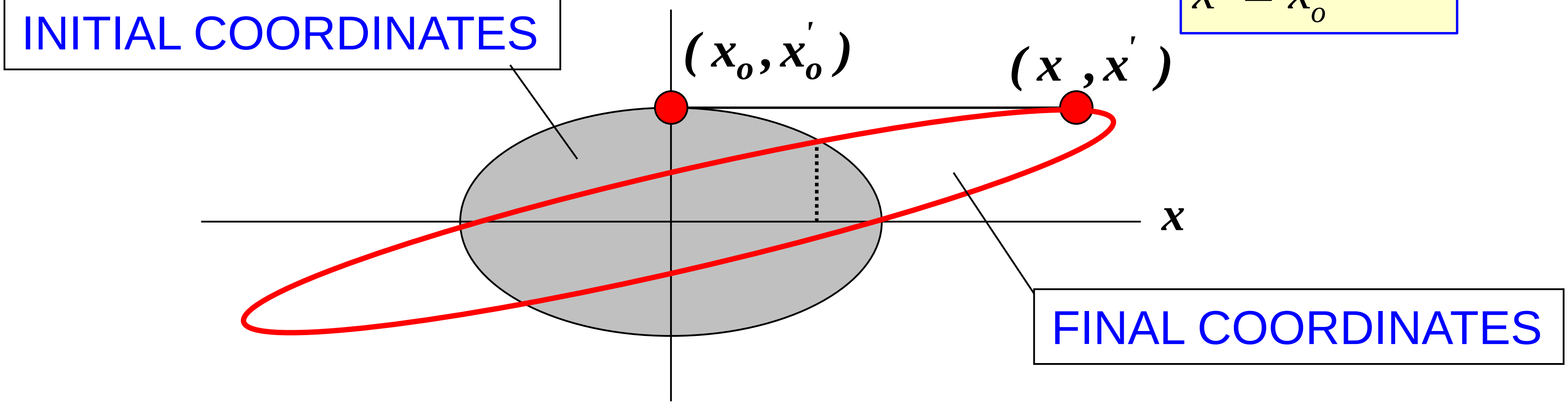
CHANGE OF PARTICLE DISTRIBUTION IN PHASE SPACE

The initial coordinates of a particle ensemble in the transverse phase plane are contained in the ellipse:

$$x' = \frac{dx}{ds}$$

$$x = x_o + s \cdot x_o'$$

$$x' = x_o'$$



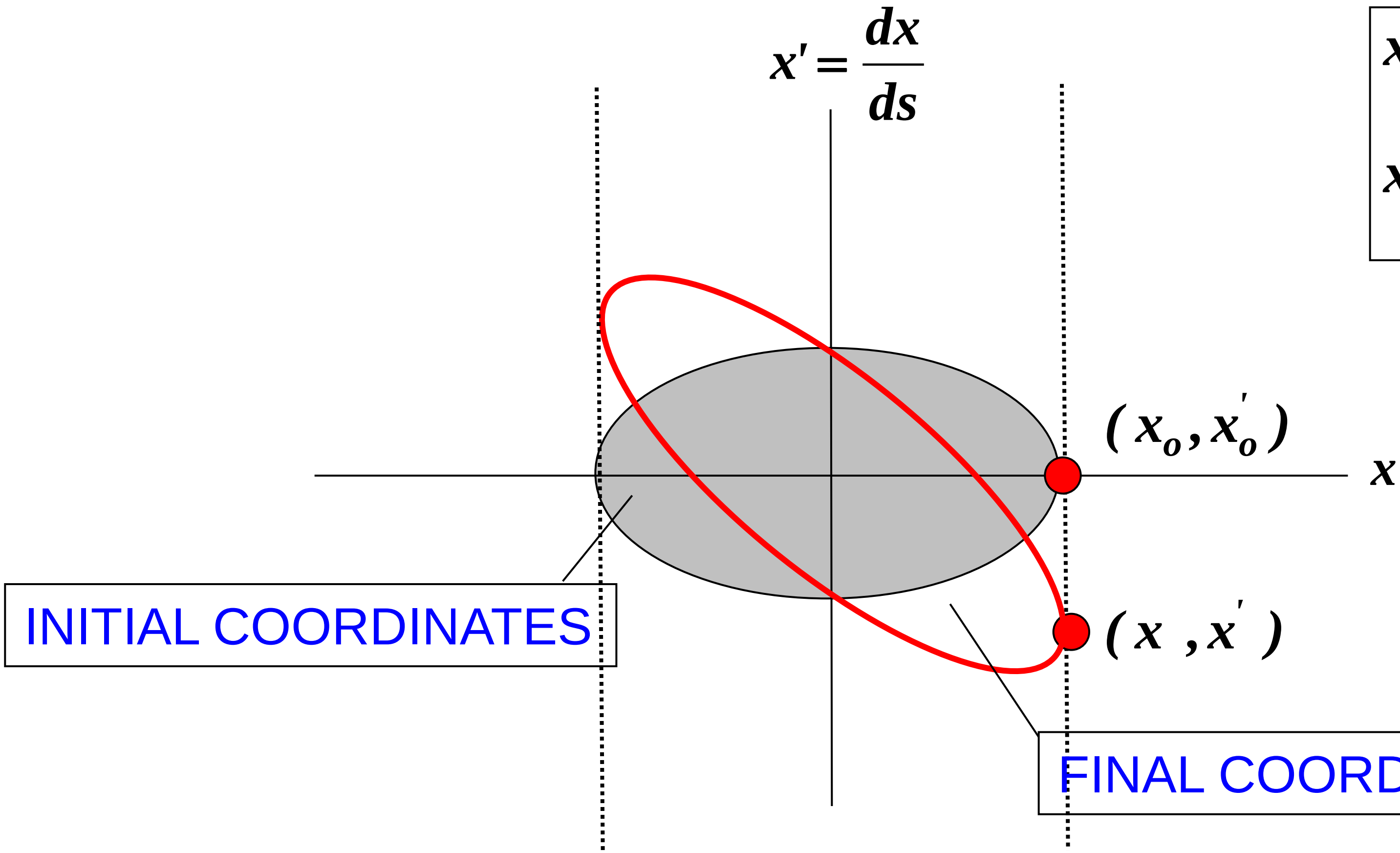
Focusing is needed to avoid beam blow up !

FOCUSING QUADRUPOLE

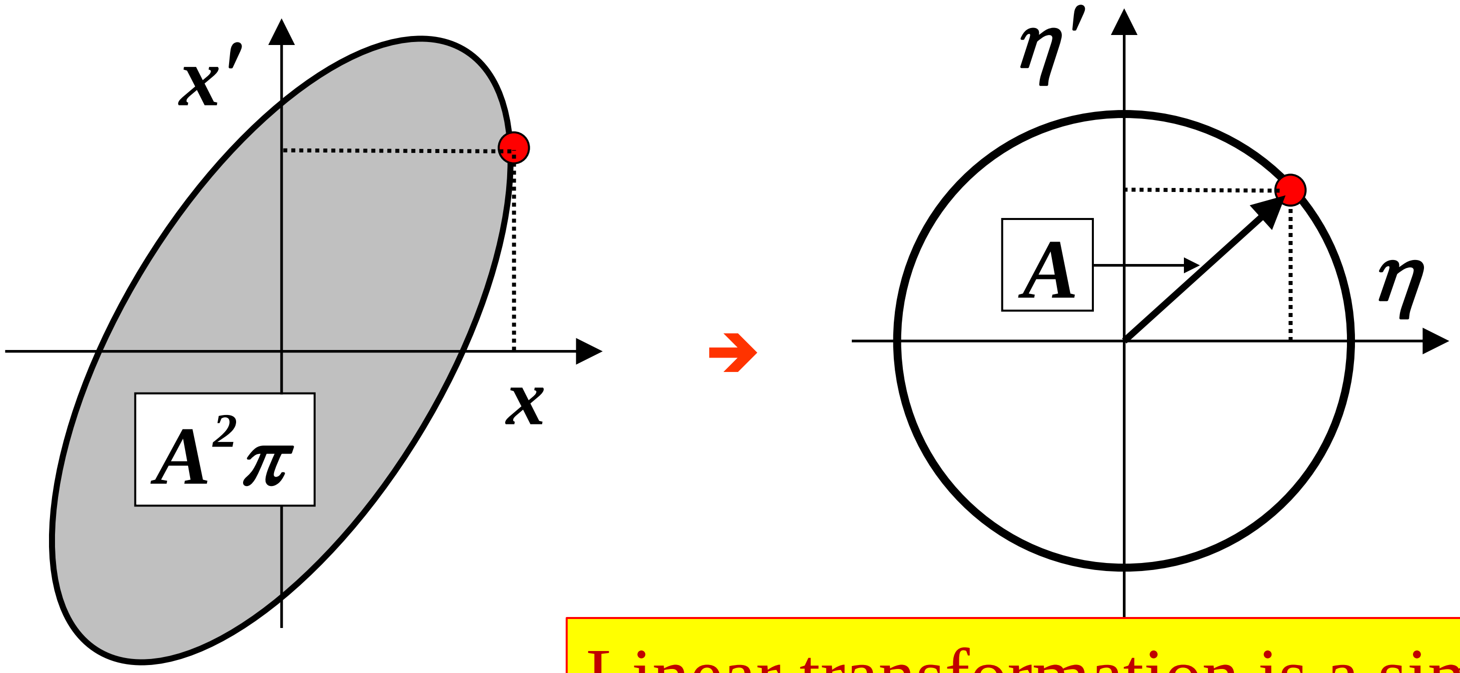
CHANGE OF PARTICLE DISTRIBUTION IN PHASE SPACE

$$x = x_o$$

$$x' = x_o' - \frac{x_o}{f}$$



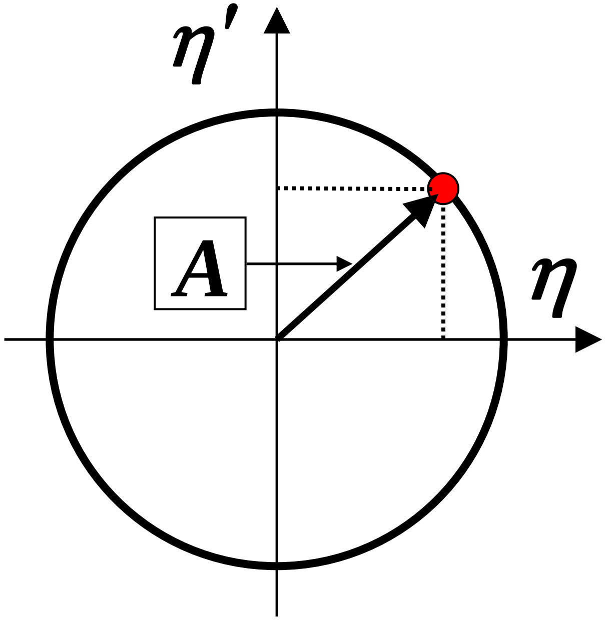
PHASE SPACE: angle – action variables

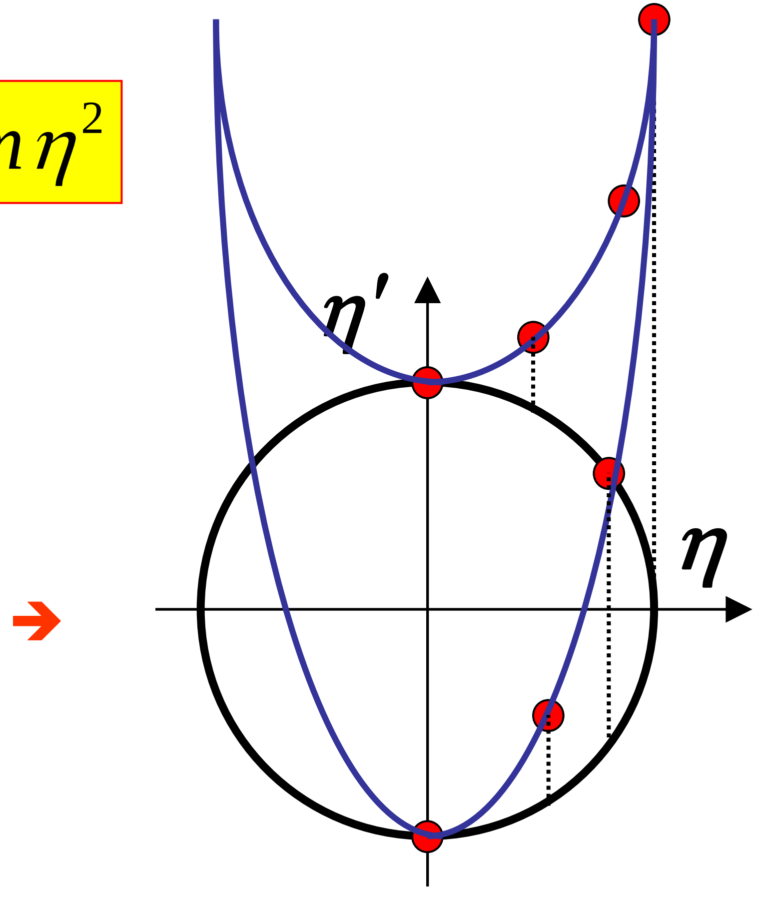


Linear transformation is a simple rotation in these coordinates

PHASE SPACE TRANSFORMATION in NON-linear element (sextupole magnet)

