From zero-temperature unitarity to quantum Boltzmann equation

Peter Maták
In collaboration with T. Blažek

[Eur. Phys. J. C 81 (2021) 1050, Eur. Phys. J. C 82 (2022) 214]

COMENIUS UNIVERSITY BRATISLAVA

Theory and Experiment in High Energy Physics

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Outline of this talk

- Holomorphic cutting rules for classical kinetic theory [Phys. Rev. D 103 (2021) L091302]
- From unitarity to quantum statistics [Eur. Phys. J. C 81 (2021) 1050]
- Anomalous thresholds and thermal masses [Eur. Phys. J. C 82 (2022) 214]

Holomorphic cutting rules

$$S = 1 + \mathrm{i}\, T \qquad\qquad T_{fi} = (2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi} \tag{1}$$

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$$|T_{fi}|^2 = -\mathrm{i}\,T^\dagger_{if} \mathrm{i}\,T_{fi} = -\mathrm{i}\,T_{if} \mathrm{i}\,T_{fi} + \sum_n \mathrm{i}\,T_{in} \mathrm{i}\,T_{nf} \mathrm{i}\,T_{fi} - \sum_{n,k} \mathrm{i}\,T_{in} \mathrm{i}\,T_{nk} \mathrm{i}\,T_{kf} \mathrm{i}\,T_{fi} + \dots \tag{3}$$

[Coster, Stapp '70, Bourjaily, Hannesdottir, *et al.* '21, Hannesdottir, Mizera '22, Blažek, Maták '21a]

Holomorphic cutting rules

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$$\begin{aligned} \Delta|T_{fi}|^2 = |T_{fi}|^2 - |T_{if}|^2 = & \sum_n \left(\mathrm{i}T_{in} \mathrm{i}T_{nf} \mathrm{i}T_{fi} - \mathrm{i}T_{if} \mathrm{i}T_{fn} \mathrm{i}T_{ni} \right) \\ & - \sum_{n,k} \left(\mathrm{i}T_{in} \mathrm{i}T_{nk} \mathrm{i}T_{kf} \mathrm{i}T_{fi} - \mathrm{i}T_{if} \mathrm{i}T_{fk} \mathrm{i}T_{kn} \mathrm{i}T_{ni} \right) \\ & + \dots \end{aligned} \tag{4}$$

[Blažek, Maták '21a, see also Roulet, Covi, Vissani '98]

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[Blažek, Maták ’21a, see also Roulet, Covi, Vissani ’98]

$$\sum_f \Delta|T_{fi}|^2 = 0 \tag{5}$$

[Dolgov ’79, Kolb, Wolfram ’80]

Holomorphic cuts and the classical Boltzmann equation

Change in # of particles $\leftrightarrow$ average # of their interactions

$$\dot{n}_{f_1} + 3Hn_{f_1} = \sum_{\text{all reactions}} \gamma_{fi} - \gamma_{if} \qquad \gamma_{fi} = \frac{1}{V_4}\int \prod_{k=1}^{p} [\mathrm{d}\boldsymbol{p}_k]\, f_{i_k}(\boldsymbol{p}_k) \int \prod_{l=1}^{q} [\mathrm{d}\boldsymbol{p}_l] |\, T_{fi}|^2 \tag{6}$$

$$[\mathrm{d}\boldsymbol{p}_k] = \frac{\mathrm{d}^3\boldsymbol{p}_k}{(2\pi)^3 2E_{\boldsymbol{p}_k}} \qquad |\, T_{fi}|^2 = V_4 (2\pi)^4 \delta^{(4)}(p_f - p_i) |M_{fi}|^2 \tag{7}$$

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[Blažek, Maták '21a]

From unitarity to quantum statistics

$$\mathcal{L} = -\mu \Phi \phi_1 \phi_2 \tag{9}$$

Which processes contribute to n_{ϕ_1} evolution at $\mathcal{O}(\mu^2)$?

