DPDs from lattice QCD

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Workshop on Double Parton Scattering and the 3D structure of hadrons

Centre Paul Langevin, Aussois

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Introduction

Double Parton Distributions (DPDs):

- Crucial piece of information for description of **Double Parton Scattering (DPS)**:

$$\sigma \sim \sigma_1 \sigma_2 \int \mathrm{d}^2\boldsymbol{y}\; F(\boldsymbol{y})\overline{F}(\boldsymbol{y}) \,,$$

  with parton level cross sections σ_i and DPDs F, $\overline{F}$.
- Non-perturbative objects: largely unknown.
- Several quark model studies: *[Chang et al. '13, Rinaldi et al. '13-'22; Broniowski et al '14-'20; Kasemets, Mukherjee '16; Courtoy et al. '19]*.
- Lack of knowledge: One often assumes (naive) factorizations in terms of GPDs or PDFs.
- Calculation from first principle ⇒ **Lattice QCD**.
- **This talk: quark-quark-DPDs for proton** (results for the pion are also available).

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Double Parton Distributions

General definition of collinear quark-quark DPDs *[Diehl et al. '12, Manohar et al. '12]*:

$$F_{jj',\ell\ell'}(x_1, x_2, y) := 2p^+ \int \mathrm{d}y^- \left[\prod_{j=1,2} \int \frac{\mathrm{d}z_j^-}{2\pi} e^{ix_j p^+ z_j^-} \right] \times$$
$$\langle p| \, \overline{\psi}_j^{\mathcal{W},q_1}(y - \tfrac{z_1}{2}) \, \psi_{j'}^{\mathcal{W},q_2}(y + \tfrac{z_2}{2}) \, \overline{\psi}_\ell^{\mathcal{W},q_3}(-\tfrac{z_1}{2}) \, \psi_{\ell'}^{\mathcal{W},q_4}(\tfrac{z_2}{2}) \, |p\rangle \Big|_{\boldsymbol{z}=\boldsymbol{0}, z_i^+=y^+=0}$$

with

$$\psi_j^{\mathcal{W},q}(x) = [\mathcal{W}(x,v)]_{jj'} \psi_{j'}^q(x)$$

- $\psi^q(x)$ quark field for flavor q.
- $\mathcal{W}(x,v)$ Wilson line in direction v.
- Light-cone coordinates: $k^\pm = (k^0 \pm k^3)/\sqrt{2}$, $\boldsymbol{k} = (k^1, k^2)$.
- Frame: Hadron momentum $p^+ \sim Q \gg \Lambda_{\mathrm{QCD}}$, $\boldsymbol{p} = \boldsymbol{0}$.

Double Parton Distributions

Color-singlet quark-quark DPDs:

$$^{1}F_{a(q)a'(q')}(x_1, x_2, \boldsymbol{y}) := 2p^+ \int \mathrm{d}y^- \left[\prod_{j=1,2} \int \frac{\mathrm{d}z_j^-}{2\pi} e^{i x_j p^+ z_j^-} \right] \times$$

$$\langle p | \, \mathcal{O}_{a(q)}(\boldsymbol{y}, z_1^-) \; \mathcal{O}_{a'(q')}(0, z_2^-) \, | p \rangle \Big|_{y^+=0}$$

with **light cone operators** (twist-2):

$$\mathcal{O}_{a(q)}(y, z) := \overline{\psi}^{\mathcal{W},q}(y - \tfrac{z}{2}) \, \Gamma_a \, \psi^{\mathcal{W},q}(y + \tfrac{z}{2}) \Big|_{\boldsymbol{z}=\boldsymbol{0}, z^+=0}$$

- a: quark polarization selected by $\Gamma_a \in \{\gamma^+, \gamma^+\gamma_5, i\sigma^{j+}\gamma_5\}$
- q: quark flavor

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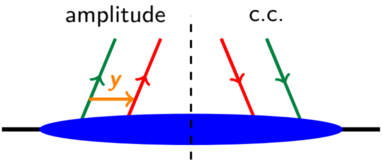
Joint probability to find quark a with momentum $x_1 p^+$ and quark b with momentum $x_2 p^+$ at transverse distance $\boldsymbol{y}$ ($|x_1| + |x_2| \leq 1$)