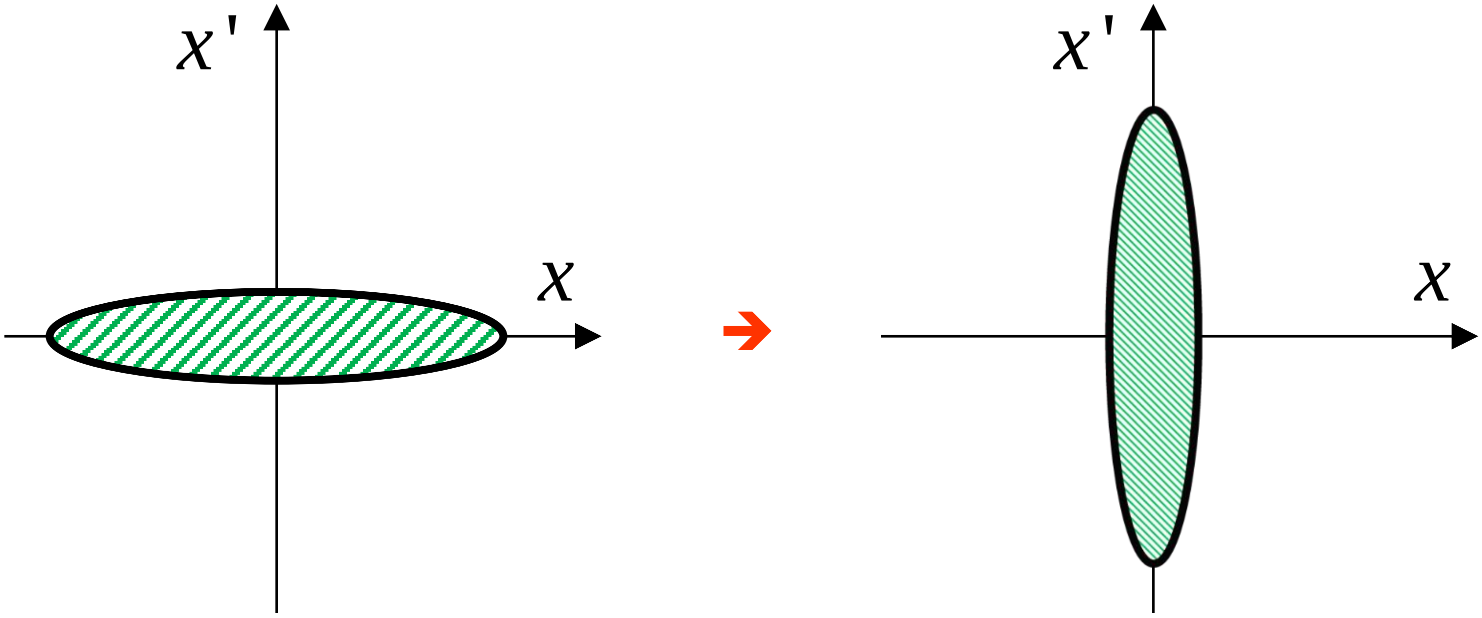
$$\Delta \eta' \propto m \eta^2$$





Conservation of phase space: emittance

Canonical transformations
preserve phase space areas



The language of Accelerator Physics

Matrix notation

Transfer matrices (as in geometric optics)

- Describe canonical transformations i.e. phase space area is preserved (symplectic matrices)

$$\det M = 1$$

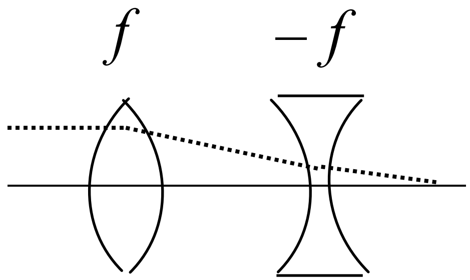
- Thin focusing lens

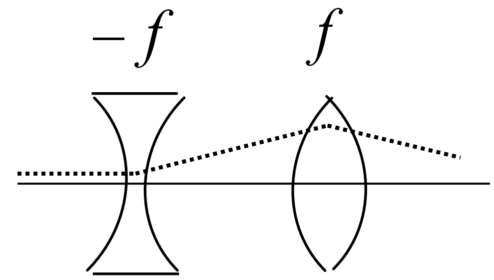
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

- Drift of length L

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

Strong focusing example





$$M_x = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$$M_z = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

$$M_x = \begin{pmatrix} 1 - L/f & L \\ -L/f^2 & 1 + L/f \end{pmatrix} \qquad \text{focusing for } L \ll f$$

This lens doublet focuses in both planes

- The focal length is, of course, the good old lens makers equation

$$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 \cdot f_2} = \frac{L}{f^2}$$

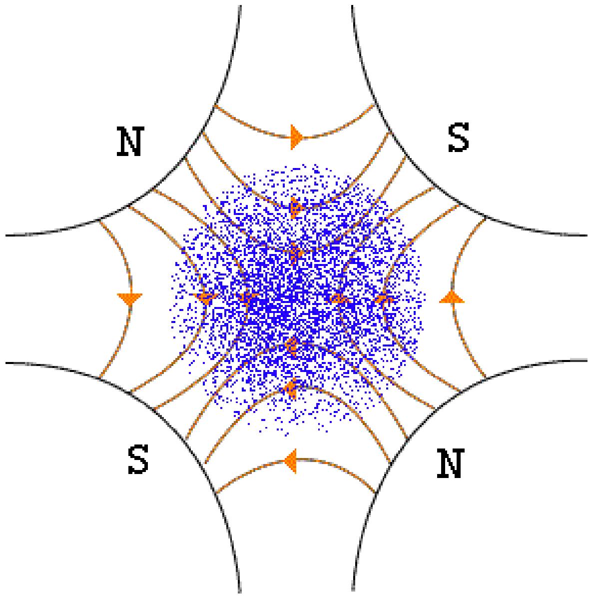
SUMMARY: Strong (Transverse)Focusing – Alternating Gradient Principle

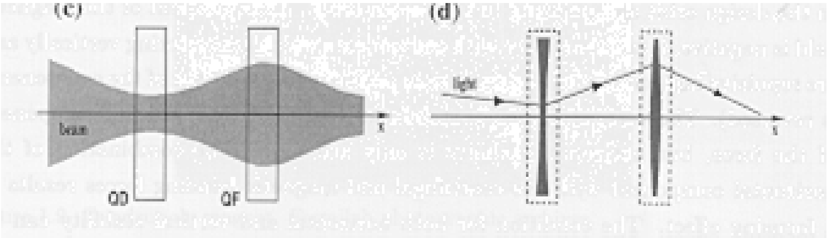
A sequence of focusing-defocusing fields provides a stronger net focusing force.

Quadrupoles focus horizontally, defocus vertically or vice versa. Forces are proportional to displacement from axis.

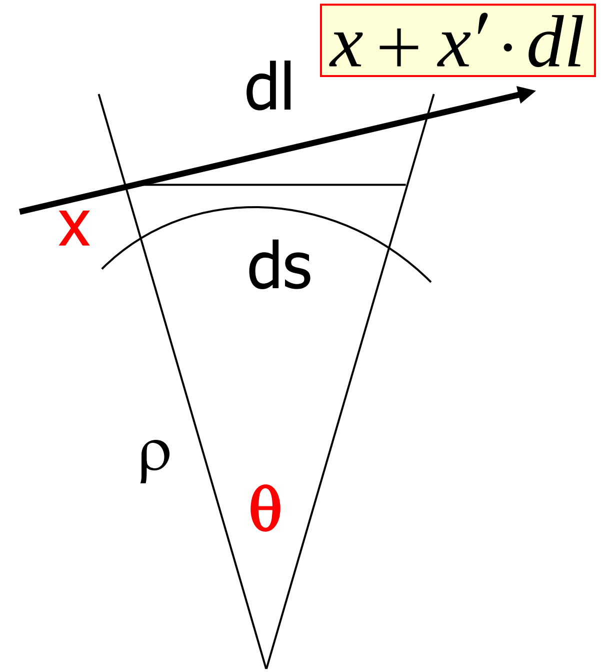
A succession of opposed elements enable particles to follow stable trajectories, making small oscillations about the design orbit.

Technological limits on magnets are high.





Equation of motion



In horizontal plane

$$x' = \theta - \theta_0 \quad and \quad x'' = \frac{d(\theta - \theta_0)}{ds}$$

$$d\theta_0 = -\frac{ds}{\rho_0}$$

$$d\theta = -\frac{dl}{\rho} = -\frac{B_y}{(B\rho)} dl$$

$$dl = ds \cdot \left(1 + \frac{x}{\rho}\right)$$

$$x'' = \frac{d(\theta - \theta_0)}{ds} = -\left(\frac{1}{\rho} + kx\right)\left(1 + \frac{x}{\rho}\right) + \frac{1}{\rho} \approx -\left(\frac{1}{\rho^2} + k\right)x$$

$$x'' + K_x \cdot x = 0 \quad \text{where} \quad K_x \equiv \frac{1}{\rho^2} + k$$

Equations of motion

In individual elements K = const. : Harmonic Oscillator
(KISS principle of accelerator building)

$$x'' + K \cdot x = 0$$
$$z'' - K \cdot z = 0$$

Overall, K(s) is a piecewise constant, periodic function

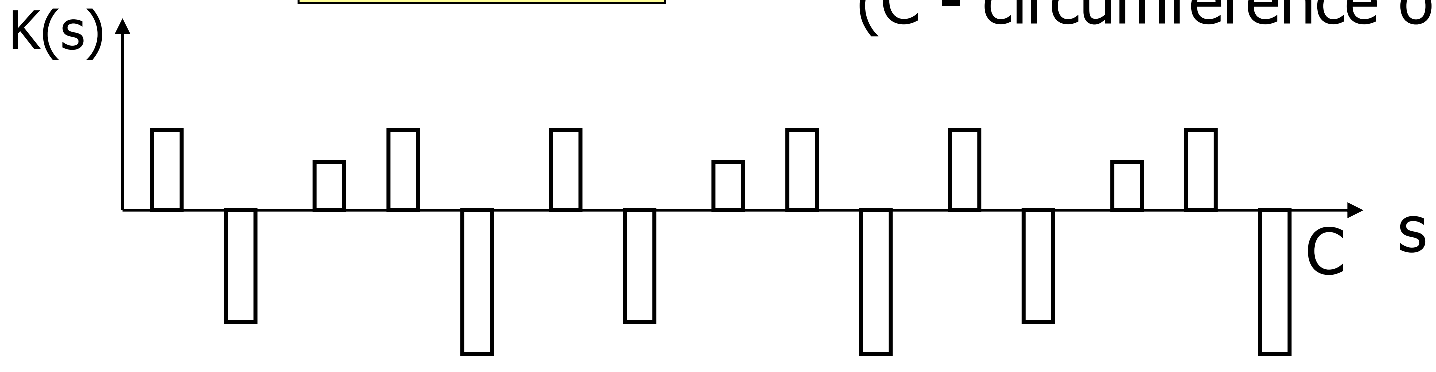
$$x'' + K(s) \cdot x = 0$$

Hill equation

$$H = \frac{p^2}{2} + \frac{K(s) \cdot x^2}{2}$$

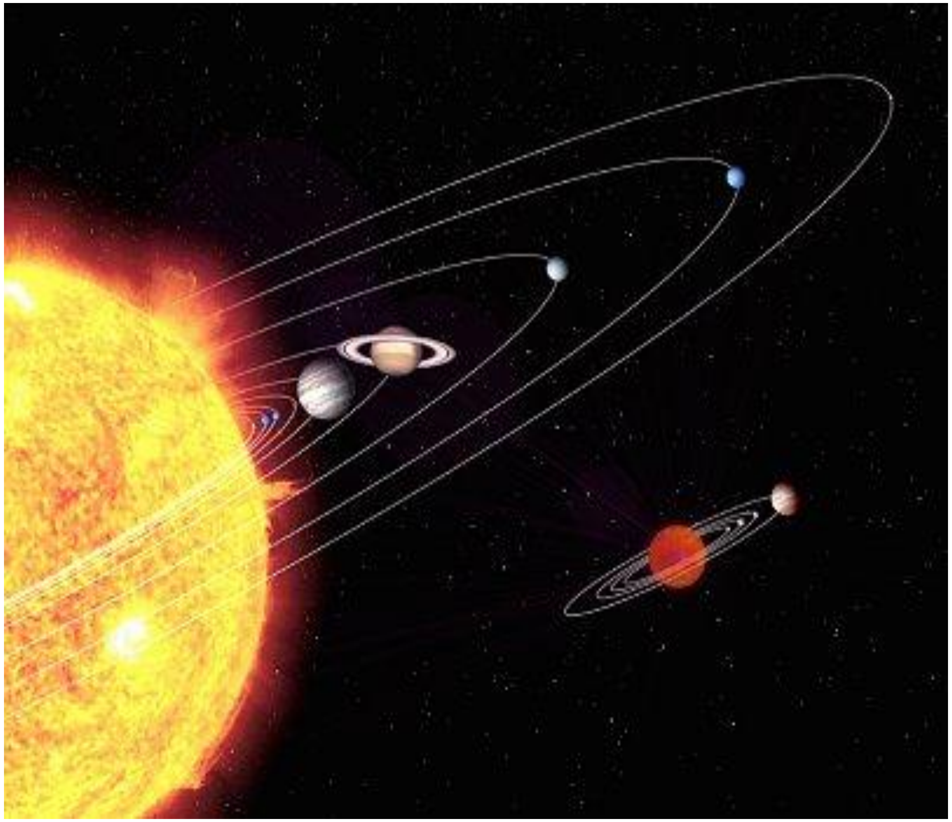
$$K(s) = K(s + C)$$

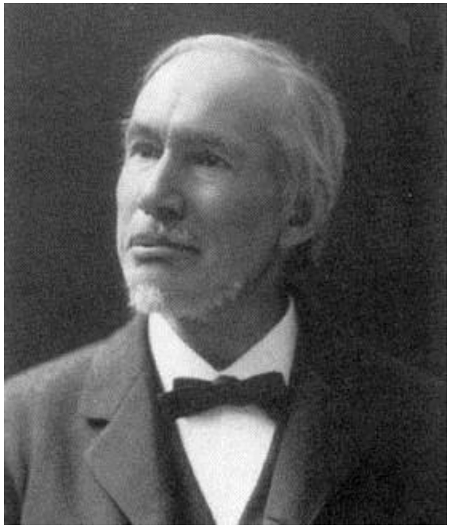
(C - circumference or period)