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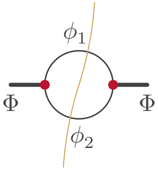

$$\int [\mathrm{d}\boldsymbol{p}_\Phi] \mathrm{e}^{-E_\Phi/T} \int [\mathrm{d}\boldsymbol{k}_1][\mathrm{d}\boldsymbol{k}_2](2\pi)^4 \delta^{(4)}(p_\Phi - k_1 - k_2)$$

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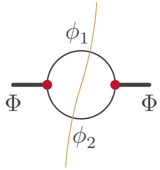

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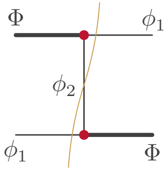
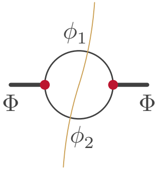


?

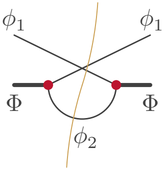
From unitarity to quantum statistics

$$\mathcal{L} = -\mu \Phi \phi_1 \phi_2 \tag{9}$$

Which processes contribute to n_{ϕ_1} evolution at $\mathcal{O}(\lambda^2)$?

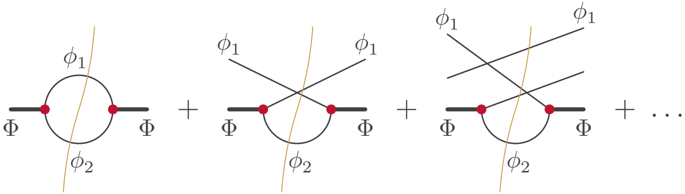


$$\int [\mathrm{d}\boldsymbol{p}_\Phi] \mathrm{e}^{-E_\Phi/T} \int [\mathrm{d}\boldsymbol{k}_1][\mathrm{d}\boldsymbol{k}_2](2\pi)^4 \delta^{(4)}(p_\Phi - k_1 - k_2)$$



$$\int [\mathrm{d}\boldsymbol{p}_\Phi] \mathrm{e}^{-E_\Phi/T} \int [\mathrm{d}\boldsymbol{k}_1][\mathrm{d}\boldsymbol{k}_2] \mathrm{e}^{-E_{k_1}/T}(2\pi)^4 \delta^{(4)}(p_\Phi - k_1 - k_2)$$

From unitarity to quantum statistics



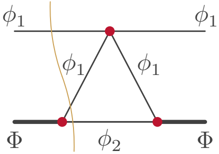
$$\int [\mathrm{d}\boldsymbol{p}_\Phi] \mathrm{e}^{-E_\Phi/T} \int [\mathrm{d}\boldsymbol{k}_1][\mathrm{d}\boldsymbol{k}_2] \left[1 + \frac{1}{\mathrm{e}^{E_{k_1}/T} - 1}\right] (2\pi)^4 \delta^{(4)}(p_\Phi - k_1 - k_2) \tag{10}$$

[Blažek, Maták '21b]

Anomalous thresholds and thermal corrections

$$\mathcal{L} = -\mu\Phi\phi_1\phi_2 - \frac{1}{4!}\lambda\phi_1^4 \tag{11}$$

$$\gamma^{\mathrm{eq}}_{\Phi\phi_1\to\phi_1\phi_1\phi_2} = -$$



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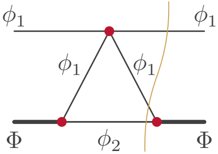

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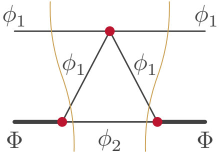

(12)

[Hannesdottir, Mizera ’22 and the references there in]

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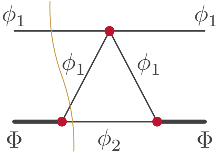

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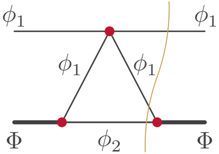

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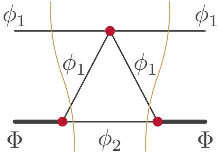
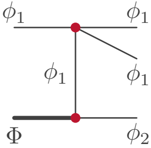


(12)

[Hannesdottir, Mizera '22 and the references there in]

$$\frac{1}{k^2 + \mathrm{i}\epsilon} = \mathrm{P.V.}\frac{1}{k^2} - \mathrm{i}\pi\delta\big(k^2\big)$$



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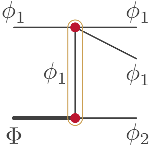
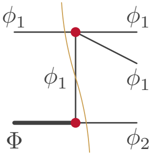


$$+ \frac{1}{2}$$



(13)

