Relativistic Quantum Particles and Fields

Some Theoretical Basics

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Outline

- Relativistic Kinematics and Special Relativity
- Quantum Dynamics
- Relativistic Quantum Particles and Fields
- Scattering and Perturbation Theory
- Feynman Rules and Cross Sections
- Lie Groups and Symmetries

Reference Material

- Lecture notes available from the Proceedings of the International Workshops and COPROMAPH Summer Schools on Contemporary Problems in Mathematical Physics (COPROMAPH, Cotonou, Benin), J. Govaerts, M. Norbert Hounkonnou *et al.*, eds. (World Scientific Publishing), Volumes 2 (2001), 3 (2003), 5 (2007); available as `arXiv:hep-th/0207276 [hep-th]`, `arXiv:hep-th/0408021 [hep-th]`, `arXiv:0812.0721 [hep-th]`.

- AIMS (African Institute for Mathematical Sciences, South Africa) Lecture notes available from the web site of the ASP2012 School (`COPRO5.AIMS2009.pdf`, `Lect.AIMS2009.Part2.pdf`); more material with solutions available on request.

- Textbooks (one amongst many others): M. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Books Publishing, Cambridge, Massachusetts, 1995).

Relativistic Kinematics and Special Relativity

Massive particle: Lorentz factor: $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\vec{\beta} = \frac{\vec{v}}{c}$

$$E = mc^2\gamma = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \simeq mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right)$$

$$\simeq mc^2 + \frac{1}{2}mv^2 + \ldots$$

$$\vec{p}c = mc^2\vec{\beta}\gamma = \frac{mc^2\vec{\beta}}{\sqrt{1-\beta^2}} \simeq mc^2\frac{\vec{v}}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right)$$

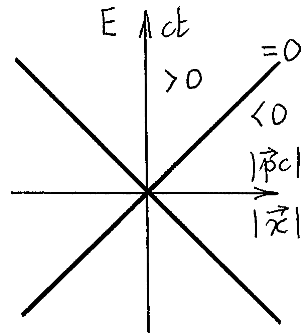
$$\frac{\vec{v}}{c} = \vec{\beta} = \frac{\vec{p}c}{E}, \qquad |\vec{\beta}| = \frac{1}{E}|\vec{p}c|$$

Relativistic invariants

Energy-momentum | Light cône | Space-time

$$E^2 - (\vec{p}c)^2 = \left(mc^2\right)^2$$

$$(ct)^2 - \vec{x}^{\,2} = s^2$$



$$E = \sqrt{(\vec{p}c)^2 + \left(mc^2\right)^2}$$

$s^2 > 0$: time-like
$s^2 = 0$: light-like
$s^2 < 0$: space-like

Massless particle: $m = 0$: light-like particle

$$E = |\vec{p}c|, \qquad |\vec{\beta}| = 1, \qquad |\vec{v}\,| = c$$

Photon: $E = \hbar\omega = h\nu$, $\vec{p} = \hbar\vec{k}$, $|\vec{k}| = \frac{2\pi}{\lambda}$, $|\vec{p}\,| = \frac{h}{\lambda}$, $\nu\lambda = c$, $\hbar = \frac{h}{2\pi}$

$$E^2 - (\vec{p}c)^2 = 0$$

Lorentz boosts:

$$\cosh^2\omega - \sinh^2\omega = 1, \quad \cosh\omega = \gamma, \quad \sinh\omega = \beta\gamma, \quad \tanh\omega = \beta$$

$$\begin{aligned} E' &= E\cosh\omega - (p_x c)\sinh\omega \\ (p'_x c) &= -E\sinh\omega + (p_x c)\cosh\omega \\ (p'_y c) &= (p_y c) \\ (p'_z c) &= (p_z c) \end{aligned} \qquad \begin{aligned} (ct') &= \gamma\,[\ (ct) - \beta x] \\ x' &= \gamma\,[-\beta(ct) + x] \\ y' &= y \\ z' &= z \end{aligned}$$

Minkowski spacetime: Pseudo-Euclidean or hyperbolic geometry

$$E^2 - (\vec{p}c)^2 = \left(mc^2\right)^2 \quad ; \quad (ct)^2 - \vec{x}^{\,2} = s^2$$

4-vectors: $x^\mu = (ct, \vec{x}\,)$, $p^\mu = (E, \vec{p}c)$, $\mu, \nu = 0, 1, 2, 3$, $i, j = 1, 2, 3$

Minkowski metric: $\eta_{\mu\nu} = \mathrm{diag}(+ - - -)$, $x_\mu = \eta_{\mu\nu}x^\nu$, $x^\mu = \eta^{\mu\nu}x_\nu$

Lorentz invariants: $x \cdot y = \eta_{\mu\nu}x^\mu y^\nu$, $x^2 = \eta_{\mu\nu}x^\mu x^\nu = (ct)^2 - \vec{x}^{\,2} = s^2$,

$$\boxed{p^2 = (mc^2)^2}$$

Henceforth: $\boxed{\boxed{c = 1}}$ (Choice of natural particle physics units)

Particle decay:

$$\begin{array}{ccc} X & \rightarrow & X_1 + X_2 \\ m & & m_1 \quad m_2 \\ p^\mu & & p_1^\mu \quad p_2^\mu \end{array}$$

Energy-momentum conservation (4 independent relations):

$$\boxed{p = p_1 + p_2}$$

Invariant relation: $m^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2$

Center-of-mass frame: Vanishing total momentum

$$p^\mu = (m, \vec{0}), \quad p_1^\mu = (E_1, \vec{p}), \quad p_2^\mu = (E_2, -\vec{p})$$

Exercise

1. Solve the center-of-mass kinematics of the $1 \rightarrow 2$ particle decay.

Particle scattering: $X_1 + X_2 \rightarrow X_3 + X_4$

$$\begin{array}{cccc} m_1 & m_2 & m_3 & m_4 \\ p_1^\mu & p_2^\mu & p_3^\mu & p_4^\mu \end{array}$$

Energy-momentum conservation (4 independent relations):

$$\boxed{p_1 + p_2 = p_3 + p_4}$$

Mandelstam variables:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \\ s + t + u &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned}$$

