Modified black hole with extra dimensions as an unusual dark matter candidate

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Modified Schwarzschild spacetime in Ricci–flat brane

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Theory and Experiment in High Energy Physics

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Contents

- Schwarzschild spacetime $\rightarrow$ black holes
- vacuum solutions with extra dimensions
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- dark matter / dark energy

Einstein gravity

Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi\kappa T_{\mu\nu}$$

without cosmological constant, $\Lambda = 0$, and in vacuum $T_{\mu\nu} = 0$:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

$$\Downarrow$$

$$\boxed{R_{\mu\nu} = 0}$$

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\rho\nu}$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right)$$

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Schwarzschild solution

Minkowski spacetime

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\
&= -dt^2 + dr^2 + r^2 \underbrace{\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)}_{d\Omega^2_{(2)}}
\end{aligned}
$$

Schwarzschild spacetime

$$
ds^2 = -\left(1 + \frac{a}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{a}{r}} + r^2 d\Omega^2_{(2)}
$$

$$
a = -2\kappa M
$$

K. **Schwarzschild**: *On the gravitational field of a mass point according to Einstein's theory*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189-196 (1916).

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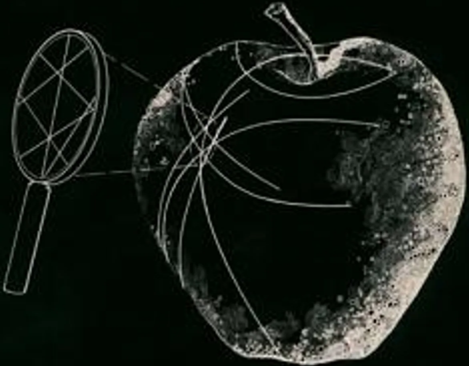
K. Schwarzschild: *On the gravitational field of a mass point according to Einstein's theory*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189-196 (1916).

GRAVITATION

Charles W. MISNER Kip S. THORNE John Archibald WHEELER



Schwarzschild solution

D. Finkelstein: *Past-Future Asymmetry of the Gravitational Field of a Point Particle*, Phys. Rev. **110**, 965-967 (1958).

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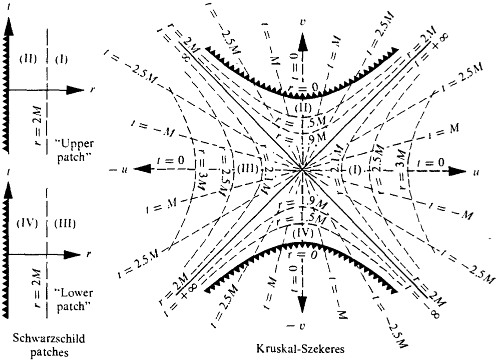
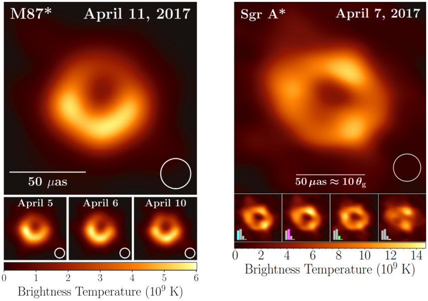


Black holes

Event Horizon Telescope Collaboration: *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, Astrophys. J. Lett. **875**, 17 (2019).

Event Horizon Telescope Collaboration: *First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way*, Astrophys. J. Lett. **930**, 21 (2022).



Extra dimensions

Kaluza–Klein theory

Th. Kaluza: *On the Unification Problem in Physics*, Int. J. Mod. Phys. D **27**, No. 14 (2018) 1870001 (translation); Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966-972 (original).

