Cosmological evolution of a Peccei-Quinn field with small self-coupling and implications for Axion Dark Matter

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based on PK, M. Olechowski, JCAP 06 (2023) 043

- **Axions (QCD axions and ALPs)**
  - **pseudo Goldstone bosons of spontaneously broken Peccei-Quinn global $U(1)_{PQ}$ symmetry**
  - **phase component of complex PQ scalar**

$$\Phi = \frac{1}{\sqrt{2}} S e^{ia/f_a} = \frac{1}{\sqrt{2}} S e^{i\theta}$$

  - **$U(1)_{PQ}$ is anomalous and axion potential is developed (non-perturbatively) long after inflation**

$$V_a \sim m_a^2 f_a^2 \left(1 - cos\frac{a}{f_a}\right) = m_a^2 f_a^2 \left(1 - cos\theta\right)$$

Axions are interesting candidates for Dark Matter

$$V(\Phi) = \lambda_\Phi \left(|\Phi|^2 - \frac{f_a^2}{2} \right)^2 = \frac{\lambda_\Phi}{4} \left(S^2 - f_a^2 \right)^2 , \qquad m_S^2 = 2\lambda_\Phi f_a^2$$

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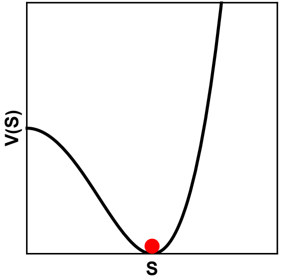
(i) $m_S \gg H_I$ **('$U(1)_{PQ}$ broken')**

- $S = f_a$ **during and after inflation**
- θ_i **determined by some stochastic process**
  (the phase transition from unbroken to broken $U(1)_{PQ}$)
- $\Rightarrow$ **If θ_i not aligned with the minimum of V_a**
  Axion starts oscillating when $H \lesssim \frac{1}{3} m_a$
  $\rightarrow$ **cold dark matter (CDM)**



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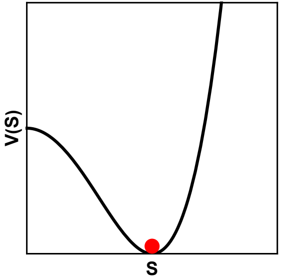

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Axion field a is massless during inflation $\Rightarrow$ it has stochastic quantum fluctuations: a changes on average by $H_I/2\pi$ during each Hubble time, in each Hubble volume

- θ **has dispersion $\langle \delta\theta_i^2 \rangle$, generated by quantum fluctuations during inflation**
  $\Rightarrow$ **Isocurvature perturbations of axion CDM $\langle \delta\theta_i^2 \rangle \propto (H_I/f_a)^2 \ll 1$**

(ii) $\underline{m_S \ll H_I}$

If saxion field S is light enough it also fluctuates during inflation

During inflation stochastic fluctuations of a light field "compete" with classical evolution caused by its potential
$\rightarrow$ After long enough time the system approaches the Fokker-Planck probability distribution: $P_{\rm eq}(\Phi) \propto \exp\left(-\frac{8\pi^2}{3}\frac{V(\Phi)}{H_I^4}\right)$ **[Starobinsky, Yokoyama '94]**

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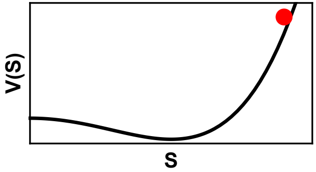
- initial value of saxion field S_i (and θ_i) is determined by stochastic quantum fluctuations (ambiguity of S_i)
- the fields have dispersions, $\langle \delta S_i^2 \rangle$ and $\langle \delta\theta_i^2 \rangle$, generated by the stochastic quantum fluctuations during the last $\sim$ 50 e-folds of inflation



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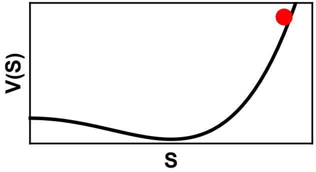

This non-trivial dynamics after inflation may lead to production of a as

- cold dark matter, e.g. kinetic misalignment mechanism e.g. [Co, Harigaya, 19']
- warm dark matter (WDM), e.g. parametric resonance production e.g. [Harigaya et al 15']
    - Bounds on isocurvature perturbations lead to very strong upper bounds on the self-coupling, $\lambda_\Phi \lesssim 10^{-20} \lll 1$ (flat potentials motivated by SUSY)

If $\lambda_\Phi \lll 1$ one should consider corrections

I. **radiative**
II. **geometric (curvature of space-time)**
III. **thermal**

In this talk I will concentrate on (ii)$m_S \ll H_I$, i.e. models in which PQ field has nontrivial dynamics during and after inflation

The goal is to check influence of I, II, III on axion contribution to CDM and WDM