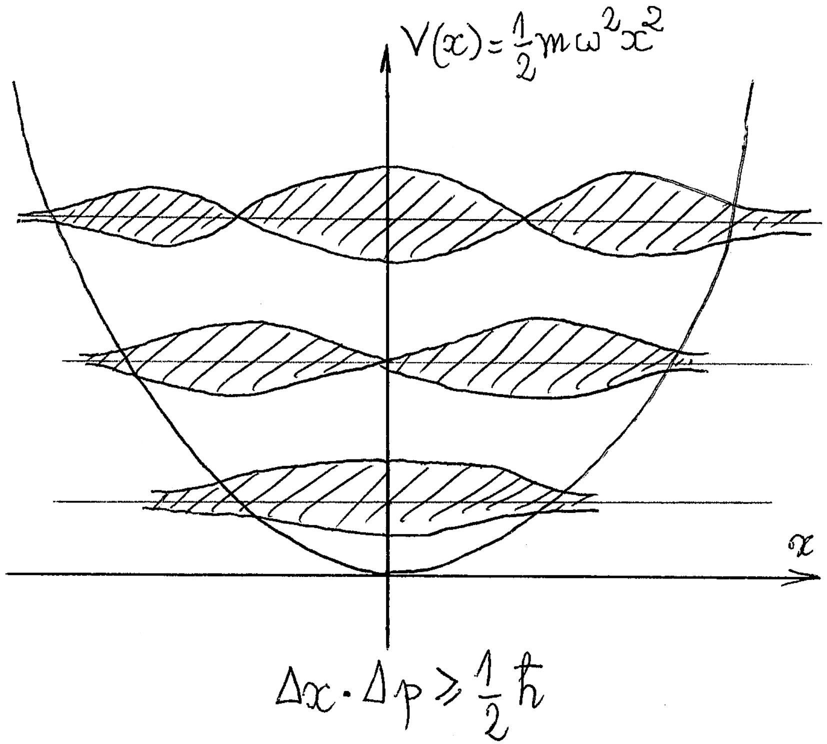
Exercises

1. Solve the center-of-mass kinematics of the $2 \rightarrow 2$ particle scattering.
2. Establish the identity $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.
3. Determine the values of the three Mandelstam variables in the case of 4 identical particles as a function of the scattering angle and the total energy in the center-of-mass frame.

Quantum Dynamics

The quantum harmonic oscillator:

mass m, angular frequency ω, position x, momentum $p = m\dot{x}$



$E_n = \hbar\omega \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$

Hilbert space:

basis of quantum states

Fock states: $|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle$

$\langle 0|0\rangle = 1$, $a|0\rangle = 0$, $\langle n|m\rangle = \delta_{n,m}$

Fock algebra: $\left[a, a^\dagger\right] = \mathbb{I}$

Number operator: $N = a^\dagger a$, $N|n\rangle = n|n\rangle$

Fundamental operators

$$\widehat{x}^\dagger = \widehat{x}, \quad \widehat{p}^\dagger = \widehat{p}, \quad \left(a^\dagger\right)^\dagger = a$$

Hamiltonian: $\widehat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) = \frac{1}{2m}\widehat{p}^2 + \frac{1}{2}m\omega^2\widehat{x}^2$

$$\widehat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right), \qquad \widehat{p} = -im\omega\sqrt{\frac{\hbar}{2m\omega}}\left(a - a^\dagger\right)$$

$$[\widehat{x}, \widehat{p}] = i\hbar\,\mathbb{I}$$

$$\left[a, a^\dagger\right] = \mathbb{I}$$

Schrödinger picture: $i\hbar\frac{d|\psi,t\rangle}{dt} = \widehat{H}\,|\psi,t\rangle$, $\quad U(t_2,t_1){=}e^{-\frac{i}{\hbar}(t_2-t_1)\widehat{H}}$

$$|\psi,t\rangle = U(t,t_0)\,|\psi,t_0\rangle, \qquad |\psi,t\rangle = \sum_{n=0}^{\infty} |n\rangle\, e^{-\frac{i}{\hbar}(t-t_0)\,E_n}\,\langle n|\psi,t_0\rangle$$

$$\sum_{n=0}^{\infty} |n\rangle\,\langle n| = \mathbb{I}$$

Heisenberg picture: $i\hbar\frac{d\widehat{A}(t)}{dt} = \left[\widehat{A}(t),\widehat{H}\right]$, $\quad \widehat{A}(t) = U^{\dagger}(t,t_0)\,\widehat{A}(t_0)\,U(t,t_0)$

$$\widehat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a\,e^{-i\omega\,t} + a^{\dagger}\,e^{+i\omega\,t}\right) \qquad (t_0 = 0)$$

Euler-Lagrange equation of motion:

$$\left(\frac{d^2}{dt^2} + \omega^2\right)\,\widehat{x}(t) = 0$$

Action principle: Variational principle

$$S[x] = m\int_{t_i}^{t_f} dt\,\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}\omega^2\,x^2\right]$$

Lagrangian formulation	$\longleftrightarrow$ (Legendre transform)	Hamiltonian formulation
$\hbar \updownarrow$	**ACTION (Symmetries)**	$\updownarrow \hbar$
Functional integral quantisation	$\longleftrightarrow$	Operator quantisation

$$S[x] = \int dt\, L(x,\dot{x}), \qquad L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2, \qquad m\ddot{x} = -m\omega^2 x$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \qquad H = \dot{x}p - L$$

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \qquad \frac{df(x,p,t)}{dt} = \{f,H\} + \frac{\partial f}{\partial t}, \qquad \{x,p\} = 1$$

$$\dot{x} = \{x,H\} = \frac{1}{m}p, \qquad \dot{p} = \{p,H\} = -m\omega^2 x$$

$$S[x,p] = \int dt\, [\dot{x}p - H(x,p)]$$

Symmetries and Noether's (First) Theorem

For each (independent) **continuous symmetry transformation** (leaving the equations of motion invariant, namely the action invariant up to a total derivative), there exists a conserved quantity, **the** corresponding so-called **Noether charge**

$$t' = t'(t), \qquad q^{n\prime}(t') = q^{n\prime}(q^n, t)$$

$$S[q^{n\prime}] = \int dt'\, L\left(q^{n\prime}, \frac{dq^{n\prime}}{dt'}\right) = \int dt \left[L\left(q^n, \frac{dq^n}{dt}\right) + \frac{d\Lambda(q^n, t)}{dt}\right] = S[q^n] + \text{t.d.}$$

