A brief introduction to TMD factorisation

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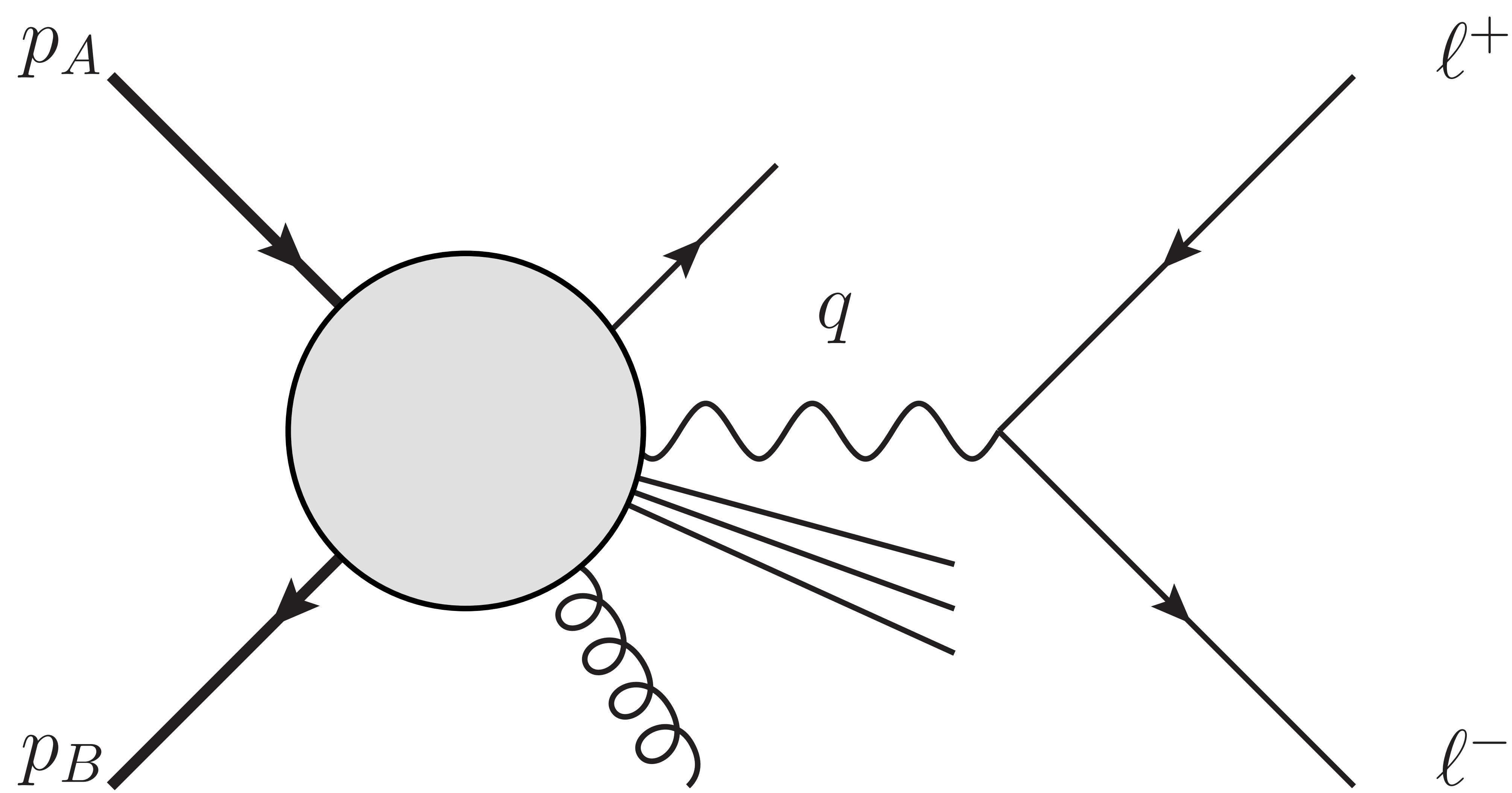
TMD factorisation

- Let us use Drell-Yan production (*i.e. inclusive* production of a lepton pair in hadron-hadron collisions) to sketch the main steps of factorisation:



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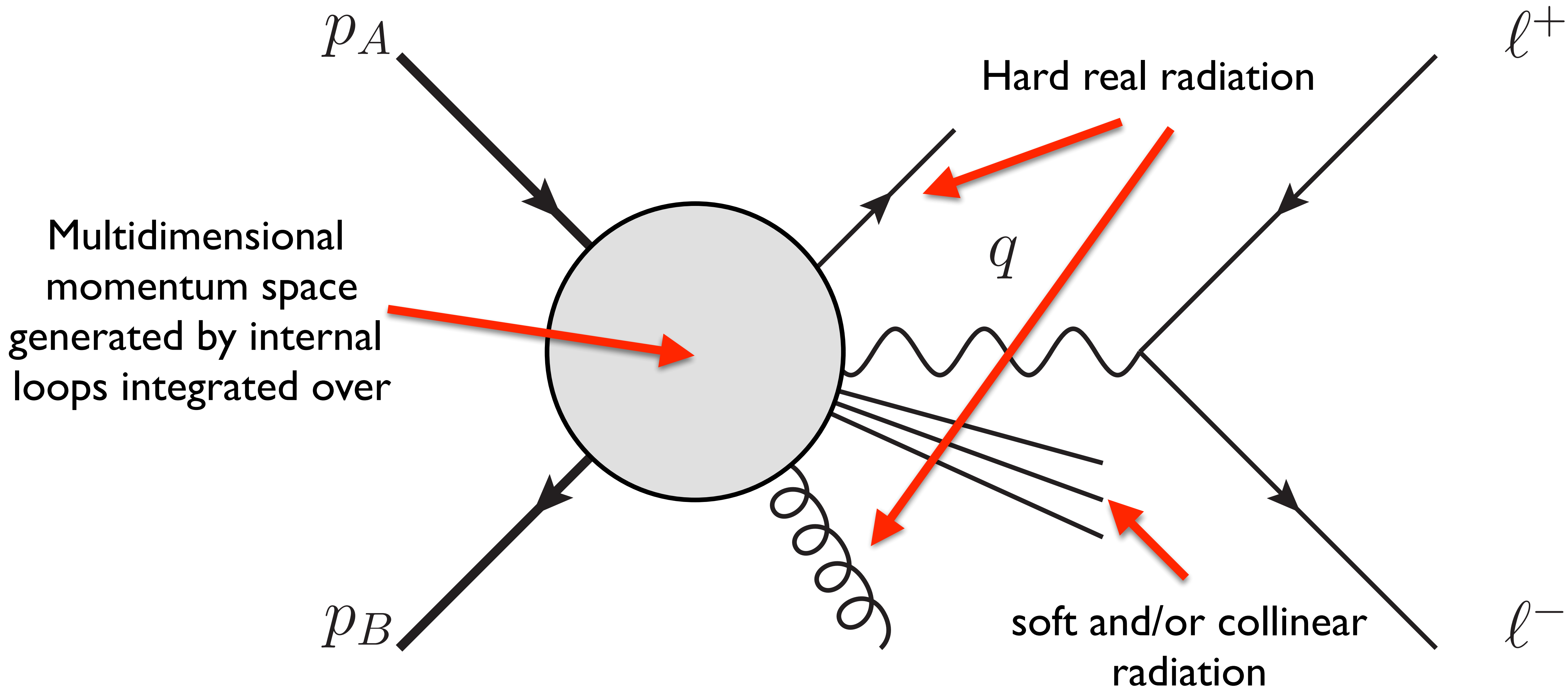


- Relevant scales:

$$Q = \sqrt{q^2}\,, \qquad \Lambda_{\rm QCD} \ll Q$$

TMD factorisation

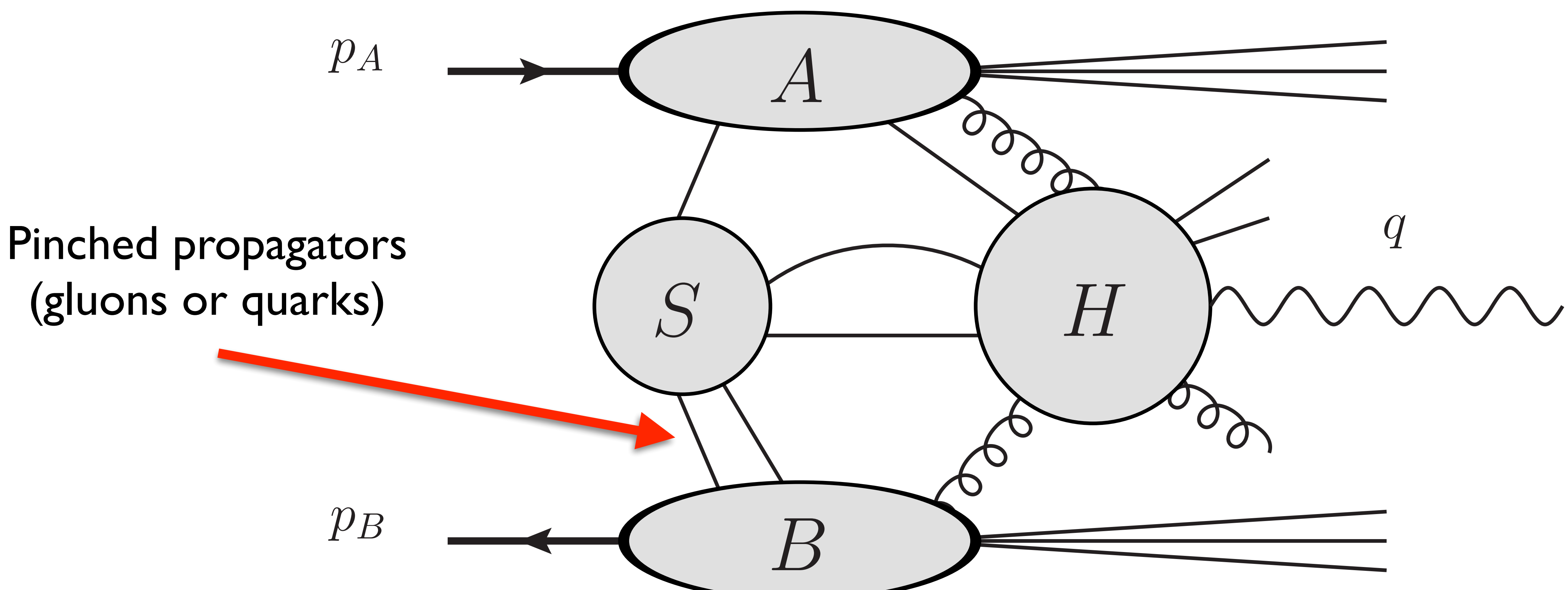
- Let us work in **massless** QCD and, for each *single* graph, identify the regions in the integration momentum space where the integrand is large:

  - **pinched** propagators $\Rightarrow$ Landau (Coleman-Norton) criterion and **reduced graphs**,

    - soft modes (**S**): $(k^+, k^-, \mathbf{k}_T) \sim (\lambda_S^2, \lambda_S^2, \lambda_S^2)Q$,

    - (anti)collinear modes (**A** and **B**): $(k^+, k^-, \mathbf{k}_T) \sim (1, \lambda^2, \lambda)Q$,

  - hard modes (**H**) (not pinched but large region): $(k^+, k^-, \mathbf{k}_T) \sim (1,1,1)Q$.