I. Radiative corrections (Coleman-Weinberg (CW) potential):
We adopt Gildener-Weinberg approach

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- **The PQ scalar Φ couples to some scalars ϕ_i and some fermions ψ_j**

$$\mathcal{L} \supset -\sum_i \left(\frac{1}{2} m_i^2 \phi_i^2 + \frac{1}{2} \lambda_i \left|\Phi\right|^2 \phi_i^2 \right) - \sum_j y_j \Phi \overline{\psi}_j \psi_j$$

$$\rightarrow V_{CW}(\Phi) = \frac{1}{64\pi^2} \sum_{\text{scalars}} M_{\phi_i}^4 \left[\ln\left(\frac{M_{\phi_i}^2}{\mu^2} \right) - \frac{3}{2} \right] - \frac{4}{64\pi^2} \sum_{\text{fermions}} M_{\psi_j}^4 \left[\ln\left(\frac{M_{\psi_j}^2}{\mu^2} \right) - \frac{3}{2} \right]$$

$$M_{\phi_i}^2 = m_i^2 + \lambda_i \left|\Phi\right|^2 , \qquad M_{\psi_j}^2 = y_j^2 \left|\Phi\right|^2$$

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Simple model:

$$y_j = y, \qquad \lambda_i = \lambda, \qquad m_i^2 = m^2, \qquad N_{scalars} = 4 N_{fermions}$$

Bosonic contribution must dominate for large values of S

$$y^2 = (1-\delta)\lambda, \qquad 0 \leq \delta \leq 1$$

- quasi-SUSY when $\delta \ll 1$
- SUSY limit: $m \rightarrow 0$ and $\delta \rightarrow 0 \Rightarrow V_{CW} \rightarrow 0$

II. Geometric corrections:

CW potential in curved space-time

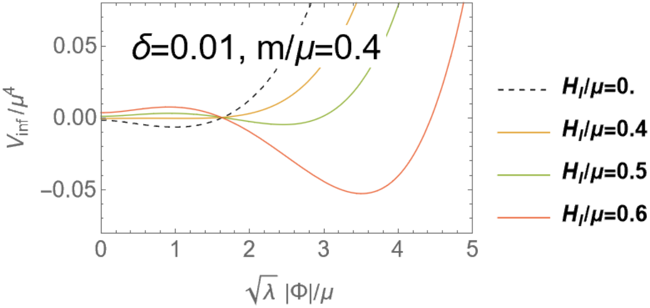
[Markkanen, Nurmi, Rajantie, Stopyra, 18'], [Hardwick, Markkanen, Nurmi, 19']

$$V(\Phi) = \frac{1}{64\pi^2}\sum_{\text{bosons}}\left\{M_{\phi_i}^4\left[\ln\left(\frac{\left|M_{\phi_i}^2\right|}{\mu^2}\right) - \frac{3}{2}\right] + \frac{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}}{90}\ln\left(\frac{\left|M_{\phi_i}^2\right|}{\mu^2}\right)\right\}$$

$$- \frac{4}{64\pi^2}\sum_{\text{fermions}}\left\{M_{\psi_j}^4\left[\ln\left(\frac{\left|M_{\psi_j}^2\right|}{\mu^2}\right) - \frac{3}{2}\right] - \frac{\frac{7}{8}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R_{\mu\nu}R^{\mu\nu}}{90}\ln\left(\frac{\left|M_{\psi_j}^2\right|}{\mu^2}\right)\right\}$$

$$M_{\phi_i}^2 = m^2 + \lambda\left|\Phi\right|^2 + (\xi - \frac{1}{6})R\,, \qquad M_{\psi_j}^2 = y^2\left|\Phi\right|^2 + \frac{1}{12}R$$

	Inflation	Matter Domination	Radiation Domination
R	$12H_I^2$	$3H^2$	0



Quasi-SUSY limit:
Potential during inflation (solid lines) is more complicated (as compared to $R = 0$ case, dashed line) and usually has second deeper minimum for bigger value of S

$$S_i^2 \sim \frac{(3 - 12\xi)H_I^2 - m^2}{\delta\lambda}$$

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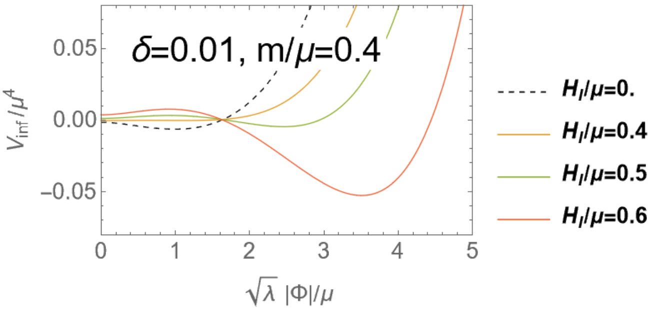