Hill equation

First used by an astronomer G. Hill in his studies of the motion of the moon, a motion under the influence of periodically changing forces





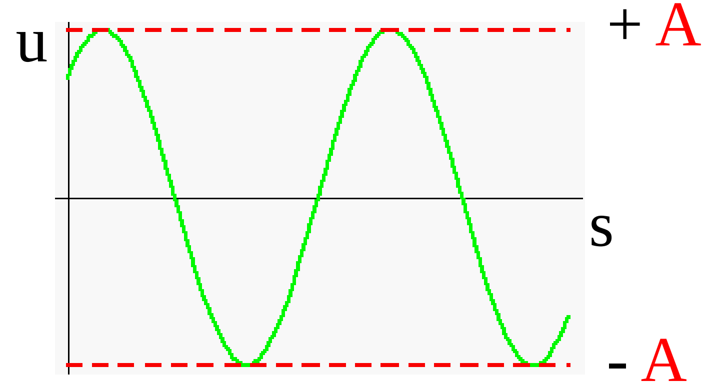
1838 -- 1914

Harmonic oscillator

$$u'' + Ku = 0$$

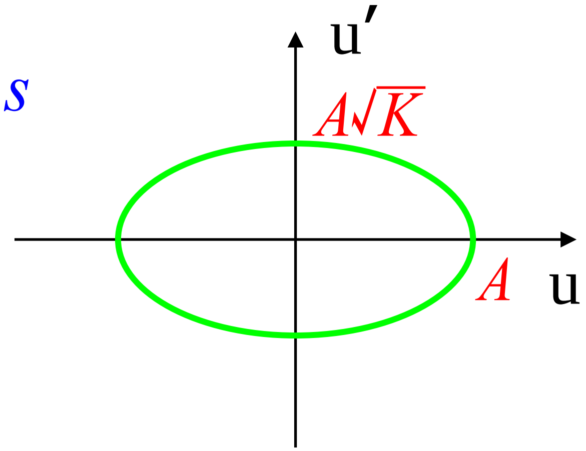
Solution:

$$u(s) = A \cdot \cos\left(\sqrt{K} \cdot s\right)$$



Amplitude: constant A

Phase advance: uniform: $\phi(s) = \sqrt{K} \cdot s$



Phase space: $u' = -A\sqrt{K} \cdot \sin\left(\sqrt{K} \cdot s\right)$

Independent of s: $A^2 = u^2 + \left(\frac{u'}{\sqrt{K}}\right)^2$

$$u'' + Ku = 0$$

Solutions of the Hill equation

$$u'' + k(s) \cdot u = 0$$

"Pseudo-harmonic oscillator" solutions (here u stands for x or z)

$$u(s) = a\sqrt{\beta(s)}\cos\left[\phi(s) - \phi_0\right]$$

$$\phi(s) = \int_0^s \frac{d\dot{s}}{\beta(\dot{s})}$$

- We have introduced a periodic "envelope" function β(s) the amplitude of the betatron oscillation is modulated by

$$\sqrt{\beta(s)}$$

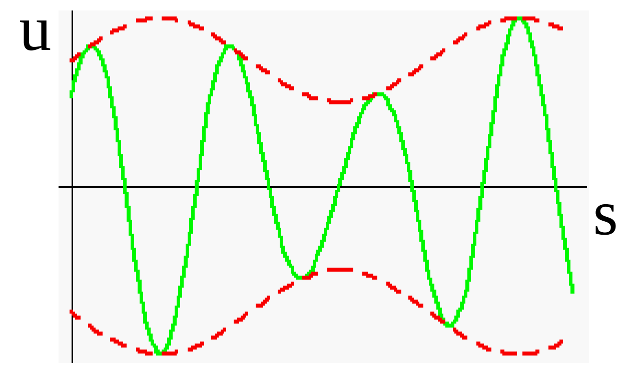
- the rate of phase advance at any point along the accelerator is related to the value of the β function at that point

$$\phi' = \frac{1}{\beta}$$

Hill equation (pseudo-harmonic oscillations)

Solution:

$$u(s) = A\sqrt{\beta(s)}\cos\left[\phi(s)\right]$$

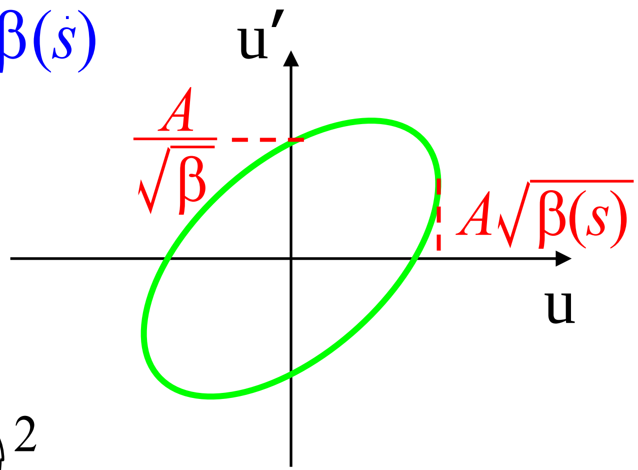


Amplitude: modulated with s: $A\sqrt{\beta(s)}$

Phase advance: non-uniform: $\phi(s) = \int_0^s \frac{d\dot{s}}{\beta(\dot{s})}$

Phase space: $u'(s) = -\frac{\alpha}{\beta}u - \frac{1}{\beta}A\sqrt{\beta}\sin\left[\phi\right]$

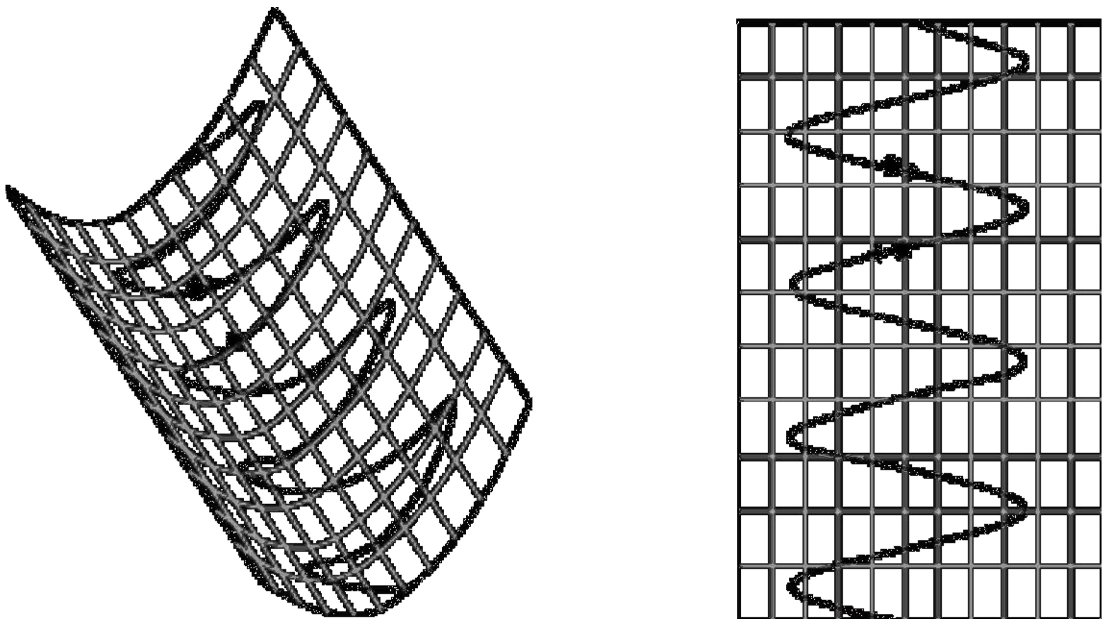
where $\alpha(s) \equiv -\frac{1}{2}\beta'(s)$



Depends on s: $A^2\beta(s) = u^2 + (\alpha u + \beta u')^2$

$$u'' + K(s)u = 0$$

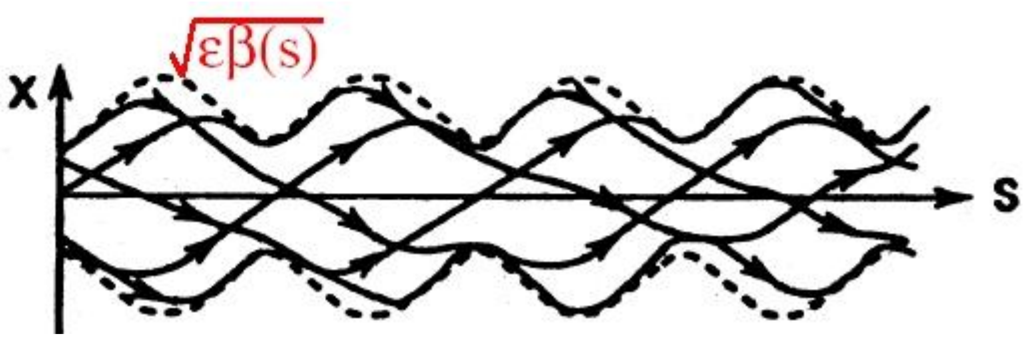
Sphere rolling in a gutter analogy



Turn, after turn, after turn…

Betatron oscillations within an envelope

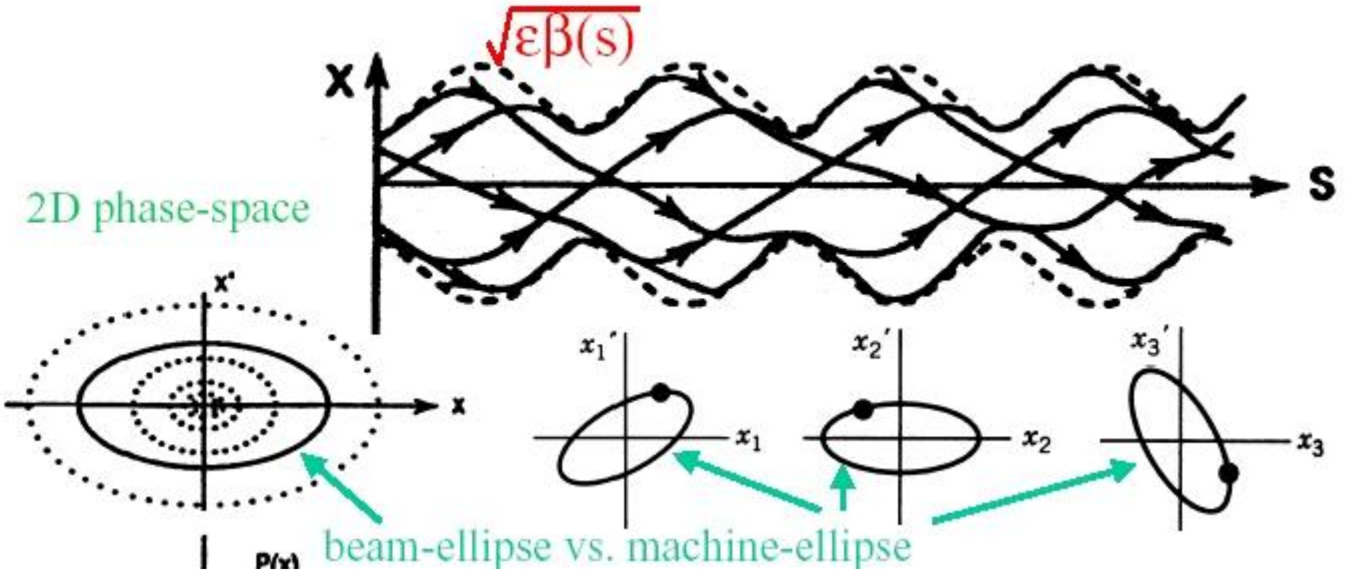
$$x(s)=\sqrt{\varepsilon\beta(s)}\sin[\varphi(s)-\varphi_0]$$



Turn, after turn, after turn…

Betatron oscillations within an envelope

$$x(s)=\sqrt{\varepsilon\beta(s)}\sin[\varphi(s)-\varphi_0]$$



Harmonic oscillator solutions (K = const.)