L. Randall, R. Sundrum: *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. **83**, 3370-3373 (1999).

string theory

E. Witten: *Strong Coupling Expansion Of Calabi-Yau Compactification*, Nucl. Phys. B **471**, 135-158 (1996).

Nuclear Physics B360 (1991) 197–209
North-Holland

BLACK STRINGS AND p-BRANES

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Received 4 March 1991

It is shown that low-energy string theory admits a variety of solutions with the structure of an extended object surrounded by an event horizon. In particular there is a family of black string solutions, labelled by the mass and axion charge per unit length, corresponding to a string in ten dimensions surrounded by an event horizon. The extremal member of this family is the known supersymmetric singular solution corresponding to a macroscopic fundamental string. A similar family of solutions is found describing a fivebrane surrounded by an event horizon, whose extremal member is a previously discovered non-singular supersymmetric fivebrane. Additional charged, extended black hole solutions are presented for each of the antisymmetric tensors that arise in heterotic and type II string theories.

In this section we find extrema of the action

$$S = \int \mathrm{d}^{10}x \sqrt{-g} \left[\mathrm{e}^{-2\phi} \left[R + 4(\nabla\phi)^2 \right] - \frac{2\,\mathrm{e}^{2\alpha\phi}}{(D-2)!} F^2 \right], \tag{1}$$

where F is a $(D-2)$-form satisfying $\mathrm{d}F = 0$. We will assume $D \geqslant 4$. For certain values of α and D this is part of the low-energy action from string theory. The

Finally, using eqs. (2), (3) and (11), one obtains black $(10-D)$-brane solutions of (1)

$$F = Q\epsilon_{D-2},$$

$$\mathrm{d}s^2 = -\left[1-(r_+/r)^{D-3}\right]\left[1-(r_-/r)^{D-3}\right]^{\gamma_x - 1}\mathrm{d}t^2$$

$$+\left[1-(r_+/r)^{D-3}\right]^{-1}\left[1-(r_-/r)^{D-3}\right]^{\gamma_r}\mathrm{d}r^2$$

$$+r^2\left[1-(r_-/r)^{D-3}\right]^{\gamma_r+1}\mathrm{d}\Omega_{D-2}^2 + \left[1-(r_-/r)^{D-3}\right]^{\gamma_x}\mathrm{d}x^i\,\mathrm{d}x_i,$$

$$\mathrm{e}^{-2\phi} = \left[1-(r_-/r)^{D-3}\right]^{\gamma_\phi}, \tag{15}$$

re the exponents are given by

Vacuum solution

ansatz with n extra dimensions $\zeta^A = \zeta^1, ...\zeta^n$:

$$ds^2 = -f(r)^\alpha dt^2 + f(r)^\beta dr^2 + r^2 d\Omega^2_{(2)} + f(r)^\gamma \delta_{AB} \underbrace{d\zeta^A d\zeta^B}_{\text{extra}}$$

Ricci tensor $R_{\mu\nu}$:

$$R_{00} = \alpha f^{\alpha-\beta} F_1 \quad R_{rr} = F_2 \quad R_{\vartheta\vartheta} = F_3$$
$$R_{\varphi\varphi} = F_3 \sin^2\vartheta \quad R_{AB} = -\gamma f^{\gamma-\beta} F_1 \delta_{AB}$$

$$F_1 = \frac{1}{r}\frac{f'}{f} + \frac{1}{4}(\alpha - \beta + n\gamma - 2)\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\frac{f''}{f}$$

$$F_2 = \beta\frac{1}{r}\frac{f'}{f} + \frac{1}{4}\left[\alpha(-\alpha + \beta + 2) + n\gamma(\beta - \gamma + 2)\right]\left(\frac{f'}{f}\right)^2 - \frac{1}{2}(\alpha + n\gamma)\frac{f''}{f}$$

$$F_3 = 1 - f^{-\beta}\left[1 + \frac{1}{2}(\alpha - \beta + n\gamma)\, r\frac{f'}{f}\right]$$