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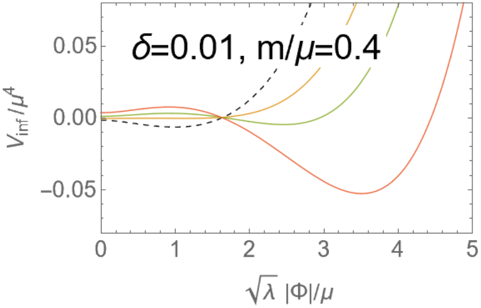
- **Just after inflation fields S and θ are almost homogeneous**
- **For some time PQ field is almost constant due to Hubble friction**
- **When $H \approx m_S^{eff}/3$, saxion field S starts to oscillate, and the energy stored transfers to particles mainly via parametric resonance**
  $\rightarrow$ produced axions contribute to WDM [the picture in $\lambda_\Phi(S^2 - f_a^2)^2$: Shtanov et al, Kofman et al '94]



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Different corrections to the simple potential may play – sometimes very important – role.

III. Thermal corrections:

Our model

- Thermal effects depend on the same fields ϕ_i, ψ_j and couplings λ, y, m as the CW potential does
- We concentrate on two kinds of thermal effects
  - thermal corrections to the potential of the PQ field
  - thermalization of oscillations

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- **Typically oscillations of the saxion field, which start at $H \approx \frac{1}{3} m_S^{eff}$, are due to the thermal mass correction**
  - **Saxion oscillates in potential dominated by thermal mass $\frac{1}{2}\frac{\alpha}{24}T^2S^2$ term**
  - **$\Rightarrow$ particle production via parametric resonance very much delayed (because $\dot{\omega}_k/\omega_k^2 \lesssim 1$)**

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  - $\Rightarrow$ particle production via parametric resonance very much delayed (because $\dot{\omega}_k/\omega_k^2 \lesssim 1$)
- Resonant production delayed at least until temperature drops below $\tilde{T}$ at which thermal mass domination fades away
- How much has the amplitude of saxion oscillations A_S decreased till such time?

- **Production of cold and warm axions depends strongly on $A_S(\tilde{T})/S_{min,0}$**

  rough estimate $\frac{A_S(\widetilde{T})}{S_{min,0}} \approx \mathcal{O}\left(\frac{1}{\sqrt{\epsilon\lambda}} \frac{m}{\mu} \frac{H_I}{10^{18}\,\mathbf{GeV}}\right)$
- **scenario "A": $A_S(\widetilde{T}) \gg S_{min,0}$**
  - **Less warm axions produced (parametric resonance)**
- **scenario "B": $A_S(\widetilde{T}) \sim S_{min,0}$**
  - **More warm axions produced (tachyonic instability)**
- **in both scenarios A and B:**
  - **Cold axions from misalignment**
  - **Saxion oscillations "remember" the initial value θ_i**

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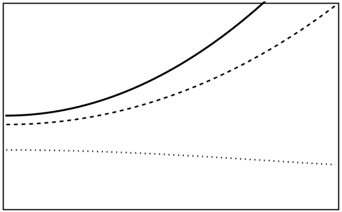
- scenario "C": $A_S(\widetilde{T}) \ll S_{min,0}$:
  - Tachyonic instability important
  - Dynamics may "forget" the initial value θ_i $\to$ if so, relic density of cold axions depends on stochastic processes after inflation $\to$ white-noise isocurvature
  - Tachyonic instability makes S oscillations decay very quickly [Felder, Garcia-Bellido, Greene, Kofman, Linde, Tkachev, 00'], [Felder, Kofman, Linde, 01']
    warm axions may be produced depending on details of the $V_{CW}(\Phi)$ potential

- **C1: no barrier between $S = 0$ and global minimum**
  potential which changes with temperature is very shallow at $T \sim \widetilde{T}$
  - **very few warm axions produced via tachyonic instability**



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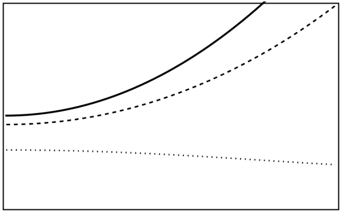
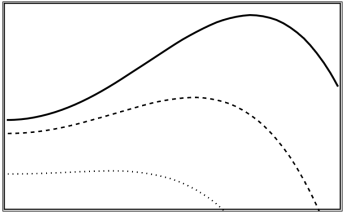


- C2: barrier between $S = 0$ and global minimum (for a range of temperatures)
  global minimum has a non-negligible depth at tachyonic instability
  - a lot of warm axions produced via tachyonic instability