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Generalization: skewed DPDs (technically relevant for lattice calculations):

$$p^+ \int \mathrm{d}y^- \to p^+ \int \mathrm{d}y^- \, e^{-i\zeta p^+ y^-} \qquad \Rightarrow F(x_1, x_2, \zeta, \boldsymbol{y})$$

Further DPD-like functions:

- Flavor-interference DPDs $F_{a(qq'),a'(q'q)}$ involving flavor-changing operators.
- Fermion interference distributions $I_{a(q_1 q_2)\bar{a}'(q_3 q_4)}$.
- Color octet DPDs ${}^8F_{a(qq'),a'(q'q)}$.

Double Parton Distributions: Factorization

Definition of quark-quark-DPDs for quarks:

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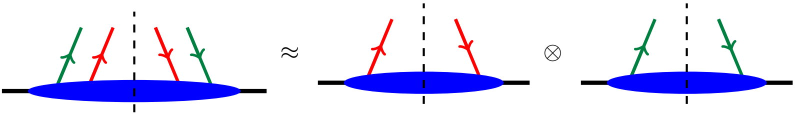
Factorization assumption I

$$\langle p | \mathcal{O}_{a(q)}(y, z_1) \; \mathcal{O}_{a'(q')}(0, z_2) \, | p \rangle \approx \int \frac{\mathrm{d}^2\boldsymbol{p}' \mathrm{d}p'^+}{(2\pi)^3 2p'^+} \langle p | \mathcal{O}_{a(q)}(y, z_1) \, | p' \rangle \, \langle p' | \, \mathcal{O}_{a'(q')}(0, z_2) \, | p \rangle$$

$$\Rightarrow {}^{1}F_{a(q)a'(q')}(x_1, x_2, \boldsymbol{y}) \approx \int \mathrm{d}^2\boldsymbol{b} \; f_{a(q)}(x_1, \boldsymbol{b} + \boldsymbol{y}) \; f_{a'(q')}(x_2, \boldsymbol{b})$$



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Factorization assumption II
Pocket formula *[arXiv:1111.0469]*:

$$
\sigma^{\mathrm{DPS}} \approx \frac{1}{C} \frac{\sigma_1^{\mathrm{SPS}} \; \sigma_2^{\mathrm{SPS}}}{\sigma_{\mathrm{eff}}}
$$

$$
\Rightarrow F_{a(q)a'(q')}(x_1, x_2, \boldsymbol{y}) \approx f_{a(q)}(x_1) f_{a'(q')}(x_2) \, T(\boldsymbol{y})
$$

with **unique** $T(\boldsymbol{y})$

Double parton distributions on the lattice

On the lattice, we have **Euclidean time**; Two approaches:

- **Mellin moments:** Integrate over x_i and calculate corresponding local matrix elements; exploit Lorentz symmetry to deal with the y^--integration ($p^+ \mathrm{d}y^- \to \mathrm{d}(py)$) *[Diehl et al. '12; Bali et al. '18-'21; Reitinger et al. '24]*:
  - \+ Can use local lattice operators (well-known).
  - \+ Renormalization trivial.
  - − Obtain only information about Mellin moments → only very limited knowledge about x-dependence. The number of Mellin moments calculable on the lattice is limited (operator mixing).
  - − Have to deal with inverse problem (py-integral).
  - − Fermion interference and octet distributions not accessible (non-local operators).
- **Quasi-DPDs (LaMET):** Instead of light-like z_i^-, y^- consider space-like z_i^3, y^3 and large hadron momentum p *[Jaarsma et al. '23; Zhang '23]*:
  - \+ Access to the x-dependence.
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  - − Renormalization more complicated.
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DPDs on the lattice: Mellin moments

$$\langle p| \mathcal{O}_a(y, z_1^-)\mathcal{O}_b(0, z_2^-)\,|p\rangle \xrightarrow[y^+=0,\ \text{twist-2}]{p^+ \int \mathrm{d}y^- \,\mathrm{d}z_i^- \, e^{-i z_i x_i p^+}} F_{ab}(x_i, \boldsymbol{y})$$

