Eötvös Loránd University
Doctoral School of Physics

Volume Concepts

in the Thermodynamics of Black Holes

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October 2, 2024

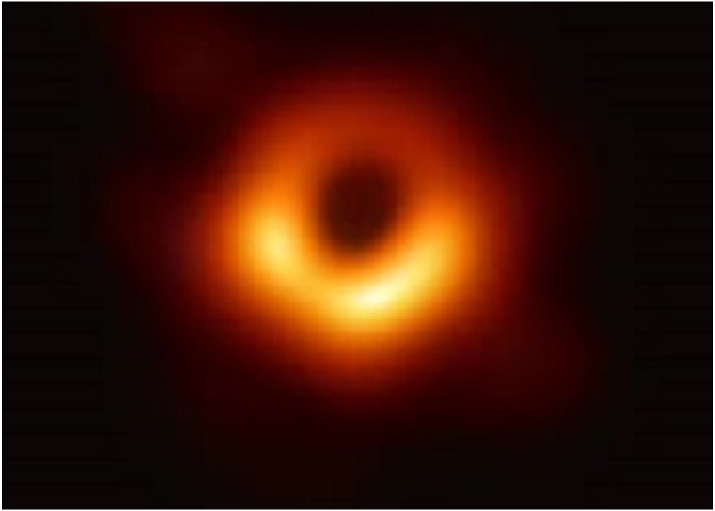


Image from Event Horizon Telescope project

Extensivity

$$S(aU, aV, aN) = aS(U, V, N)$$

Generalized extensivity - Quevedo et al., 2017 [1]:

$$\Psi(\lambda^{\beta_1} E^1, ..., \lambda^{\beta_n} E^n) = \lambda^{\beta_\psi} \Psi(E^1, ..., E^n)$$

Stability criteria

$$\frac{\partial T(U, V)}{\partial U} = \frac{1}{C_V} > 0 \qquad \frac{\partial p(T, V)}{\partial V} \leq 0$$

$$\frac{1}{C_P} = \frac{\partial T(U, p)}{\partial U}\Big|_p > 0$$

Categorizing black holes - No hair theorem

	Uncharged	**Charged (Q)**
Not rotating	Schwarzschild	Reissner–Nordström
Rotating (J)	Kerr	Kerr–Newman

Can all be embedded in Anti-de Sitter spacetime

Volume concepts
Where can we go?

Image from KindPNG

Negative heat capacity - Biró et al., 2018 [2]

$$r_s = 2M = 2U$$

$$S(U) = \pi r_S^2 = 4\pi U^2$$

$$\frac{\partial^2 S(U)}{\partial U^2} = -\frac{1}{C_V T^2} = 8\pi$$

therefore $C_V < 0$?

Volume concepts
Where can we go?

Image from KindPNG

3 dimension sphere volume - Spallucci & Smailagic, 2013 [3]

$$\Lambda = -8\pi P \implies V = \frac{4}{3}\pi r_S^3$$

$$S = \pi R^2$$

$$H(S, P, J, Q) = M = \frac{1}{2}\sqrt{\frac{\left(S + \pi Q^2 + \frac{8PS^2}{3}\right)^2 + 4\pi^2\left(1 + \frac{8PS}{3}\right)J^2}{\pi S}}$$

$$c_P = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_{P,J,Q}}$$

Volume concepts

In the steps of Dolan [4]

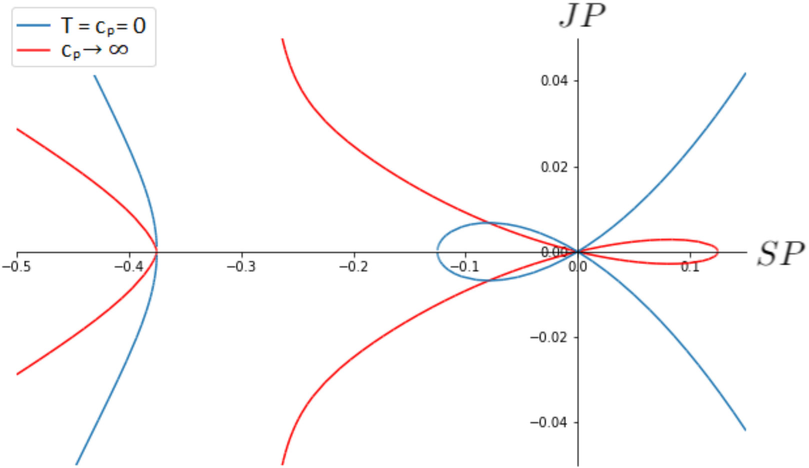


Figure: The full JP-SP phase space (Q = 0)