TMD factorisation

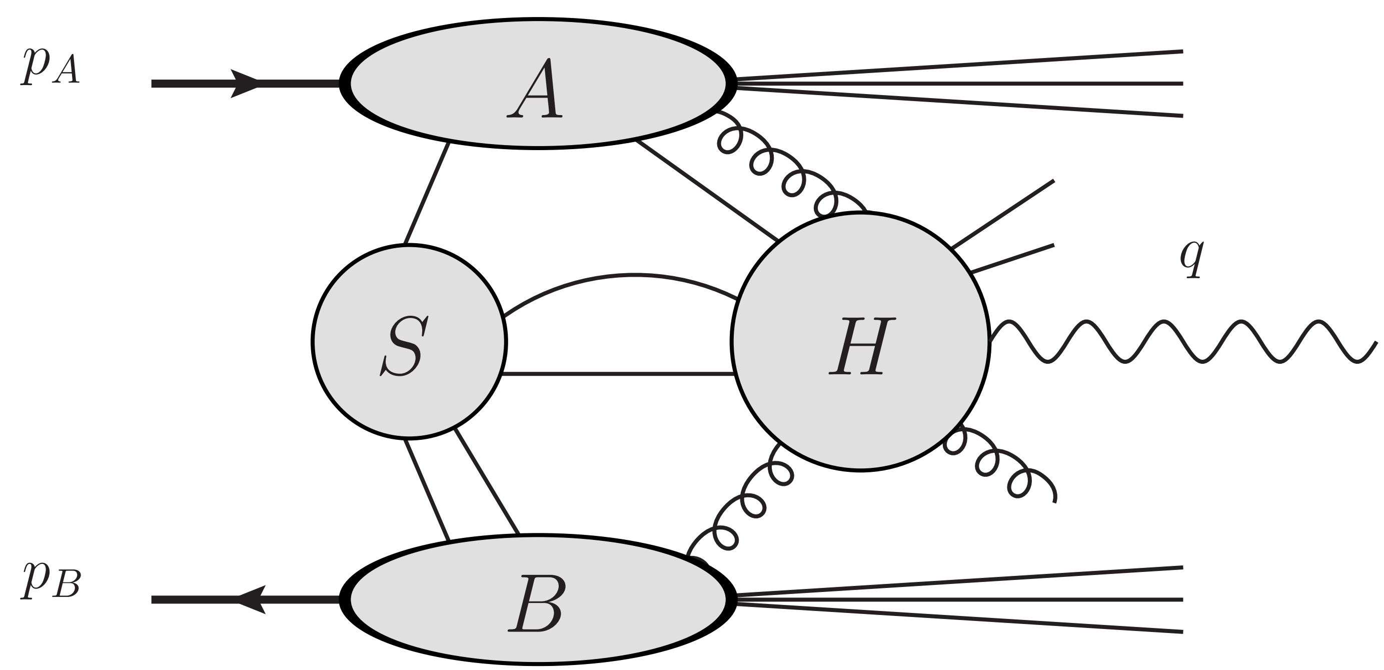
- Scaling of the reduced graph (in massless QCD):

$$Q^p \left(\frac{\lambda}{Q}\right)^\alpha \left(\frac{\lambda_S}{Q}\right)^\beta$$

Basic scaling
(depends on the process)

Collinear scaling

Soft scaling



$p = 4 - \#(\text{ext. lines})$

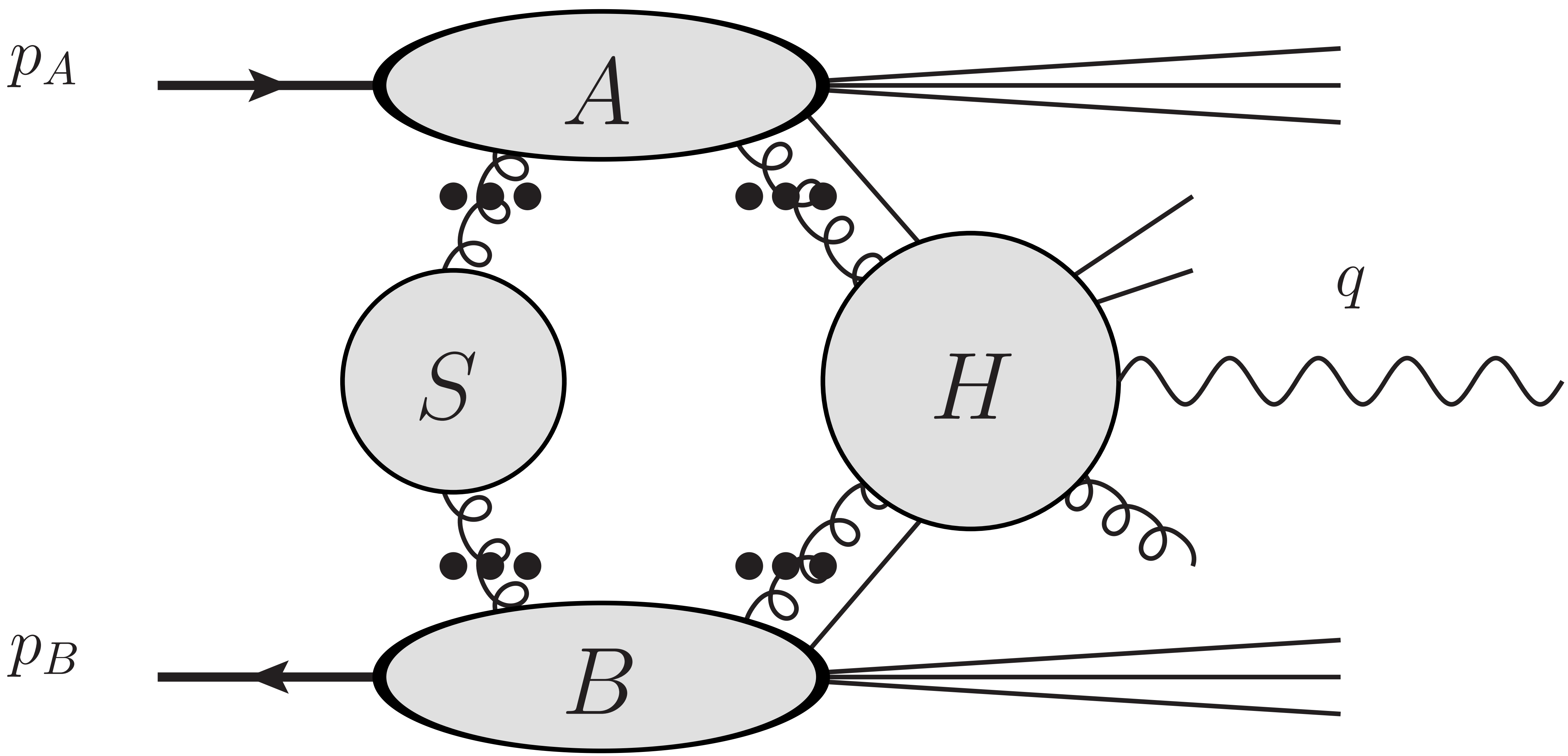
$\alpha = \#(\text{quarks and } \textit{transverse} \text{ gluons connecting A(B) to H}) - \#(\text{ext. lines})$

$\beta = \#(\text{gluons connecting S to H}) + \frac{3}{2}\#(\text{quarks connecting S to H}) + \frac{1}{2}\#(\text{quarks connecting S to A(B)})$

TMD factorisation

[G. Sterman, Phys. Rev. D 17, 2773]

- Apply Libby-Sterman power counting to identify the **asymptote** in $\boldsymbol{Q}$.
- The result (in a **covariant gauge**) is strikingly simple:



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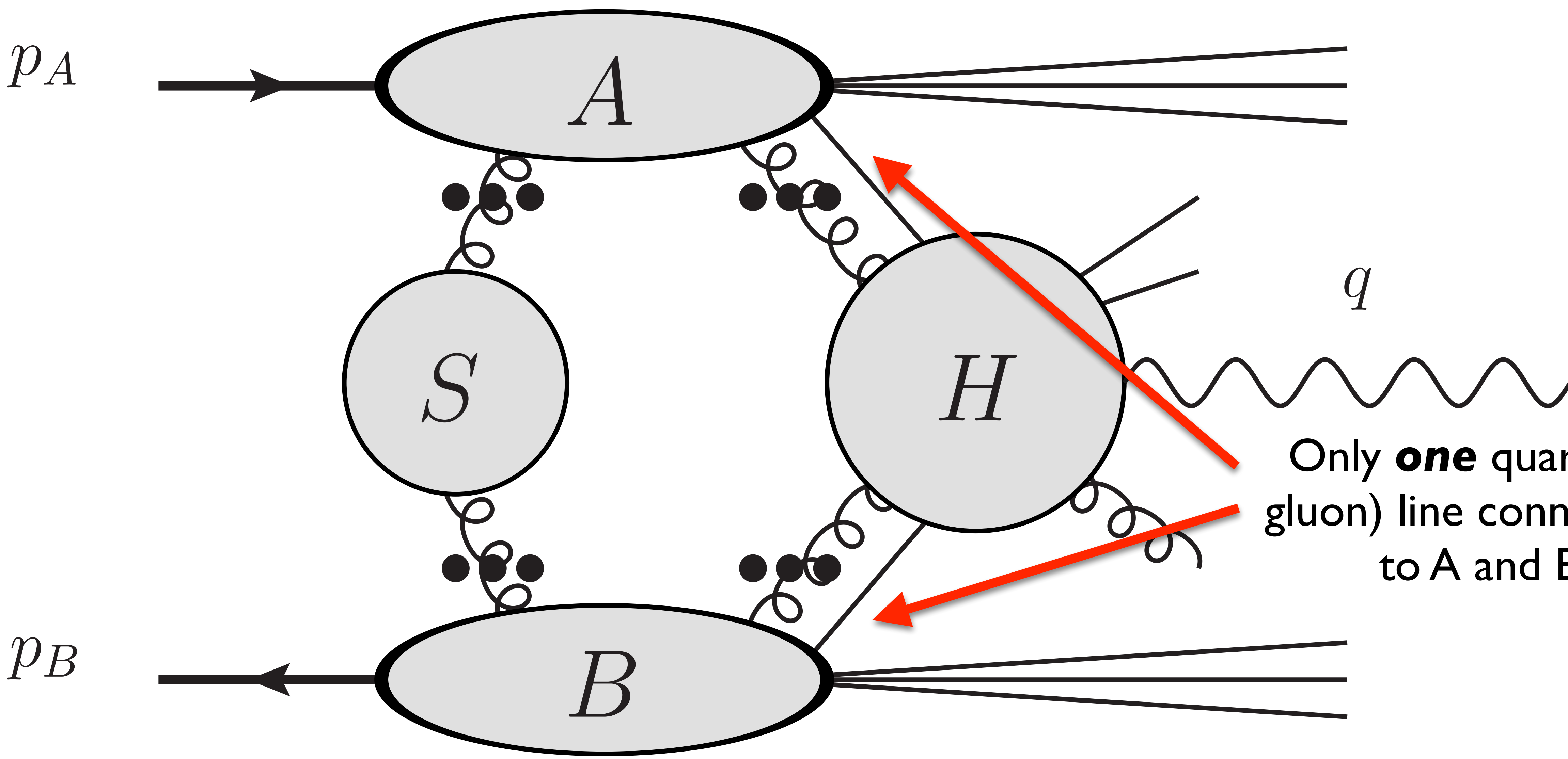

