Self-Excited Gravitational Instantons

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Motivation

- In QFT we usually consider Wick rotated $t \to i\tau$ Euclidean path integrals

$$Z = \int_{\mathcal{M}} d[\psi] e^{-\mathcal{S}^E}$$

- Path integral is dominated by a finite Euclidean action solutions: **instantons**
- In Yang-Mills case very well-studied
  - non-trivial vacuum structure (...and more...)
  - link between physics of gauge fields and topology
- Can we do the same in a case of gravity?
  - Yes: (Hawking) Eguchi, Hanson 1970s
  - Can we do it better?

Based on

M. Kršš́ák: *Self-Excited Gravitational Instantons*, ArXiv:2408.01140

Yang-Mills Theory

- Non-abelian gauge theory:
  - **Connection 1-form:** $A = A_\mu dx^\mu$, where $A_\mu = A^a{}_\mu \tau_a$ and τ_a are generators of $SU(N)$ group
  - **Field strength 2-form** $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$

$$F = DA = dA + A \wedge A$$

- Satisfies sourceless field equations

$$\begin{aligned} DF &= 0 \qquad\qquad \text{(Bianchi Identity)} \\ D \star F &= 0 \qquad\qquad \text{(Field Equations)} \end{aligned}$$

- Bianchi identity is automatic, and FE comes from the YM action

$$\mathcal{S}_{\text{YM}} = \int_{\mathcal{M}} \operatorname{Tr} F \wedge \star F$$

BPST Instanton (Belavin-Polyakov-Schwarz-Tyupkin 1975)

- Consider an ansatz with (anti)-self dual field strength

$$F = \pm \star F$$

- Since any F obeys the Bianchi identity $DF = 0$, then self-dual F satisfies $D \star F = 0$ as well
- Self-dual F are **automatic solutions** of YM equations; instead of set of 2nd ODE, we solve only the self-duality condition (1st ODE)
- The YM action reduces to

$$\tilde{\mathcal{S}}_{\mathrm{YM}} = \pm \int_{\mathcal{M}} \mathrm{Tr}\, F \wedge F$$

which is a total derivative

BPST Instanton Action and BPS Bound

- The YM action for the BPST self-dual solutions $F = \pm \star F$ is

$$\boxed{\tilde{\mathcal{S}}_{\mathrm{YM}} = \pm \int_{\mathcal{M}} \mathrm{Tr}\, F \wedge F = \pm \int_{\mathcal{M}} dK = \pm 8\pi^2 k}$$

- Where K is the **Chern-Simons form**

$$K = \frac{1}{8\pi^2} \mathrm{Tr} \left(F \wedge A - \frac{1}{3} A \wedge A \wedge A \right)$$

  and k is not only finite but an integer, known as the **winding number** or the **second Chern number**
- Moreover, this is the absolute (global) minimum of the YM action known as the Bogomol'nyi–Prasad–Sommerfield (BPS) bound

$$\boxed{\mathcal{S}_{YM} \geq \pi^2 |k|}$$

Vacuum Structure in Yang-Mills Theory

- Instantons are the finite Euclidean action solutions of YM field equations
- BPST (anti) self-dual instanton $F = \pm \star F$ has three **"nice properties"**:
  1. **Automatic solution:** simplifies solving field equations
  2. **Topological solution:** makes the action related to a topological invariant
  3. **Absolute minimum:** not just a critical point of the action but the global minimum. Therefore, they are the **dominant contribution** to the Euclidean path integral

$$Z = \int_{\mathcal{M}} d[\psi] e^{-\mathcal{S}^E}$$

- Play important role in
  - Many important insights about the non-trivial vacuum structure,
  - Possibility of tunneling between different vacua 't Hooft 1976
  - Deep connection between physics and topology of gauge fields
  - Making the topological term dynamical–> **axion**

$$\int_{\mathcal{M}} \mathrm{Tr} F \wedge \star F + \theta \mathrm{Tr} F \wedge F$$

Gravitational Instantons

- **Hawking 1970s**: Euclidean path integral approach to quantization of gravity
- Many important results
  - total gravitational action[1] Gibbons, Hawking 1977

$$\mathcal{S}_{\text{grav}} = \int_{\mathcal{M}} \sqrt{-g}\,\overset{\circ}{R} + 2\oint_{\partial\mathcal{M}} \sqrt{-\gamma}(\mathcal{K} - \mathcal{K}_0)$$