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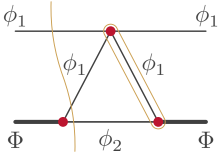
$$\gamma^{\mathrm{eq}}_{\Phi\phi_1 \to \phi_1\phi_1\phi_2} = -\,2$$



$$= \mathring{m}^2_{\phi_1}(T) \times \frac{\partial \gamma^{\mathrm{eq}}_{\Phi \to \phi_1\phi_2}}{\partial m^2_{\phi_1}} \tag{14}$$

[Blažek, Maták '22]

$$2\delta_+\left(k^2\right)\mathrm{P.V.}\frac{1}{k^2} = -\frac{1}{\left(k^0 + |\boldsymbol{k}|\right)^2} \frac{\partial \delta\left(k^0 - |\boldsymbol{k}|\right)}{\partial k^0} \tag{15}$$

[Frye, *et al.* '19, Racker '19]

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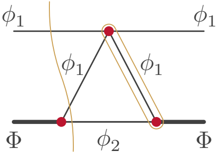

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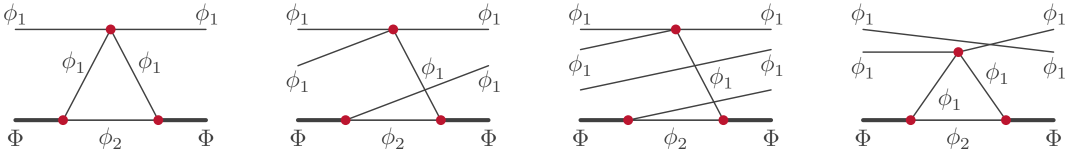
[Blažek, Maták '22]

$$\mathring{m}^2_{\phi_1}(T) = \lambda \int [\mathrm{d}\boldsymbol{k}_1] \mathrm{e}^{-E_1/T} = \frac{\lambda}{4\pi^2} T^2 \tag{16}$$

Anomalous thresholds and thermal corrections



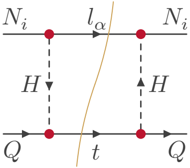
$\downarrow$

$$m^2_{\phi_1}(T) \times \frac{\partial \gamma^{\text{eq}}_{\Phi\to\phi_1\phi_2}}{\partial m^2_{\phi_1}} \text{ with } m^2_{\phi_1}(T) = \frac{\lambda}{24}T^2 \text{ and quantum statistics in } \gamma^{\text{eq}}_{\Phi\to\phi_1\phi_2}$$

[Blažek, Maták '22]

Anomalous thresholds and IR finiteness

$$\gamma^{\text{eq}}_{NQ\to lt} \quad \leftarrow \quad -$$



$$\propto \quad \left[\frac{1}{(p_Q - p_t)^2}\right]^2 \tag{17}$$

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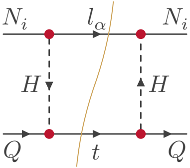

$$\propto \quad \int_{-1}^{1} \mathrm{d}\cos\theta \left[\frac{1}{1-\cos\theta}\right]^2 \qquad (17)$$

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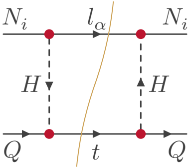
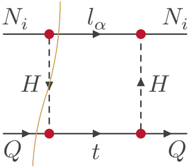
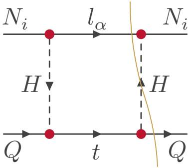
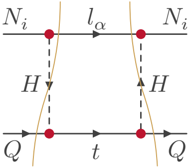
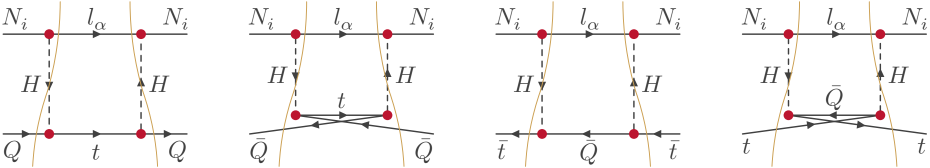
$\gamma^{\mathrm{eq}}_{NQ\to lt} \leftarrow -$  $\propto \int_{-1}^{1} \mathrm{d}\cos\theta \left[\frac{1}{1-\cos\theta}\right]^2$ (17)