	λ	δ	m/μ	μ [GeV]	H_I [GeV]	n_{CW}	n_{CW+G}	n_{CW+T}	n_{CW+T+G}
P_1	10^{-7}	0.1	0.1	$\mathbf{10^9}$	$\mathbf{10^{11}}$	0.042	0.20	$4.57 \cdot 10^5$	$4.57 \cdot 10^5$
P_2	10^{-7}	0.1	0.1	$\mathbf{10^{10}}$	$\mathbf{10^{13}}$	34	195	$4.33 \cdot 10^5$	$3.18 \cdot 10^5$
P_3	10^{-7}	0.1	0.1	$\mathbf{10^{12}}$	$\mathbf{10^{13}}$	56	223	$4.28 \cdot 10^5$	$3.04 \cdot 10^5$
P_4	10^{-7}	0.1	**0.1**	10^{11}	10^{13}	41.6	206	$4.3 \cdot 10^5$	$3.1 \cdot 10^5$
P_5	10^{-7}	0.1	**0.5**	10^{11}	10^{13}	41.9	207	$5.0 \cdot 10^3$	$2.2 \cdot 10^3$
P_6	10^{-7}	0.1	**0.7**	10^{11}	10^{13}	42.0	207	95	5.9
P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

Number densities n of warm axions are evaluated at a common T and in units T^3

$$n \to n(T)/T^3$$

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I. Radiative corrections: may change the amount of warm axions by factor of a few compared to n_{tree}
(depending on m/μ, δ and H_I/μ)

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P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

II. Geometric corrections: $n_{CW+G} \gtrsim n_{CW}$ **(effect increases with decreasing** δ**)**

- $n_{CW+G}/n_{CW} \sim O(5)$ **for** $\delta = 0.1$
- $n_{CW+G}/n_{CW} \sim O(30)$ **for** $\delta = 0.001$

	λ	δ	m/μ	μ [GeV]	H_I [GeV]	n_{CW}	n_{CW+G}	n_{CW+T}	n_{CW+T+G}
P_1	10^{-7}	0.1	0.1	$\mathbf{10^{9}}$	$\mathbf{10^{11}}$	0.042	0.20	$4.57 \cdot 10^5$	$4.57 \cdot 10^5$
P_2	10^{-7}	0.1	0.1	$\mathbf{10^{10}}$	$\mathbf{10^{13}}$	34	195	$4.33 \cdot 10^5$	$3.18 \cdot 10^5$
P_3	10^{-7}	0.1	0.1	$\mathbf{10^{12}}$	$\mathbf{10^{13}}$	56	223	$4.28 \cdot 10^5$	$3.04 \cdot 10^5$
P_4	10^{-7}	0.1	**0.1**	10^{11}	10^{13}	41.6	206	$4.3 \cdot 10^5$	$3.1 \cdot 10^5$
P_5	10^{-7}	0.1	**0.5**	10^{11}	10^{13}	41.9	207	$5.0 \cdot 10^3$	$2.2 \cdot 10^3$
P_6	10^{-7}	0.1	**0.7**	10^{11}	10^{13}	42.0	207	95	5.9
P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

III. Thermal corrections: Scenario C2(barrier) $n_{CW+T},\ n_{CW+T+G} \gg n_{CW}$

	λ	δ	m/μ	μ [GeV]	H_I [GeV]	n_{CW}	n_{CW+G}	n_{CW+T}	n_{CW+T+G}
P_1	10^{-7}	0.1	0.1	$\mathbf{10^{9}}$	$\mathbf{10^{11}}$	0.042	0.20	$4.57 \cdot 10^5$	$4.57 \cdot 10^5$
P_2	10^{-7}	0.1	0.1	$\mathbf{10^{10}}$	$\mathbf{10^{13}}$	34	195	$4.33 \cdot 10^5$	$3.18 \cdot 10^5$
P_3	10^{-7}	0.1	0.1	$\mathbf{10^{12}}$	$\mathbf{10^{13}}$	56	223	$4.28 \cdot 10^5$	$3.04 \cdot 10^5$
P_4	10^{-7}	0.1	**0.1**	10^{11}	10^{13}	41.6	206	$4.3 \cdot 10^5$	$3.1 \cdot 10^5$
P_5	10^{-7}	0.1	**0.5**	10^{11}	10^{13}	41.9	207	$5.0 \cdot 10^3$	$2.2 \cdot 10^3$
P_6	10^{-7}	0.1	**0.7**	10^{11}	10^{13}	42.0	207	95	5.9
P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