  - Derivation of the area law of BH entropy
- Schwarzschild case: total gravitational action $\mathcal{S}^E_{\text{grav}} = 4\pi M^2$ is finite and leads to area law (**Gibbons-Hawking instanton**), but
  - is not self-dual
  - is not a topological invariant
  - is not a global minimum Gibbons, Hawking, Perry 1978 *Indefiniteness of the gravitational action...*
- Can we find self-dual gravitational analogue of BPST? Hawking 1977, Eguchi and Hanson 1978

[1] Riemannian quantities are denoted with $\circ$ above them

General Relativity vs Yang-Mills: Problem

Yang-Mills Theory

- Action

$$\mathcal{S}_{\mathrm{YM}} = \int_{\mathcal{M}} \mathrm{Tr}\, F \wedge \star F$$

- Field Equations

$$DF = 0$$
$$D \star F = 0$$

General Relativity

- Action

$$\mathcal{S}_{\mathrm{GR}} = -\int_{\mathcal{M}} \sqrt{-g}\,\mathring{R}$$

- Bianchi identities

$$\mathring{R}^{\rho}{}_{[\sigma\mu\nu]} = 0 \qquad \mathring{\nabla}_{[\lambda}\mathring{R}_{\rho\sigma]\mu\nu} = 0$$

- Field Equations

$$\mathring{R}_{\mu\nu} = 0$$

Do not look alike!

Cartan Formalism and General Relativity

- To make them look alike follow Cartan
- Introduce
  - **tetrad 1-form** $h^a = h^a{}_\mu dx^\mu$ related to the (Euclidean) metric $g_{\mu\nu} = \delta_{ab} h^a{}_\mu h^b{}_\nu$
  - **connection 1-form** $\mathring{\omega}^a{}_b = \mathring{\omega}^a{}_{b\mu} dx^\mu$
- Defining
  - **Curvature 2-form** $\mathring{\mathcal{R}}^a{}_b = \frac{1}{2} \mathring{R}^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$ with $\mathring{R}^\alpha{}_{\beta\mu\nu} = h_a{}^\alpha h^b{}_\beta \mathring{R}^a{}_{b\mu\nu}$ being components of Riemann curvature
  - **Torsion 2-form** $\mathring{\mathcal{T}}^a = \frac{1}{2} \mathring{T}^a{}_{\mu\nu} dx^\mu \wedge dx^\nu$
- Cartan structure equations

$$
\begin{aligned}
0 &= dh^a + \mathring{\omega}^a{}_b \wedge h^b, && \text{(zero torsion)} \\
\mathring{\mathcal{R}}^a{}_b &= d\mathring{\omega}^a{}_b + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b && \text{(non-zero curvature)}
\end{aligned}
$$

define Riemannian geometry

General Relativity

- (Einstein-)Hilbert action[2]

$$\mathcal{S}_{\text{EH}} = -\int_{\mathcal{M}} h \overset{\circ}{R} d^4x = -\int_{\mathcal{M}} \overset{\circ}{\mathcal{R}}_{ab} \wedge \star(h^a \wedge h^b)$$

- Introduce the "tangent" dual as (distinct from the Hodge one!)

$$\star \overset{\circ}{\mathcal{R}}_{ab} = \frac{1}{2}\epsilon_{abcd}\overset{\circ}{\mathcal{R}}_{cd}$$

- Then GR equations written through forms as

$$\overset{\circ}{\mathcal{R}}{}^a{}_b \wedge h^b = 0 \quad \text{(First Bianchi identity)}$$
$$\star\overset{\circ}{\mathcal{R}}{}^a{}_b \wedge h^b = 0 \quad \text{(Einstein field equations)}$$

[2] In units $16\pi G/c^4 = 1$

Gravitational Self-Dual Instanton

- Eguchi-Hanson self-dual solution (1978)
- Solutions with (anti) self-dual curvature

$$\overset{\circ}{\mathcal{R}}_{ab} = \pm \star \overset{\circ}{\mathcal{R}}_{ab}$$

