The Charged Liquid Drop Model
Binding Energy
and
Fission

The Liquid Drop Model

This is a simple model for the *binding energy* of a nucleus

This model is also important to understand
fission and how energy is obtained from fission.

The Liquid Drop Model

A simple model for the binding energy of a nucleus

In the case of a large nucleus a fair number of nucleons are assembled and we know that they *attract each other*.

[As we have seen, the binding energy *per nucleon* is (very approximately) a constant ~8 MeV.]

The volume effect

If nucleons attract each other then if a nucleon is surrounded by other nucleons, its energy will be lower than if it is isolated.

Therefore the binding energy is proportional to the number of particles in the nucleus (i.e. the volume) so $E_b \propto C_1 A$

The mass number

Some constant

In other words, if the binding energy *per nucleon* is a constant, then the *total* binding energy will be directly proportional to the total number of nucleons.

The Liquid Drop Model

A simple model for the binding energy of a nucleus

This is reduced because *not all* of the nucleons are surrounded by other nucleons (i.e. those on the surface).

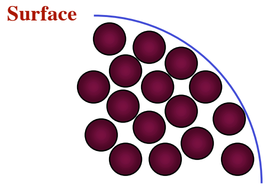
The surface effect

The binding energy will therefore be decreased in proportion to the area of the surface. $E_b \propto -C_2 A^{2/3}$

The volume is proportional to A so the area of the surface is proportional to $A^{2/3}$

Some constant

The binding energy is reduced by this effect



The nucleons at the surface are not surrounded

so their energy is higher

The (Charged) Liquid Drop Model

A simple model for the binding energy of a nucleus

Taking into account volume and surface effects gives the following expression:

$$E_b \propto C_1 A - C_2 A^{2/3}$$

According to this formula, the binding energy per nucleon should increase monotonically as the nucleus gets larger and larger, while we have seen that it goes through a maximum when A is around 56.

Something else must be included. What is it?
Well we know that nuclei contain protons and we haven't taken the repulsion between these protons into account.

The (Charged) Liquid Drop Model

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So …the binding energy is also reduced by Coulomb repulsion between the protons.

Coulomb repulsion

The nucleus contains Z protons. These are all positively charged and therefore repel each other.

If two charged particles are at a distance, *r*, then their potential energy is $\frac{k_e q_1 q_2}{r}$**.**

The total Coulomb energy of the nucleus is therefore proportional to the number of pairs of charged particles, i.e. $Z(Z$-1). It is also inversely proportional to the radius of the nucleus $A^{1/3}$ **.**

Thus the binding energy is reduced by an amount: $-C_3 \frac{Z(Z-1)}{A^{1/3}}$ **, because of the Coulomb repulsion between the protons.**

The (Charged) Liquid Drop Model

A simple model for the binding energy of a nucleus

The binding energy is also reduced by Coulomb repulsion

Coulomb repulsion continued

We have just waved our hands around and seen that it is reasonable that the Coulomb repulsion reduces the binding energy by an amount:

$$-C_3 \frac{Z(Z-1)}{A^{1/3}}$$

We can however derive this.
Let's do it.

The (Charged) Liquid Drop Model

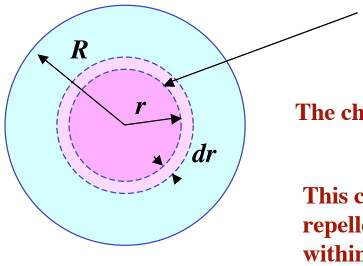
A simple model for the binding energy of a nucleus

Calculation of the Coulomb energy term from first principles.

If we consider the electrostatic self-energy of a nucleus we can *calculate* the Coulomb repulsion term in the liquid drop model.

$$-C_3 \frac{Z(Z-1)}{A^{1/3}}$$

Consider a nucleus containing *Z* protons. The radius of the nucleus is *R*.



Take a spherical shell at radius *r* from the centre and of thickness *dr*.

The charge in this shell is $4\pi r^2 dr\rho$

Charge density

$$\rho = \frac{Ze}{V}\ \mathrm{C\cdot m^{-3}}$$

This charge is repelled by the charge within the sphere, i.e. : $\frac{4}{3}\pi r^3 \rho$

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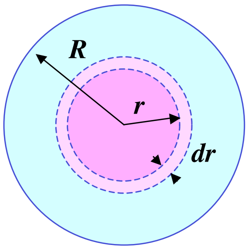
Calculation of the Coulomb energy term from first principles.



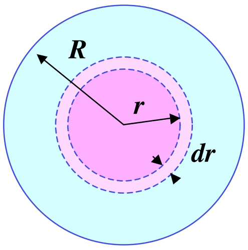
The nucleus is made up of a whole lot of such spherical shells.

The potential energy of the charge in the nucleus can be found by seeing what the potential energy of a typical shell at radius *r*, and then summing over all the shells (i.e. forming the integral).