Vacuum solution

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Vacuum solution

$$F_3 = 1 - f^{-\beta}\left[1 + \frac{1}{2}\left(\alpha - \beta + n\gamma\right) r \frac{f'}{f}\right] = 0$$

$$\Downarrow$$

$$\boxed{f = \left(1 + ar^q\right)^{-1/\beta}} \qquad q = \frac{2\beta}{\alpha - \beta + n\gamma}$$

exception: $\alpha - \beta + n\gamma = 0 \Rightarrow f = 1 \Rightarrow$ Minkowski spacetime

the rest is then

$$F_1 = C_1\Phi \quad F_2 = \left(C_2 + C_3 ar^q\right)\Phi \quad \text{where:}$$

$$\Phi = \frac{1}{\left(\alpha - \beta + n\gamma\right)^2} \frac{ar^{q-2}}{\left(1 + ar^q\right)^2}$$

$$C_1 = -\alpha - \beta - n\gamma$$

$$C_2 = 2\beta^2 + \beta\left(\alpha + n\gamma\right) - \left(\alpha + n\gamma\right)^2$$

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Vacuum solution(s)

algebraic equations

$$C_1 = -\alpha - \beta - n\gamma$$
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can be solved by $\alpha = -\beta - n\gamma \Rightarrow C_1 = 0, C_2 = 0 \Rightarrow$

$$C_3 = -n\gamma\left[2\beta + \left(n+1\right)\gamma\right]$$

possible solutions are:

• $\gamma = 0 \rightarrow \alpha = -\beta$:
$f^\alpha = 1 + \frac{a}{r} \quad f^\beta = \left(1 + \frac{a}{r}\right)^{-1} \quad f^\gamma = 1$
trivial extension of the Schwarzschild spacetime

• $2\beta + \left(n+1\right)\gamma = 0 \rightarrow \alpha/\beta = (n-1)/(n+1), \gamma/\beta = -2/(n+1)$:
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nontrivial extension

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Vacuum solutions

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nontrivial extension

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existence of horizon for $a < 0$ at $r = |a|$!!!

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trivial extension

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Size of extra dimensions

trivial extension

$$g_{AB} = \delta_{AB}$$

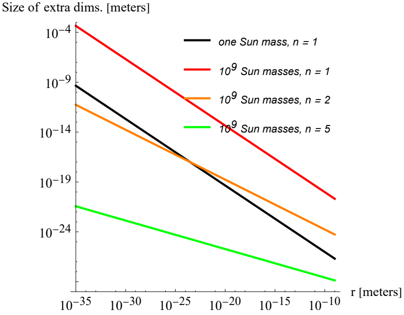
nontrivial extension

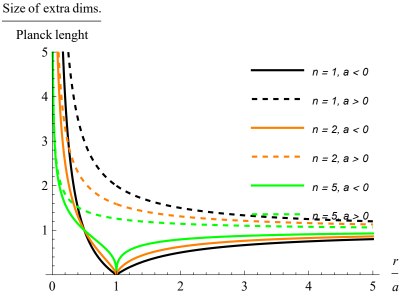
$$\boxed{g_{AB} = \left(1 + \frac{a}{r}\right)^{\frac{2}{n+1}} \delta_{AB}}$$


NASA

Size of extra dimensions

$$\left(1+\frac{a}{r}\right)^{\frac{2}{n+1}}$$





Newtonian limit

$$ds^2 \approx -(1+2\phi)dt^2 + (1-2\psi)\delta_{ij}dx^i dx^j$$

$$g_{00} \approx -(1+2\phi) \quad \boxed{\phi = -\frac{\kappa M}{r}}$$

trivial extension

$$g_{00} = -\left(1+\frac{a}{r}\right) \Rightarrow \phi = \frac{1}{2}\frac{a}{r} \Rightarrow \boxed{M = -\frac{a}{2\kappa}}$$

nontrivial extension

$$g_{00} = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}} \Rightarrow \phi = -\frac{n-1}{n+1}\frac{a}{2}\frac{1}{r} \Rightarrow \boxed{M = \frac{n-1}{n+1}\frac{a}{2\kappa}}$$