- Lead to (anti) self-dual connection

$$\overset{\circ}{\omega}_{ab} = \pm \star \overset{\circ}{\omega}_{ab}$$

- Automatically solve the Einstein field equations
- Eguchi-Hanson instanton

$$ds^2 = f^{-1} dr^2 + \frac{r^2}{4}\left[f(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right], \qquad f = 1 - \frac{a^4}{r^4}$$

Gravitational and YM Instantons

	YM Theory	General Relativity
Basic variables	A	h^a
Field strength	$F = DA$	$\mathring{\mathcal{R}}^a{}_b = d\mathring{\omega}^a{}_b + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b$
Action	$\int \mathrm{Tr} F \wedge \star F$	$-\int \mathring{\mathcal{R}}_{ab} \wedge \star(h^a \wedge h^b)$
Bianchi identity	$DF = 0$	$\mathring{\mathcal{R}}^a{}_b \wedge h^b = 0$
Field equations	$D \star F = 0$	$\star\mathring{\mathcal{R}}^a{}_b \wedge h^b = 0$
Self-dual field strength	$F = \pm \star F$	$\mathring{\mathcal{R}}_{ab} = \pm \star \mathring{\mathcal{R}}_{ab}$
Self-dual solution	$F = \pm \star F$	$\mathring{\omega}^a{}_b = \pm \star \mathring{\omega}^a{}_b$
Topological term(s)	$\int \mathrm{Tr} F \wedge F$	$\int \epsilon_{abcd} \mathring{\mathcal{R}}^{ab} \wedge \mathring{\mathcal{R}}^{cd}$ $\int \mathring{\mathcal{R}}^a{}_b \wedge \mathring{\mathcal{R}}^b{}_a$
Topological charges	k	χ, P_1

There ARE some similarities, but also MANY DIFFERENCES!!!

Solution: Do Exactly Opposite (... aka Teleparallel Gravity)

- Instead of Riemannian geometry

$$\begin{aligned} 0 &= dh^a + \overset{\circ}{\omega}{}^a{}_b \wedge h^b, && \text{(zero torsion)} \\ \overset{\circ}{\mathcal{R}}{}^a{}_b &= d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b && \text{(non-zero curvature)} \end{aligned}$$

- Consider the **teleparallel geometry** (exactly opposite)

$$\begin{aligned} 0 &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b && \text{(zero curvature)} \\ T^a &= dh^a + \omega^a{}_b \wedge h^b && \text{(non-zero torsion)} \end{aligned}$$

- **Ricci theorem** relates Riemmanian $\overset{\circ}{\omega}{}^a{}_b$ and teleparallel $\omega^a{}_b$ connections

$$\boxed{\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b}$$

where $T^a = K^a{}_b \wedge h^b$

Teleparallel Equivalent of General Relativity

- Using Ricci theorem $\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b + K^a{}_b$ we can rewrite the Einstein-Hilbert action as

$$\mathcal{S}_{\mathrm{EH}} = \mathcal{S}_{\mathrm{TG}} + \int \partial_\mu \left(\frac{h}{\kappa} T^{\nu\mu}{}_\nu \right)$$

- **Teleparallel action**

$$\mathcal{S}_{\mathrm{TG}} = \int h \left[\frac{1}{4} T^a{}_{\mu\nu} T_a{}^{\mu\nu} + \frac{1}{2} T^a{}_{\mu\nu} T^{\nu\mu}{}_a - T^\mu T_\mu \right] = \int_{\mathcal{M}} h T$$

- yields **teleparallel equivalent of general relativity**, which is
  - **dynamically equivalent theory** (same solutions as GR: same Schwarzschild, Kerr, etc...) but differs from GR by boundary terms
  - my recent suggestions: it is a **fully equivalent** to the full GR action
    - MK: Bulk Action Growth for Holographic Complexity, 2308.04354, PRD
    - MK: Teleparallel Gravity, Covariance and their Geometrical Meaning, 2401.08106
    - MK: Einstein Gravity from Einstein Action: Counterterms and Covariance, 2406.08452