Only ***one*** quark (or gluon) line connects H to A and B

TMD factorisation

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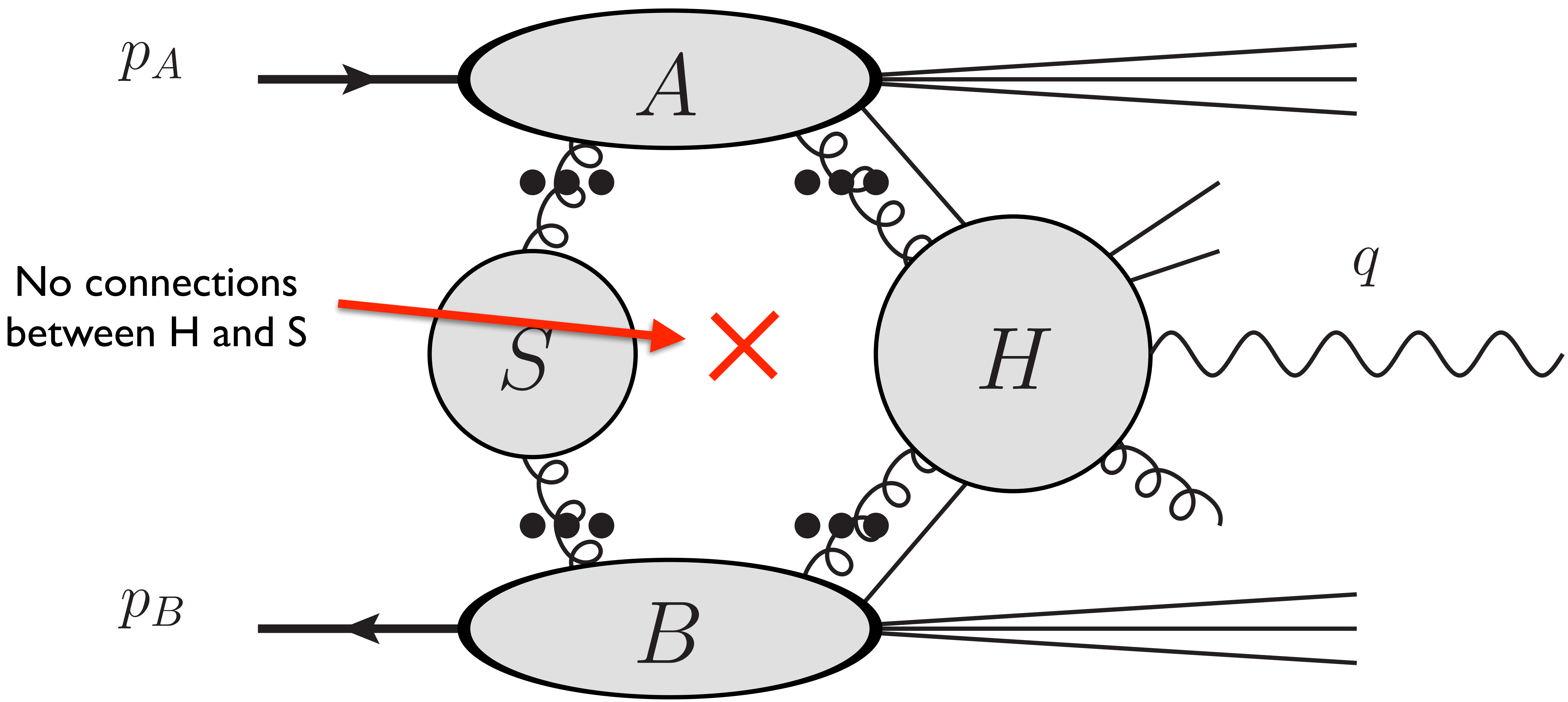

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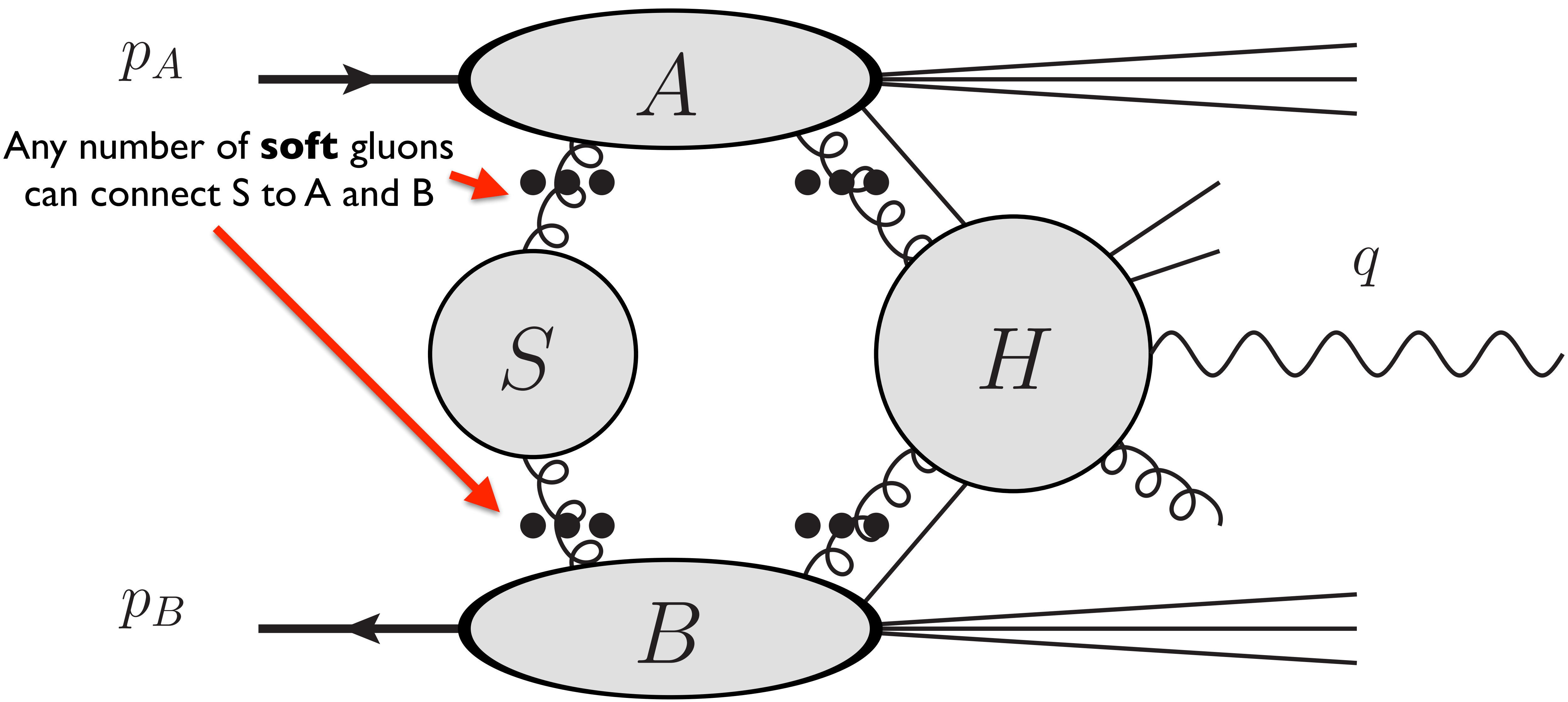

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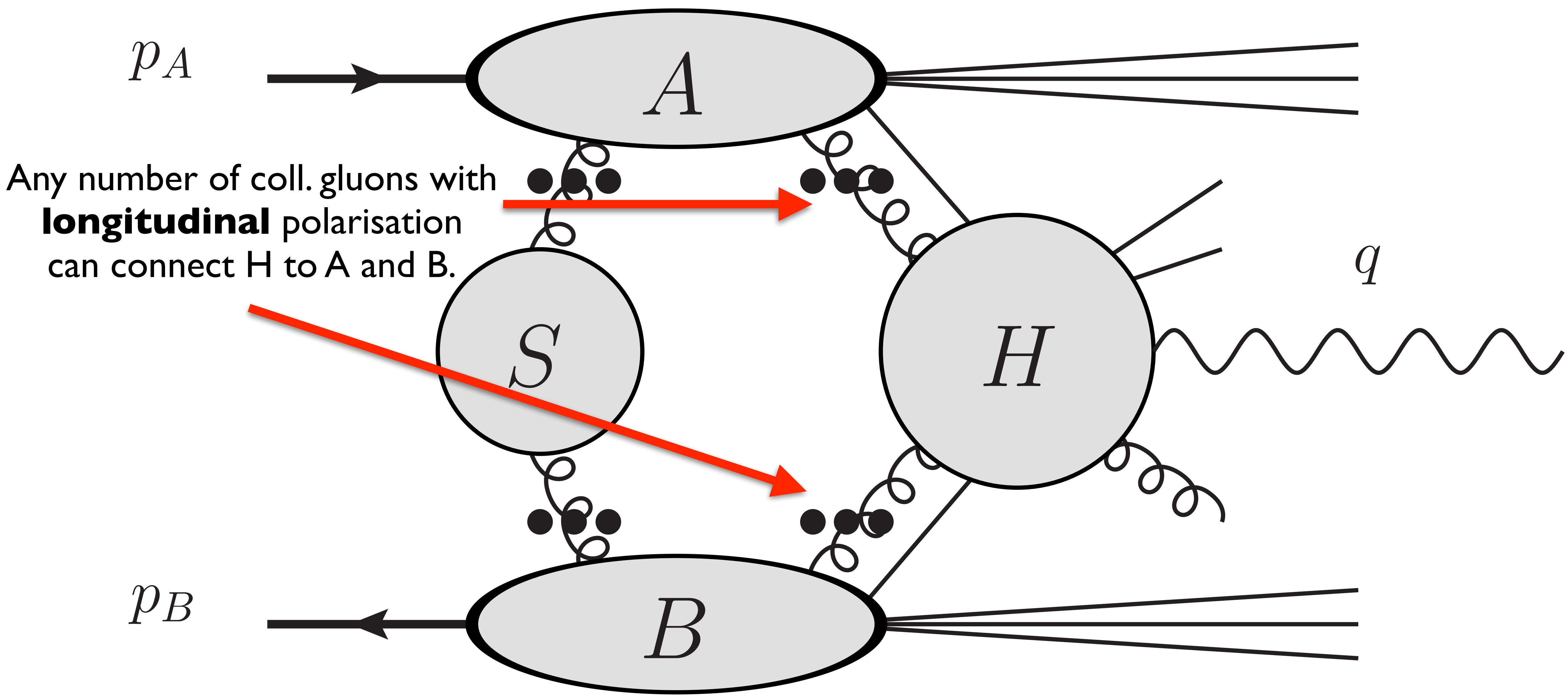