Space translations	$\longleftrightarrow$	Total momentum
Space rotations	$\longleftrightarrow$	Total angular-momentum
Physical time translations	$\longleftrightarrow$	Total energy
Internal space symmetries	$\longleftrightarrow$	Additional conserved charges

Gauged symmetries: A secret of the fundamental interactions

Spherically symmetric harmonic oscillator ($d = 2$)

$$L = \frac{1}{2}m\dot{\vec{x}}^{\,2} - \frac{1}{2}m\omega^2\vec{x}^{\,2} = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}m\omega^2\left(x_1^2 + x_2^2\right)$$

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_1 \mp i a_2\right), \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger \pm i a_2^\dagger\right)$$

$$\left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}, \qquad \left[a_\pm, a_\mp^\dagger\right] = 0$$

(Helicity) Fock basis: $|n_+, n_-\rangle = \frac{1}{\sqrt{n_+!\, n_-!}}\left(a_+^\dagger\right)^{n_+}\left(a_-^\dagger\right)^{n_-}|0\rangle$

$$\hat{H} = \hbar\omega\left(a_1^\dagger a_1 + \frac{1}{2} + a_2^\dagger a_2 + \frac{1}{2}\right) = \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right)$$

SO(2) = U(1) symmetry : $\hat{L} = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$, $\quad \left[\hat{L}, \hat{H}\right] = 0$

$$\hat{H}|n_+, n_-\rangle = E(n_+, n_-)\,|n_+, n_-\rangle, \qquad \hat{L}|n_+, n_-\rangle = \hbar(n_+ - n_-)\,|n_+, n_-\rangle$$

$$E(n_+, n_-) = \hbar\omega(n_+ + n_- + 1)$$

Energy degeneracies?

Particle physics natural units

$$\boxed{c = 1 \qquad \hbar = 1}$$

Conversion factors: $\hbar c \simeq 197$ MeV·fm, $\quad c \simeq 3 \times 10^8$ m·s^{-1}

$$\boxed{\text{space} \overset{c}{\leftrightarrow} \text{time} \overset{\hbar}{\leftrightarrow} (\text{energy})^{-1} \overset{c}{\leftrightarrow} (\text{mass})^{-1}}$$

Exercise

1. Consider the evaluation of Heisenberg's uncertainty relation $\Delta x \, \Delta p \geq \frac{1}{2}\hbar$ for each of the Fock states of the quantum harmonic oscillator.

Relativistic Quantum Particles and Fields

Free relativistic particles:

$$\omega(\vec{k}\,) = \sqrt{\vec{k}^{\,2} + m^2} \qquad [E(\vec{p}\,) = \sqrt{(\vec{p}c)^2 + (mc^2)^2}]$$

$$\left[a(\vec{k}), a^\dagger(\vec{\ell})\right] = (2\pi)^3\, 2\omega(\vec{k})\, \delta^{(3)}\left(\vec{k} - \vec{\ell}\right) \mathbb{I}$$

1-particle states: $|\vec{k}\rangle = a^\dagger(\vec{k})\,|0\rangle$

Energy-momentum (operators): [quantum vacuum energy]

$$\begin{pmatrix} \hat{H} \\ \hat{\vec{P}} \end{pmatrix} = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})} \begin{pmatrix} \omega(\vec{k}) \\ \vec{k} \end{pmatrix} a^\dagger(\vec{k})a(\vec{k})$$

$$\hat{H}\,|\vec{k}\rangle = \omega(\vec{k})\,|\vec{k}\rangle, \qquad \hat{\vec{P}}\,|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle$$

Heisenberg picture: **Relativistic invariance:** $k \cdot x = k^0 x^0 - \vec{k} \cdot \vec{x}$

$$\begin{aligned} \widehat{\phi}(x^\mu) &= \widehat{\phi}(t, \vec{x}\,) \\ &= \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})} \left[a(\vec{k})\, e^{-i\left(\omega(\vec{k})\, t - \vec{k}\cdot\vec{x}\right)} + a^\dagger(\vec{k})\, e^{+i\left(\omega(\vec{k})\, t - \vec{k}\cdot\vec{x}\right)} \right] \end{aligned}$$

Euler-Lagrange equation of motion:

$$\left(\Box + m^2\right) \widehat{\phi}(x) = \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2 \right) \widehat{\phi}(x) = 0$$

Klein-Gordon equation (wave dynamics)

Action principle:

$$\begin{aligned} S[\phi] &= \int d^4x^\mu \left\{ \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} \left(\vec{\nabla} \phi \right)^2 - \frac{1}{2} m^2 \phi^2 \right\} \\ &= \int d^4x^\mu \left\{ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \right\} = \int d^4x^\mu\, \mathcal{L}(\phi, \partial_\mu \phi) \end{aligned}$$

$$\phi^*(x) = \phi(x)$$

Real scalar field $\longleftrightarrow$ neutral spin 0 (scalar) (massive) particle

Fundamental unification: $\hbar \; + \; c$ [Spacetime symmetries]

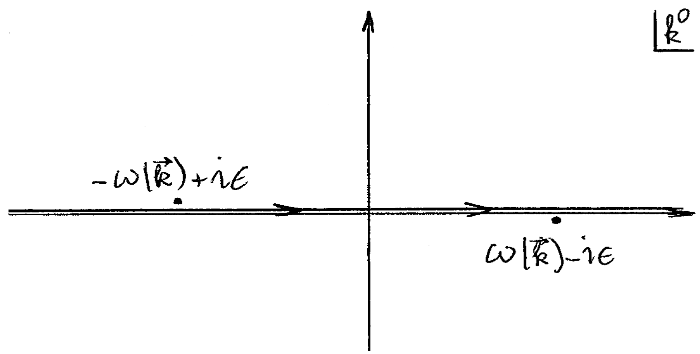
The Feynman propagator

Spacetime localised 1-particle quantum states:

$$\hat{\phi}(x^\mu)\,|0\rangle = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})}\, e^{ik\cdot x}\big|_{k^0=\omega(\vec{k})}\, |\vec{k}\rangle$$

Causal propagation: time-ordered 2-point function or
Feynman propagator

$$\begin{aligned}\langle 0|T\phi(x)\phi(y)|0\rangle &= \theta(x^0-y^0)\,\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|\phi(y)\phi(x)|0\rangle \\ &= \int_{(\infty)} \frac{d^4k^\mu}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\, e^{-ik\cdot(x-y)}\end{aligned}$$



Exercise

1. Through contour integration in the complex k^0 plane, confirm this manifest spacetime invariant expression of the Feynman propagator.