III. Thermal corrections: Scenario C2(barrier) $n_{CW+T},\ n_{CW+T+G} \gg n_{CW}$

- **Strong dependence on** m/μ
- **scenarios** $C2 \to C1$ **(e.g.** P_6**)**

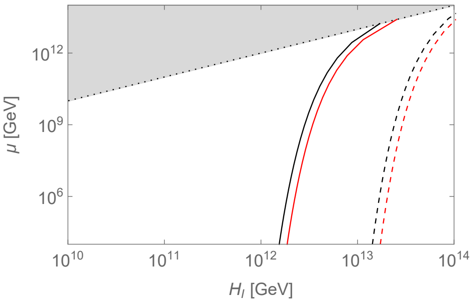
	λ	δ	m/μ	μ [GeV]	H_I [GeV]	n_{CW}	n_{CW+G}	n_{CW+T}	n_{CW+T+G}
P_1	10^{-7}	0.1	0.1	$\mathbf{10^{9}}$	$\mathbf{10^{11}}$	0.042	0.20	$4.57 \cdot 10^5$	$4.57 \cdot 10^5$
P_2	10^{-7}	0.1	0.1	$\mathbf{10^{10}}$	$\mathbf{10^{13}}$	34	195	$4.33 \cdot 10^5$	$3.18 \cdot 10^5$
P_3	10^{-7}	0.1	0.1	$\mathbf{10^{12}}$	$\mathbf{10^{13}}$	56	223	$4.28 \cdot 10^5$	$3.04 \cdot 10^5$
P_4	10^{-7}	0.1	**0.1**	10^{11}	10^{13}	41.6	206	$4.3 \cdot 10^5$	$3.1 \cdot 10^5$
P_5	10^{-7}	0.1	**0.5**	10^{11}	10^{13}	41.9	207	$5.0 \cdot 10^3$	$2.2 \cdot 10^3$
P_6	10^{-7}	0.1	**0.7**	10^{11}	10^{13}	42.0	207	95	5.9
P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

III. Thermal corrections: Scenario C1(no barrier) $n_{CW+T},\ n_{CW+T+G} \approx 0$

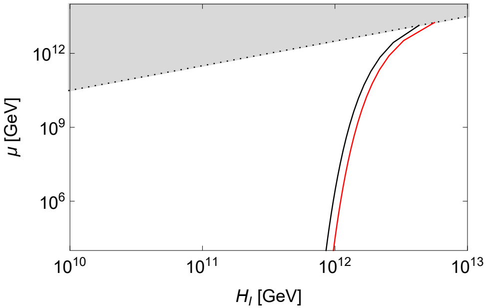
- **Non-trivial dynamics of Peccei-Quinn requires extremely small self-coupling**
- **Crucial role is played by various corrections:**
  - **radiative, geometric, and thermal**
- **During inflation**
  - **saxion potential has (second) minimum at** $S \gg S_{min,0}$
  - $\langle S_i \rangle$, $\langle \theta_i \rangle$, $\langle \delta S_i^2 \rangle$, $\langle \delta\theta_i^2 \rangle$ **determined by quantum fluctuations during inflation**
  - $\langle S_i \rangle$ **close to the position of the minimum at** $S \gg S_{min,0}$
    - **constraints from DM isocurvature relaxed**
    - **very long inflation not needed?**
- **After inflation**
  - **thermal corrections are very important for the evolution of saxion field** S
  - **production of axions can be via parametric resonance or tachyonic instability, and it depends quite strongly on details of a model**
    - **axion contribution to WDM may vary by many orders of magnitude**
    - **isocurvature perturbations of axion CDM may be standard (scale-invariant generated during inflation) or may have the form of white noise**
- **Numerical simulations necessary for precise predictions**

BACK UP

MD($\lambda = 10^{-7}$, $\delta = 0.1$)



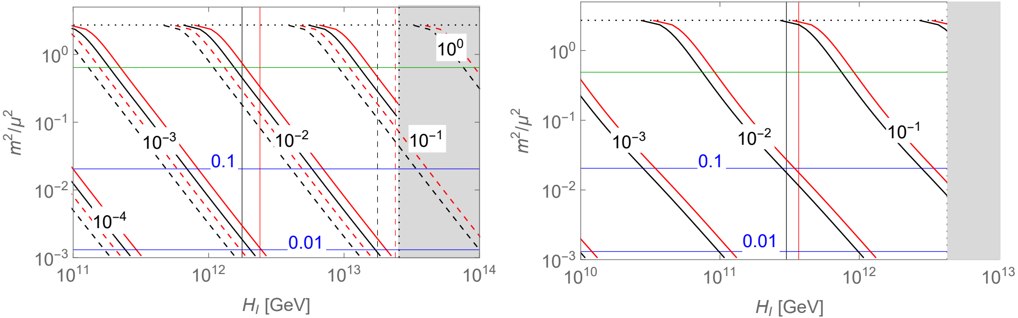
RD($\lambda = 10^{-10}$, $\delta = 10^{-2}$)



For parameters to the left from the black/red solid lines, the S oscillations start due to the thermal mass