Volume concepts
In the steps of Dolan [4]

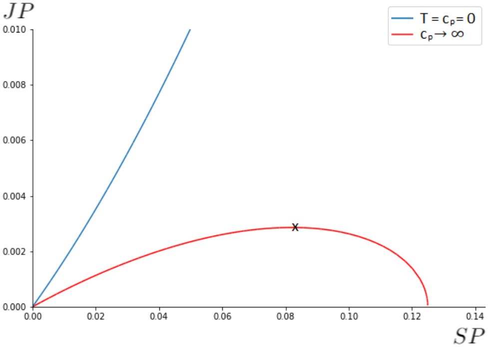


Figure: First plane quarter of the JP-SP phase space with the critical point.

$$(SP)_{crit} \approx 0.08204 \qquad \text{and} \qquad (JP)_{crit} \approx 0.002857$$

Image from KindPNG

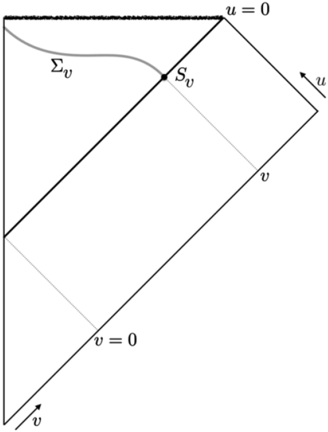


Figure: Conformal or Penrose diagram of a collapsing object spacetime

Christodoulou & Rovelli, 2015 [5]

Christodoulou–Rovelli volume [5]

$$\Lambda = -\frac{6PS}{\pi} \quad (Previously : \Lambda = -8\pi P) \quad \Longrightarrow \quad V = r_S^5$$

$$S = \pi R^2$$

Christodoulou–Rovelli volume [5]

$$\Lambda = -\frac{6PS}{\pi} \quad (Previously : \Lambda = -8\pi P) \quad \Longrightarrow \quad V = r_S^5$$

$$S = \pi R^2$$

$$H(S, P, J, Q = 0) = M = \frac{1}{2\sqrt{\pi S}} \sqrt{\left(S + \frac{2PS^3}{\pi^2}\right)^2 + 4\pi^2 J^2 \left(1 + \frac{2PS^2}{\pi^2}\right)}$$

$$c_P = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_{P,J}}$$

Phase space

With the thermodynamic volume

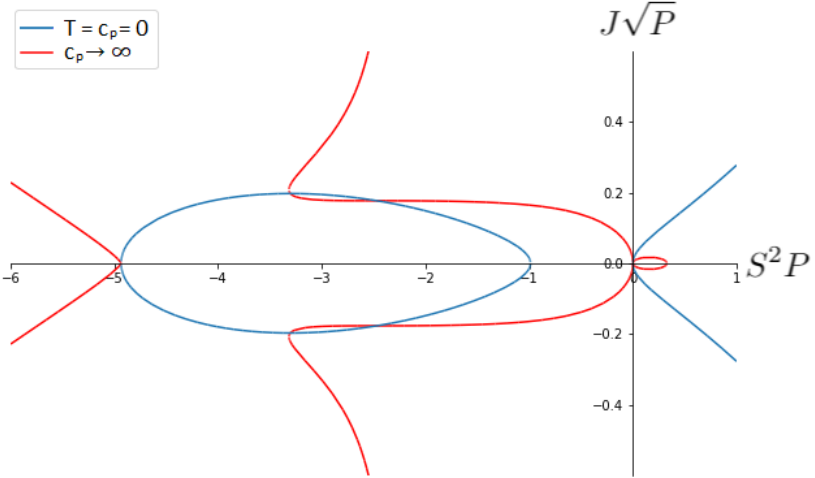
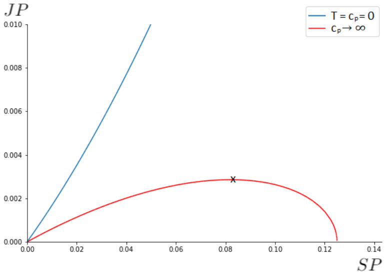
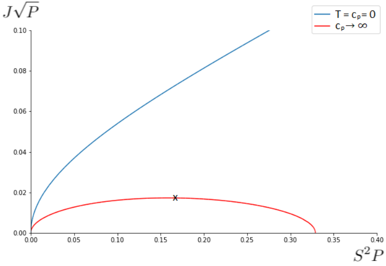


Figure: The full $J - S - P$ phase space in reduced coordinates (Q=0).