The complex scalar field

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2} m^2 \phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m^2 \phi_2^2 = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right)$$

Internal SO(2)=U(1) [global] symmetry: $\phi(x) \rightarrow e^{-i\alpha}\,\phi(x)$

$$\phi(x) \;=\; \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})} \left[a(\vec{k})\, e^{-i\left(\omega(\vec{k})\, t - \vec{k}\cdot\vec{x}\right)} + b^\dagger(\vec{k})\, e^{+i\left(\omega(\vec{k})\, t - \vec{k}\cdot\vec{x}\right)}\right]$$

$$a(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) + i a_2(\vec{k})\right], \qquad b(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) - i a_2(\vec{k})\right]$$

$$\left[a(\vec{k}), a^\dagger(\vec{\ell})\right] = (2\pi)^3\, 2\omega(\vec{k})\, \delta^{(3)}\left(\vec{k} - \vec{\ell}\right) \mathbb{I} = \left[b(\vec{k}), b^\dagger(\vec{\ell})\right]$$

$$H = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})}\, \omega(\vec{k}) \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right]$$

$$Q = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})} \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right]$$

Particles: $Q = +1$, $\quad Q\, a^\dagger(\vec{k})\, |0\rangle = +\, a^\dagger(\vec{k})\, |0\rangle$

Antiparticles: $Q = -1$, $\quad Q\, b^\dagger(\vec{k})\, |0\rangle = -\, b^\dagger(\vec{k})\, |0\rangle$

Fundamental unification: $\hbar + c$ [Internal symmetries]

Gauging the U(1) symmetry: $\phi(x) \rightarrow e^{-i\alpha(x)}\, \phi(x)$

$\Longrightarrow$ Fundamental interactions governed by the symmetry

Exercise

1. Establish the expressions for the 2-point functions, *i.e.*, the Feynman propagator of the complex scalar field. Explain the outcome of the analysis in terms of the conserved U(1) quantum number.

The real vector field: Neutral spin/helicity 1 particles

$$A_\mu(x) \;=\; \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})} \sum_\lambda \left[a(\vec{k},\lambda)\,\epsilon_\mu(\vec{k},\lambda)\,e^{-ik\cdot x} + a^\dagger(\vec{k},\lambda)\,\epsilon^*_\mu(\vec{k},\lambda)\,e^{+ik\cdot x}\right]$$

$$k^\mu\,\epsilon_\mu(\vec{k},\lambda) = 0, \qquad \sum_\lambda \epsilon_\mu(\vec{k},\lambda)\,\epsilon^*_\nu(\vec{k},\lambda) = -\eta_{\mu\nu}$$

$$\left[a(\vec{k},\lambda), a^\dagger(\vec{\ell},\lambda')\right] = (2\pi)^3\; 2\omega(\vec{k})\,\delta_{\lambda\lambda'}\,\delta^{(3)}\left(\vec{k}-\vec{\ell}\right)\,\mathbb{I}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}m^2\,A_\mu A^\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial_\mu A^\mu = 0$$

[Three polarisations states in the massive case]

Massless case: two polarisation states (helicity ± 1)

Local gauge symmetry: Vector gauge boson

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x), \qquad F'_{\mu\nu}(x) = F_{\mu\nu}(x)$$

The Dirac spinor field: U(1) Charged spin/helicity $\frac{1}{2}$ (anti)particles

$$\psi_\alpha(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})} \sum_{\lambda=\pm} \left[b(\vec{k},\lambda)\, u_\alpha(\vec{k},\lambda)\, e^{-ik\cdot x} + d^\dagger(\vec{k},\lambda)\, v_\alpha(\vec{k},\lambda)\, e^{+ik\cdot x} \right]$$

$$(\gamma^\mu k_\mu - m)\; u(\vec{k},\lambda) = 0, \qquad (\gamma^\mu k_\mu + m)\; v(\vec{k},\lambda) = 0$$

$$\sum_{\lambda=\pm} u_\alpha(\vec{k},\lambda)\, \overline{u}_\beta(\vec{k},\lambda) = (\not{k} + m)_{\alpha\beta}\,, \qquad \sum_{\lambda=\pm} v_\alpha(\vec{k},\lambda)\, \overline{v}_\beta(\vec{k},\lambda) = (\not{k} - m)_{\alpha\beta}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma^0\, \gamma_\mu\, \gamma^0, \quad \overline{\psi} = \psi^\dagger\, \gamma^0, \quad \alpha, \beta = 1,2,3,4$$

$$\left\{ b(\vec{k},\lambda), b^\dagger(\vec{\ell},\lambda') \right\} = (2\pi)^3\; 2\omega(\vec{k})\, \delta_{\lambda\lambda'}\, \delta^{(3)}\left(\vec{k} - \vec{\ell}\right)\, \mathbb{I} = \left\{ d(\vec{k},\lambda), d^\dagger(\vec{\ell},\lambda') \right\}$$

$$\mathcal{L} = \overline{\psi}\,(i\not{\partial} - m)\,\psi, \qquad \mathcal{L} = \frac{1}{2} i \left(\overline{\psi}\gamma^\mu \partial_\mu \psi - \partial_\mu \overline{\psi} \gamma^\mu \psi \right) - m\,\overline{\psi}\psi$$

Dirac equation: $\quad (i\not{\partial} - m)\; \psi(x) = 0$

The Feynman propagator

$$\langle 0|T\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle = \theta(x^0-y^0)\langle 0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle - \theta(y^0-x^0)\langle 0|\overline{\psi}_\beta(y)\psi_\alpha(x)|0\rangle$$

$$= \int_{(\infty)} \frac{d^4k^\mu}{(2\pi)^4} e^{-ik\cdot(x-y)} \left(\frac{i}{\not{k} - m + i\epsilon}\right)_{\alpha\beta}$$

$$= \int_{(\infty)} \frac{d^4k^\mu}{(2\pi)^4} e^{-ik\cdot(x-y)} \left(\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}\right)_{\alpha\beta}$$