For K(s) constant, "principal" solutions are (harmonic oscillator)

- case of K > 0

$$C(s) = \cos(\sqrt{K}s) \quad \text{and} \quad S(s) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \qquad \text{for } K > 0$$

- case of K < 0

$$C(s) = \cosh(\sqrt{|K|}s) \quad \text{and} \quad S(s) = \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \qquad \text{for } K < 0$$

- these are linearly independent solutions with initial conditions:

$$C(0) = 1\,; \quad C'(0) = \frac{dC}{ds} = 0 \quad \text{and} \quad S(0) = 0\,; \quad S'(0) = \frac{dS}{ds} = 1$$

- any other solution is a linear combination of these:

$$u(s) = C(s)u_0 + S(s)u_0'$$
$$u'(s) = C'(s)u_0 + S'(s)u_0'$$

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

Harmonic oscillator solutions (matrix form)

$$x''+K_x(s)x=0$$
$$y''+K_y(s)y=0$$

Transfer matrices for particular cases:

- drift space (K=0)

$$\begin{pmatrix} u \\ u' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{\text{in}}$$

- focusing magnet (K > 0, const.) of length l

$$\begin{pmatrix} u \\ u' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cos(\sqrt{K}l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}l) \\ -\sqrt{K}\sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{\text{in}}$$

- defocusing magnet (K < 0, const.) of length l

$$\begin{pmatrix} u \\ u' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{\text{in}}$$

- the thin lens limit:

$$\lim \quad l \to 0 \quad \text{keeping} \quad K \cdot l = \frac{1}{f} = \text{const.}$$

TRANSVERSE MOTION SUMMARY

$$x'' + K_x(s)x = 0$$
$$y'' + K_y(s)y = 0$$

DRIFT:

$k = 0$

$$\begin{pmatrix} u \\ u' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{\text{in}}$$

QUADRUPOLE:

$K > 0$ → focusing

$$\begin{pmatrix} u \\ u' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cos(\sqrt{K}l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}l) \\ -\sqrt{K}\sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{\text{in}}$$

$K < 0$ → defocusing

$$\begin{pmatrix} u \\ u' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{\text{in}}$$

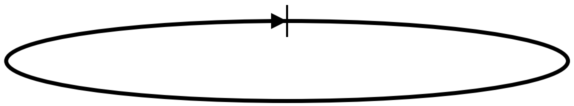
Stability of transverse (betatron) oscillations

The transfer matrix of a beamline that consists of elements with individual matrices M_1 , M_2 , ... M_n

$$\mathbf{M_{tot} = M_n \cdot \ \dots \ \cdot M_2 \cdot M_1}$$

(N.B. the order in which matrices are multiplied!)

- Full turn matrix M



$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

- After n turns must remain finite for arbitrarily large n

Stability condition

Let v_1 and v_2 be eigenvectors and λ_1 and λ_2 eigenvalues of M

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = A\mathrm{v}_1 + B\mathrm{v}_2 \qquad\qquad M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0 = A\lambda_1^n \mathrm{v}_1 + B\lambda_2^n \mathrm{v}_2$$

- For stability λ_1^n, λ_2^n must not grow with n
- since the product of eigenvalues is unity:

$$\det M = 1 \quad \Rightarrow \quad \lambda_1 \cdot \lambda_2 = 1$$

we can write in general $\lambda_1 = e^{i\mu}, \lambda_2 = e^{-i\mu}$

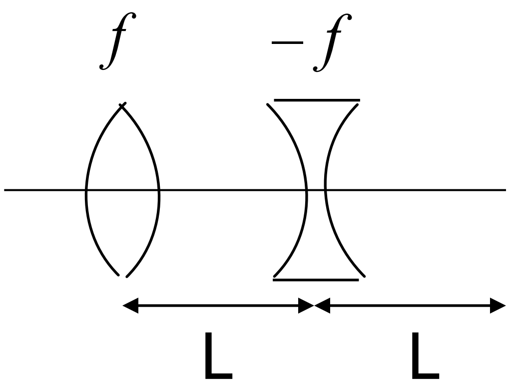
- For stability μ should be real!

$$\mathrm{Tr}\, M = \lambda_1 + \lambda_2 = 2\cos\mu$$

$$-1 \le \frac{1}{2}\mathrm{Tr}\, M \le 1$$

Example

Consider one period of FODO lattice:



$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{pmatrix}$$

- applying the stability condition

$$-1 \leq 1 - \frac{1}{2}\left(\frac{L}{f}\right)^2 \leq 1$$

- The motion is stable, provided the focal length > 1/2 the lens spacing

$$\left|\frac{L}{2f}\right| \leq 1$$

Solutions of the Hill equation

$$u'' + k(s) \cdot u = 0$$

"Pseudo-harmonic oscillator" solutions (here u stands for x or z)

$$u(s) = a\sqrt{\beta(s)}\cos\left[\phi(s) - \phi_0\right]$$

$$\phi(s) = \int_0^s \frac{d\dot{s}}{\beta(\dot{s})}$$

- We have introduced a periodic "envelope" function β(s) the amplitude of the betatron oscillation is modulated by

$$\sqrt{\beta(s)}$$

- the rate of phase advance at any point along the accelerator is related to the value of the β function at that point

$$\phi' = \frac{1}{\beta}$$

Courant - Snyder invariant

At any point s along the accelerator for a given betatron oscillation the following combination of u and u′ has the same value

$$a^2 = \frac{u^2}{\beta} + \beta \cdot \left(u' - \frac{\beta'}{2\beta} u \right)^2$$

$$\alpha \equiv -\frac{1}{2}\beta'$$

Introducing some additional notation:

$$\gamma \equiv \frac{1+\alpha^2}{\beta}$$

$$\varepsilon = a^2 = \gamma u^2 + 2\alpha u u' + \beta u'^2$$

Describing an ellipse in phase space {u,u′} with area $\pi \cdot \varepsilon$

- The parameters β, α, γ vary along the machine
- The phase space area remains constant

Betatron oscillation solution

- Displacement $$x(s)=\sqrt{\varepsilon\beta(s)}\cos[\phi(s)-\phi_0]$$
- Slope $$x'(s)=-\frac{\alpha}{\beta}x(s)-\sqrt{\varepsilon\beta(s)}\sin[\phi(s)-\phi_0]$$
- combining the two $$x^2+(\alpha x+\beta x')^2=\varepsilon\beta$$

$$\varepsilon=\gamma x^2+2\alpha x x'+\beta x'^2$$

equation of an ellipse with **area = $\pi\varepsilon$**

Phase space ellipse: Courant – Snyder Invariant

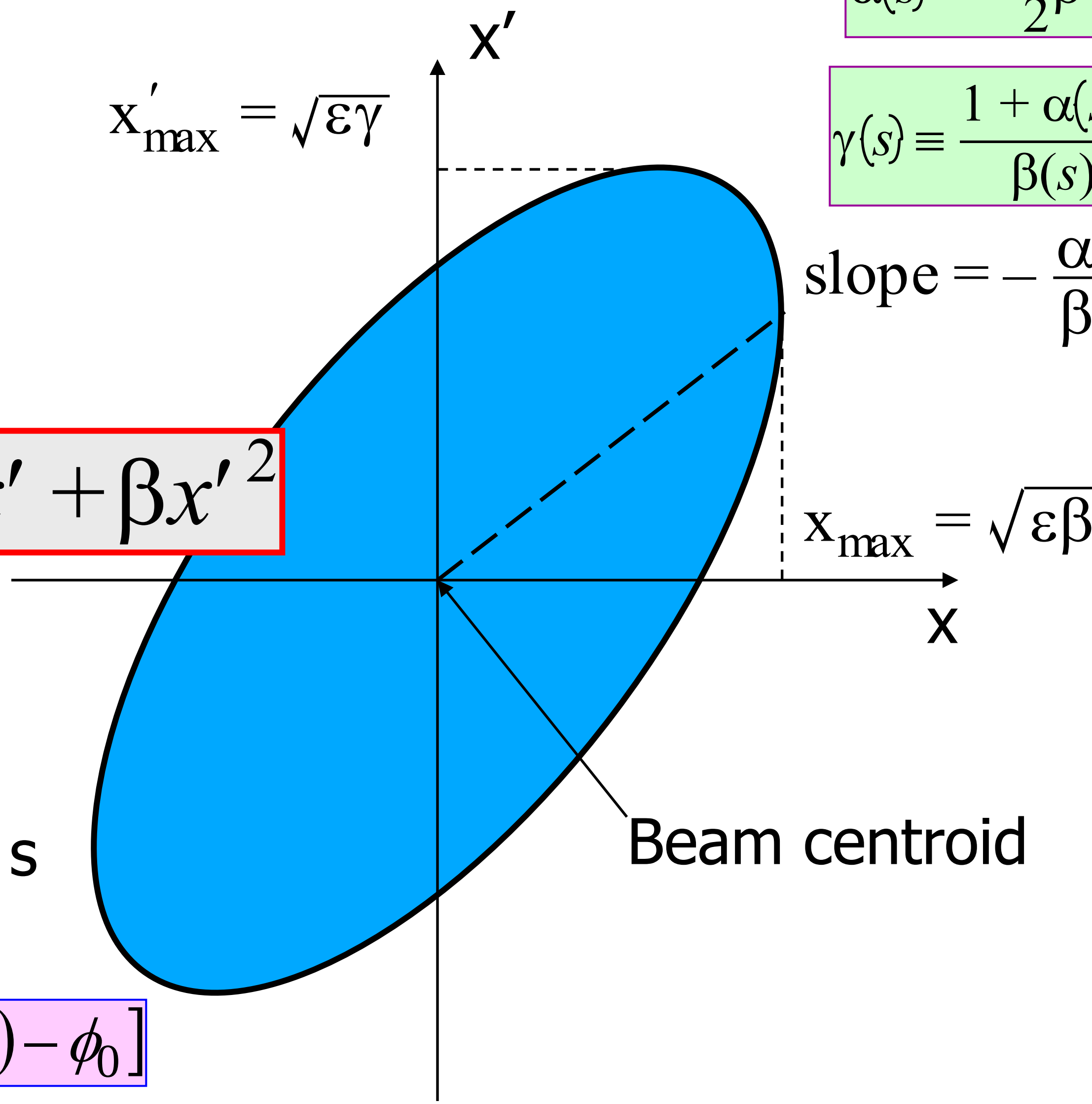
Single particle motion

- At a place with Courant-Snyder parameters (β, α, γ)

$$\alpha(s) \equiv -\frac{1}{2}\beta'(s)$$

$$\gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$



- at a given point s

$$x_n = \sqrt{\varepsilon\beta}\cos\left[\phi(s + n \cdot L) - \phi_0\right]$$

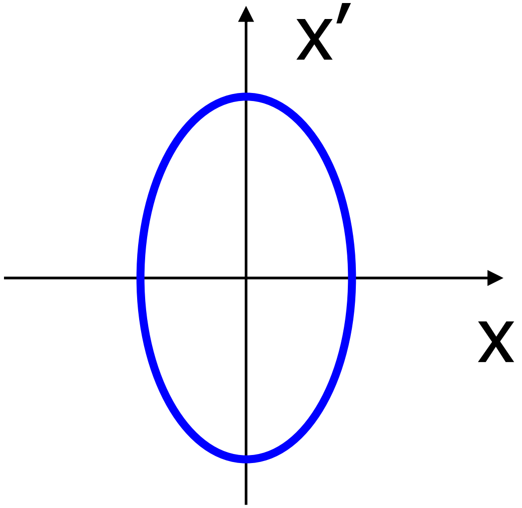
Simple case: Upright ellipse

For the simple case when

$$\alpha = 0 \quad \rightarrow \quad \gamma = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta}$$

$$\varepsilon = \frac{x^2}{\beta} + \beta x'^2$$

the ellipse is upright



Beam emittance

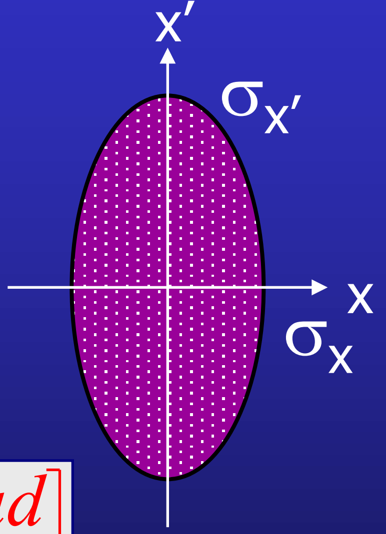
$$\text{Area} = \pi \cdot \varepsilon$$

Betatron oscillations

- Particles in the beam execute betatron oscillations with different amplitudes.

Transverse beam distribution

- Gaussian (electrons)
- "Typical" particle: 1 - σ ellipse
  (in a place where $\alpha = \beta' = 0$)



$$\text{Units of } \varepsilon \quad [m \cdot rad]$$

$$\text{Emittance} \equiv \frac{\sigma_x^2}{\beta}$$

$$\varepsilon = \sigma_x \cdot \sigma_{x'}$$

$$\sigma_x = \sqrt{\varepsilon\, \beta}$$

$$\sigma_{x'} = \sqrt{\varepsilon / \beta}$$

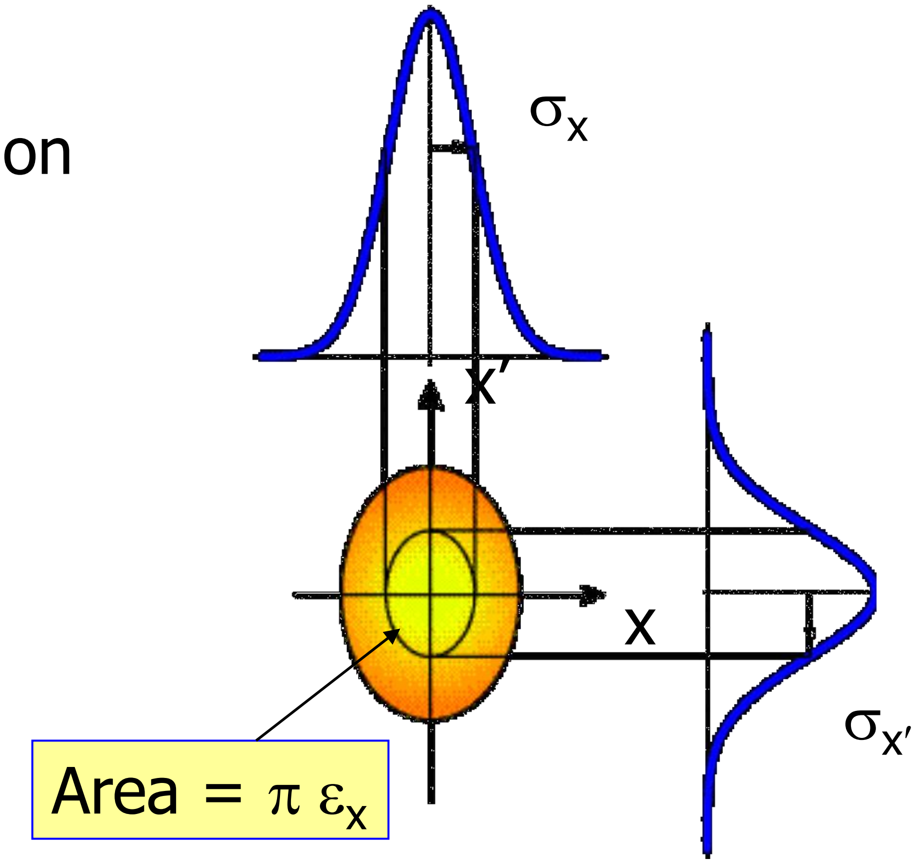
$$\beta = \frac{\sigma_x}{\sigma_{x'}}$$