(*) into basis tensors and scalar functions

DPDs on the lattice: Mellin moments

$F_{ab}(x_i, \boldsymbol{y})$

$p^+ \int \mathrm{d}y^- \, \mathrm{d}z_i^- \, e^{-iz_i x_i p^+}$ ↗ $y^+=0$, twist-2

$\langle p | \, \mathcal{O}_a(y, z_1^-) \mathcal{O}_b(0, z_2^-) \, | p \rangle$

not accessible on the lattice
if $z_i^- > 0$

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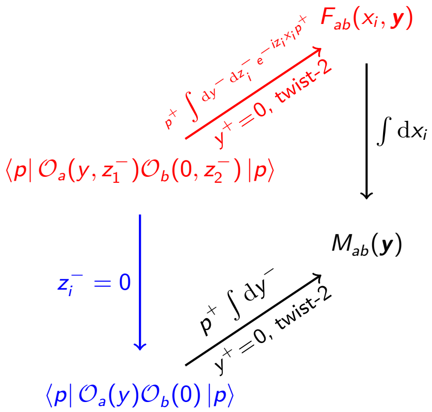

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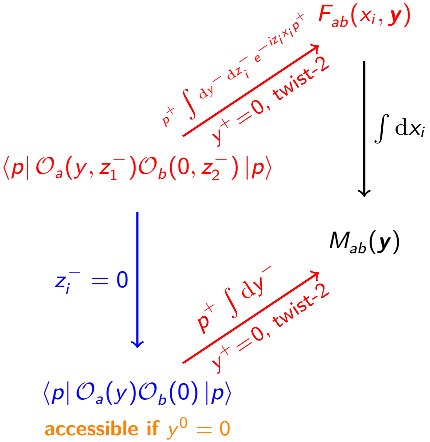

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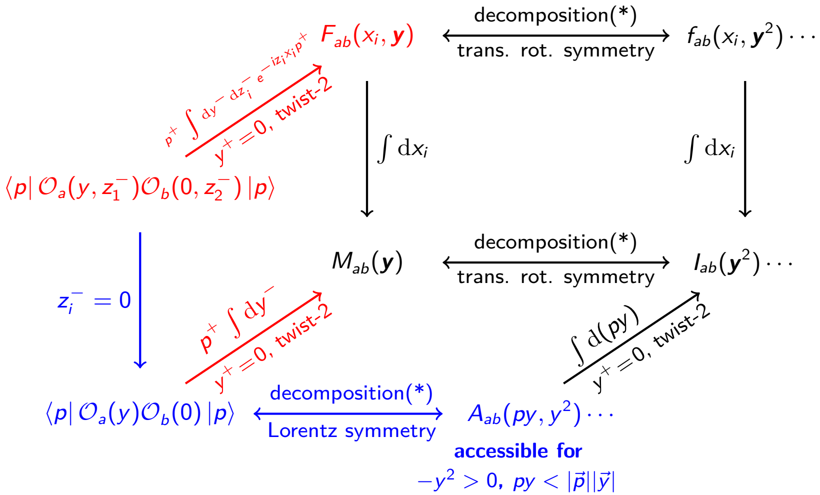

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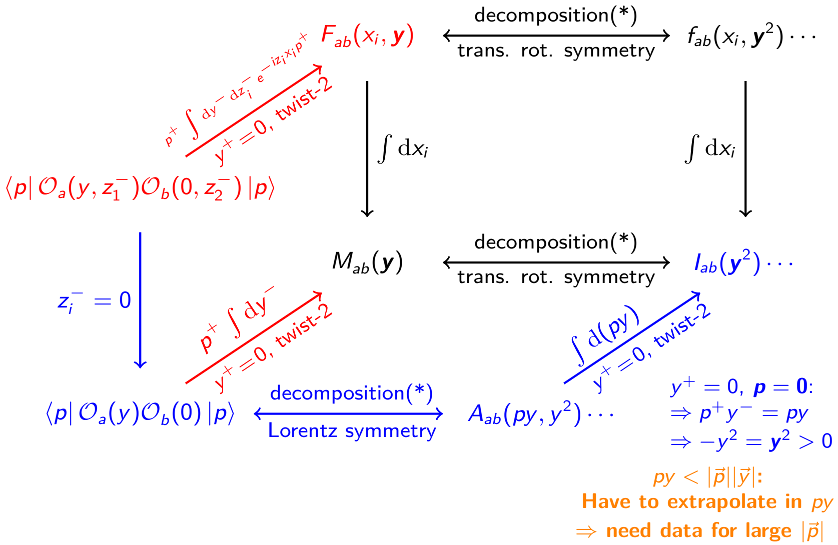