$\gamma^{\mathrm{eq}}_{NQ\to lHQ} \leftarrow -$  $-$  $+$  (18)

$$\gamma^{\mathrm{eq}}_{NQ\to lt} + \gamma^{\mathrm{eq}}_{NQ\to lHQ} = \text{IR finite}$$

[Racker '19, Frye, *et al.* '19]

Anomalous thresholds and IR finiteness

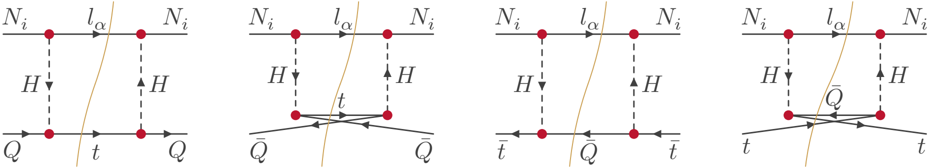


$$\gamma^{\rm eq}_{NQ\to lHQ} + \gamma^{\rm eq}_{N\bar{Q}\to lH\bar{Q}} + \gamma^{\rm eq}_{Nt\to lHt} + \gamma^{\rm eq}_{N\bar{t}\to lH\bar{t}} = \mathring{m}^2_H(T) \times \frac{\partial \gamma^{\rm eq}_{N\to lH}}{\partial m^2_H} \tag{19}$$

$$\mathring{m}^2_H(T) = 12\, Y_t^2 \int [d\boldsymbol{p}] \mathrm{e}^{-E/T} \tag{20}$$

[Blažek, Maták ’22, see also Salvio, Lodone, Strumia ’11]

Anomalous thresholds and IR finiteness



Sum up to IR finite result for $(p_N + p_{\bar{Q}})^2, (p_N + p_t)^2 \leq 2M_N^2$.

[Blažek, Maták '22, compare to Czarnecki, *et al.* '12]

What else can be done?

- Resonances beyond narrow-width approximation with no double counting [Maták '24, see also Tkachov '98]

$$\left|\frac{1}{s - M^2 + \mathrm{i}\epsilon}\right|^2 \quad \rightarrow \quad -\frac{\partial}{\partial s}\mathrm{P.V.}\frac{1}{s - M^2}$$

- CPT and unitarity constraints at finite-temperature [Blažek, Maták, Zaujec '22]

Where it all comes from?

$$\rho = \prod_p \rho_p = \frac{1}{Z} \exp\left\{ -\sum_p F_p a_p^\dagger a_p \right\} \quad \leftarrow \quad Z = \prod_p Z_p = \prod_p \frac{\exp F_p}{\exp F_p - 1} \tag{21}$$

$$\exp\{-E_p/T\} \quad \to \quad \exp\{-F_p\} = \frac{f_p}{1 \pm f_p} \qquad\qquad f_p = \mathrm{Tr}\left[a_p^\dagger a_p \rho\right] \tag{22}$$

[Wagner '91]

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[Wagner '91]

$$\rho' = S\rho S^\dagger \quad \to \quad (1 + \mathrm{i}\,T)\rho(1 - \mathrm{i}\,T + \mathrm{i}\,T\mathrm{i}\,T - \ldots) \tag{23}$$

The collision term for the Boltzmann equation is obtained as $\mathrm{Tr}\left[a_p^\dagger a_p (\rho - \rho')\right]/V_4$.

[McKellar, Thomson '94, Blažek, Maták '21b]

Summary

- Unitarity may help in calculating reaction rates for the Boltzmann equation.

$$\gamma_{fi}^{\text{eq}} = \frac{1}{V_4} \int \prod_{k=1}^{p} [\mathrm{d}\boldsymbol{p}_k]\, f_{i_k}^{\text{eq}}(\boldsymbol{p}_k) \int \prod_{l=1}^{q} [\mathrm{d}\boldsymbol{p}_l] \Big(-\mathrm{i}\, T_{if} \mathrm{i}\, T_{fi} + \sum_n \mathrm{i}\, T_{in} \mathrm{i}\, T_{nf} \mathrm{i}\, T_{fi} + \dots \Big)$$

- Completing diagrams by all possible winding numbers accounts for quantum statistics.

- Anomalous thresholds approximate thermal-mass effects in lower-order process kinematics.

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"Language is a way we cut reality into pieces."

[Kvasz '15]

Thank you for your attention!