2-D Gaussian distribution

Electron rings emittance definition

- 1 - σ ellipse

$$n(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} dx$$



- Probability to be inside 1-σ ellipse $P_1 = 1 - e^{-1/2} = 0.39$

- Probability to be inside n-σ ellipse $P_n = 1 - e^{-n^2/2}$

Courant - Snyder parameters: transfer matrices

Consider a transfer matrix M for a full turn starting at some point

- We know that it is symplectic, i.e. det M = 1
- Any such matrix with unit determinant can be parameterized:

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

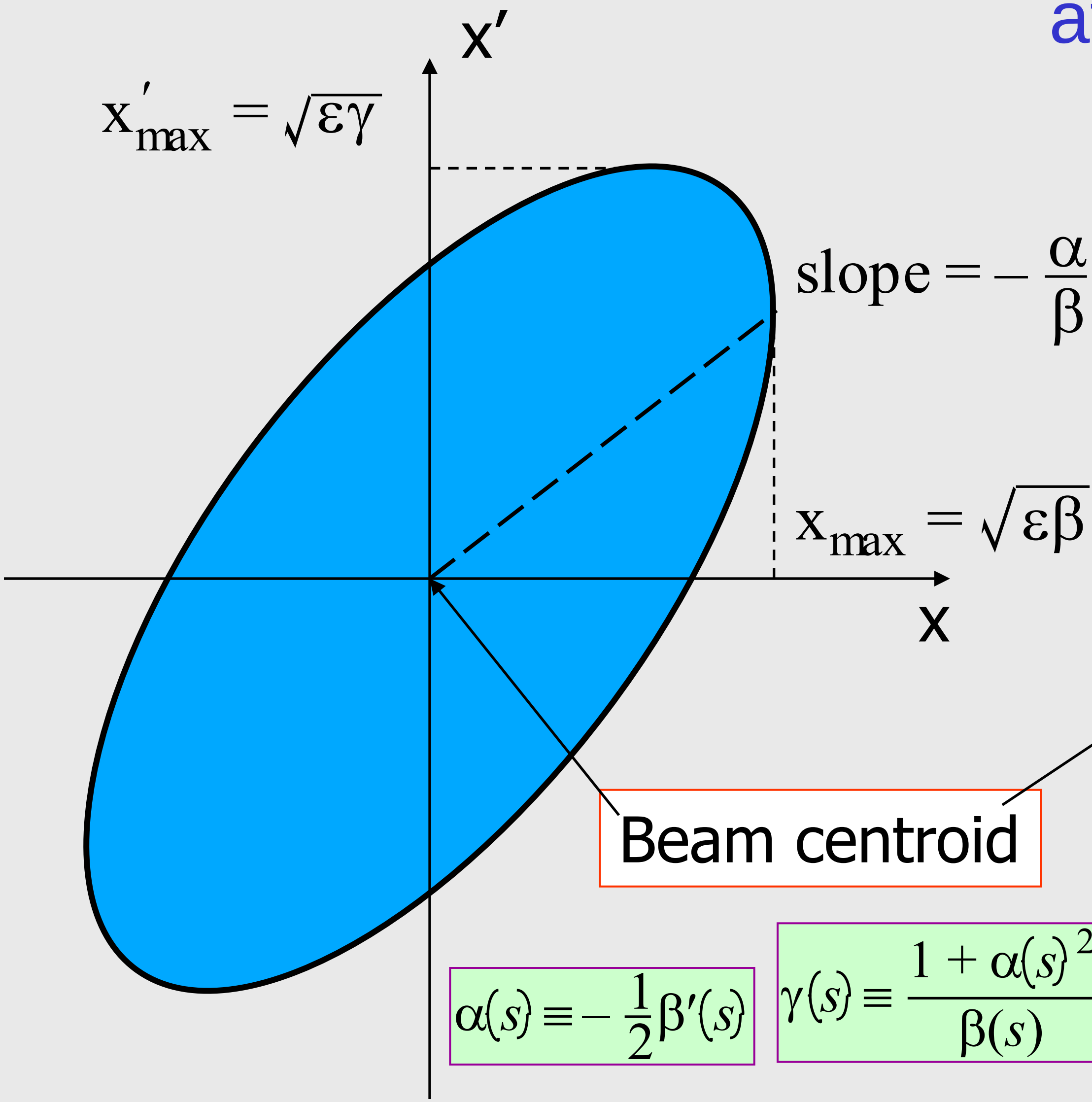
where in order to satisfy the condition of unit determinant

$$\beta\gamma - \alpha^2 = 1$$

we can regard this now as just a formal parameterization

Phase space ellipse

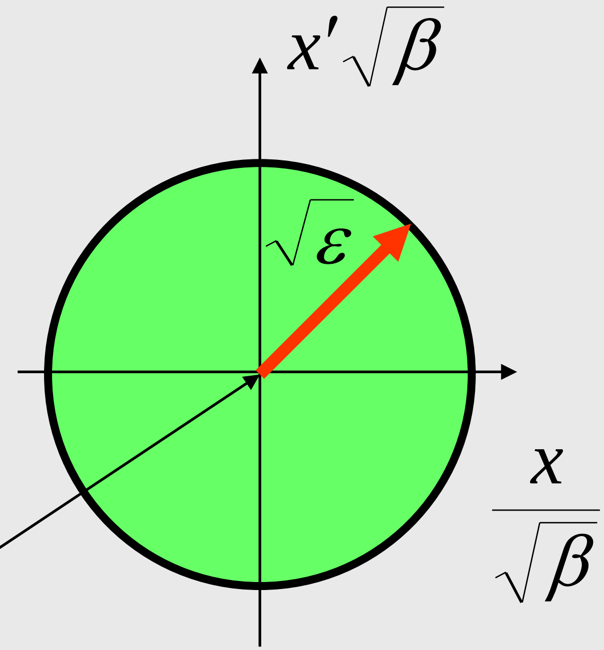
$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$



$$\alpha(s) \equiv -\frac{1}{2}\beta'(s)$$

$$\gamma(s) \equiv \frac{1+\alpha(s)^2}{\beta(s)}$$

at $s_{max/min}$ where $\alpha = 0$



circle of radius $\sqrt{\varepsilon}$
everywhere
in the machine

Tune

Transfer matrix for one complete turn:

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

Phase advance over one turn is independent of location

$$\mu = \oint \frac{ds}{\beta(s)}$$

Tune **Q** is the number of betatron oscillations in one revolution

$$Q \equiv \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

Transfer matrix between two points

The transfer matrix between two arbitrary point in the machine

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{1\to 2} \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

in terms of Courant - Snyder parameters at these points and the phase advance between them

$$\begin{pmatrix} \sqrt{\dfrac{\beta_2}{\beta_1}}\left(\cos\Delta\mu + \alpha_1 \sin\Delta\mu\right) & \sqrt{\beta_1\beta_2}\sin\Delta\mu \\ -\dfrac{(1+\alpha_1\alpha_2)\sin\Delta\mu + (\alpha_2 - \alpha_1)\cos\Delta\mu}{\sqrt{\beta_1\beta_2}} & \sqrt{\dfrac{\beta_1}{\beta_2}}\left(\cos\Delta\mu - \alpha_2\sin\Delta\mu\right) \end{pmatrix}$$

Transforming C-S parameters between two points

The transfer matrix between two arbitrary point in the machine

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{1\to 2} \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

$$M_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The Courant - Snyder parameters at those points are related by

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11} \cdot m_{12} & m_{12}^2 \\ -m_{11} \cdot m_{21} & 1 + 2m_{12} \cdot m_{21} & -m_{12} \cdot m_{22} \\ m_{21}^2 & -2m_{21} \cdot m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}$$

And the phase advance between the points

$$\Delta\mu = \int_1^2 \frac{ds}{\beta(s)}$$

$$\tan\Delta\mu = \frac{m_{12}}{m_{11} \cdot \beta_1 - m_{12} \cdot \alpha_1}$$

or

$$\sin\Delta\mu = \frac{m_{12}}{\sqrt{\beta_1 \beta_2}}$$

Full turn transfer matrix

Transfer matrix for one complete turn:

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Tune **Q** is the number of betatron oscillations in one revolution

$$Q \equiv \frac{\mu}{2\pi} = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

If the tune is an integer, i.e. $\mu = 2\pi \cdot n$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Some simple cases and their phase advance

Thin lens $s_1 = s_2$ $\Delta\mu = 0$

'point to point imaging' $r_{12} = 0$

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} r_{11} & 0 \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} 0 \\ x'_1 \end{pmatrix} = \begin{pmatrix} 0 \\ r_{22} x'_1 \end{pmatrix}$$

$\Delta\mu = n \cdot \pi$

'parallel to point imaging' $r_{11} = 0$

$$\tan \Delta\mu = -\frac{1}{\alpha_1}$$

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 0 & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ r_{21} x_1 \end{pmatrix}$$

for $\alpha_{1=0}$ $\Delta\mu = n \cdot \frac{\pi}{2}$

drift of length L $r_{12} = L$

$$\sin \Delta\mu = \frac{L}{\sqrt{\beta_1 \beta_2}}$$

Proton therapy Gantry at PSI: point to parallel!

Courant - Snyder parameters: transfer matrices

The eigenvalues of this matrix are related to μ:

$$\lambda = e^{\pm i\mu} \quad \Rightarrow \quad \mathrm{Tr}\, M = 2\cos\mu$$

We can also write our matrix M as

$$M = I\cos\mu + J\sin\mu$$

$$M = e^{J\mu}$$

where $J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$ and $J^2 = -I$

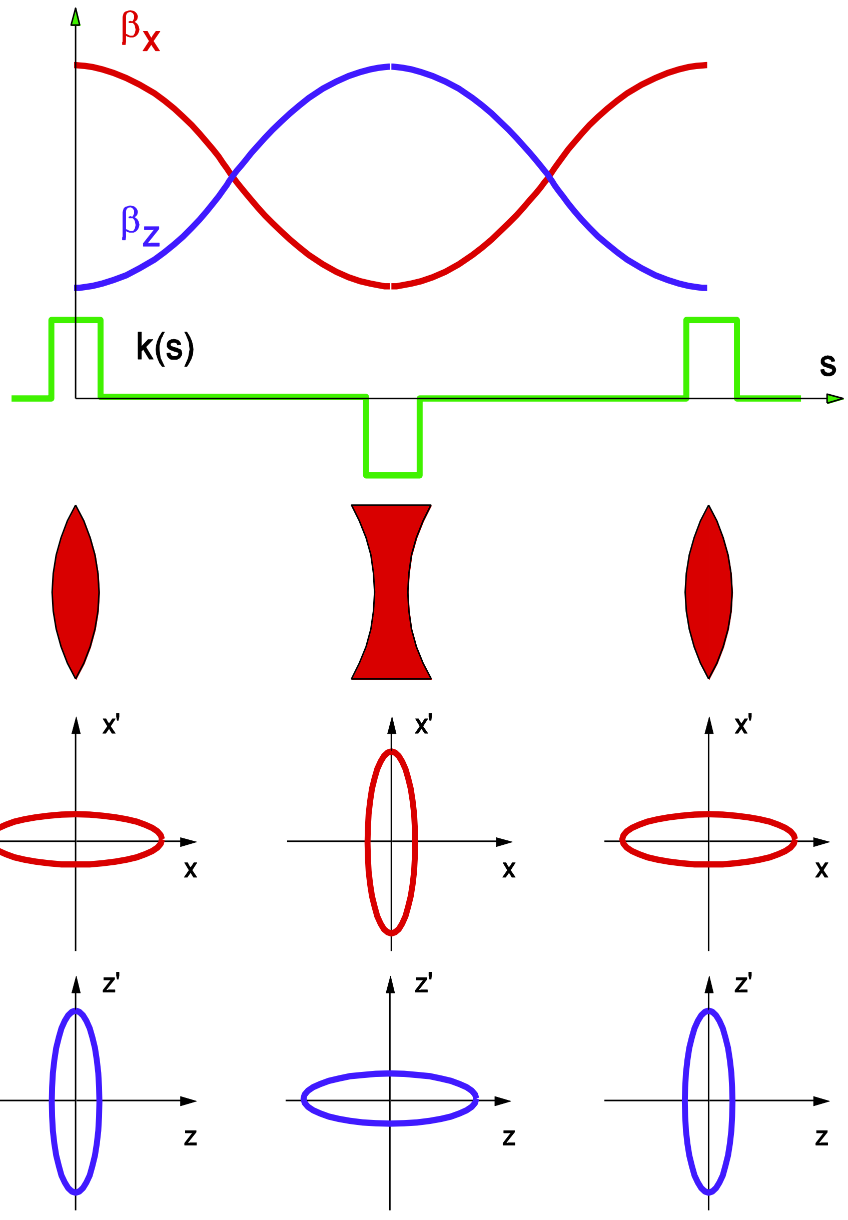
The powers of matrix M can be written simply as:

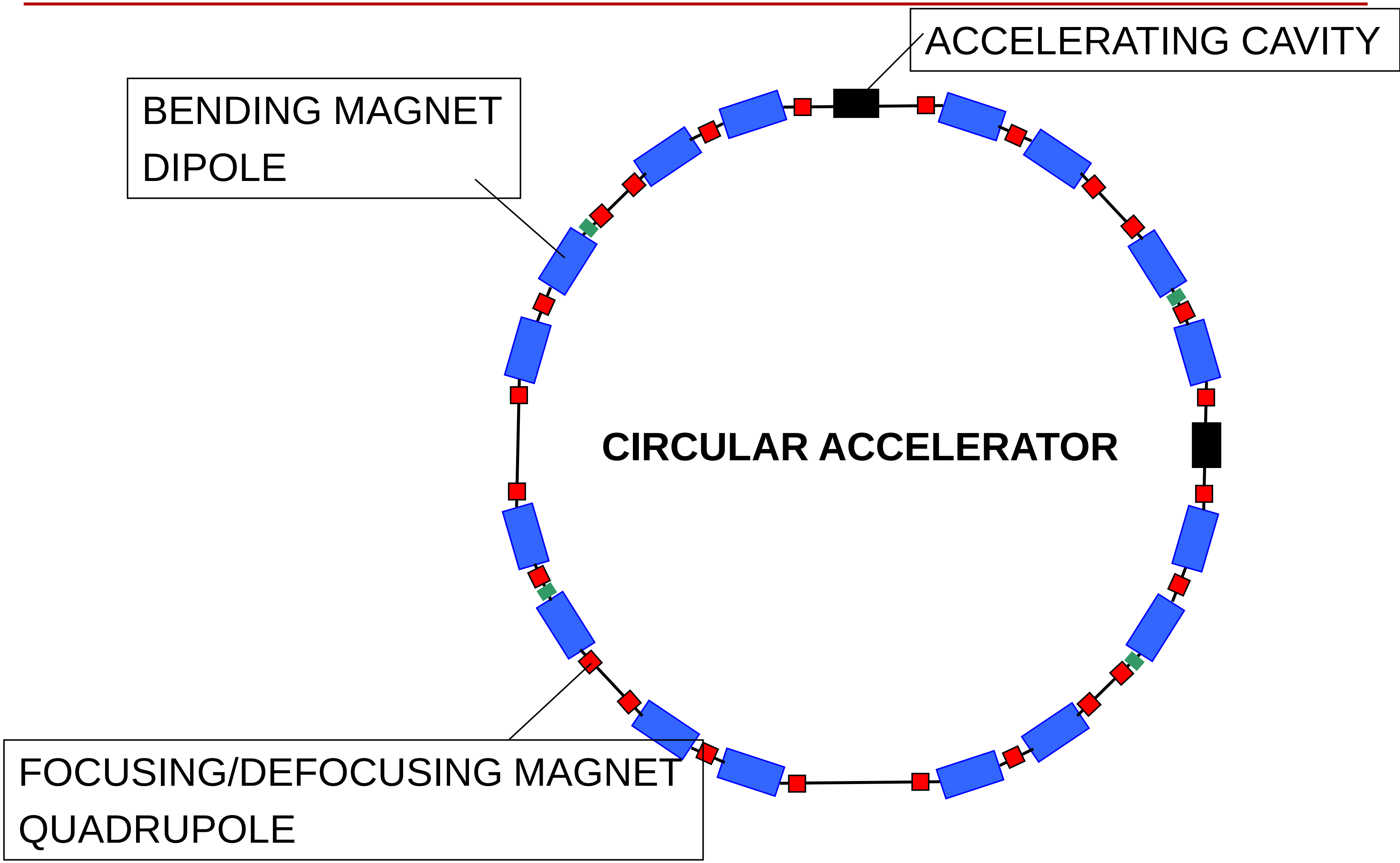
$$M^k = I\cos k\mu + J\sin k\mu$$

And the elements of M^k are bounded for all k if μ is real

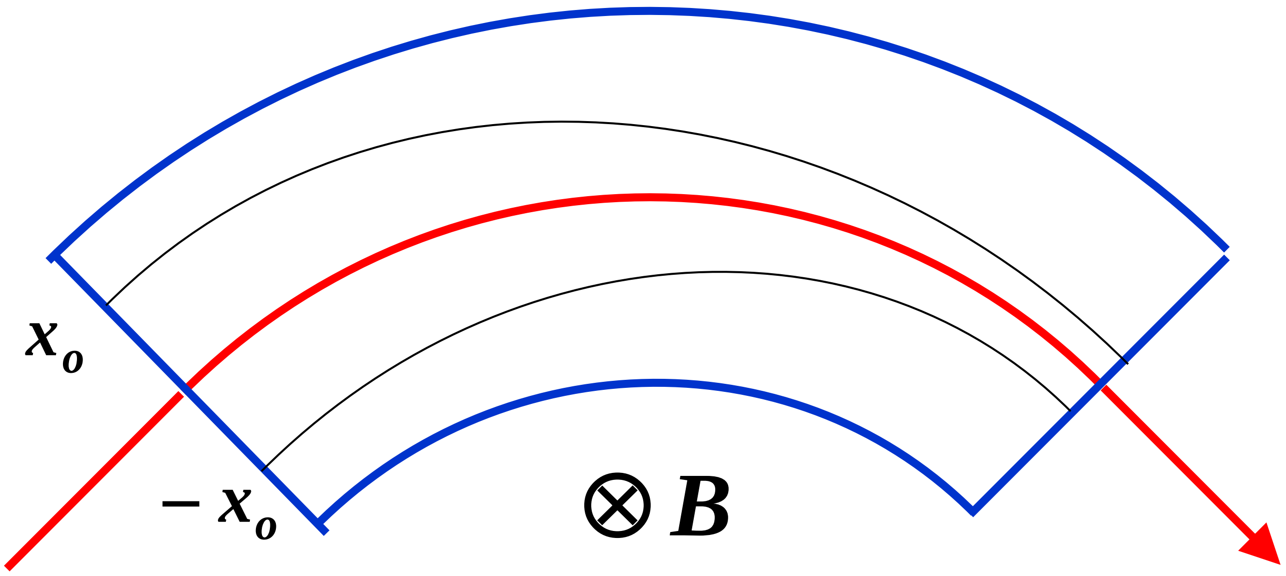
$$|\mathrm{Tr}\, M| \le 2$$

FODO cell lattice




ACCELERATING CAVITY
BENDING MAGNET
DIPOLE
CIRCULAR ACCELERATOR
FOCUSING/DEFOCUSING MAGNET
QUADRUPOLE

SECTOR BENDING MAGNET



Positive displacement x_o of the initial coordinate from the center axis leads to a longer path inside the magnet, i.e. more deflection

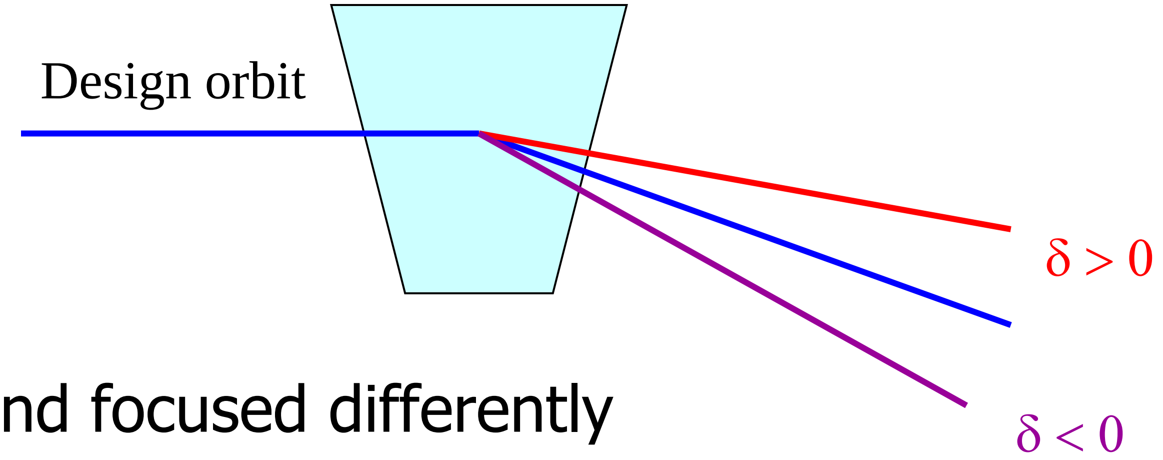
Negative displacement $-x_o$ of the initial coordinate from the center axis leads to a shorter path inside the magnet, i.e. less deflection

In both cases the trajectory comes closer to the central orbit → **FOCUSING**

Off-energy particles

Off-momentum particles

Particle with an energy deviation δ



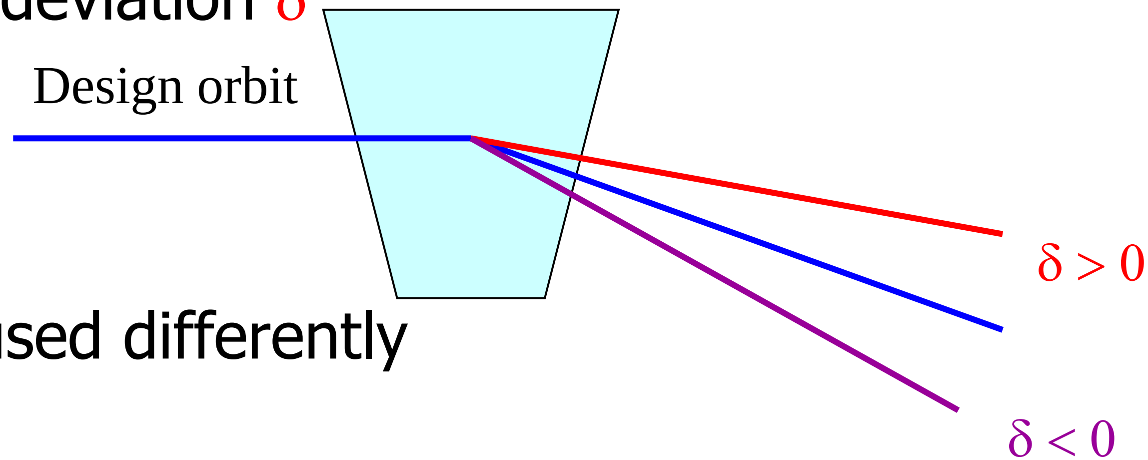
- Will be bent and focused differently

- The equation of motion: non-homogeneous Hill equation

$$x'' + k(s)\,x = \frac{\delta}{\rho}$$

Off-momentum particles

Particle with an energy deviation δ



- Will be bent and focused differently
- The equation of motion: non-homogeneous Hill equation

$$x'' + k(s)\,x = \frac{\delta}{\rho}$$

- The motion is a sum of the
solution of homogeneous equation + a particular solution

Dispersion function

Particle deviation from ideal orbit

$$x = x_\beta + x_\varepsilon = x_\beta + D(s) \cdot \delta$$

D(s) - dispersion function

- Periodic solution of the inhomogeneous Hill equation

$$D'' + k(s)D = \frac{1}{\rho(s)} \begin{pmatrix} = 0 \text{ in straights} \\ = \frac{1}{\rho} \text{ in bends} \end{pmatrix}$$

- New equilibrium orbit of a particle with energy deviation δ
- Betatron oscillations are executed around this new equilibrium

Matrix notation: extended to 3 by 3 case

Taking into account particle energy deviation, particle position

$$\vec{x} = \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}$$ and $$\vec{x}(s) = \begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ x'_0 \\ \delta \end{pmatrix}$$

we usually assume that δ does not change

$$M = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

where D and D′ are the solutions of inhomog. equation

Examples of 3 by 3 transfer matrices

For simple cases of piece-wise constant K(s), ρ(s)

$$\begin{pmatrix} \cos\phi & \frac{1}{\sqrt{K}}\sin\phi & \frac{1}{\rho K}(1-\cos\phi) \\ -\sqrt{K}\sin\phi & \cos\phi & \frac{1}{\rho\sqrt{K}}\sin\phi \\ 0 & 0 & 1 \end{pmatrix}$$

$$K > 0$$
$$\phi \equiv \sqrt{K}(s-s_0)$$

$$\begin{pmatrix} \cosh\psi & \frac{1}{\sqrt{-K}}\sinh\psi & \frac{1}{\rho(-K)}(\cosh\psi-1) \\ \sqrt{-K}\sinh\psi & \cosh\psi & \frac{1}{\rho\sqrt{-K}}\sinh\psi \\ 0 & 0 & 1 \end{pmatrix}$$

$$K < 0$$
$$\psi \equiv \sqrt{-K}(s-s_0)$$

Bending magnet transfer matrix

Pure dipole field:

- k = 0
- θ – bending angle

$$K = \frac{1}{\rho^2} \quad \Rightarrow \quad \phi = \sqrt{K} \cdot s = \frac{s}{\rho} = \theta$$