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nontrivial extension

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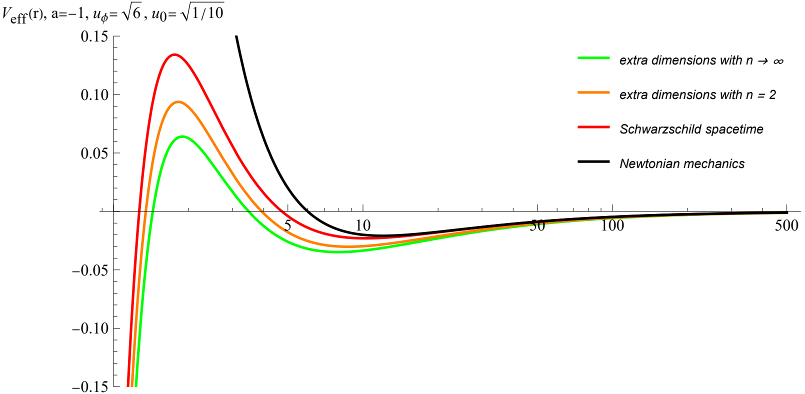
nontrivial extension

$$g_{00} = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}} \Rightarrow \phi = -\frac{n-1}{n+1}\frac{a}{2}\frac{1}{r} \Rightarrow \boxed{M = \frac{n-1}{n+1}\frac{a}{2\kappa}}$$

Motion of a test particle

geodesic equation: $\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = E_{\text{eff}}$

$$V_{\text{eff}}(r) = \frac{1}{2}\left\{\frac{a}{r} + \frac{u_\phi^2}{r^2} + \frac{a u_\phi^2}{r^3} + u_0^2\left[1 - \left(1 + \frac{a}{r}\right)^{\frac{2n}{n+1}}\right]\right\}$$



Horizon

Schwarzschild spacetime

$$ds^2 = -\left(1 + \frac{a}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{a}{r}} + r^2 d\Omega^2_{(2)}$$

$$a = -2\kappa M$$

has only the central singularity

Kretschmann scalar proves it

$$\boxed{K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{12a^2}{r^6}}$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\mu\rho,\nu\sigma} - g_{\nu\sigma,\mu\rho}\right) + g_{\alpha\beta}\left(\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\nu\rho} - \Gamma^{\alpha}_{\mu\rho}\Gamma^{\beta}_{\nu\sigma}\right)$$

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Kretschmann scalar

trivial extension

$$ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \delta_{AB}d\zeta^A d\zeta^B$$

$$K = \frac{12a^2}{r^6} \quad \text{the same as original Schwarzschild}$$

nontrivial extension

$$ds^2 = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}}dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega_{(2)}^2 + \left(1+\frac{a}{r}\right)^{\frac{2}{n+1}}\delta_{AB}d\zeta^A d\zeta^B$$

$$K = \frac{12a^2}{r^6}\left[1 - \frac{1}{3}\frac{n(n-1)}{(n+1)^3}\left(1+\frac{r}{a}\right)^{-2}\left(\frac{3}{4}n + 1 + (n+1)\frac{r}{a}\right)\right]$$

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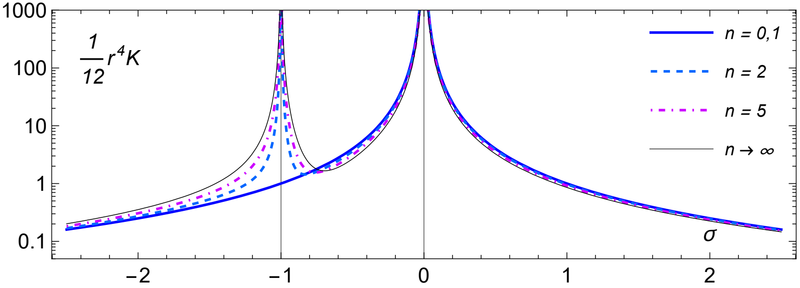
dimensionless radial coordinate $\sigma = \dfrac{r}{a}$ for both $a > 0$ and $a < 0$