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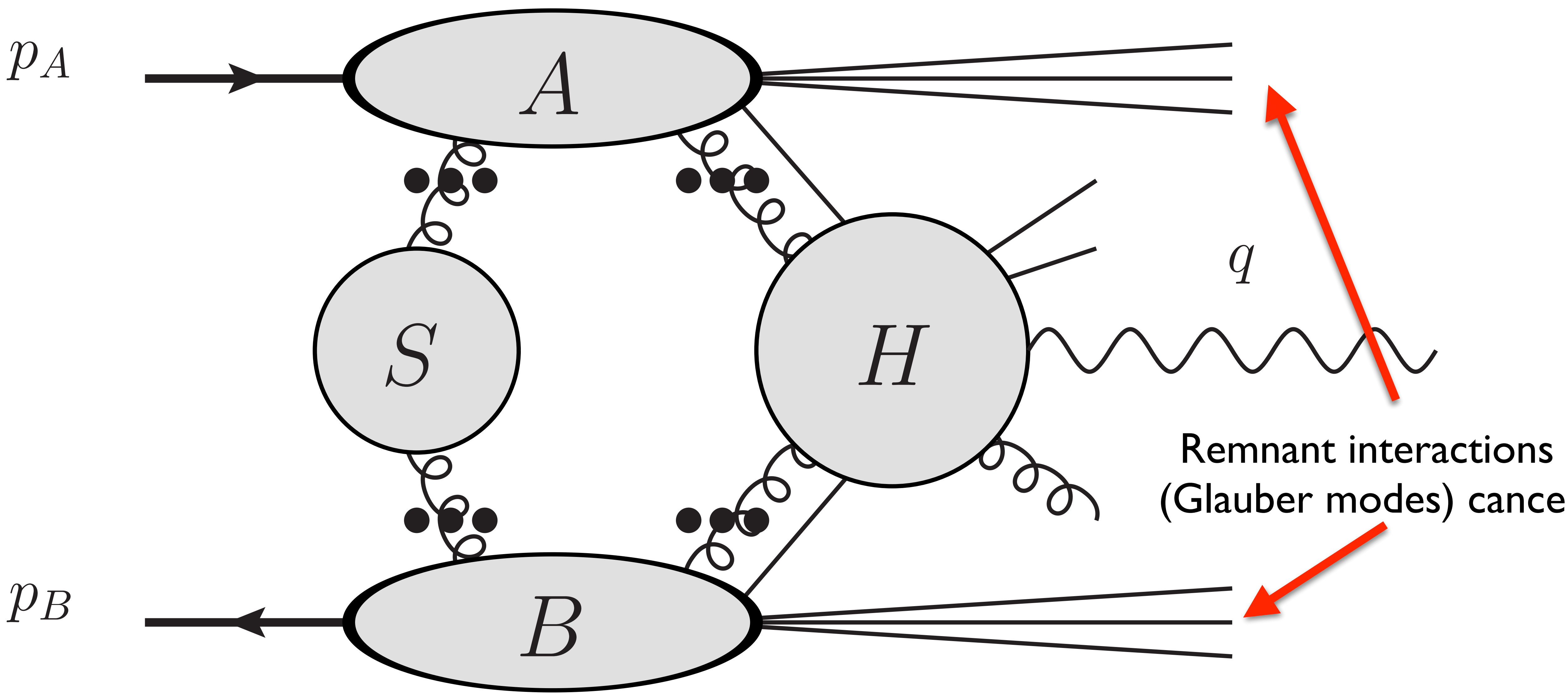


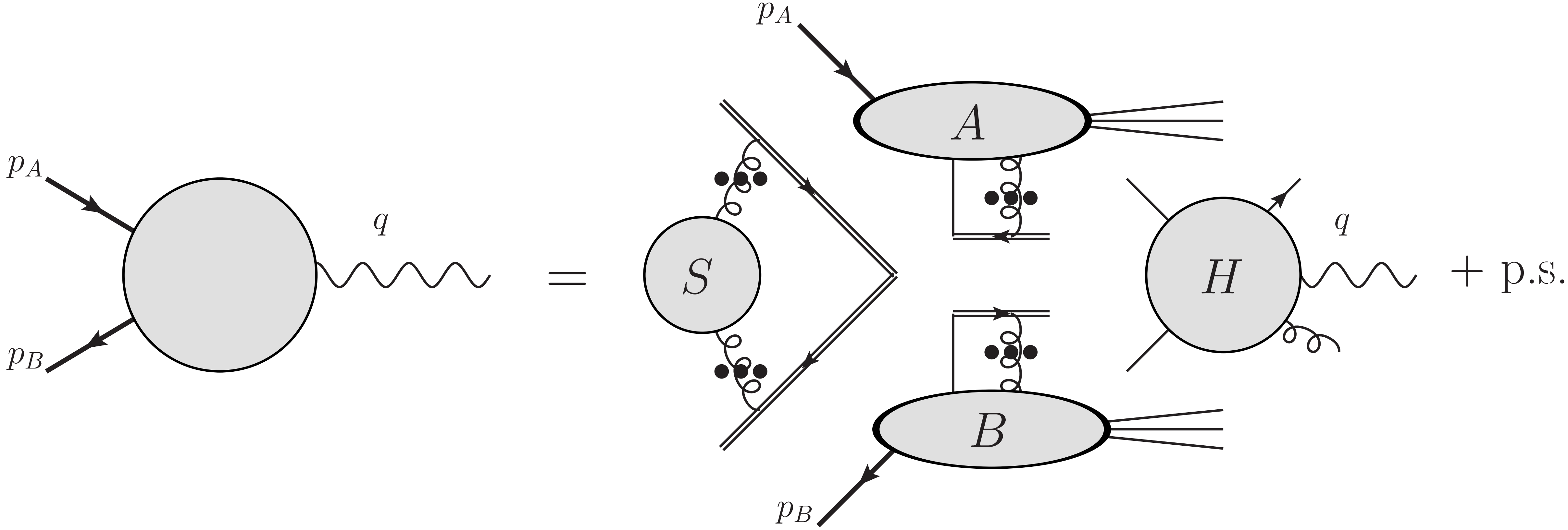
TMD factorisation

- Finally, factorisation is achieved by:
    - using **eikonal approximations** for each soft/collinear gluon attachment with momentum k to write, for example:

$$S^{\mu\cdots} g_{\mu\nu} A^{\nu\cdots} \sim (k \cdot S^{\cdots}) \frac{1}{n \cdot k} (n \cdot A^{\cdots})$$

    - this allows one to use **Ward identities** and introduces **Wilson lines**.

- A recursive application leads to factorisation of the amplitude:

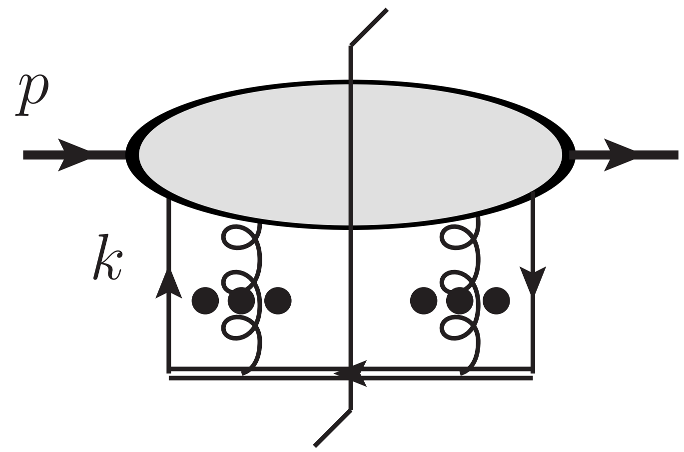


$$\mathcal{A} \sim H \cdot A \cdot B \cdot S$$

TMD factorisation

- Upon squaring, factorisation leads to the operator definition of:
  - gauge invariant (unsubtracted) parton-distribution function (**PDF**):

$$f^{(0)}(x, \mathbf{k}_T) \propto \int dk^-$$



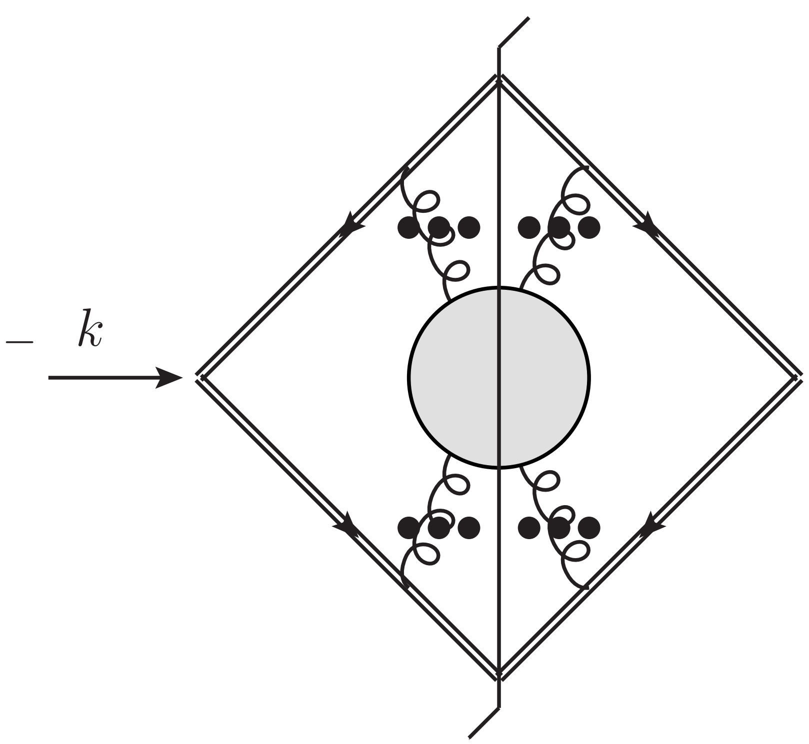
light-cone coordinates:

$$k = (k^+, k^-, \mathbf{k})$$

$$k^\pm = \frac{E \pm k_z}{\sqrt{2}}$$

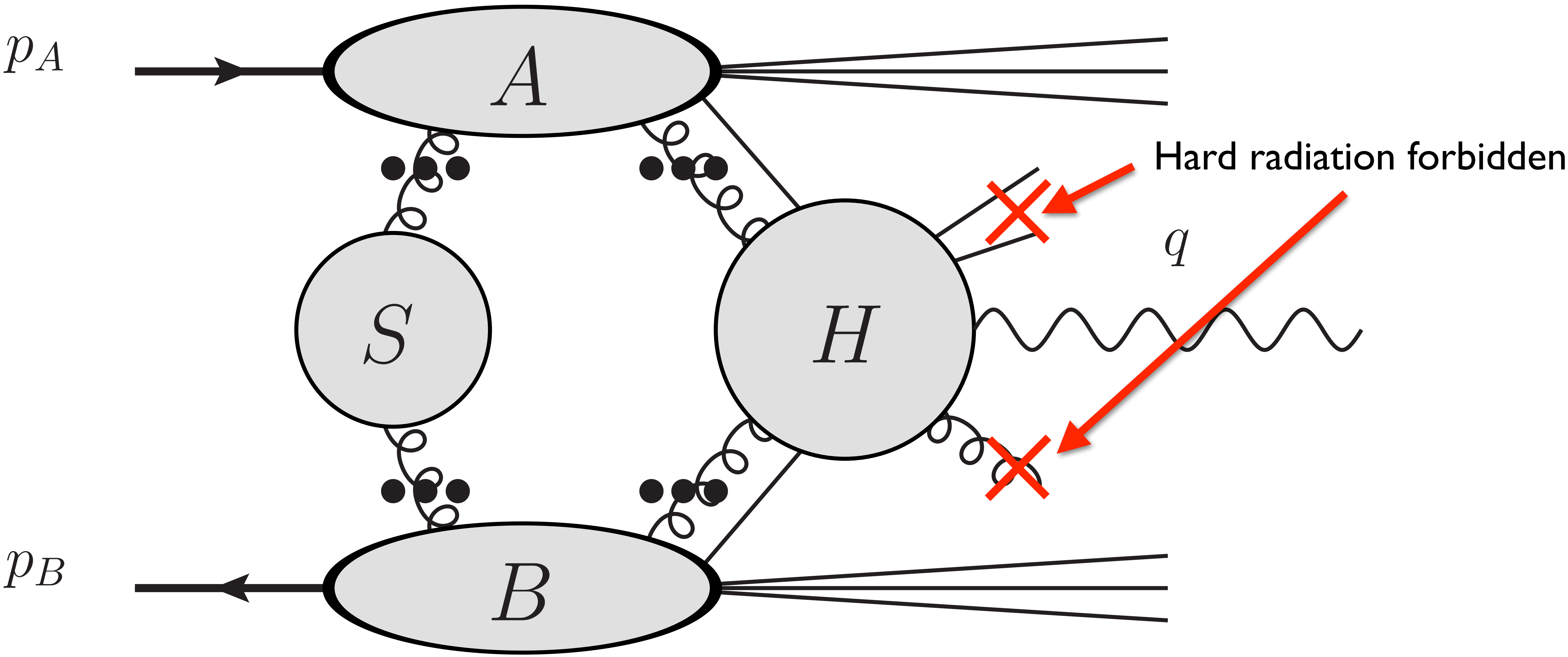
  - and **soft function:**

$$S^{(0)}(\mathbf{k}_T) \propto \int dk^+ dk^-$$



TMD factorisation

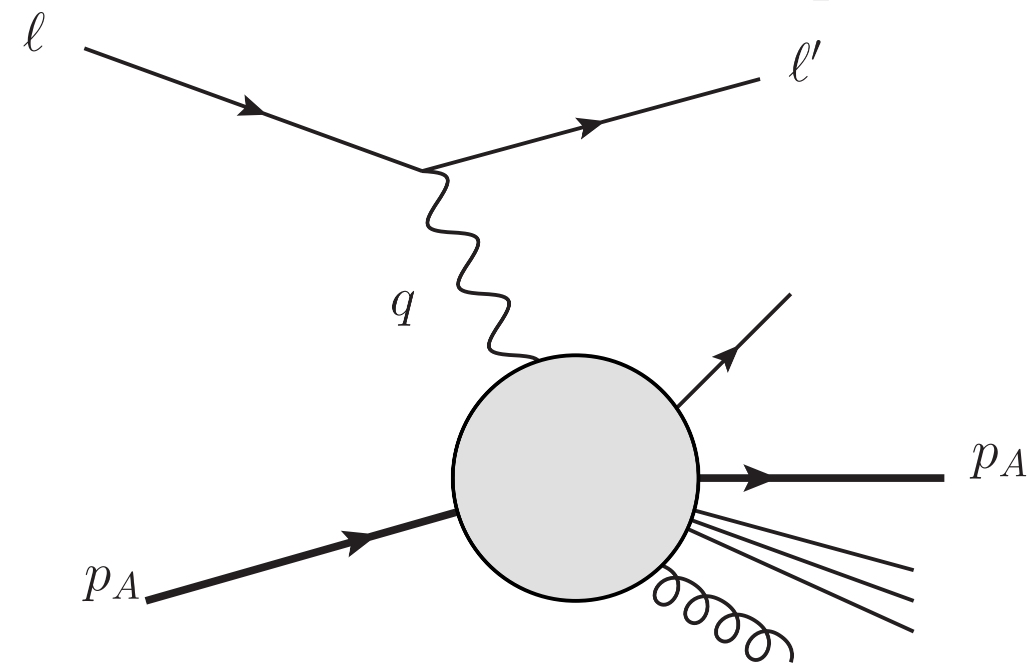
- If $q_T \ll Q$ …



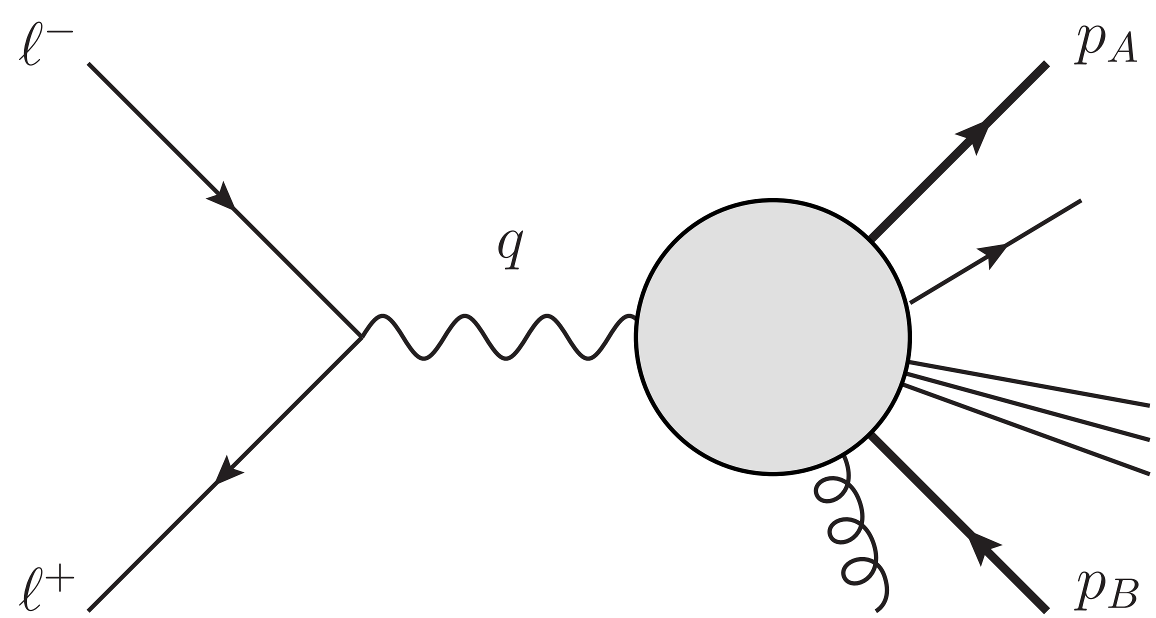
- ***H*** becomes a "simple" scalar function that only accounts for **virtual** corrections.
- As opposed to $q_T \sim Q$ (collinear fact.), k_T dependence of A, B, S cannot be integrated out. Therefore, we need to **treat $\boldsymbol{k_T}$ exactly**: this is the essence of TMD factorisation.

TMD factorisation

- Leading-power TMD factorisation has been **proven** also for:
  - Semi-inclusive DIS al low values of the $\boldsymbol{p}_\mathbf{T}$ of the produced hadron:



  - double-inclusive e^+e^- annihilation where the two hadrons are almost back-to-back:



- Necessary to replace PDFs with **fragmentation functions** (FFs) as appropriate.

TMD renormalisation

- Both PDF and soft function are affected by **UV** divergences that are removed through renormalisation:
  - UV renormalisation introduces the **scale** μ and the **scale** ζ.
- PDF and soft function are *individually* affected by **rapidity** divergences:
  - $k^+ = (1-x)p^+ \to 0$, with $\mathbf{k}_T$ left unconstrained ($\eta = \frac{1}{2}\ln(k^+/k^-) \to 0$).
- The following combination is free of divergences:

$$f(x, k_T, \mu, \zeta) = Z^{\mathrm{UV}}(\mu) f^{(0)}(x, k_T, \mu, \zeta)\sqrt{S(k_T, \mu, \zeta)}$$

- The final TMD factorised formula for the cross sections takes the form:

$$\begin{aligned}\frac{d\sigma}{dq_T} &\propto H(Q,\mu)\int d^2k_{TA}d^2k_{TB} f_A(x, k_{TA}, \mu, \zeta_A) f_B(x, k_{TB}, \mu, \zeta_B)\delta^{(2)}(q_T - k_{TA} - k_{TB}) \\ &\propto H(Q,\mu)\int_0^\infty db_T\, b_T J_0(b_T q_T) f_A(x, b_T, \mu, \zeta_A) f_B(x, b_T, \mu, \zeta_B)\end{aligned}$$

- Important constraints on the scales:
  - $\Lambda_{\mathrm{QCD}} \ll q_T \ll Q$,
  - $\mu \sim Q$ and $\zeta_A\zeta_B = Q^4$.

TMD evolution

- Removal of UV divergences allows one to derive **two** evolution equations for the TMD, along with the cross derivative:

$$\left\{\begin{array}{l}\dfrac{d\ln f}{d\ln\mu} = \gamma(\mu,\zeta) \\[2ex] \dfrac{d\ln f}{d\ln\sqrt{\zeta}} = K(\mu)\end{array}\right. , \qquad \frac{d^2\ln f}{d\ln\mu\, d\ln\sqrt{\zeta}} = \left\{\begin{array}{l}\dfrac{d\gamma}{d\ln\sqrt{\zeta}} \\[2ex] \dfrac{dK}{d\ln\mu}\end{array}\right. = \gamma_K(\alpha_s(\mu))$$

- Solving these equations requires fixing **two pairs of scales**:
  - **initial** scales: (μ_0, ζ_0)
  - **final** scales: (μ, ζ)
- The solution reads:

$$f(\mu,\zeta) = R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right] f(\mu_0,\zeta_0)$$

$$R\left[(\mu,\zeta)\leftarrow(\mu_0,\zeta_0)\right] = \exp\left\{K(\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}} + \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu'))\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}$$

- We know how to choose (μ, ζ) to compute a cross section.
- A question remains: **how do we *sensibly* choose** (μ_0, ζ_0)**?**

TMD evolution

[Collins, Soper, Sterman, *Nucl.Phys.B* 250 (1985) 199-224]

- A sensible choice of the scales is important to **allow truncated perturbation theory to be reliable**:
  - **no large unresummed logarithms** should be introduced,
  - each scale has to be set in the **vicinity of its "natural" value**.
- In TMD factorisation for DY the relevant scales are $\boldsymbol{q_T}$ and $\boldsymbol{Q}$:
  - natural to expect $\mu_0 \sim \sqrt{\zeta_0} \sim q_T \sim b_T^{-1}$ and $\mu \sim \sqrt{\zeta} \sim Q$
- In the $\overline{\text{MS}}$ scheme, **central scales** are usually chosen to be:

$$\mu_0 = \sqrt{\zeta_0} = \frac{2e^{-\gamma_E}}{b_T} \equiv \mu_b \quad \text{and} \quad \mu = \sqrt{\zeta} = Q$$

- This particular choice **nullifies** all unresummed (fixed-order) logs.
- Modest variations around these values give an estimate of higher-order corrections.

