Slowing down the neutrons

Clearly, an obvious way to make a reactor work, and to make use of this characteristic of the ^{235}U(n,f) cross-section, is to slow down the fast, fission neutrons. This can be accomplished, for example, when the neutrons collide with nuclei and *scatter* in some substance (a *moderator)*.

It *moderates* the neutrons (slows them down).

If this could be achieved, neutron-induced fission would be very much more likely.

Slowing down the neutrons

Say that the neutrons could be slowed down until they are in *thermal equilibrium* with the surrounding material in the reactor.

They would then have the same distribution of energies as the molecules in the material.

This is the same distribution of energies as for the atoms or molecules of a gas, say, in thermal equilibrium at the same temperature.

What would such a distribution look like?

Thermal neutron distribution

Originally Boltzmann analysed the statistics of a number of particles at different energies in thermal equilibrium (actually for the molecules in a gas).

This produced the Maxwell-Boltzmann distribution.

$$n(E) = n\left(\frac{4}{\pi k_B T}\right)^{1/2} E^{1/2} \exp\left(-\frac{E}{k_B T}\right)$$

E is the neutron energy in Joules

$n(E)dE$ is the *number* of neutrons with energies between E and $E + dE$.

T is the temperature of the moderator and hence of the neutron distribution in equilibrium with the moderator.

Thus one talks of "300 K neutrons" for example.

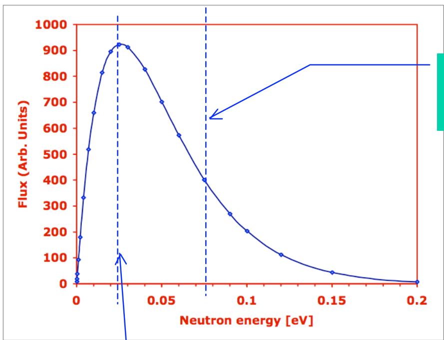
The total number of neutrons, n.

k_B is Boltzmann's constant.
$k_B = 1.38 \times 10^{-23}$ J.K^{-1}

Thermal equilibrium

Once again, to express this as a flux distribution we multiply by v.

Maxwell-Boltzmann flux distribution at 300K.



The *average energy* at an absolute temperature T is $3/2kT$

The most probable energy at an absolute temperature T is $1/2kT$. (0.025eV for T = 300 K)

Slowing neutrons down

So, we have decided that we have to slow the neutrons down from the fission spectrum energy distribution to something like the thermal distribution.

In order to see how this can be done let's consider what happens in an elastic collision (elastic scattering).

Let's work out the energy of a particle of mass *m* after it collides with a stationary particle of mass, *M*.

We will examine this process in some detail remembering that the momentum and the kinetic energy are both conserved in such an elastic collision.

Elastic scattering

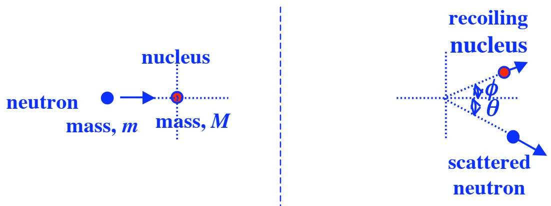
Before collision After collision

The energy of the scattered neutron depends on the scattering angle, θ, and the mass of the scattering nucleus.

We apply the equations for conservation of momentum and energy.

This is the physics

Elastic scattering



Before collision *After collision*

Conservation of momentum in the x direction

$$mv_m = mv'_m\cos\theta + Mv'_M\cos\phi$$

Conservation of momentum in the y direction

$$0 = mv'_m\sin\theta + Mv'_M\sin\phi$$

Conservation of energy

$$\frac{1}{2}mv_m^2 = \frac{1}{2}mv'^2_m + \frac{1}{2}Mv'^2_M \quad \text{or}$$

$$E_0 = E_f + E_r$$

Remember we want to find the final energy of the neutron, E_f in terms of the masses, the scattering angle θ and, of course, the initial energy, E_0.

Now, doing the maths

Elastic scattering

Notice that we can get rid of the angle ϕ by squaring and adding the momentum equations, because $\cos^2\phi + \sin^2\phi = 1$.