$$\frac{1}{12} r^4 K = \frac{1}{\sigma^2} \left[1 - \frac{1}{3} \frac{n(n-1)}{(n+1)^3} \frac{1}{(1+\sigma)^2} \left(\frac{3}{4} n + 1 + (n+1)\sigma \right) \right]$$



for $a < 0$ there is **horizon singularity** at $r = |a|$!!!

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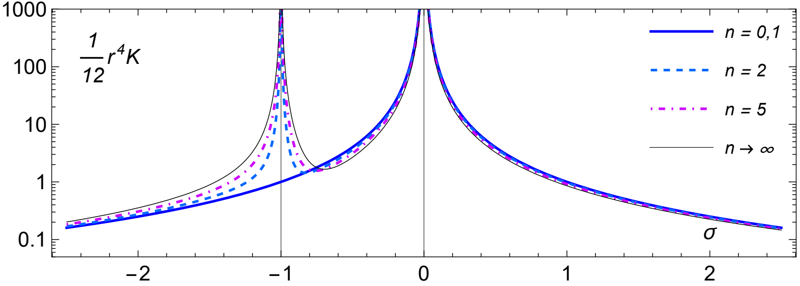

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Conserved energy
Landau–Lifshitz stress-energy pseudotensor

conserved D-momentum in $D-1$ dimensional space region Ω

$$P^{\mu} = \oint_{\partial\Omega} h^{\mu 0 \nu} d\Sigma_{\nu}$$

where $\quad h^{\mu\nu\lambda} = \frac{1}{16\pi\kappa}\left[(-g)\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma}\right)\right]_{,\sigma}$

choice of $\partial\Omega$ such that $d\Sigma_i = dS_i$ and $d\Sigma_A = 0$

only $P^0 = \mathcal{E}$ is nonzero

$$\mathcal{E} = \oint_{\partial\Omega} h^{00i} dS_i$$

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Isotropic coordinates

new radial coordinate ρ: $r = \rho \left(1 - \frac{a}{4\rho}\right)^2$, $\rho \in [1/4, \infty)$

trivial extension

$$ds^2 = -\left(\frac{\chi_+}{\chi_-}\right)^2 dt^2 + (\chi_-)^4 \delta_{ij} dx^i dx^j + \delta_{AB} d\zeta^A d\zeta^B$$

nontrivial extension

$$ds^2 = -\left(\frac{\chi_+}{\chi_-}\right)^{-2\frac{n-1}{n+1}} dt^2 + (\chi_-)^4 \delta_{ij} dx^i dx^j + \left(\frac{\chi_+}{\chi_-}\right)^{\frac{2}{n+1}} \delta_{AB} d\zeta^A d\zeta^B$$

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$$\mathcal{E}(\rho) = -\frac{a}{2\kappa}(\chi_-)^7 \xrightarrow{\rho\to\infty} \boxed{-\frac{a}{2\kappa}}$$

nontrivial extension

$$\mathcal{E}(\rho) = \frac{a\,(\chi_+)^{\frac{4n}{n+1}}\,(\chi_-)^{4\frac{n+2}{n+1}}}{4(n+1)\kappa}\left(\frac{n}{\chi_+} - \frac{n+2}{\chi_-}\right) \xrightarrow{\rho\to\infty} \boxed{-\frac{a}{2(n+1)\kappa}}$$