Fermionic Fock algebra: Pauli exclusion principle

$$\left\{b, b^\dagger\right\} = \mathbb{I}$$

$$b|0\rangle = 0 \quad , \quad b^\dagger|0\rangle = |1\rangle$$
$$b|1\rangle = |0\rangle \quad , \quad b^\dagger|1\rangle = 0$$

$$\langle 0|0\rangle = 1 = \langle 1|1\rangle, \qquad \langle 0|1\rangle = 0 = \langle 1|0\rangle$$

Other spin 1/2 spinors:

Left- and right-handed Weyl spinors, Majorana spinor

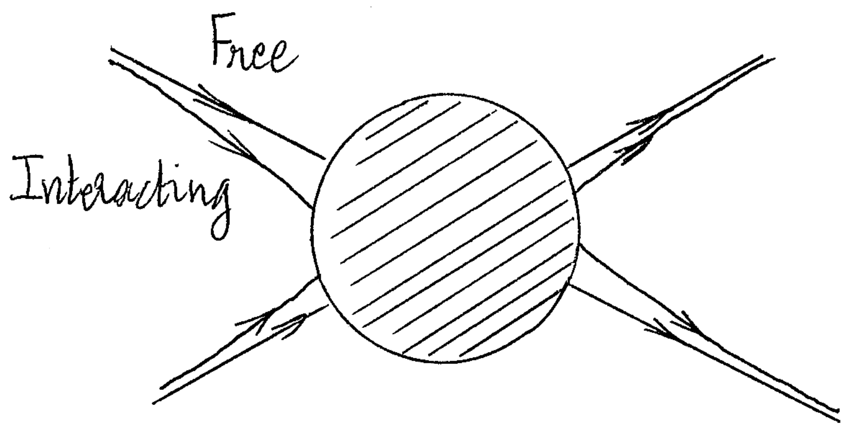
Scattering and Perturbation Theory

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4, \qquad \lambda \geq 0$$

$$\pi(t,\vec{x}\,) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi(t,\vec{x}\,)} = \partial_0\phi(t,\vec{x}\,), \qquad \{\phi(t,\vec{x}\,), \pi(t,\vec{y}\,)\} = \delta^{(3)}(\vec{x}-\vec{y}\,)$$

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$$

$$\int_{(\infty)} d^3\vec{x}\,\mathcal{H} = H = H_0 + H_{\text{int}} = \int_{(\infty)} d^3\vec{x}\,\mathcal{H}_0 + \int_{(\infty)} d^3\vec{x}\,\mathcal{H}_{\text{int}}$$



In-state: $\lim_{t\to-\infty} \underbrace{e^{-i(t-t_0)H}|\psi, t_0\rangle}_{|\psi,t\rangle} = \lim_{t\to-\infty} \underbrace{e^{-i(t-t_0)H_0}|\psi_{\text{in}}, t_0\rangle}_{|\psi_{\text{in}},t\rangle}$

Out-state: $\lim_{t\to+\infty} \underbrace{e^{-i(t-t_0)H}|\chi, t_0\rangle}_{|\chi,t\rangle} = \lim_{t\to+\infty} \underbrace{e^{-i(t-t_0)H_0}|\chi_{\text{out}}, t_0\rangle}_{|\chi_{\text{out}},t\rangle}$

Transition probability amplitude

$$\langle\chi, t|\psi, t\rangle = \langle\chi, t_0|\psi, t_0\rangle =$$

$$= \lim_{t_\mp\to\mp\infty} \langle\chi_{\text{out}}, t_0| \underbrace{e^{i(t_+-t_0)H_0}e^{-i(t_+-t_0)H}}_{\Omega(t_+,t_0)} \underbrace{e^{i(t_--t_0)H}e^{-i(t_--t_0)H_0}}_{\Omega^\dagger(t_-,t_0)} |\psi_{\text{in}}, t_0\rangle$$

$$= \langle\chi_{\text{out}}, t_0|S|\psi_{\text{in}}, t_0\rangle$$

S: **Scattering operator, S matrix**

$$S = \lim_{t_\mp\to\mp\infty} \Omega(t_+, t_0)\, \Omega^\dagger(t_-, t_0)$$

$$\Omega(t,t_0) = e^{i(t-t_0)H_0}\, e^{-i(t-t_0)H}$$

Differential equation:

$$i\,\partial_t\,\Omega(t,t_0) = e^{i(t-t_0)H_0}\,(H - H_0)\,e^{-i(t-t_0)H} = e^{i(t-t_0)H_0}\,H_{\text{int}}\,e^{-i(t-t_0)H}$$

Interaction picture: $A_{(I)}(t) = e^{i(t-t_0)H_0}\,A(t_0)\,e^{-i(t-t_0)H_0}$

$$i\partial_t\Omega(t,t_0) = H^{(I)}_{\text{int}}(t)\;\Omega(t,t_0)$$

$$\Omega(t,t_0) = Te^{-i\int_{t_0}^{t} dt'\,H^{(I)}_{\text{int}}(t')}$$ [Time ordered product]

$$S = Te^{-i\int_{t_0}^{+\infty} dt H^{(I)}_{\text{int}}(t)}\,Te^{-i\int_{-\infty}^{t_0} dt H^{(I)}_{\text{int}}(t)} = Te^{-i\int_{-\infty}^{+\infty} dt H^{(I)}_{\text{int}}(t)}$$

S matrix: $$\boxed{S = Te^{-i\int_{(\infty)} d^4x^\mu\,\mathcal{H}^{(I)}_{\text{int}}} = Te^{i\int_{(\infty)} d^4x^\mu\,\mathcal{L}^{(I)}_{\text{int}}}}$$

[Non derivative couplings]