First, it probably makes it easier if we express everything in terms of energies using:

$$E = \frac{1}{2}mv^2 \qquad p = mv$$

for non-relativistic speeds and energies.

so $$p^2 = m^2v^2 = 2m\frac{1}{2}mv^2 = 2mE$$

and $$p = mv = \sqrt{2mE}$$

Using this we get for the two momentum equations

$$\sqrt{2mE_0} = \sqrt{2mE_f}\cos\theta + \sqrt{2ME_r}\cos\phi$$

Conservation of momentum in the *x* direction

$$0 = \sqrt{2mE_f}\sin\theta + \sqrt{2ME_r}\sin\phi$$

Conservation of momentum in the *y* direction

We want to get rid of the ϕ terms, so lets put them on the left hand side.

$$\sqrt{2ME_r}\cos\phi = \sqrt{2mE_0} - \sqrt{2mE_f}\cos\theta$$

$$\sqrt{2ME_r}\sin\phi = -\sqrt{2mE_f}\sin\theta$$

Squaring and adding: $$2ME_r\left(\cos^2\phi + \sin^2\phi\right) = \left(\sqrt{2mE_0} - \sqrt{2mE_f}\cos\theta\right)^2 + 2mE_f\sin^2\theta$$

Elastic scattering

Doing the maths

Our last equation was: $2ME_r\left(\cos^2\phi+\sin^2\phi\right)=\left(\sqrt{2mE_0}-\sqrt{2mE_f}\cos\theta\right)^2+2mE_f\sin^2\theta$

Now since: $\cos^2\phi+\sin^2\phi=1$ ***for any angle.***

We have:

$$2ME_r=\left(2mE_0-2\sqrt{2mE_0}\sqrt{2mE_f}\cos\theta+2mE_f\cos^2\theta\right)+2mE_f\sin^2\theta$$
$$=2mE_0-2\times 2m\sqrt{E_0E_f}\cos\theta+2mE_f$$

using $\cos^2\theta+\sin^2\theta=1$ **as well**

Remember, we want E_f in terms of E_0 and θ (as well as M and m) so we have to get rid of E_r.

We do this by going back to the original equations and using the one for conservation of energy: $E_0=E_f+E_r$ **so** $E_r=E_0-E_f$

Then substituting for E_r and cancelling out the 2, we have: $M\left(E_0-E_f\right)=mE_0-2m\sqrt{E_0E_f}\cos\theta+mE_f$

Now we have an equation that only contains E_f, E_0 and $\cos\theta$ (as well as M and m) but it is not in a convenient form – we would like to have E_f by itself on one side.

Elastic scattering

Doing the maths

In order to do this, we remember that if we have a quadratic equation: $ax^2+bx+c=0$ **We can find x in terms of a, b and c:** $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

So let's get $M\left(E_0-E_f\right)=mE_0-2mE_0-2m\sqrt{E_0E_f}\cos\theta+mE_f$ **into this form with $\sqrt{E_f}$ playing the part of x.**

Move the terms to the left and group the factors

$$-(M+m)E_f+2m\sqrt{E_0}\cos\theta\sqrt{E_f}+(M-m)E_0=0$$

or

$$(M+m)E_f-2m\sqrt{E_0}\cos\theta\sqrt{E_f}-(M-m)E_0=0$$

We can now apply the above formula for a quadratic to write:

$$\sqrt{E_f}=\frac{2m\sqrt{E_0}\cos\theta\pm\sqrt{4m^2E_0\cos^2\theta+4\left(M^2-m^2\right)E_0}}{2(M+m)}$$
$$=\frac{2m\sqrt{E_0}\cos\theta\pm 2\sqrt{m^2E_0\cos^2\theta+\left(M^2-m^2\right)E_0}}{2(M+m)}$$

Elastic scattering

Doing the maths

Continuing to simplify this equation:

step-by-step.

$$\sqrt{E_f} = \frac{m\sqrt{E_0}\cos\theta \pm \sqrt{m^2 E_0 \cos^2\theta + (M^2 - m^2)E_0}}{(M+m)}$$

$$= \frac{m\sqrt{E_0}\cos\theta \pm \sqrt{E_0}\sqrt{m^2 \cos^2\theta + (M^2 - m^2)}}{(M+m)}$$