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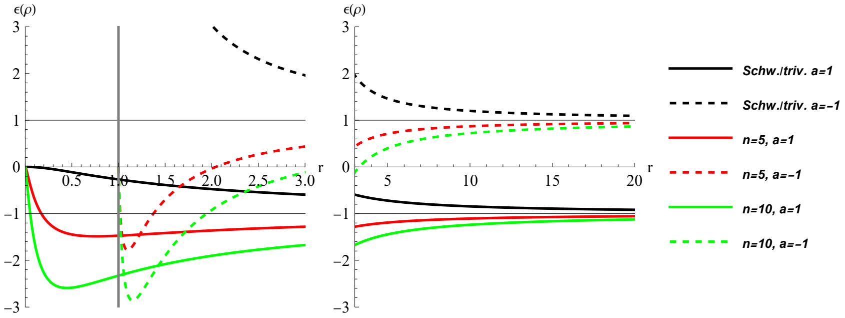
Conserved energy

partial energy in spherical volume with coordinate radius r:

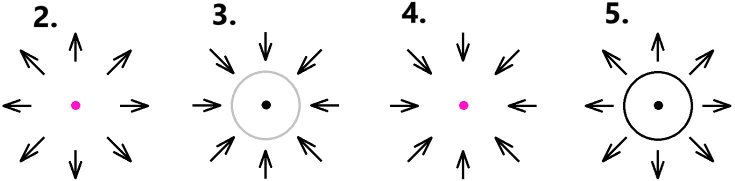


$a > 0$ - starting at center $r = 0$

$a < 0$ - starting on horizon $r = |a|$

Physical properties

	horizons	singularities	Newt. lim. mass	cons. energy
1. Mink. ($a = 0$)	none	none	$M = 0$	$\mathcal{E} = 0$
2. trivial ($a > 0$)	none	$r = 0$	$M = \frac{-a}{2\kappa} < 0$	$\mathcal{E} = M < 0$
3. trivial ($a < 0$)	$r = -a$	$r = 0$	$M = \frac{-a}{2\kappa} > 0$	$\mathcal{E} = M > 0$
4. nontriv. ($a > 0$)	none	$r = 0$	$M = \frac{n-1}{n+1}\frac{a}{2\kappa} > 0$	$\mathcal{E} = \frac{-a}{2(n+1)\kappa} < 0$
5. nontriv. ($a < 0$)	$r = -a$	$r = \{0, -a\}$	$M = \frac{n-1}{n+1}\frac{a}{2\kappa} < 0$	$\mathcal{E} = \frac{-a}{2(n+1)\kappa} > 0$



Special case

nontrivial case with $n = 1$

$$ds^2 = -dt^2 + \left(1 + \frac{a}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} + \left(1 + \frac{a}{r}\right) d\zeta^2$$

gives $\quad M = 0 \quad$ and $\quad \mathcal{E} = -\frac{a}{4\kappa}$

no gravitational force (in Newt. lim.)
but nonzero conserved energy $\mathcal{E}$!

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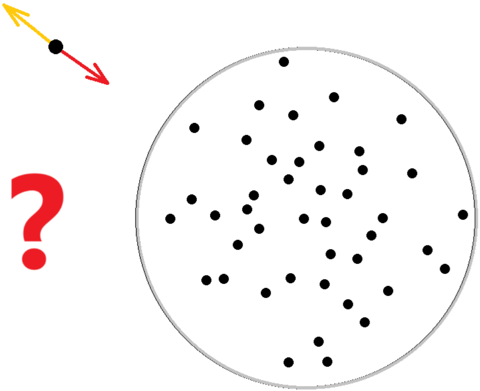
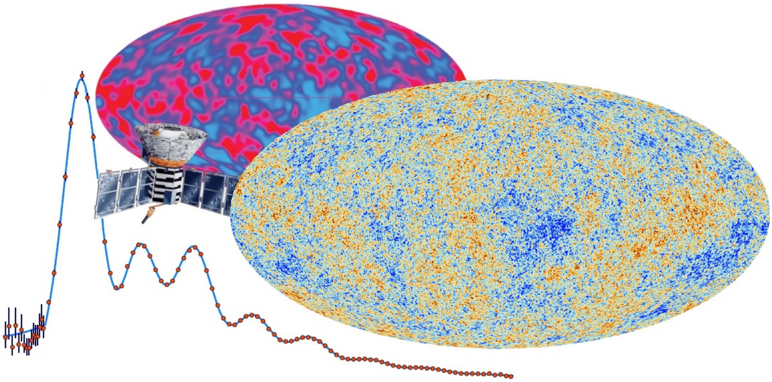
Dark matter?