Field operators in the interaction picture

$$\phi_{(I)}(t,\vec{x}\,) = e^{i(t-t_0)H_0}\,\phi(t_0,\vec{x}\,)\,e^{-i(t-t_0)H_0}$$

$$\pi_{(I)}(t,\vec{x}\,) = e^{i(t-t_0)H_0}\,\pi(t_0,\vec{x}\,)\,e^{-i(t-t_0)H_0}$$

$$[\phi(t_0,\vec{x}\,),\pi(t_0,\vec{y}\,)] = i\delta^{(3)}(\vec{x}-\vec{y}\,), \qquad \left[\phi_{(I)}(t,\vec{x}\,),\pi_{(I)}(t,\vec{y}\,)\right] = i\delta^{(3)}(\vec{x}-\vec{y}\,)$$

$$\phi_{(I)}(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\,2\omega(\vec{k})}\left[a(\vec{k})\,e^{-ik\cdot x} + a^\dagger(\vec{k})\,e^{+ik\cdot x}\right]$$

$$\pi_{(I)}(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\,2\omega(\vec{k})}\left(-i\omega(\vec{k}\,)\right)\left[a(\vec{k})\,e^{-ik\cdot x} - a^\dagger(\vec{k})\,e^{+ik\cdot x}\right]$$

$$\langle 0|T\phi_{(I)}(x)\phi_{(I)}(y)|0\rangle = \int_{(\infty)} \frac{d^4k^\mu}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\,e^{-ik\cdot(x-y)}$$

$|0\rangle$: perturbative Fock vacuum

Fock space quantization:

appropriate for the particle picture of particle interactions

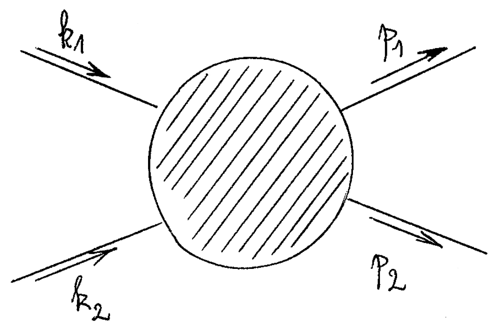
Perturbation theory in $\mathcal{H}_{\text{int}}$ or $\mathcal{L}_{\text{int}}$.

Feynman Rules and Cross Sections

Simplest Example: $\mathcal{L}_{\text{int}} = -\frac{1}{4!}\lambda\phi^4, \quad \lambda > 0$

$$\mathcal{H}_{\text{int}}^{(I)} = \frac{1}{4!}\lambda : \phi_{(I)}^4 : \qquad [\text{normal} - \text{ordering}]$$

Process: $k_1 + k_2 \rightarrow p_1 + p_2$



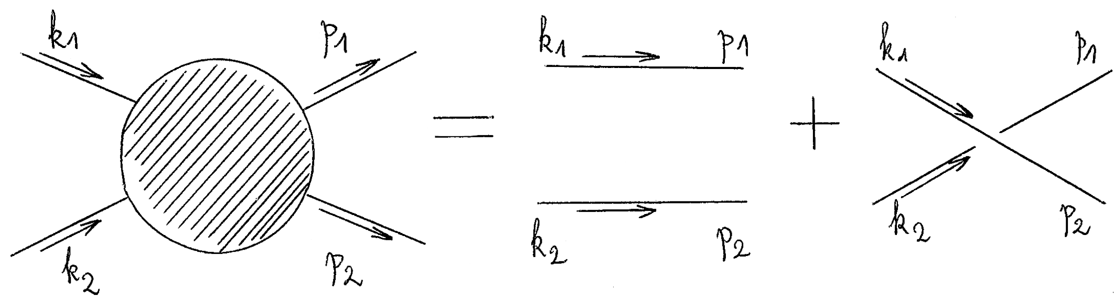
$$|\psi_{\text{in}}, t_0\rangle = a^\dagger(k_1)a^\dagger(k_2)\,|0\rangle$$
$$|\chi_{\text{out}}, t_0\rangle = a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle$$

Transition amplitude: $\langle\chi_{\text{out}}, t_0|S|\psi_{\text{in}}, t_0\rangle$

$$S = Te^{-i\int d^4x^\mu \mathcal{H}_{\text{int}}^{(\mathcal{I})}} = \mathbb{I} + T\left(-i\int d^4x^\mu \mathcal{H}_{\text{int}}^{(I)}\right) + \frac{1}{2!}T\left(-i\int d^4x^\mu \mathcal{H}_{\text{int}}^{(I)}\right)^2 + \ldots$$

Lowest order contribution

$$\langle 0|a(p_2)a(p_1)\ \mathbb{I}\ a^\dagger(k_1)a^\dagger(k_2)|0\rangle =$$
$$= (2\pi)^3 2\omega(k_1)\,(2\pi)^3 2\omega(k_2)\,\left[\delta_{p_1k_1}\delta_{p_2k_2} + \delta_{p_1k_2}\delta_{p_2k_1}\right]$$



First order contribution

$$\left(-i\frac{\lambda}{4!}\right)\int d^4x^\mu \langle 0|a(p_2)a(p_1) : \phi^4(x) : a^\dagger(k_1)a^\dagger(k_2)|0\rangle$$

$$= 6\left(-i\frac{\lambda}{4!}\right)\int \prod_{i=1}^{4} \frac{d^3\vec{\ell}_i}{(2\pi)^3 2\omega(\ell_i)} \int d^4x^\mu e^{i(\ell_1+\ell_2-\ell_3-\ell_4)\cdot x} \times$$

$$\times \langle 0|a(p_2)a(p_1)\, a^\dagger(\ell_1)a^\dagger(\ell_2)a(\ell_3)a(\ell_4)\, a^\dagger(k_1)a^\dagger(k_2)|0\rangle$$

The matrix element is

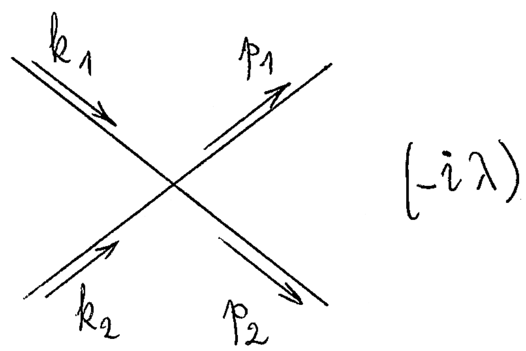
$$\langle 0|a(p_2)a(p_1)a^\dagger(\ell_1)a^\dagger(\ell_2)a(\ell_3)a(\ell_4)a^\dagger(k_1)a^\dagger(k_2)|0\rangle =$$

$$= \prod_{i=1}^{4}(2\pi)^3 2\omega(\ell_i)\left[\delta_{\ell_1 p_1}\delta_{\ell_2 p_2}+\delta_{\ell_1 p_2}\delta_{\ell_2 p_1}\right]\left[\delta_{\ell_4 k_1}\delta_{\ell_3 k_2}+\delta_{\ell_4 k_2}\delta_{\ell_3 k_1}\right]$$

hence

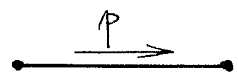
$$\begin{aligned}\langle \chi_{\text{out}}, t_0|S_2|\psi_{\text{in}}, t_0\rangle &= 4\times 6\left(-i\frac{\lambda}{4!}\right)\int d^4x^\mu e^{i(p_1+p_2-k_1-k_2)\cdot x}\\ &= (-i\lambda)\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-k_1-k_2)\end{aligned}$$