Take $\sqrt{E_0}$ out of the square root.

$$= \frac{\sqrt{E_0}}{M+m}\left[m\cos\theta \pm \left(m^2\cos^2\theta + M^2 - m^2\right)^{1/2}\right]$$

$$= \frac{\sqrt{E_0}}{M+m}\left[m\cos\theta \pm \left(m^2\{1-\sin^2\theta\} + M^2 - m^2\right)^{1/2}\right]$$

$$= \frac{\sqrt{E_0}}{M+m}\left[m\cos\theta \pm \left(m^2 - m^2\sin^2\theta + M^2 - m^2\right)^{1/2}\right]$$

$$= \frac{\sqrt{E_0}}{M+m}\left[m\cos\theta \pm \left(M^2 - m^2\sin^2\theta\right)^{1/2}\right]$$

$$= \frac{m\sqrt{E_0}}{M+m}\left[\cos\theta \pm \left(\left\{\frac{M}{m}\right\}^2 - \sin^2\theta\right)^{1/2}\right]$$

Elastic scattering

Finally, the result

Thus we obtain:

$$\sqrt{E_f} = \frac{m\sqrt{E_0}}{M+m}\left[\cos\theta \pm \left(\left\{\frac{M}{m}\right\}^2 - \sin^2\theta\right)^{1/2}\right]$$

or

$$E_f = \frac{m^2 E_0}{(M+m)^2}\left[\cos\theta + \left(\left\{\frac{M}{m}\right\}^2 - \sin^2\theta\right)^{1/2}\right]^2$$

Squaring both sides and dropping the minus sign because it is not meaningful

For a neutron scattering off a nucleus with mass number *A* we can take:

$m = 1$

$M = A$

Because the masses always occur in ratios.

$$E_f = \frac{E_0}{(A+1)^2}\left[\cos\theta + \left(A^2 - \sin^2\theta\right)^{1/2}\right]^2$$

Elastic scattering

$$E_f = \frac{E_0}{(A+1)^2}\left[\cos\theta + \left(A^2 - \sin^2\theta\right)^{1/2}\right]^2$$

Initial energy of the neutron. (E_0)

Final energy of the neutron, after scattering. (E_f)

Mass number of the scattering nucleus (A)

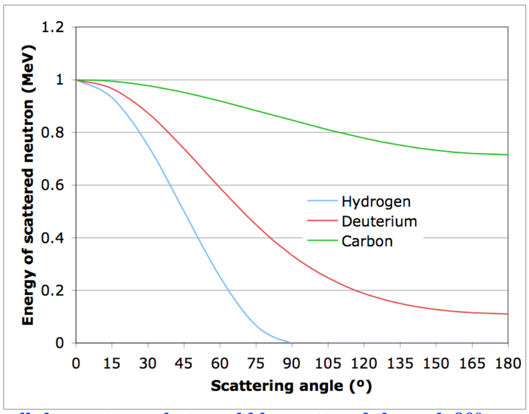
Scattering angle – angle through which the neutron is scattered (θ)

Elastic scattering

If we plot:

$$E_f = \frac{E_0}{(A+1)^2}\left[\cos\theta + \left(A^2 - \sin^2\theta\right)^{1/2}\right]^2$$

for a 1 MeV neutron and different scattering substances we obtain:



As you can see all the neutrons that would be scattered through 90° or more are basically stopped (and thermalised) in one collision with a hydrogen nucleus (proton). In the case of carbon the maximum energy loss is only about 300 keV even for a scattering angle of 180° (π) so several collisions would be necessary to thermalise the neutrons.

Elastic scattering

Exercise 36

A neutron with an energy of 1 MeV is scattered by the moderator in a light water reactor. In other words in a reactor where the moderator is ordinary H_2O.

Suppose that it is scattered through an angle of 45°.

Calculate:

1. **Its energy after scattering if it were scattered by the oxygen.**
2. **Its energy after scattering if it were scattered by the hydrogen.**

In the second case, what is the energy of the *proton* after the scattering?