Some examples – lines of constant $A_S(\widetilde{T})/S_{\min,0}$

MD$(\lambda = 10^{-7},\ \delta = 0.1)$ **RD**$(\lambda = 10^{-10},\ \delta = 10^{-2})$



$\Rightarrow$ **scenario B typically only when both H_I and m/μ are $\sim$maximal allowed**
$\Rightarrow$ **scenario C typically when either H_I or m/μ is smaller**

	λ	δ	m/μ	μ [GeV]	H_I [GeV]	n_{CW}	n_{CW+G}	n_{CW+T}	n_{CW+T+G}
P_1	10^{-7}	0.1	0.1	$\mathbf{10^9}$	$\mathbf{10^{11}}$	0.042	0.20	$4.57 \cdot 10^5$	$4.57 \cdot 10^5$
P_2	10^{-7}	0.1	0.1	$\mathbf{10^{10}}$	$\mathbf{10^{13}}$	34	195	$4.33 \cdot 10^5$	$3.18 \cdot 10^5$
P_3	10^{-7}	0.1	0.1	$\mathbf{10^{12}}$	$\mathbf{10^{13}}$	56	223	$4.28 \cdot 10^5$	$3.04 \cdot 10^5$
P_4	10^{-7}	0.1	**0.1**	10^{11}	10^{13}	41.6	206	$4.3 \cdot 10^5$	$3.1 \cdot 10^5$
P_5	10^{-7}	0.1	**0.5**	10^{11}	10^{13}	41.9	207	$5.0 \cdot 10^3$	$2.2 \cdot 10^3$
P_6	10^{-7}	0.1	**0.7**	10^{11}	10^{13}	42.0	207	95	5.9
P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

P_8, P_9, P_{11} – **scenario B**, $n_{CW+T}, n_{CW+T+G} \gg n_{CW}$

	λ	δ	m/μ	μ [GeV]	H_I [GeV]	n_{CW}	n_{CW+G}	n_{CW+T}	n_{CW+T+G}
P_1	10^{-7}	0.1	0.1	$\mathbf{10^{9}}$	$\mathbf{10^{11}}$	0.042	0.20	$4.57 \cdot 10^5$	$4.57 \cdot 10^5$
P_2	10^{-7}	0.1	0.1	$\mathbf{10^{10}}$	$\mathbf{10^{13}}$	34	195	$4.33 \cdot 10^5$	$3.18 \cdot 10^5$
P_3	10^{-7}	0.1	0.1	$\mathbf{10^{12}}$	$\mathbf{10^{13}}$	56	223	$4.28 \cdot 10^5$	$3.04 \cdot 10^5$
P_4	10^{-7}	0.1	**0.1**	10^{11}	10^{13}	41.6	206	$4.3 \cdot 10^5$	$3.1 \cdot 10^5$
P_5	10^{-7}	0.1	**0.5**	10^{11}	10^{13}	41.9	207	$5.0 \cdot 10^3$	$2.2 \cdot 10^3$
P_6	10^{-7}	0.1	**0.7**	10^{11}	10^{13}	42.0	207	95	5.9
P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

$\mathrm{P}_8, \mathrm{P}_9, \mathrm{P}_{11}$– **scenario B**, **geometric correction increase** n **:** $n_{CW+T} \lesssim n_{CW+T+G}$

	λ	δ	m/μ	μ [GeV]	H_I [GeV]	n_{CW}	n_{CW+G}	n_{CW+T}	n_{CW+T+G}
P_1	10^{-7}	0.1	0.1	$\mathbf{10^9}$	$\mathbf{10^{11}}$	0.042	0.20	$4.57 \cdot 10^5$	$4.57 \cdot 10^5$
P_2	10^{-7}	0.1	0.1	$\mathbf{10^{10}}$	$\mathbf{10^{13}}$	34	195	$4.33 \cdot 10^5$	$3.18 \cdot 10^5$
P_3	10^{-7}	0.1	0.1	$\mathbf{10^{12}}$	$\mathbf{10^{13}}$	56	223	$4.28 \cdot 10^5$	$3.04 \cdot 10^5$
P_4	10^{-7}	0.1	**0.1**	10^{11}	10^{13}	41.6	206	$4.3 \cdot 10^5$	$3.1 \cdot 10^5$
P_5	10^{-7}	0.1	**0.5**	10^{11}	10^{13}	41.9	207	$5.0 \cdot 10^3$	$2.2 \cdot 10^3$
P_6	10^{-7}	0.1	**0.7**	10^{11}	10^{13}	42.0	207	95	5.9
P_7	10^{-7}	0.1	**0.8**	10^{11}	10^{13}	42.0	207	0	0
P_8	10^{-7}	**0.03**	0.5	10^{11}	10^{13}	84	$6.9 \cdot 10^2$	$1.9 \cdot 10^3$	$3.0 \cdot 10^3$
P_9	10^{-7}	**0.01**	0.5	10^{11}	10^{13}	$1.6 \cdot 10^2$	$2.0 \cdot 10^3$	$1.2 \cdot 10^3$	$8.9 \cdot 10^3$
P_{10}	10^{-6}	**0.01**	0.5	10^{11}	10^{13}	9.0	120	490	47
P_{11}	10^{-6}	**0.001**	0.5	10^{11}	10^{13}	36	$1.1 \cdot 10^3$	370	940
P_{12}	$\mathbf{10^{-7}}$	0.1	0.2	10^{10}	10^{12}	1.3	6.5	$8.4 \cdot 10^4$	$8.3 \cdot 10^4$
P_{13}	$\mathbf{10^{-8}}$	0.1	0.2	10^{10}	10^{12}	23	120	$2.6 \cdot 10^5$	$2.5 \cdot 10^5$
P_{14}	$\mathbf{10^{-9}}$	0.1	0.2	10^{10}	10^{12}	$4.2 \cdot 10^2$	$2.1 \cdot 10^3$	$7.9 \cdot 10^5$	$5.8 \cdot 10^5$