TMD matching

- **At small values of $\boldsymbol{b}_\mathbf{T}$** the TMD can be related to the *collinear* PDF.

- The main observation is that the TMD obeys a (non-local) **OPE**:

$$f(x, b_T, \mu, \zeta) = C(x, b_T, \mu, \zeta) \otimes f(x, \mu) + \mathcal{O}(b_T^2)$$

- The coefficient of leading-power matching are perturbative:

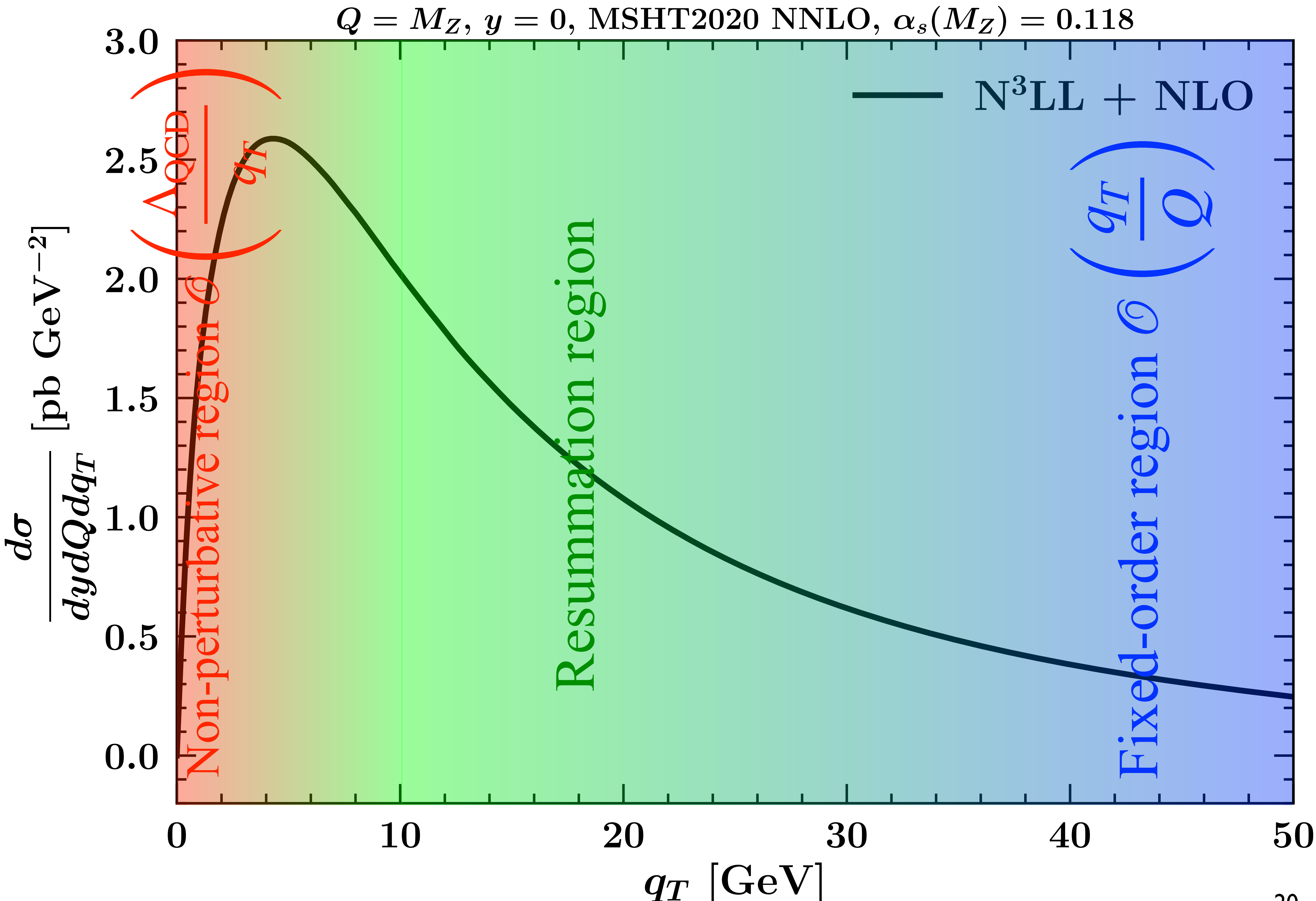
$$C(x, b_T, \mu, \zeta) = \sum_{n=0}^{\infty} \alpha_s^n C^{(n)}(x, b_T, \mu, \zeta)$$

- In conclusion, provided that $\boldsymbol{b}_\mathbf{T}$ is small enough, one obtains the formula:

$$f(x, b_T, \mu, \zeta) = R\left[(\mu, \zeta) \leftarrow (\mu_0, \zeta_0); b_T\right] C(x, b_T, \mu_0, \zeta_0) \otimes f(x, \mu_0)$$

- This allows one to relate perturbative ingredients and the collinear PDF to obtain the TMD.

- Ready for phenomenology? Not quite ($\Lambda_{\rm QCD} \ll q_T \ll Q$).

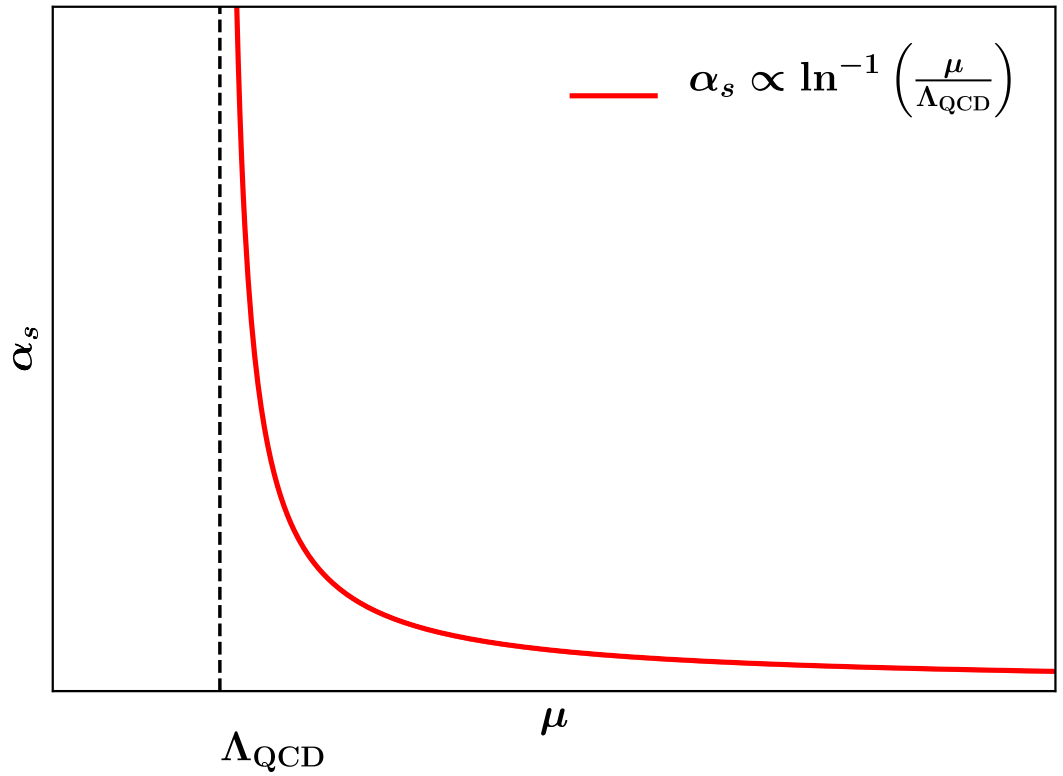


$Q = M_Z$, $y = 0$, MSHT2020 NNLO, $\alpha_s(M_Z) = 0.118$
$\frac{d\sigma}{dy dQ dq_T}$ [pb GeV^{-2}]
q_T [GeV]
N^{3}LL + NLO
Non-perturbative region $\mathcal{O}\left(\frac{\Lambda_{QCD}}{q_T}\right)$
Resummation region
Fixed-order region $\mathcal{O}\left(\frac{q_T}{Q}\right)$

Landau pole regularisation

Origin of ***TMD*** *non-perturbative corrections*

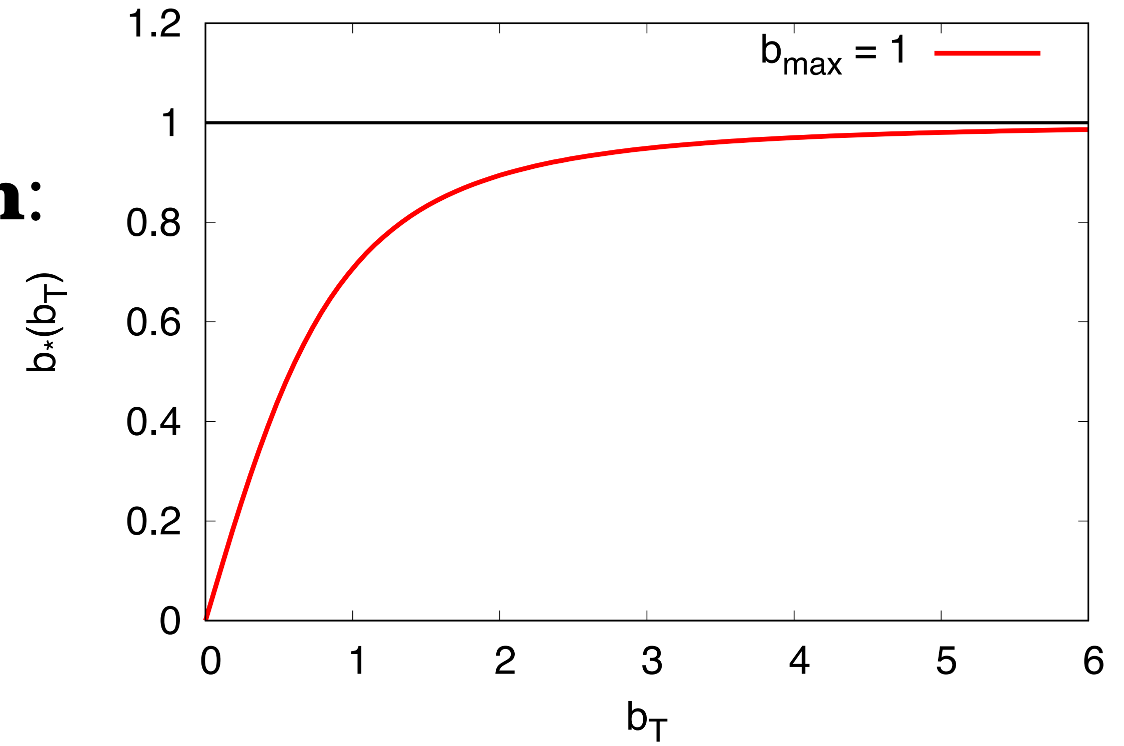
$$\sigma \propto \int_0^\infty db_T\, \alpha_s^p\left(\frac{1}{b_T}\right)\cdots \sim \int_0^Q dk_T\, \alpha_s^p\left(k_T\right)\ldots$$



- Integrating over the full phase space gives a **divergent** result.
- **Prescriptions** to avoid integrating over the **Landau pole** exist:
  - they all introduce **power corrections** $(\Lambda_{QCD}\,/\,Q)^n$.
- TMD non-perturbative effects thus connected to **low-energy** region:
  - no matter how large Q is.
- In addition, matching on PDF only valid for small b_T.