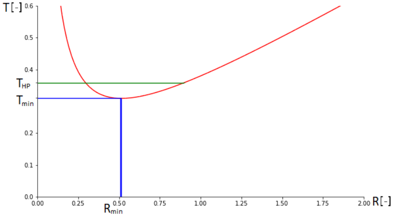
$(SP)_{crit} \approx 0.08204$ and $(JP)_{crit} \approx 0.002857$

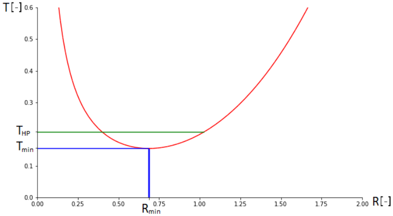
$(S^2P)_{crit} \approx 0.1584$ and $(J\sqrt{P})_{crit} \approx 0.01729$

Critical point with using geometric and thermodynamic volumes.

Hawking-Page phase boundaries

Hawking & Page, 1983 [6]





Radial dependence of temperature in dimensionless coordinates, left: geometric volume, right: thermodynamic volume. Hawking–Page temperature marked in green.

Summary

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[3] E. Spallucci and A. Smailagic.
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Communications in Mathematical Physics, 87(4):577–588, 1983.

Entropy

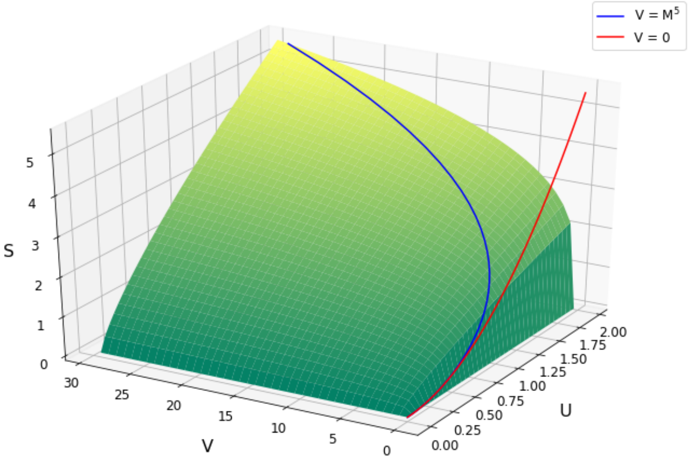

Figure: Concavity of entropy - Biró et al., 2018 [2]

Equilibrium of pressures

$$p_{id} = \frac{nR_u T}{V} \qquad p_{rad} = \frac{\rho_r}{3} = \frac{\sigma}{3} T^4$$

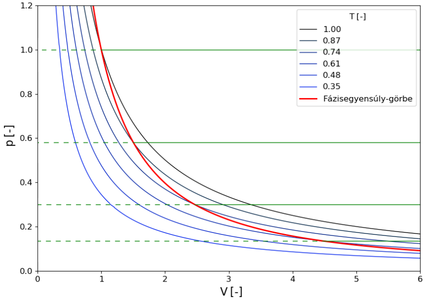

Figure: Phase equilibrium of radiation field and ideal gas in dimensionless coordinates.

Different metrics

Schwarzschild

$$ds^2 = c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \frac{1}{1 - \frac{2GM}{rc^2}} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

Reissner–Nordström

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} - r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

Different metrics

AdS–Kerr

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{1-\frac{a^2}{L^2}}d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{r^2+a^2}{1-\frac{a^2}{L^2}}d\phi\right)^2,$$

where

$$\Delta = \frac{(r^2+a^2)(L^2+r^2)}{L^2} - 2mr, \qquad \Delta_\theta = 1 - \frac{a^2}{L^2}\cos^2\theta,$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \text{és} \qquad \Lambda = -\frac{3}{L^2} = -8\pi P.$$

Different metrics

Kerr–Newman

$$ds^2 = -\frac{r^2 - 2Mr + Q^2 + a^2(1 - \sin^2\theta)}{r^2 + a^2\cos^2\theta} dv^2 + 2dvdr + r^2 + a^2\cos^2\theta d\theta^2$$

$$+ \frac{A \sin^2\theta}{(r^2 + a^2\cos^2\theta)} d\phi^2 - 2a \sin^2\theta \; drd\phi - 2\frac{2Mr - Q^2}{r^2 + a^2\cos^2\theta} a \sin^2\theta \; dvd\phi,$$

where

$$A = (r^2 + a^2)^2 - (r^2 + a^2 - 2Mr + Q^2)a^2\sin^2\theta$$