$P_{1\div 6,10,12\div 14}$**– scenario C2**, **geometric correction decreases** n **:** $n_{CW+T} \gtrsim n_{CW+T+G}$

- **thermal corrections neglected – (s)axions produced by parametric resonance:**
  - $n_{CW+G} \gtrsim n_{CW}$

$$n_{CW} \propto \delta^{-5/8}\lambda^{-5/4}H_I^{3/2}T^3, \qquad n_{CW+G} \propto \delta^{-1}\lambda^{-5/4}H_I^{3/2}T^3$$

dependence on m,μ very weak

- **thermal corrections accounted –(s)axions produced by tachyonic instability:**
  - **scenario C more natural than B and especially A**
  - **C**$\rightarrow$ $C1$(**no barrier**), $C2$(**barrier for some** T)
  - **scenario C1 :** $n_{CW+T}, n_{CW+T+G} \approx 0$
  - **scenario C2 :** $n_{CW+T}, n_{CW+T+G} \ggg n_{CW}$

$$n_{CW+T}, n_{CW+T+G} \propto \lambda^{-1/2}\left(\frac{m}{\mu}\right)^{-2} T^3$$

dependence on δ, μ and H_I much weaker, $n_{CW+T+G} \lesssim n_{CW+T}$

- **dedicated numerical computations for** $C1 - C2$ **transition region, scenario** B **...**

scenario D: early thermalization : $T_{\rm th} > \widetilde{T}$

- **interactions with thermal plasma – modification to EOM:**

$$\ddot{S} + (3H + \Gamma_{\rm th})\,\dot{S} + \frac{\partial V_{\rm eff}}{\partial S} = 0$$

$$\Gamma_{\rm th}^{(\phi)} \sim \frac{\lambda^2 S^2}{\alpha_{\rm th} T}\,, \qquad \Gamma_{\rm th}^{(\psi)} \sim y^2 \alpha_{\rm th} T$$

$$\Rightarrow \qquad T_{\rm th} \sim \sqrt{\frac{45}{4\pi^3 g_*}}\,\alpha_{\rm th}\lambda M_{Pl}$$

early thermalization if:

$$\frac{m^2}{\mu^2}\ln\left(\frac{e\mu^2}{m^2}\right) \lesssim \frac{15 n_{\rm eff}\left(\widetilde{T}\right)}{\pi g_* N_s}\,\alpha_{\rm th}^2 \lambda^2 \frac{M_{Pl}^2}{\mu^2}$$

$\alpha_{\rm th} = \alpha_s \sim 0.1 \Rightarrow$ **RHS bigger than 1, if** $\mu \lesssim \lambda \cdot 10^{17}$ **GeV**