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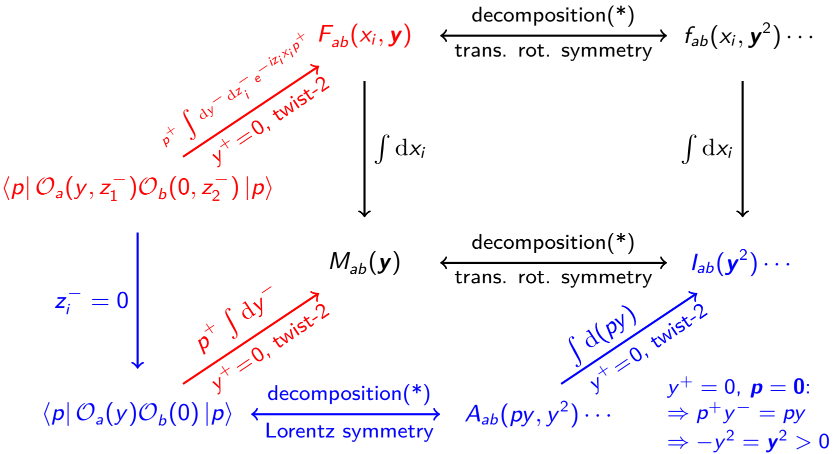

$py < |\vec{p}||\vec{y}|$:

Have to extrapolate in py

$\Rightarrow$ **need data for large $|\vec{p}|$**

INVERSE PROBLEM!

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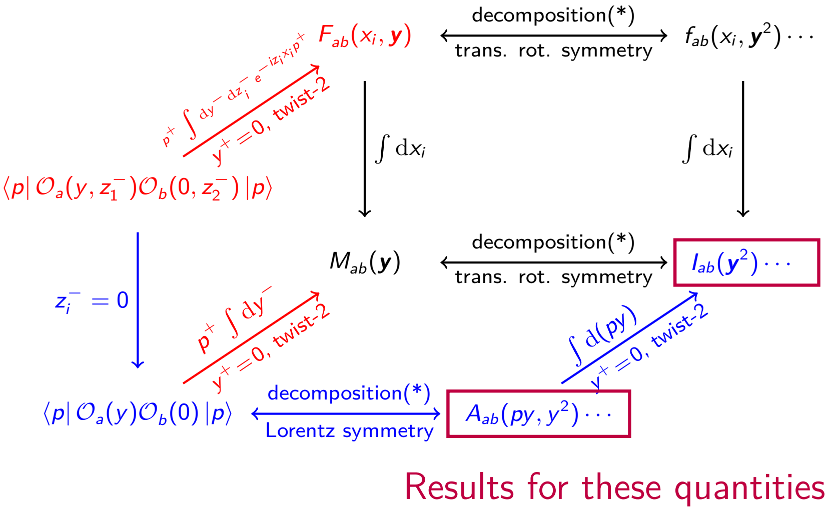

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DPDs on the lattice: Quasi-DPDs