Teleparallel Gravity a la Yang-Mills

- Many interesting aspects, but here crucial that the teleparallel action

$$\mathcal{S}_{\text{TG}} = \int_{\mathcal{M}} hT = \int h\left[\frac{1}{4}T^{a}{}_{\mu\nu}T_{a}{}^{\mu\nu} + \frac{1}{2}T^{a}{}_{\mu\nu}T^{\nu\mu}{}_{a} - T^{\mu}T_{\mu}\right]$$

- Can be written as

$$\boxed{\mathcal{S}_{\text{TG}} = \int_{\mathcal{M}} T^{a} \wedge H_{a}}$$

where we have introduced the **excitation** 2-form with components

$$H^{a}{}_{\rho\sigma} = h\epsilon_{\rho\sigma\alpha\beta}\left(\frac{1}{4}T^{a\alpha\beta} + \frac{1}{2}T^{\alpha a\beta} - h^{a\beta}T^{\alpha}\right),$$

Teleparallel Gravity a la Yang-Mills

Yang-Mills Theory

- Action

$$\mathcal{S}_{\mathrm{YM}} = \int_{\mathcal{M}} \operatorname{Tr} F \wedge \star F$$

- Field Equations

$$DF = 0$$
$$D \star F = 0$$

Teleparallel Gravity

- Action

$$\mathcal{S}_{\mathrm{TG}} = \int_{\mathcal{M}} T^a \wedge H_a$$

- Field Equations

$$DT^a = 0$$
$$DH^a + E^a = 0$$

Self-Excited Solutions in Teleparallel Gravity

- Self-duality is not important for instanton construction ("only" for proving the BPS bound), important is that the action is a exterior product of two forms

$$\mathcal{S}_{\mathrm{TG}} = \int_{\mathcal{M}} T^a \wedge H_a \qquad \Longleftrightarrow \qquad \mathcal{S}_{\mathrm{YM}} = \int_{\mathcal{M}} F^a \wedge \star F_a$$

- Premetric/axiomatic approach Itin, Hehl, Obukhov 2017

$$\begin{aligned} dF &= 0 \\ dH &= 0 \end{aligned}$$

  Maxwell electrodynamics is a special case $H = \star F$
- **(Anti) Self-excited solutions**

$$\boxed{T^a = \pm H^a}$$

Topological Self-Excited Action

- The action for (anti) self-excited solutions $T^a = \pm H^a$ is

$$\tilde{\mathcal{S}}_{\text{TG}} = \pm \int_{\mathcal{M}} T^a \wedge T_a$$

- **Nieh-Yan identity** (for $R = 0$)

$$d(h^a \wedge T^a) = dh^a \wedge T^a + h^a \wedge dT^a = T^a \wedge T_a$$

- (Anti) self-excited action is then

$$\boxed{\tilde{\mathcal{S}}_{\text{TG}} = \pm \int_{\mathcal{M}} T^a \wedge T_a = \pm \int_{\mathcal{M}} d(h^a \wedge T^a) = \pm \oint_{\partial\mathcal{M}} h^a \wedge T^a}$$

Axial Torsion as a Topological Current

- Nieh-Yan topological charge

$$\boxed{\tilde{\mathcal{S}}_{\mathsf{TG}} = \pm \oint_{\partial \mathcal{M}} h^a \wedge T^a = \pm \mathcal{N}}$$