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

In the vertical plane - drift

Dispersion: periodic solution

Let the matrix for one full period be

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

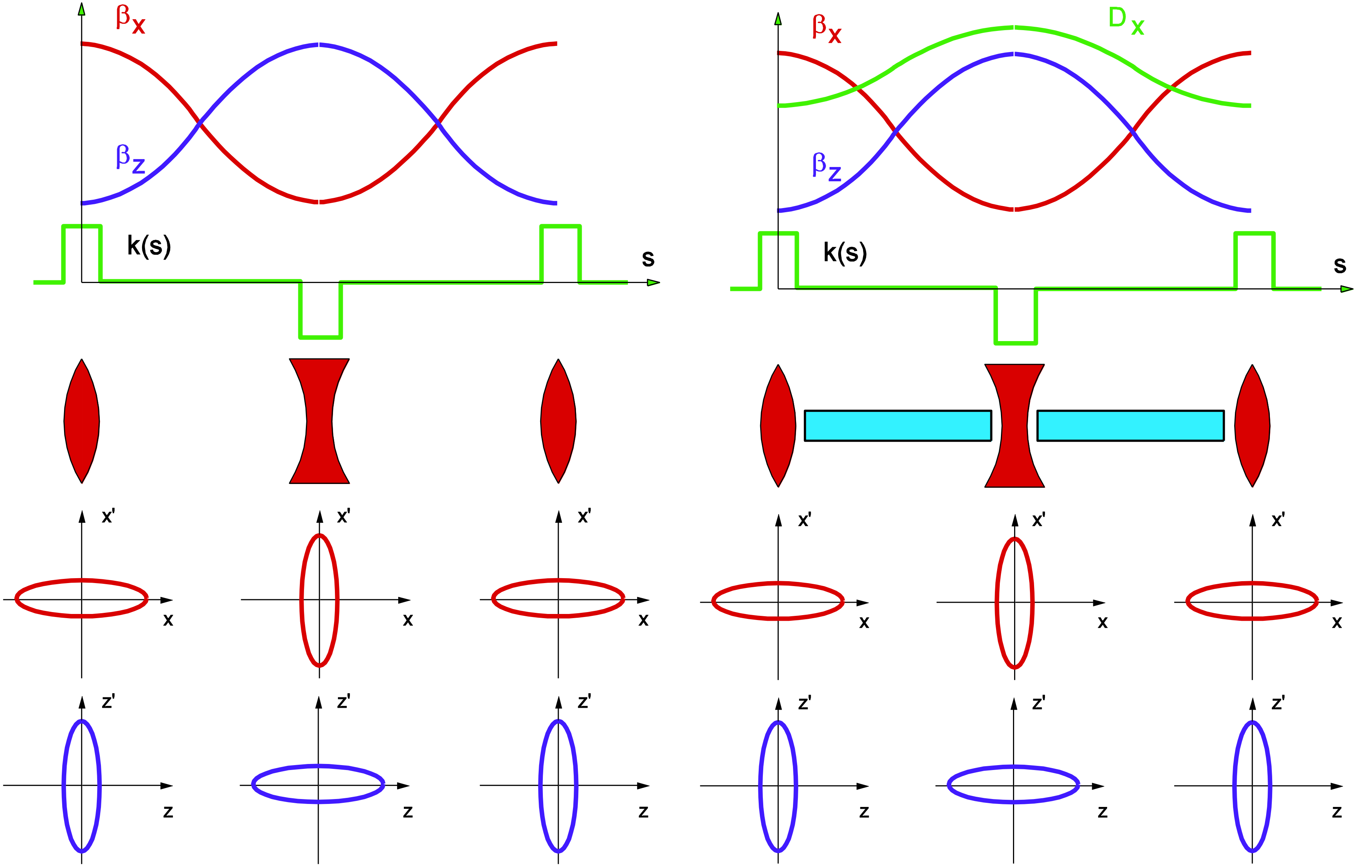
Dispersion being a periodic solution:

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

$$D' = \frac{m_{13}m_{21} + (1 - m_{11})m_{23}}{(1 - m_{11})(1 - m_{22}) - m_{21}m_{12}}$$

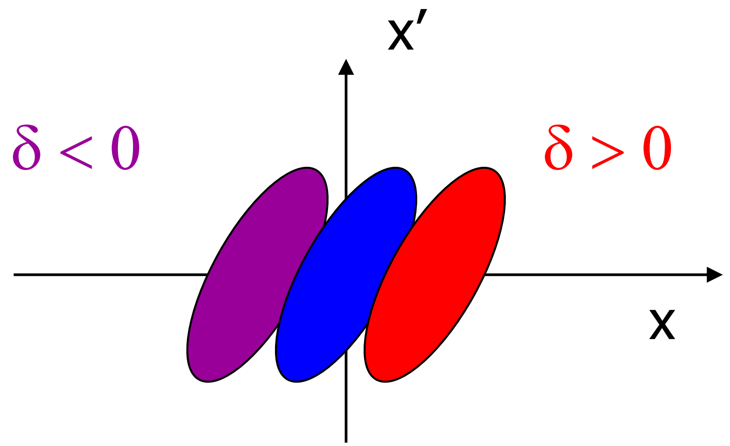
$$D = \frac{m_{12}}{1 - m_{11}} D' + \frac{m_{13}}{1 - m_{11}}$$

FODO cell lattice



Beam size

- When the beam energy spread is δ



$$\sigma^2 = \sigma_\beta^2 + \sigma_\varepsilon^2 = \varepsilon \cdot \beta + D^2 \delta^2$$

Full turn transfer matrix

Transfer matrix for one complete turn:

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

Tune **Q** is the number of betatron oscillations in one revolution

$$Q \equiv \frac{\mu}{2\pi} = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

If the tune is an integer, i.e. $\mu = 2\pi \cdot n$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Full turn transfer matrix: special cases

The tune is **half-integer**, i.e. $\{Q\} = 0.5$

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$M^2 = I$$

The tune is quarter-integer, i.e. $\{Q\} = 0.25$

$$M = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = J$$

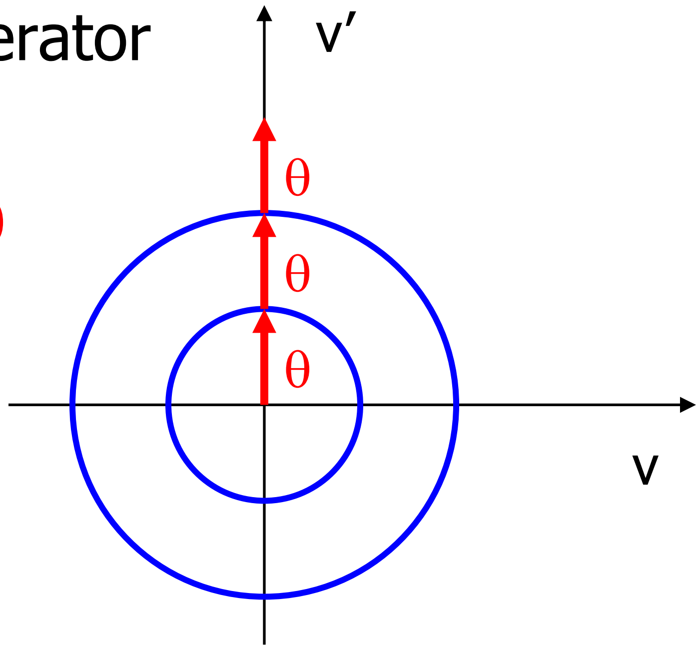
$$M^2 = J^2 = -I$$

$$M^4 = I$$

Errors, errors, errors

Suppose at some point along the accelerator

- extra field ΔB over some length l
- it will kick a particle by an angle θ



$$\theta = \frac{\Delta B \cdot l}{(B\rho)}$$

If the tune Q is close to an integer, $M \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

the kicks will add up in phase each turn

driving the particle out of the machine

Integer resonance

Betatron oscillation solution

- Displacement $$x(s)=\sqrt{\varepsilon\beta(s)}\cos[\phi(s)-\phi_0]$$
- Slope $$x'(s)=-\frac{\alpha}{\beta}x(s)-\sqrt{\varepsilon\beta(s)}\sin[\phi(s)-\phi_0]$$
- combining the two $$x^2+(\alpha x+\beta x')^2=\varepsilon\beta$$

$$\varepsilon=\gamma x^2+2\alpha xx'+\beta x'^2$$

- e.g. if we start a particle with x_0\` at a place whith β_0

$$\varepsilon=\beta_0 x_0'^2$$

$$x(s)=x_0'\sqrt{\beta_0\beta(s)}\cos[\phi(s)-\phi_0]$$

Field error

In the presence of such a kick

- x = 0 is no longer a solution
- there will be a new closed orbit

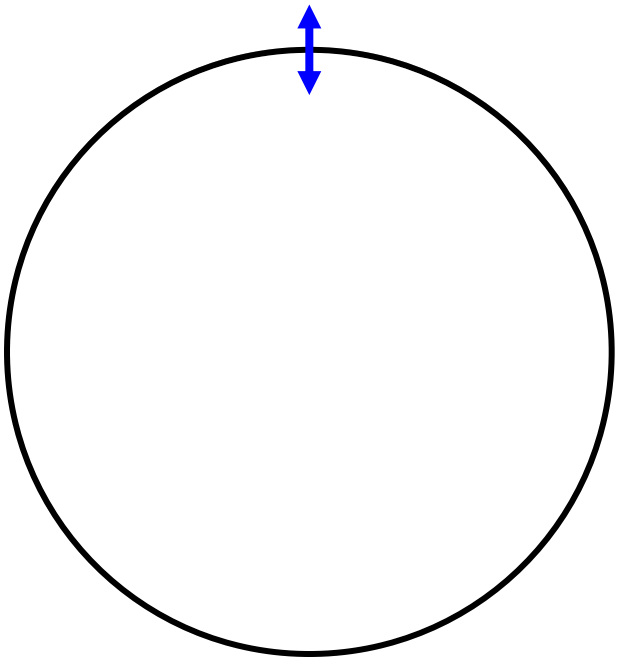
$$x(s) = \frac{\theta\sqrt{\beta(s)\beta_\theta}}{2\sin\pi Q}\cos[\phi(s) - \pi Q]$$

Particles perform betatron oscillations around this new closed orbit

- β function is a measure of sensitivity to errors
- when Q approaches an integer value, the new closed orbit becomes very large

Focusing error

A gradient error over a short distance



- a thin lens

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

- transfer matrix for full turn becomes

$$M = M_0 \cdot \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Focusing error (algebra)

$$M = M_0 \cdot \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta_0\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}$$

Comparing the traces of the two matrices, the new tune:

$$\cos 2\pi Q = \cos 2\pi Q_0 - \frac{1}{2}\frac{\beta_0}{f}\sin 2\pi Q_0$$

Stability

The motion remains stable in the presence of focussing errors, if the new tune remains a real number, i.e.

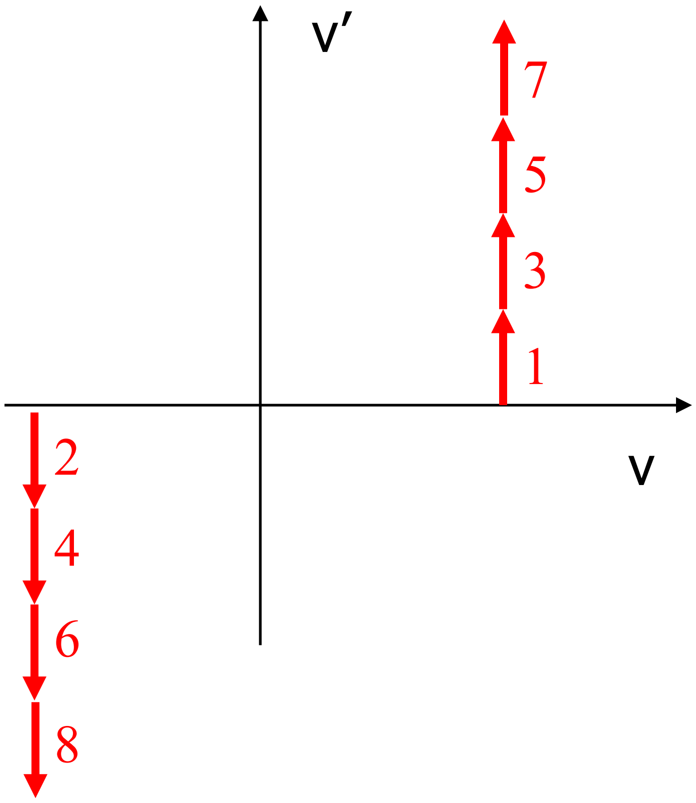
$$\left|\cos 2\pi Q\right| = \left|\cos 2\pi Q_0 - \frac{1}{2}\frac{\beta_0}{f}\sin 2\pi Q_0\right| < 1$$

and when the unperturbed tune is not near an integer or half-integer resonance and the perturbation is sufficiently small

$$Q = Q_0 + \delta Q = Q_0 + \frac{1}{4\pi} \cdot \frac{\beta}{f}$$

Focusing error

Tune near half-integer: $\{Q\} = 0.5$



$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

the kicks will add up in phase every two turns
driving the particle out of the machine

Half-integer resonance

Tune shift

- A small gradient error leads to a change in tune:

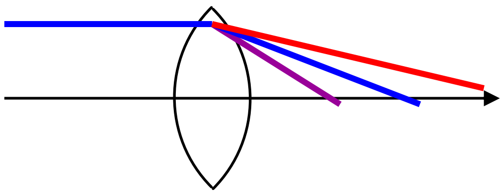
$$\delta Q = \frac{1}{4\pi} \cdot \frac{\beta}{f}$$

- A distribution of gradient errors leads to a tune shift

$$\Delta Q = \frac{1}{4\pi} \oint \frac{g(s) \cdot \beta(s)}{(B\rho)} ds$$

This is how the tunes are adjusted

Chromaticity



Focusing depends on particle energy

Equivalent to an error in gradient

$$k = \frac{g}{(B\rho)} = \frac{g}{(B\rho)_0(1+\delta)} \cong k(1-\delta)$$

$$\Delta k = -k \cdot \delta$$

Causes a tune shift of:

$$\Delta Q = \frac{1}{4\pi}\oint \Delta k \cdot \beta ds = \left(-\frac{1}{4\pi}\oint k \cdot \beta ds\right) \cdot \delta$$

We define chromaticity ξ

$$\Delta Q = -\xi \cdot \delta$$

In strong focusing rings ξ ~ -100 ! For energy spread δ ~1%

$$\Delta Q \sim 1!!$$

Need positive chromaticity to prevent "head-tail" instability

Chromaticity correction

How can we adjust chromaticity? We need gradients (focusing) that changes with energy deviation δ