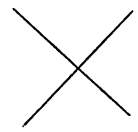
Diagrammatic representation: **Vertex Feynman rule**

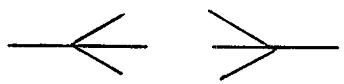


Combinatorial factor: $4!\left(-i\frac{\lambda}{4!}\right) = (-i\lambda)$

Feynman rules in momentum space

Propagator:  $\frac{i}{p^2-m^2+i\epsilon}$

Vertex:  $(-i\lambda)$

External lines:  1

Momentum conservation at each vertex:
overall factor $(2\pi)^4\delta^{(4)}\left(\sum_i p_i\right)$

Integration over undetermined loop momenta: $\int \frac{d^4p^\mu}{(2\pi)^4}$

Divide by symmetry factors
to be determined by combinatorics

Cross sections

General structure: $S = \mathbb{I} + iT$

$$\langle \vec{p}_1, \vec{p}_2, \ldots | iT | \vec{k}_A, \vec{k}_B \rangle = (2\pi)^4\, \delta^{(4)}(k_A + k_B - \sum_f p_f)\, i\mathcal{M}(k_A + k_B \to p_f)$$

Cross-section differential element in the final state phase space

$$\begin{aligned} d\sigma \;=\; & \frac{1}{4\sqrt{(k_A \cdot k_B)^2 - m_A^2 m_B^2}} \times \left(\prod_f \frac{d^3\vec{p}_f}{(2\pi)^3 2\omega(\vec{p}_f)} \right) \times \\ & \times (2\pi)^4\, \delta^{(4)}(k_A + k_B - \sum_f p_f) \times \\ & \times \left| i\mathcal{M}(k_A + k_B \to p_f) \right|^2 \end{aligned}$$

Extra symmetry factor for the total cross section
if there are identical particles in the final state

Decay rates

In the decay rest frame

$$
\begin{aligned}
d\Gamma &= \frac{1}{2m_A} \times \left(\prod_f \frac{d^3\vec{p}_f}{(2\pi)^3 2\omega(\vec{p}_f)} \right) \times \\
&\quad \times (2\pi)^4 \, \delta^{(4)}(k_A + k_B - \sum_f p_f) \times \\
&\quad \times \left| i\mathcal{M}(k_A \to p_f) \right|^2
\end{aligned}
$$

Exercise

1. Consider a model with two species of neutral scalar particles ϕ and χ of masses m and M, respectively, such that $M > 2m$, with the following coupling,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}M^2\chi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}g\chi\phi^2$$

the real coupling constant g having a dimension of mass in particle physics units. Compute to first order in perturbation theory the lifetime τ of the particle χ, $\tau = 1/\Gamma(\chi \to 2\phi)$.

Lie Groups and Symmetries

Continuous Lie groups and transformations

Translation in time: $|\psi, t + t_0\rangle = e^{-\frac{i}{\hbar} t_0 \hat{H}} |\psi, t\rangle$

Continuous parameter: t_0; (Infinitesimal) Generator: $\hat{H}$

Translation in space:

$$|x + a\rangle = e^{-\frac{i}{\hbar} a\hat{p}} |x\rangle, \quad \langle x| e^{\frac{i}{\hbar} a\hat{p}} |\psi\rangle = e^{a\frac{d}{dx}} \psi(x) = \psi(x + a)$$

Continuous parameter: a; (Infinitesimal) Generator: $\hat{p}$

Rotation in a plane: SO(2) or U(1)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad z' = x' + iy' = e^{-i\theta} (x + iy) = e^{-i\theta} z$$

$$T = -i \frac{dR(\theta)}{d\theta}_{|\theta=0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad R(\theta) = e^{i\theta T}$$

Continuous parameter: θ; (Infinitesimal) Generator: T

Rotation in (three dimensional euclidean) space: SO(3)

$$R_1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}, \quad R_2(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

$$R_3(\theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_i(\theta_i) = e^{i\theta_i T_i}, \quad T_i = -i\frac{dR_i(\theta_i)}{d\theta_i}_{|\theta_i=0}, \quad (T_i)_{jk} = -i\epsilon_{ijk}, \quad i,j,k = 1,2,3$$

General SO(3) rotation: $R(\theta_1,\theta_2,\theta_3) = R_1(\theta_1)\,R_2(\theta_2)\,R_3(\theta_3)$

$$R(\alpha_i) = e^{i(\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3)}$$

Non-abelian Lie group

Non-abelian Lie algebra (three dimensional)

$$\left[T_i, T_j\right] = i\,\epsilon_{ijk}\,T_k$$

General (compact) Lie group and Lie algebra

Generators: $T_a,\ T_a^\dagger = T_a,\ a = 1, 2, \ldots, d$

Lie algebra: $[T_a, T_b] = i f_{abc} T_c$, f_{abc}: real structure constants

Lie group: $g(\alpha) = e^{i\alpha_a T_a},\ g^\dagger(\alpha) = g^{-1}(\alpha)$

Symmetries

Noether charges (no induced surface terms): $\left[\hat{Q}_a, \hat{Q}_b\right] = i\hbar\, f_{abc}\, \hat{Q}_c$

Finite symmetry transformations: $e^{\frac{i}{\hbar}\alpha_a \hat{Q}_a}$

In the case of field theories: [current algebra]

$$Q_a = \int_{(\infty)} d^3\vec{x}\, J_a^{\mu=0}, \qquad \partial_\mu J_a^\mu = 0$$