Non-perturbative component

- When integrating over b_T, **large values of** b_T give raise to low scales in the **non-perturbative** region.



- Introduce the so-called ***b****- prescription**:

e.g. $$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{max}^2}}$$

- with $\boldsymbol{b_{max}}$ such that $\boldsymbol{1/b_{max} \gg \Lambda_{QCD}}$.

- Then rewrite the TMD as:

Perturbative

$$f(x, b_T, \mu, \zeta) = \left[\frac{f(x, b_T, \mu, \zeta)}{f(x, b_*(b_T), \mu, \zeta)}\right] f(x, b_*(b_T), \mu, \zeta) \equiv f_{NP}(x, b_T, \zeta)\, f(x, b_*(b_T), \mu, \zeta)$$

Non-perturbative

- Properties of $\boldsymbol{f_{NP}}$:
  - dependence on $\boldsymbol{\mu}$ drops,
  - dependence on ζ is know: $f_{NP}(x, b_T, \zeta) = \widetilde{f}_{NP}(x, b_T) \exp\left[g_K(b_T) \ln\left(\frac{\sqrt{\zeta}}{Q_0}\right)\right]$
  - non-perturbative origin but strictly connected to the particular choice of $\boldsymbol{b_*}$

TMD factorisation

The operational version

- At $q_T \ll Q$ (including $q_T \lesssim \Lambda_{\rm QCD}$), TMD factorisation reads:

$$\frac{d\sigma}{dq_T} = \sigma_0 H(Q,Q) \int_0^\infty db_T\, b_T J_0(q_T b_T) f_A(x_1, b_T, Q, Q^2) f_B(x_2, b_T, Q, Q^2)$$

- Defining $\mu_{b_*} = \dfrac{2e^{-\gamma_E}}{b_*(b_T)}$, the single TMD distributions are given by:

$$f(x, b_T, Q, Q^2) = C(x, \mu_{b_*}, \mu_{b_*}^2) \otimes f_{\rm coll}(x, \mu_{b_*})$$

Matching onto collinear distributions

$$\times \exp\left\{K(\mu_{b_*}) \ln\frac{Q}{\mu_{b_*}} + \int_{\mu_{b_*}}^{Q} \frac{d\mu'}{\mu'}\left[\gamma_F(\alpha_s(\mu')) - \gamma_K(\alpha_s(\mu'))\ln\frac{Q}{\mu'}\right]\right\}$$

Perturbative evolution

$$\times \widetilde{f}_{\rm NP}(x, b_T) \exp\left[g_K(b_T)\ln\left(\frac{Q}{Q_0}\right)\right]$$

TMD non-perturbative contribution

- Matching functions and anomalous dimensions are perturbatively known,
- collinear distributions are (accurately) known from dedicated fits,
- TMD non-pert. contribution needs to be **determined from data** at small $\boldsymbol{q}_{\mathbf{T}}$.

Including $\mathcal{O}(q_T/Q)$ corrections

- Accurate predictions for all q_T's can finally be obtained by **matching**, order by order in perturbation theory:

$$\left(\frac{d\sigma}{dq_T}\right)_{\rm match.} = \left(\frac{d\sigma}{dq_T}\right)_{\rm res.} + \left(\frac{d\sigma}{dq_T}\right)_{\rm f.o.} - \left(\frac{d\sigma}{dq_T}\right)_{\rm d.c.}$$

- In order for the matching to actually take place one needs:

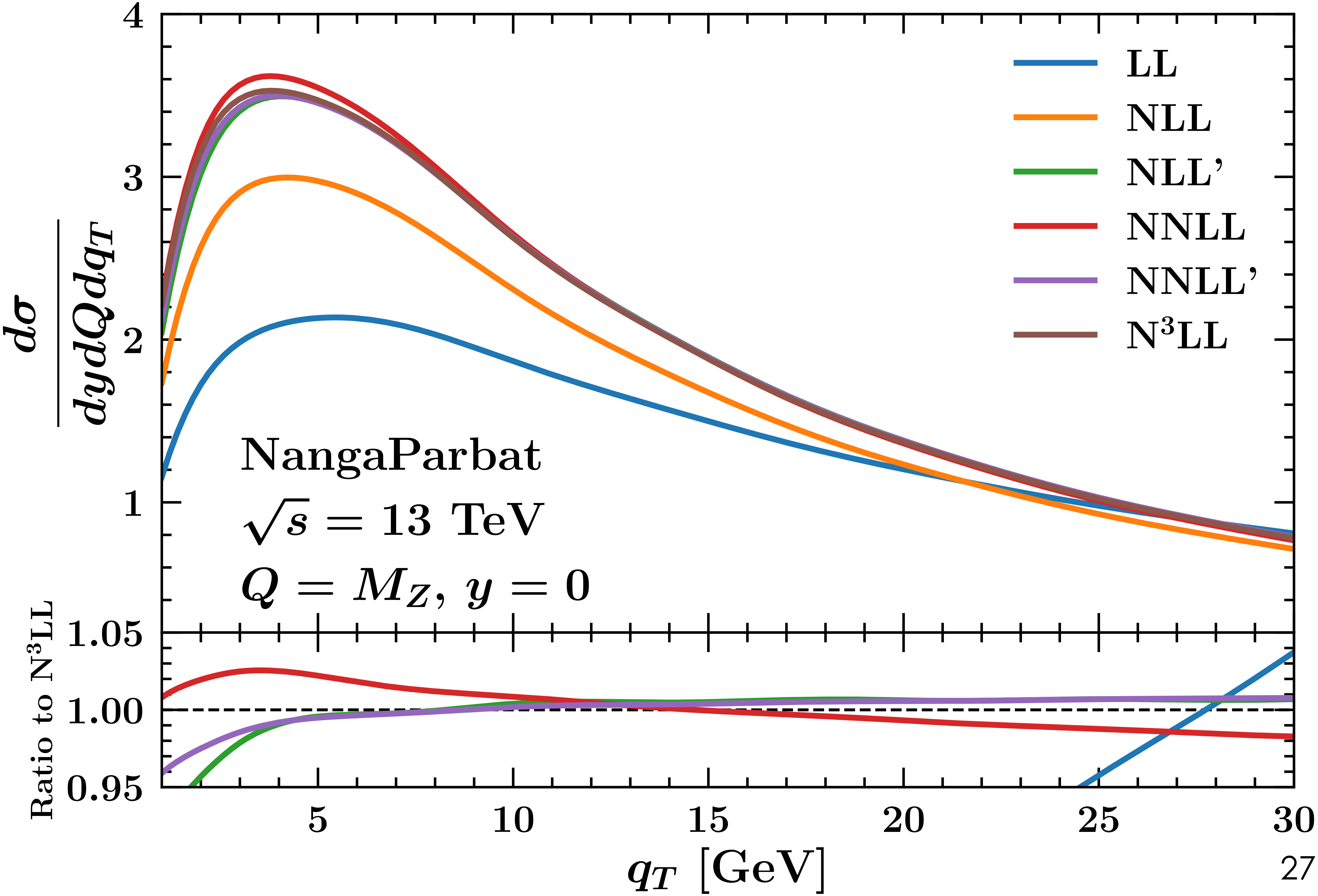
$$\left(\frac{d\sigma}{dq_T}\right)_{\rm res.} \xrightarrow[\rm f.o.]{} \left(\frac{d\sigma}{dq_T}\right)_{\rm d.c.} \xleftarrow[q_T \ll Q]{} \left(\frac{d\sigma}{dq_T}\right)_{\rm f.o.}$$

Conclusions

- **TMD factorisation** provides a valuable tool to describe q_T spectra at small values of q_T ($q_T \ll Q$):
  - written in terms of **TMD distributions** (TMDs),
  - TMDs obey evolution equations that can easily be solved (in b_T space) effectively providing an all-order resummation of all large $\ln(q_T/Q)$,
  - TMDs can be matched onto collinear distributions (PDFs).
  - Bottomline: TMDs in the region $\Lambda_{\mathrm{QCD}} \ll q_T \ll Q$ are fully determine by pQCD and PDFs.
- For $q_T \lesssim \Lambda_{\mathrm{QCD}}$ non-perturbative contributions become relevant and have to be determined from **data**:
  - see Giuseppe's seminar.
- The region $q_T \simeq Q$ can be treated in collinear factorisation and matched to the TMD description obtaining predictions for *all* q_T's.

Backup

TMDs at the LHC



Logarithmic counting

- TMD factorisation provides **resummation** of large logs $L = \log(q_T/Q)$:
    - implemented through the **Sudakov** form fact R.
- A **perturbative expansion** in powers of α_s of R would give:

One Sudakov for each TMD → R^2 ; Double-log expansion → $2n$

$$R^2 = \sum_{n=0}^{\infty} a_s^n \sum_{k=1}^{2n} \widetilde{S}^{(n,k)} L^k$$

- that can be rearranged as:

$$R^2 = \sum_{m=0}^{\infty} R^2_{\mathrm{N^mLL}} \quad \text{with} \quad R^2_{\mathrm{N^mLL}} = \sum_{n=[m/2]}^{\infty} \widetilde{S}^{(n,2n-m)} a_s^n L^{2n-m}$$

Integer part of m/2

- Therefore, multiplying R by a power p of α_s gives:

$$a_s^p R^2_{\mathrm{N^mLL}} = \sum_{j=[(m+2p)/2]}^{\infty} \widetilde{S}^{(j-p,2j-(m+2p))} a_s^j L^{2j-(m+2p)} \sim R^2_{\mathrm{N^{m+2p}LL}}$$

- Bottom line: any additional power of α_s causes a shift of **two units** in the logarithmic ordering.