Sextupole magnets

In horizontal plane

$$B = m \cdot x^2$$

$$B' = 2m \cdot x = 2mD \cdot \delta$$

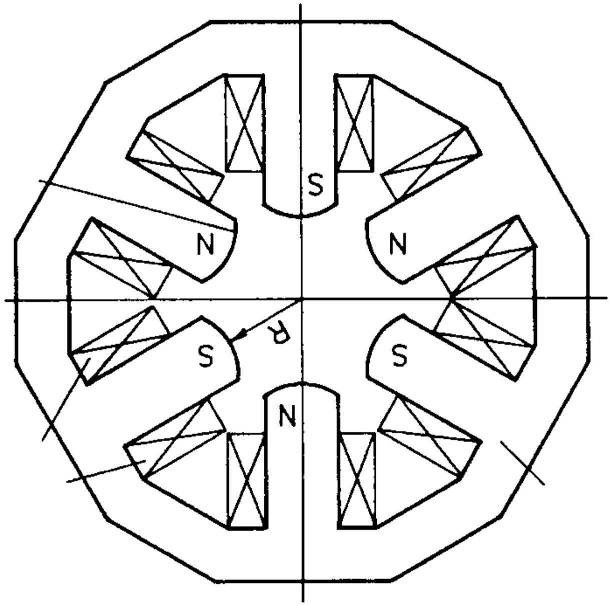
$$m = \frac{1}{2}\frac{B''L}{(B\rho)} = \frac{1}{2}g'L \qquad m[m^{-2}] = 0.2998\frac{B''[T/m^2]L[m]}{p_o[GeV/c]}$$

Sextupoles to correct chromaticity

Two ingredients are needed:

- Sextupoles placed in a region of finite dispersion: sort particles according to their energy deviation

$$x = x_\beta + D\delta$$

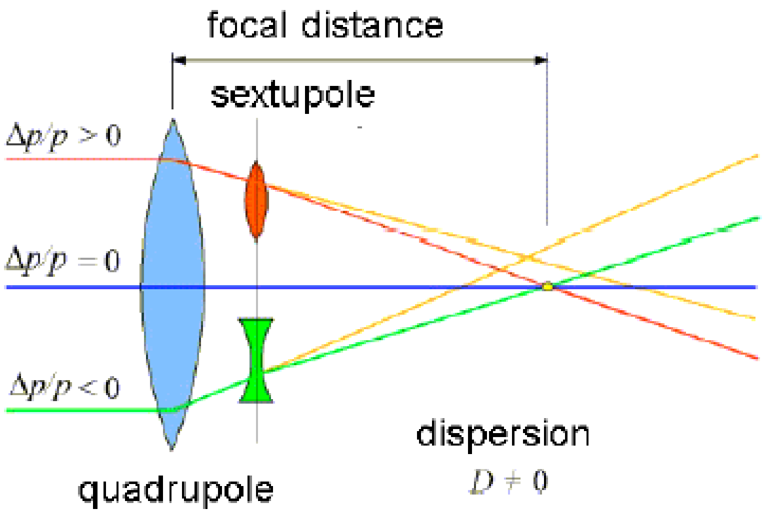


- Gradients that depend on particle position

$$\Delta x' = mx^2 = m(x_\beta + D\delta)^2$$

$$= mx_\beta^2 + 2mD\delta \cdot x_\beta + mD^2\delta^2$$

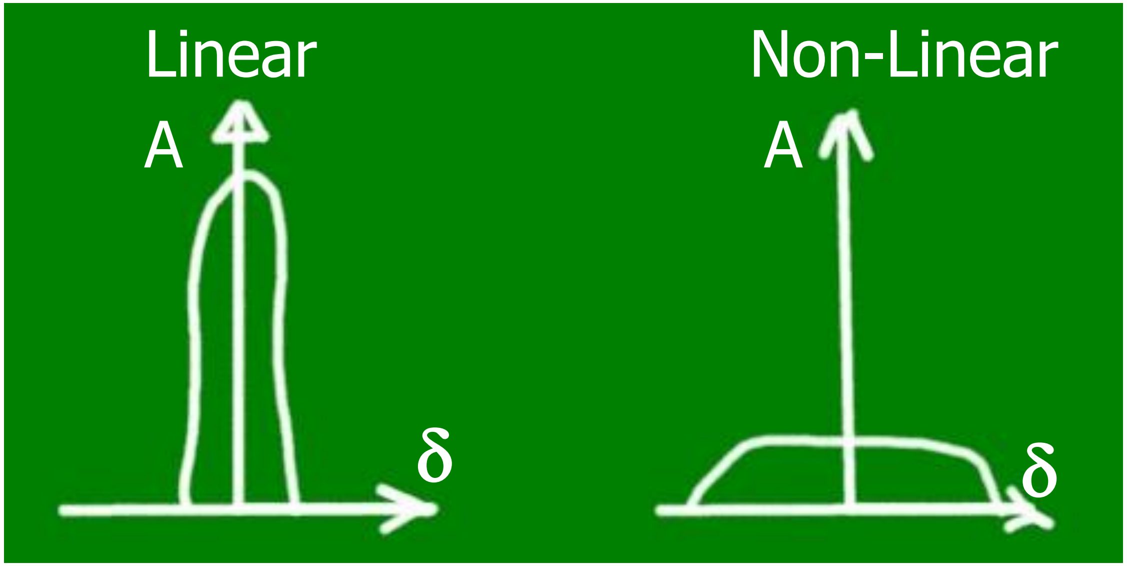
quadrupole term



DYNAMIC APERTURE

Having corrected chromatic aberrations we introduced geometric aberrations:

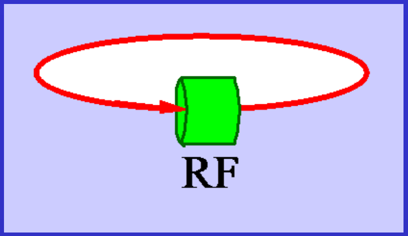
- we increased energy acceptance
- but particles with large transverse amplitudes are no longer stable!



Longitudinal dynamics

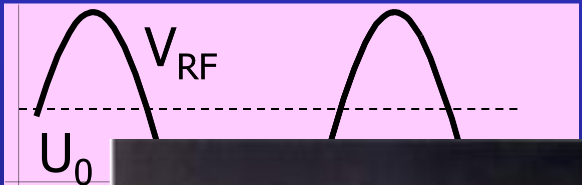
Phase stability

Longitudinal motion: compensating radiation loss U_0



- RF cavity provides accelerating field with frequency
  - h – harmonic number

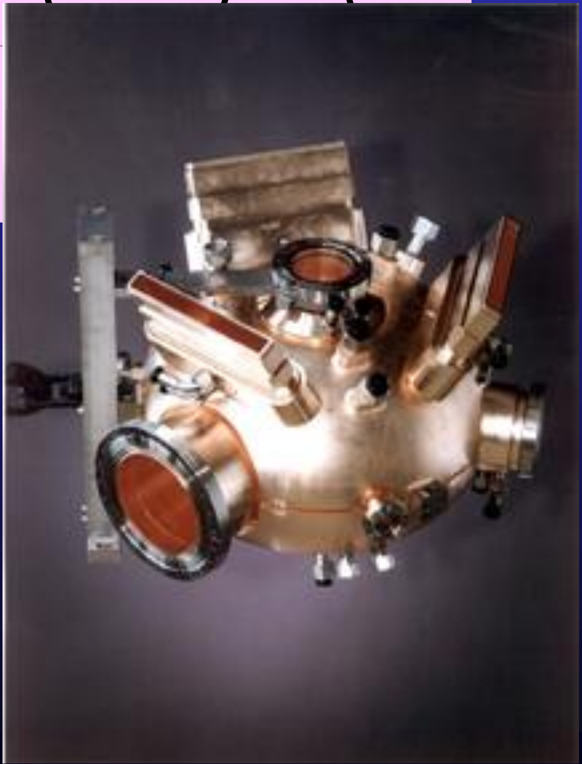
$$f_{RF} = h \cdot f_0$$



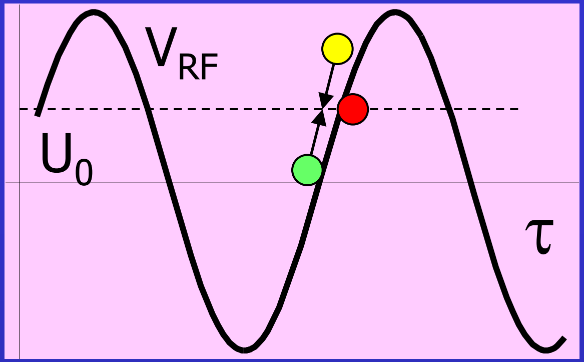
- The energy gain:

$$U_{RF} = eV_{RF}(\tau)$$

- Synchronous particle:
  - has design energy
  - gains from the RF on the average as much as it loses per turn U_0



Longitudinal motion: phase stability

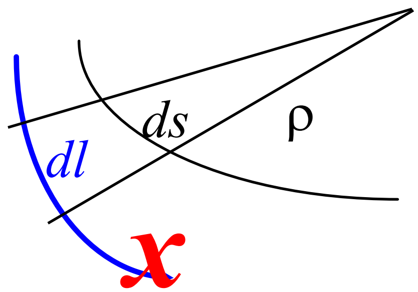


- Particle ahead of synchronous one
  - gets too much energy from the RF
  - goes on a longer orbit (not enough B)
    >> takes longer to go around
  - comes back to the RF cavity closer to synchronous part.
- Particle behind the synchronous one
  - gets too little energy from the RF
  - goes on a shorter orbit (too much B)
  - catches-up with the synchronous particle

Orbit Length

Length element depends on x

$$dl = \left(1 + \frac{x}{\rho}\right) ds$$



Horizontal displacement has two parts:

$$x = x_\beta + x_\varepsilon$$

- To first order x_β does not change L
- x_ε – has the same sign around the ring

Length of the off-energy orbit

$$L_\varepsilon = \oint dl = \oint \left(1 + \frac{x_\varepsilon}{\rho}\right) ds = L_0 + \Delta L$$

$$\Delta L = \delta \cdot \oint \frac{D(s)}{\rho(s)} ds \quad \text{where} \quad \delta = \frac{\Delta p}{p} = \frac{\Delta E}{E}$$

$$\frac{\Delta L}{L} = \alpha \cdot \delta$$

Something funny happens on the way around the ring...

Revolution time changes with energy

$$T_0 = \frac{L_0}{c\beta}$$

$$\frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta\beta}{\beta}$$

- Particle goes faster (not much!)

$$\frac{d\beta}{\beta} = \frac{1}{\gamma^2} \cdot \frac{dp}{p} \quad \text{(relativity)}$$

- while the orbit length increases (more!)

$$\frac{\Delta L}{L} = \alpha \cdot \frac{dp}{p}$$

- The "slip factor" $\eta \cong \alpha$ since $\alpha >> \frac{1}{\gamma^2}$

$$\frac{\Delta T}{T} = \left(\alpha - \frac{1}{\gamma^2}\right) \cdot \frac{dp}{p} = \eta \cdot \frac{dp}{p}$$

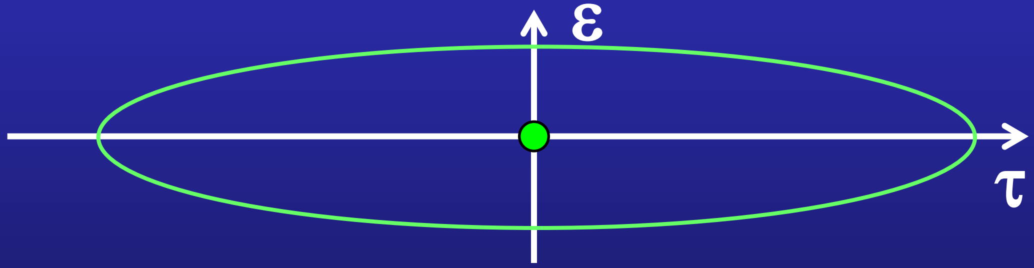
- Ring is above "transition energy"

$$\alpha \equiv \frac{1}{\gamma_{tr}^2}$$

isochronous ring: $\eta = 0 \text{ or } \gamma = \gamma_{tr}$

Longitudinal motion: energy-time oscillations

energy deviation from the design energy, or
the energy of the synchronous particle



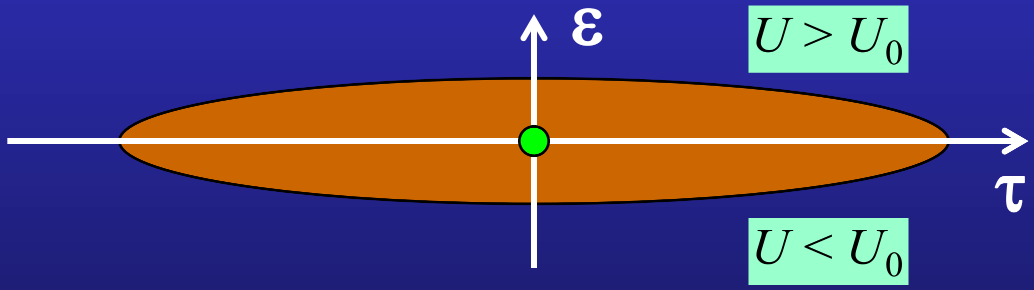
longitudinal coordinate measured from the
position of the synchronous electron

Longitudinal motion: damping of synchrotron oscillations

$$P_\gamma \propto \boldsymbol{E}^2 B^2$$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The synchrotron motion is damped

- the phase space trajectory is spiraling towards the origin