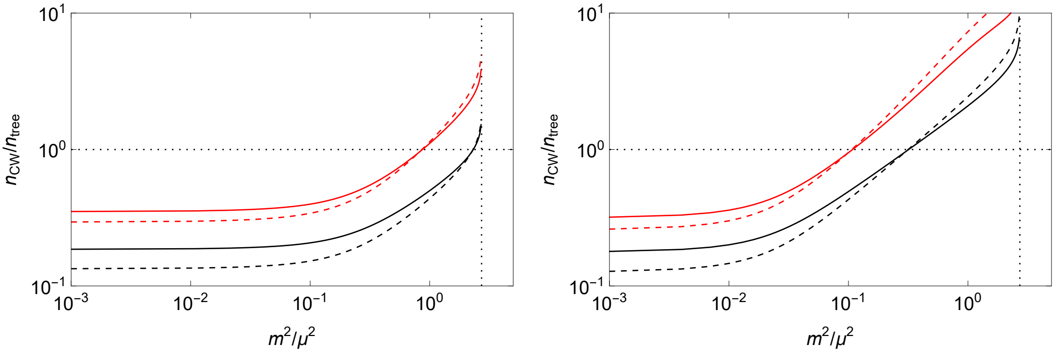


Figure: The ratio of densities of warm axions produced via a parametric resonance with (n_{CW}) and without (n_{tree}) radiative corrections taken into account as a function of m^2/μ^2. The relevant parameters are fixed as: $\delta = 0.1$ (left panel) and $\delta = 0.01$ (right panel); $H_I/\mu = 5$ (red curves) and $H_I/\mu = 10^3$ (black curves). The solid (dashed) lines correspond to situation when the axions are produced after (before) the end of the reheating process.

- We will focus on
  - CW with only geometric corrections ($CW + G$)
  - CW with only thermal corrections ($CW + T$)
  - CW with both geometric and thermal corrections ($CW + T + G$)

  n_{CW}, n_{CW+G}, n_{CW+T} and n_{CW+T+G} are rescaled to a common T (in units T^3)

n_{CW} vs n_{tree}: $n_{CW}/n_{tree} \lessgtr 1$ depending on m/μ, δ and H_I/μ

- Two different approximations:
  - $$n_{CW}, n_{CW+G} \sim \frac{1}{2}\frac{V_{CW}}{m_S}\bigg|_{S=S_i} \qquad \text{(parametric resonance)}$$
  - $$n_{CW+T}, n_{CW+T+G} \sim \frac{1}{2}\frac{\Delta V_{tot}}{m_S}\bigg|_{T\sim\widetilde{T}} \qquad \text{(tachyonic instability)}$$

  $\Delta V_{tot} =$(available potential energy), $m_S =$(mass at the global minimum)

- **Cold axions produced via the conventional misalignment mechanism**
- $n_{a,\ cold}$ **often determined by stochastic processes during inflation...**
  ...despite $V_{full}(\Phi)$ **is in the unbroken phase for some time**

saxion keeps oscillating and carries the initial PQ phase θ_i

- **in the end one should compare**

$$\rho_{a,\, warm} + \rho_{a,\, cold} \leftrightarrow \rho_{DM,\ observed}$$

$\Rightarrow$ **extra flexibility:**
if $\rho_{a,\, warm}$ **too small (or vanishing...), often possible to complement with** $\rho_{a,\, cold}$ **(choice of** θ_i**)**

Number densities (n.d.) of ALP WDM n_i (in units T^3):

1. **radiative corrections: n_{CW} vs n_{tree}: $n_{CW}/n_{tree} \lesssim 1$ (depending on m/μ, δ and H_I/μ)**
   - n_{CW} $\equiv$**(n.d. of warm ALP for "pure" CW potential)**
   - n_{tree} $\equiv$**(n.d. of warm ALP for the *corresponding* Mexican hat potential)**
2. **We will take n_{CW} as the reference and compare it with:**
   - n_{CW+G} $\equiv$**(CW with only geometric corrections)**
   - n_{CW+T} $\equiv$**(CW with only thermal corrections)**
   - n_{CW+T+G} $\equiv$**(CW with both geometric and thermal corrections)**

n_{CW}, n_{CW+G}, n_{CW+T} **and** n_{CW+T+G} **are rescaled to a common T**

- **We use the full thermal potential**

$$V_T(\Phi) = \frac{T^4}{2\pi^2}\left[\sum_{\text{bosons}} J_+\left(\frac{M_{\phi_i}}{T}\right) + 4\sum_{\text{fermions}} J_-\left(\frac{M_{\psi_j}}{T}\right)\right]$$

$$J_\pm(y) = \pm\int_0^\infty x^2 \ln\left[1 \mp \exp\left(-\sqrt{x^2+y^2}\right)\right] \mathrm{d}x$$

Number densities (n.d.) of ALP WDM n_i:

1. **radiative corrections: n_{CW} vs n_{tree}: $n_{CW}/n_{tree} \lesssim 1$ (depending on m/μ, δ and H_I/μ)**
   - $n_{CW} \equiv$**(n.d. of warm ALP for "pure" Gildener-Weinberg potential)**
   - $n_{tree} \equiv$**(n.d. of warm ALP for the *corresponding* Mexican hat potential)**
2. **We will take n_{CW} as the reference and compare it with:**
   - $n_{CW+G} \equiv$**(CW with only geometric corrections)**
   - $n_{CW+T} \equiv$**(CW with only thermal corrections)**
   - $n_{CW+T+G} \equiv$**(CW with both geometric and thermal corrections)**

n_{CW}**,** n_{CW+G}**,** n_{CW+T} **and** n_{CW+T+G} **are rescaled to a common** T **(and in units** T^3**)**