Spherically symmetric harmonic oscillator $(d = 2)$

$$L = \frac{1}{2} m \dot{\vec{x}}^{\,2} - \frac{1}{2} m\omega^2 \vec{x}^{\,2} = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} m\omega^2 \left(x_1^2 + x_2^2\right)$$

$$a_\pm = \frac{1}{\sqrt{2}} (a_1 \mp i a_2)\,, \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}} \left(a_1^\dagger \pm i a_2^\dagger\right)$$

$$\left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}, \quad \left[a_\pm, a_\mp^\dagger\right] = 0, \quad |n_+, n_-\rangle = \frac{1}{\sqrt{n_+!\, n_-!}} \left(a_+^\dagger\right)^{n_+} \left(a_-^\dagger\right)^{n_-} |0\rangle$$

$$\hat{H} = \hbar\omega \left(a_1^\dagger a_1 + \frac{1}{2} + a_2^\dagger a_2 + \frac{1}{2}\right) = \hbar\omega \left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right)$$

SO(2) = U(1) symmetry : $\hat{L} = \hbar \left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$, $\quad \left[\hat{L}, \hat{H}\right] = 0$

$$\hat{H} |n_+, n_-\rangle = E(n_+, n_-)\, |n_+, n_-\rangle, \quad \hat{L} |n_+, n_-\rangle = \hbar (n_+ - n_-)\, |n_+, n_-\rangle$$

$$E(n_+, n_-) = \hbar\omega (n_+ + n_- + 1)$$

Energy degeneracies?

$$\widehat{H} = \hbar\omega \left(a_+^\dagger \; a_-^\dagger\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_+ \\ a_- \end{pmatrix} + \hbar\omega, \qquad \widehat{L} = \hbar \left(a_+^\dagger \; a_-^\dagger\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

SU(2) invariance [dynamical symmetry]:

$$U \in SU(2), \quad U^\dagger = U^{-1}, \quad \det U = 1 : \quad \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \rightarrow \begin{pmatrix} a'_+ \\ a'_- \end{pmatrix} = U \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

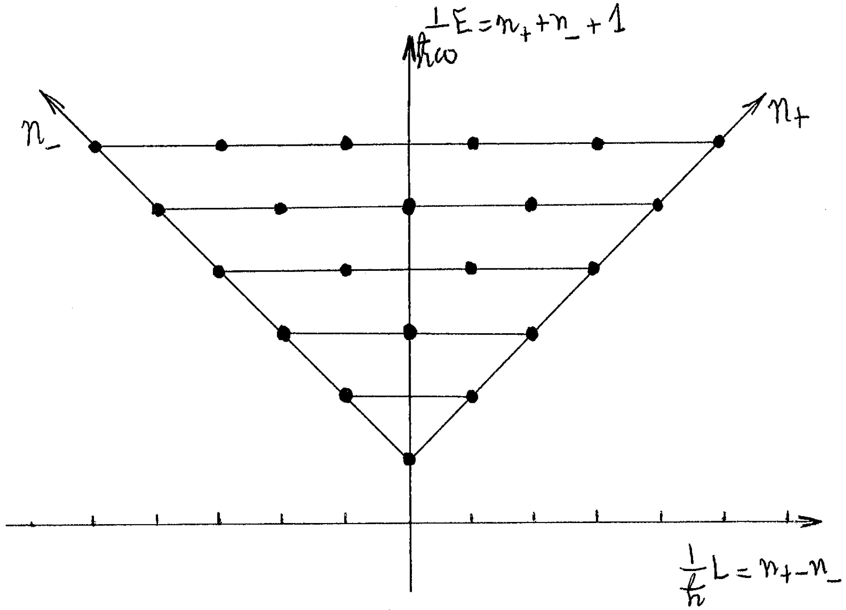
$$T_+ = a_+^\dagger a_-, \quad T_- = a_-^\dagger a_+ = T_+^\dagger, \quad T_3 = \frac{1}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) = \frac{1}{2\hbar}\widehat{L}$$

$$T_1 = \frac{1}{2}\left(T_+ + T_-\right), \quad T_2 = -\frac{i}{2}\left(T_+ - T_-\right), \quad T_\pm = T_1 \pm iT_2$$

$$\left[T_+, T_-\right] = 2T_3, \qquad [T_3, T_\pm] = \pm T_\pm, \qquad \left[T_\pm, \widehat{H}\right] = 0$$

$$\left[T_i, T_j\right] = i\epsilon_{ijk} T_k, \qquad \left[T_i, \widehat{H}\right] = 0$$

Lie algebras: $su(2) = so(3)$ [integer and half-integer spin]



SU(2) irreducible representations: $j = 0, 1, 2, \ldots, \quad -j \le m \le j$

$$|j, m\rangle = |n_+, n_-\rangle$$

$$j = \frac{1}{2}\left(n_+ + n_-\right), \qquad m = \frac{1}{2}\left(n_+ - n_-\right), \qquad n_\pm = j \pm m$$

$$E(j, m) = \hbar\omega(2j + 1)$$

The first excited level: $j = \frac{1}{2}$

In the basis $\left\{ |j = \frac{1}{2}, m = \frac{1}{2}\rangle, |j = \frac{1}{2}, m = -\frac{1}{2}\rangle \right\}$

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\tau_1$$

$$T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\tau_2$$

$$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\tau_3$$

τ_i: **Pauli matrices**

Internal symmetries for field theories

Compact Lie symmetry group G and its algebra:
generators $(T_a)_{ij}$ in some (irreducible) representation R

Symmetry transformations:
[say, complex scalar fields, in a complex representation]

$$\phi'_i(x) = \left(e^{i\alpha_a T_a}\right)_{ij} \phi_j(x)$$

$$\phi_i(x) = \int_{(\infty)} \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega(\vec{k})} \left[a_i(\vec{k})\, e^{-ik\cdot x} + b_i^\dagger(\vec{k})\, e^{+ik\cdot x}\right]$$

$$\left[a_i(\vec{k}), a_j^\dagger(\vec{\ell})\right] = (2\pi)^3\, 2\omega(\vec{k})\, \delta_{ij}\, \delta^{(3)}\left(\vec{k}-\vec{\ell}\right) \mathbb{I} = \left[b_i(\vec{k}), b_j^\dagger(\vec{\ell})\right]$$

1-particle and 1-antiparticle states: representations R and $\overline{R}$

Gauged symmetries?

$$\boxed{\phi'_i(x) = \left(e^{i\alpha_a(x) T_a}\right)_{ij} \phi_j(x)}$$