Logarithmic counting

$$\left(\frac{d\sigma}{dq_T}\right)_{\text{res.}} \stackrel{\text{TMD}}{=} \sigma_0 H(Q) \int d^2\mathbf{b}_T e^{i\mathbf{b}_T\cdot\mathbf{q}_T} f_A(x_1,\mathbf{b}_T,Q,Q^2) f_B(x_2,\mathbf{b}_T,Q,Q^2)$$

$$f_i = \sum_j \left(C_{i/j}\otimes f_j\right)\exp\left\{K\,\ln\frac{\sqrt{\zeta}}{\mu_b} + \int_{\mu_b}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F - \gamma_K\ln\frac{\sqrt{\zeta}}{\mu'}\right]\right\}$$

Accuracy	γ_K	γ_F	K	$C_{f/j}$	H	FFs/PDFs/α_s
LL	α_s	-	-	1	1	-
NLL	α_s^2	α_s	α_s	1	1	LO
NLL'	α_s^2	α_s	α_s	α_s	α_s	LO
N^{2}LL	α_s^3	α_s^2	α_s^2	α_s	α_s	NLO
N^{2}LL'	α_s^3	α_s^2	α_s^2	α_s^2	α_s^2	NLO
N^{3}LL	α_s^4	α_s^3	α_s^3	α_s^2	α_s^2	NNLO
N^{3}LL'	α_s^4	α_s^3	α_s^3	α_s^3	α_s^3	NNLO

TMD evolution (*à la* SV)

- In [Scimemi, Vladimirov, *JHEP* 08 (2018) 003] the concept of **optimal TMD** was introduced.

- Motivation: within **truncated perturbation theory**, the **path-independence** of the TMD evolution is **violated**.

- This derives from the violation by *subleading* terms of the equation:

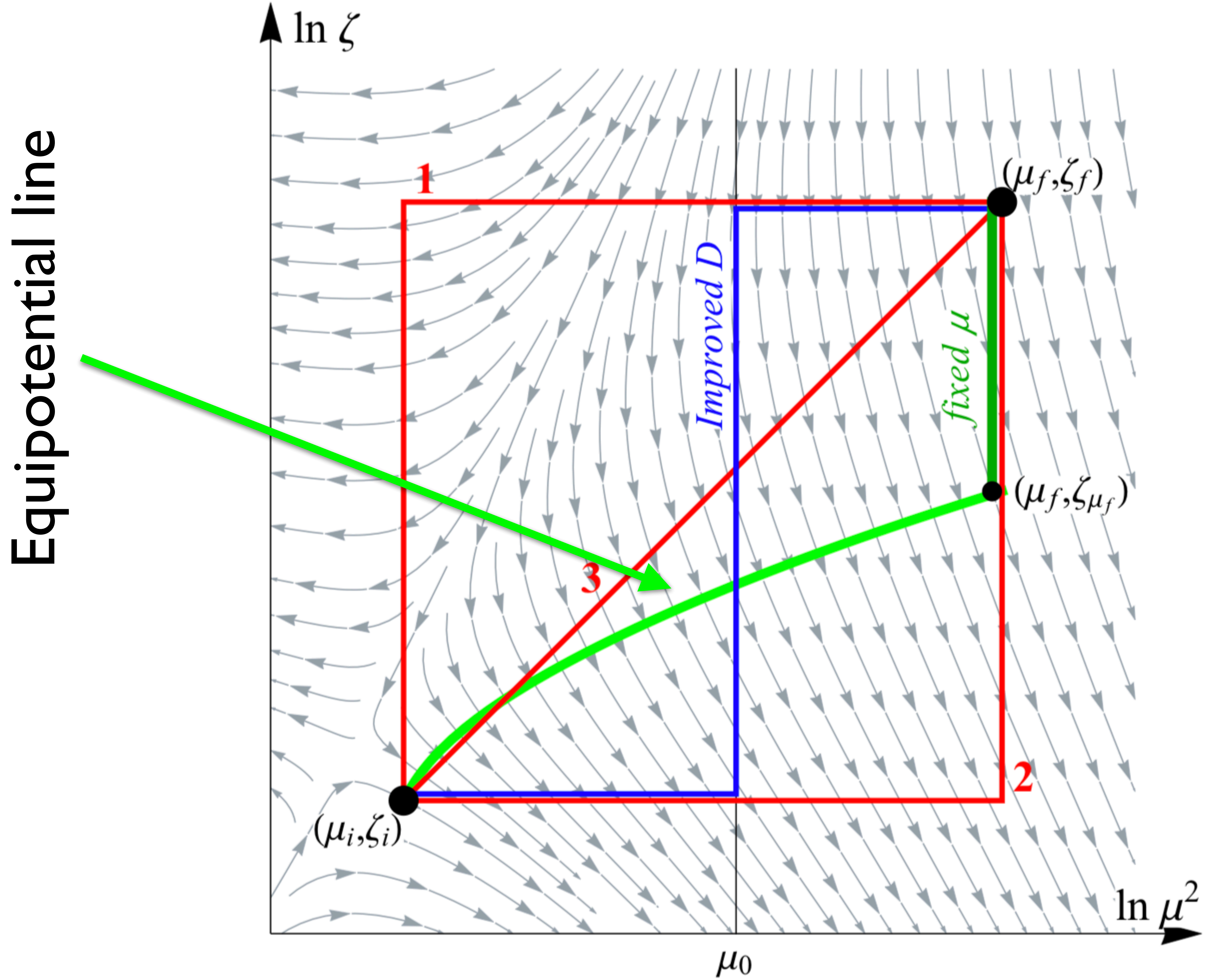
$$\frac{d\boldsymbol{K}}{d\ln\mu} \neq \frac{d\gamma}{d\ln\sqrt{\zeta}} = \gamma_K(\alpha_s(\mu))$$

descending from the *choice* to compute $\boldsymbol{K}$ *analytically*:

$$\gamma_K(\alpha_s(\mu)) = \sum_{n=0}^{N} \alpha_s^{n+1}(\mu)\gamma_K^{(0)} \quad \text{and} \quad K(\mu) = \sum_{n=0}^{N} \alpha_s^{n+1}(\mu) \sum_{k=0}^{n} \ln^k(\mu b_T)\, d^{(n,k)}$$

- This observation led SV to define the **optimal TMD** as the TMD on the plane (μ^2, ζ) at the saddle point of the vector field $\boldsymbol{E} = (\gamma, \boldsymbol{K})$.
  - for any given $\boldsymbol{b_T}$, the optimal TMD is *by definition* scale independent.
  - The TMD at any scale can be computed as a shift along an equipotential line together with a single evolution evolution in ζ.

TMD evolution (*à la* SV)



[Scimemi, Vladimirov, *JHEP* 08 (2018) 003]

TMD matching (*à la* SV)

- Another ingredient of the TMD evolution *à la* SV is the **ζ-prescription**.
- Purpose: eliminate **double logs** from the ***matching functions C***:

$$f(\mu_0, \zeta_0) = C(\mu_0, \zeta_0) \otimes f_{\rm coll}(\mu_0)$$

by exploiting the arbitrariness of ζ_0 assuming $\zeta_0 \doteq \zeta_0(\mu_0)$ and imposing:

$$\frac{d}{d \ln \mu_0} f(\mu_0, \zeta_0(\mu_0)) = 0$$

- This differential equation is the solved for $\ln(\zeta_0)$ order-by-order in α_s:

quark:

$$\begin{aligned} \mathbf{l}_{\zeta_\mu} &= \frac{\mathbf{L}_\mu}{2} - \frac{3}{2} + a_s \Big[\frac{11C_A - 4T_F N_f}{36} \mathbf{L}_\mu^2 \\ &+ C_F \left(-\frac{3}{4} + \pi^2 - 12\zeta_3 \right) + C_A \left(\frac{649}{108} - \frac{17\pi^2}{12} + \frac{19}{2}\zeta_3 \right) + T_F N_f \left(-\frac{53}{27} + \frac{\pi^2}{3} \right) \Big] + \mathcal{O}(a_s^3). \end{aligned}$$

gluon:

$$\begin{aligned} \mathbf{l}_{\zeta_\mu} &= \frac{\mathbf{L}_\mu}{2} - \frac{11}{6} + \frac{2}{3}\frac{T_F N_f}{C_A} + a_s \Big[\frac{11C_A - 4T_F N_f}{36} \mathbf{L}_\mu^2 \\ &+ C_A \left(\frac{247}{54} - \frac{11\pi^2}{36} - \frac{5\zeta_3}{2} \right) + T_F N_f \left(-\frac{16}{3} + \frac{\pi^2}{9} \right) + \left(2C_F + \frac{40}{27} T_F N_f \right) \frac{T_f N_f}{C_A} \Big] + \mathcal{O}(a_s^3). \end{aligned}$$

[Scimemi, Vladimirov, Eur.Phys.J.C 78 (2018)]

Resummation formalisms

- Different formulations of the q_T spectrum:

$$\left(\frac{d\sigma}{dq_T}\right)_{\text{res.}} \propto \begin{cases} e^{2S}\left[f_1 \otimes \mathcal{H} \otimes f_2\right] & : \text{Resum.} \\ H \times F_1 \times F_2 & : \text{TMD} \\ H \times B_1 \times B_2 \times S & : \text{SCET} \end{cases} + \mathcal{O}\left[\left(\frac{q_T}{Q}\right)^m\right]$$

- Dictionary:

$$\mathcal{H} = H C_1 C_2$$

$$F_i = e^S C_i \otimes f_i$$

$$F_i = \sqrt{S} \times B_i$$

- All **equivalent** for *exponentiating* processes.

Additive matching and counting

- Accurate predictions for all q_T's by **additive matching**, order by order in perturbation theory, of collinear and TMD calculations:

$$\left(\frac{d\sigma}{dq_T}\right)_{\mathbf{add.match.}} = \left(\frac{d\sigma}{dq_T}\right)_{\mathbf{res.}} + \left(\frac{d\sigma}{dq_T}\right)_{\mathbf{f.o.}} - \left(\frac{d\sigma}{dq_T}\right)_{\mathbf{d.c.}}$$

- In order for the match to actually take place:

$$\left(\frac{d\sigma}{dq_T}\right)_{\mathbf{res.}} \xrightarrow[\mathbf{f.o.}]{} \left(\frac{d\sigma}{dq_T}\right)_{\mathbf{d.c.}} \xleftarrow[q_T \ll Q]{} \left(\frac{d\sigma}{dq_T}\right)_{\mathbf{f.o.}}$$

- Therefore, the "fixed-order" parts have to match in the relevant limits:

Log Accuracy	Minimal f.o. accuracy
NLL'	α_s (LO)
N^{2}LL	α_s (LO)
N^{2}LL'	α_s^2 (NLO)
N^{3}LL	α_s^2 (NLO)
N^{3}LL'	α_s^3 (NNLO)