  which is an integer and hence plays a role of the winding number
- In components

$$d(h^a \wedge T^a) = \frac{1}{2}\partial_\mu a^\mu d^4 x$$

- **Axial torsion** plays the role of the Chern-Simons current

$$a^\mu = \epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}$$

Gravitational and YM Instantons

Similarities are much closer now

	YM Theory	General Relativity	Teleparallel Gravity
Basic variables	A	h^a	$h^a, \omega^a{}_b$
Field strength	$F = DA$	$\overset{\circ}{\mathcal{R}}{}^a{}_b = d\overset{\circ}{\omega}{}^a{}_b + \overset{\circ}{\omega}{}^a{}_c \wedge \overset{\circ}{\omega}{}^c{}_b$	$T^a = Dh^a = dh^a + \omega^a{}_b \wedge h^b$
Action	$\int \mathrm{Tr} F \wedge \star F$	$-\int \overset{\circ}{\mathcal{R}}_{ab} \wedge \star(h^a \wedge h^b)$	$\int T^a \wedge H_a$
Bianchi identity	$DF = 0$	$\overset{\circ}{\mathcal{R}}{}^a{}_b \wedge h^b = 0$	$DT^a = 0$
Field equations	$D \star F = 0$	$\star \overset{\circ}{\mathcal{R}}{}^a{}_b \wedge h^b = 0$	$DH^a + E^a = 0$
Self-dual f. strength	$F = \pm \star F$	$\overset{\circ}{\mathcal{R}}_{ab} = \pm \star \overset{\circ}{\mathcal{R}}_{ab}$	$T^a = \pm H^a$
Self-dual solution	$F = \pm \star F$	$\overset{\circ}{\omega}{}^a{}_b = \pm \star \overset{\circ}{\omega}{}^a{}_b$	$T^a = \pm H^a$
Topological term(s)	$\int \mathrm{Tr} F \wedge F$	$\int \epsilon_{abcd} \overset{\circ}{\mathcal{R}}{}^{ab} \wedge \overset{\circ}{\mathcal{R}}{}^{cd}$ $\int \overset{\circ}{\mathcal{R}}{}^a{}_b \wedge \overset{\circ}{\mathcal{R}}{}^b{}_a$	$\int T^a \wedge T_a$
Topological charges	k	χ, P_1	$\mathcal{N}$

Example

- Consider $SU(2)$ Cartan-Maurer forms on S^3

$$\sigma_x = \frac{1}{2}(\sin\psi\, d\theta - \sin\theta\cos\psi\, d\phi), \qquad \sigma_y = \frac{1}{2}(-\cos\psi\, d\theta - \sin\theta\sin\psi\, d\phi), \qquad \sigma_z = \frac{1}{2}(d\psi + \cos\theta\, d\phi),$$

- Ansatz tetrad

$$h^a = (fdr, g\,\sigma_x, g\,\sigma_y, g\,\sigma_z),$$

- (Anti) self-excited solution $T^a = \pm H^a$ is then $f = \pm g'$
- Leads to Nieh-Yan charge

$$\boxed{\mathcal{N} = \oint h^a \wedge T_a = \oint 6g^2\,\sigma_x \wedge \sigma_y \wedge \sigma_z = 12\,\pi^2,}$$

  for all $g \to 1$ as $r \to \infty$
- (Eguchi-Hanson instanton is a solution as well but with $\mathcal{N} = 0$)

Self-Excited Instantons: Overview and Conclusions

- Teleparallel gravity allows us to write gravity action as $\int T^a \wedge H_a$
- Analogously to BPST we can consider self-excited solutions $T^a = \pm H^a$ for which the action reduces to a topological Nieh-Yan term

$$\boxed{\tilde{\mathcal{S}}_{\mathsf{TG}} = \pm \int_{\mathcal{M}} T^a \wedge T_a = \pm \oint_{\partial\mathcal{M}} h^a \wedge T^a = \pm\mathcal{N}}$$

- Axial torsion a^μ and Nieh-Yan charge $\mathcal{N}$ play the roles of Chern-Simons current and winding number in YM theory
- We have 2 out of 3 "nice properties" of BPST instantons
  - **Automatic solutions** (same as in YM and GR) $DT^a = DH^a = 0$ (with $E^a = 0$)
  - Relates the action to a **topological term** (same as in the BPST case)
  - But **no BPS bound** since $H_{\rho\mu\nu}H^{\rho\mu\nu} \neq T_{\rho\mu\nu}T^{\rho\mu\nu}$ (compatible with indefiniteness of the gravitational action Gibbons, Hawking, Perry 1978)

Self-Excited Instantons: Applications and Conclusions

- Teleparallel geometry seems to be better-suited for understanding instanton structure of gravity than Riemannian geometry
- Allows us to explore topological structure of the gravitational action
- Hints of non-trivial vacuum structure of gravity
- Better understanding of the bound on gravitational action
- Possibly gives us another try for Euclidean quantum gravity
- Nieh-Yan “axion-like” modifications of gravity Mielke, Li, ...

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MARIE CURIE ACTIONS