problem with clustering, rotation of galaxies, formation of cosmological structures, etc.



Dark matter?

effect on cosmic microwave background



Instability

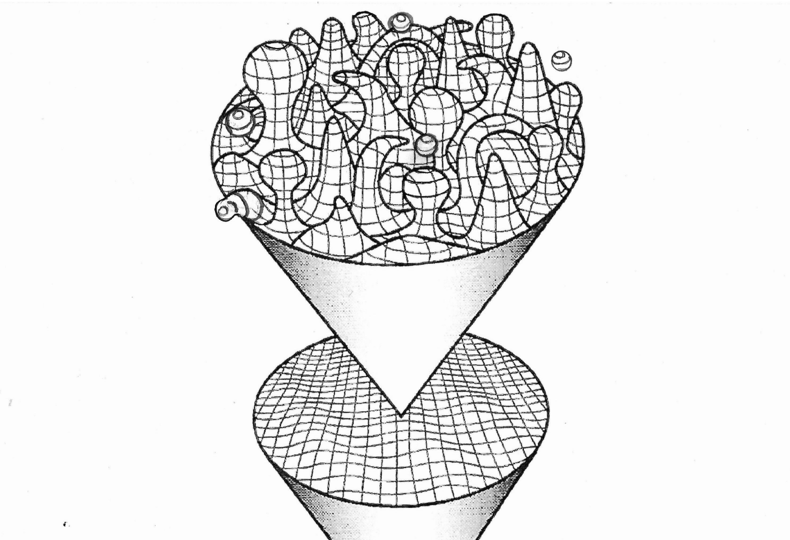
R. Gregory, R. Laflamme: *Black Strings and p-Branes are Unstable*, Phys. Rev. Lett. **70**, 2837-2840 (1993).

R. Gregory, R. Laflamme: *The Instability of Charged Black Strings and p-Branes*, Nucl. Phys. B **428**, 399-434 (1994).

G. W. Gibbons, S. A. Hartnoll, A. Ishibashi: *On the stability of naked singularities*, Prog. Theor. Phys. **113**, 963-978 (2005).

V. Cardoso, M. Cavaglia: *Stability of naked singularities and algebraically special modes*, Phys. Rev. D **74**, 024027 (2006).

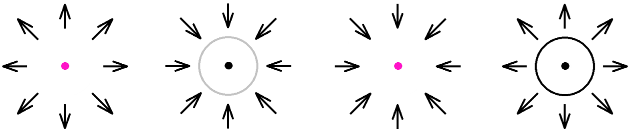
Quantum foam?



Summary

- Schwarzschild: $ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \left(1+\frac{a}{r}\right)^{-1}dr^2 + r^2 d\Omega^2_{(2)}$
- nontrivial vacuum solution with n extra dimensions:

$$ds^2 = -\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}} dt^2 + \left(1+\frac{a}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)} + \left(1+\frac{a}{r}\right)^{\frac{2}{n+1}} \delta_{AB}\, d\zeta^A d\zeta^B$$



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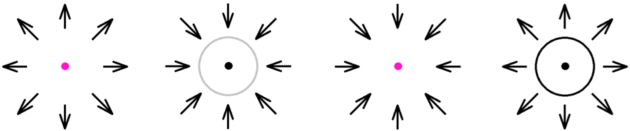

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Thank You for listening!