The Liquid Drop Model

A simple model for the binding energy of a nucleus

Calculation of the Coulomb energy term from first principles (continued)



The potential energy of a particular shell at a radius *r* is just:

k_e x the charge in the shell (of radius *r)* x charge *inside r* x 1/*r*

$$dE_C = k_e 4\pi r^2 dr\rho \times \frac{4}{3}\pi r^3 \rho \times \frac{1}{r}$$

$$= k_e 4\pi\rho \times \frac{4}{3}\pi\rho \times r^4 dr$$

The total potential energy of the charge in the nucleus is then the integral of this over all radii from 0 to *R*.

$$E_C = k_e 4\pi\rho \times \frac{4}{3}\pi\rho \times \int_0^R r^4 dr$$

$$= k_e 4\pi\rho \times \frac{4}{3}\pi\rho \times \frac{R^5}{5}$$

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Calculation of the Coulomb energy term from first principles (continued)

Substituting the value $\rho = \frac{Ze}{V}\ \mathrm{C\cdot m^{-3}}$ **for the charge density, r, gives:**

And then setting the volume, $V = \frac{4}{3}\pi R^3$ **yields:**

$$E_C = k_e 4\pi\left(\frac{Ze}{V}\right) \times \frac{4}{3}\pi\left(\frac{Ze}{V}\right) \times \frac{R^5}{5}$$

$$= \frac{k_e(4\pi)^2 Z^2 e^2}{(\frac{4}{3}\pi R^3)^2} \times \frac{1}{3} \times \frac{R^5}{5}$$

$$= \frac{3k_e}{5}\frac{Z^2e^2}{R}$$

So that: $$E_C = \frac{3k_e}{5}\frac{Z^2e^2}{R}$$

We have calculated directly, from first principles, the reduction of the binding energy that is the result of Coulomb repulsion.

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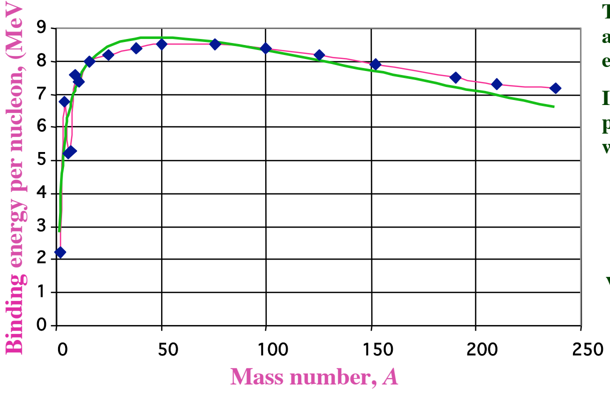
Therefore, taking into account volume, surface and Coulomb repulsion effects gives the following expression: $E_b \propto C_1 A - C_2 A^{2/3} - C_3 \frac{Z^2}{A^{1/3}}$

How does this fit the experimental data?



This shows the above formula in a fit to the experimental binding energy data (shown previously).

In this fit just one free parameter, $C_1 = C_2 = 14.7$ MeV was used with $C_3 = 0.6$ MeV..

The fitted curve is pretty good overall, but, for example, it does not reproduce the variations of the light nuclei at all - and the region of maximum binding energy is not correct.

The Liquid Drop Model

A simple model for the binding energy of a nucleus

As we have just seen: $E_b = C_1 A - C_2 A^{2/3} - C_3 \frac{Z^2}{A^{1/3}}$ Gives a reasonable if rough fit to the data.

But, something is very wrong.
What is wrong?

Well, why do nuclei contain protons at all?

So far, all that the protons do is to *reduce* the binding energy.

In the above model, (so far) a nucleus with $Z = 0$, i.e. with no protons, only containing neutrons, would actually be very much more stable.

The Liquid Drop Model

A simple model for the binding energy of a nucleus

Once again, why do nuclei contain protons at all?

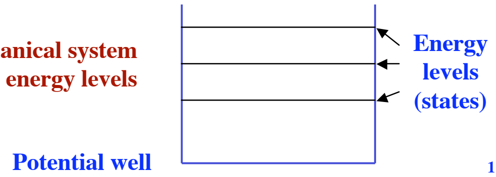
In order to explain this we need quantum mechanics.

Specifically, we need the Pauli exclusion principle.

Nucleons have an intrinsic angular momentum (spin) of $\frac{\hbar}{2}$. Such particles are *fermions* and only two fermions of the same type (i.e. two neutrons or two protons) can occupy any energy level, one with spin up and one with spin down.

That is the *Pauli exclusion principle*.

The nucleus is a quantum mechanical system and it contains discrete states or energy levels

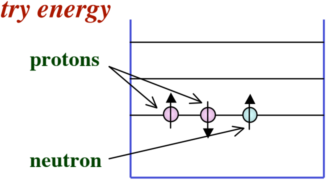


The Liquid Drop Model

A simple model for the binding energy of a nucleus

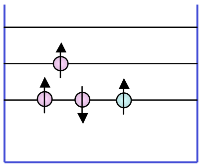
The asymmetry energy

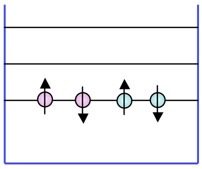
A schematic, energy level diagram of ^{3}He



A nucleus containing two neutrons and one proton in the lowest energy level

An additional proton must go into the next energy level





An additional neutron can go into the same energy level

So, if you add another proton (to make ^{4}Li), you have a higher energy than if you add another neutron (to make ^{4}He).

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Exercise 25

1. **Derive the Coulomb self-energy term for a nucleus from first principles.**

   (Make sure you understand it thoroughly.)