- Generalize the **large momentum effective theory (LaMET)** approach *[Ji '14]* that has been already extensively used for lattice calculations of PDFs, GPDs and TMDs.
- Instead of working with light-like distances z_i^-, y^-, consider separations in 3-direction
- Applying an infinite boost is understood to recover the original quantity
- Have to calculate **perturbative matching coefficients:** Relation between quasi-DPDs at finite boost and renormalized physical DPDs.
- Approach good as long as $p_z^2 \gg M^2$ and $x^2 p_z^2, (1-x)^2 p_z^2 \gg \Lambda_{\mathrm{QCD}}^2$.
- Framework established for color-singlet DPDs *[Jaarsma et al. '23; Zhang '23]* (re-use matching coefficients from quasi-PDF) and in general *[Jaarsma et al. '23]* (matching coefficients for flavor-non-singlet at one-loop order)
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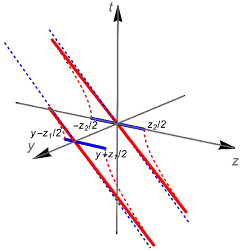

Figure from arXiv:2304.12481

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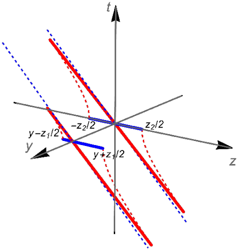

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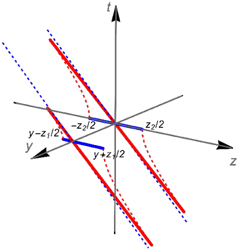

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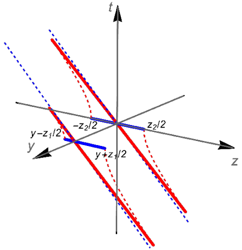

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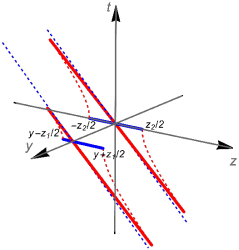

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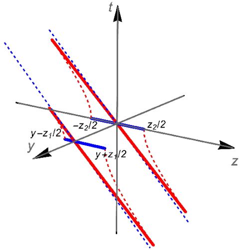

Figure from arXiv:2304.12481

DPDs on the lattice: Quasi-DPDs

- Generalize the **large momentum effective theory (LaMET)** approach *[Ji '14]* that has been already extensively used for lattice calculations of PDFs, GPDs and TMDs.
- Instead of working with light-like distances z_i^-, y^-, consider separations in 3-direction
- Applying an infinite boost is understood to recover the original quantity
- Have to calculate **perturbative matching coefficients:** Relation between quasi-DPDs at finite boost and renormalized physical DPDs.
- Approach good as long as $p_z^2 \gg M^2$ and $x^2 p_z^2, (1-x)^2 p_z^2 \gg \Lambda_{\rm QCD}^2$.
- Framework established for color-singlet DPDs *[Jaarsma et al. '23; Zhang '23]* (re-use matching coefficients from quasi-PDF) and in general *[Jaarsma et al. '23]* (matching coefficients for flavor-non-singlet at one-loop order)
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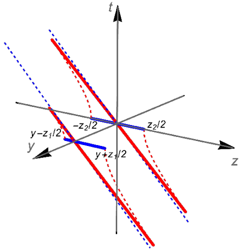


Figure from arXiv:2304.12481

Content

Lattice QCD

$$\langle \mathcal{O}[q, \bar{q}, U] \rangle = \frac{1}{Z} \int \left[\prod_{x \in \Lambda} \mathrm{d}\bar{q}(x) \mathrm{d}q(x) \mathrm{d}U(x) \right] \mathcal{O}[q, \bar{q}, U] e^{-S_F[\bar{q}, q, U] - S_G[U]}$$

$$Z = \int \left[\prod_{x \in \Lambda} \mathrm{d}\bar{q}(x) \mathrm{d}q(x) \mathrm{d}U(x) \right] e^{-S_F[\bar{q}, q, U] - S_G[U]}$$

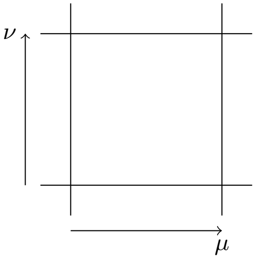
- Reduce spacetime to a lattice
- Finite volume $\Rightarrow$ IR regularization
- Finite lattice spacing $\Rightarrow$ UV regularization
- Evaluate the fermionic part (Grassmann variables) using Wick's theorem $\Rightarrow$ **Wick contractions (graphs)**
- Euclidean spacetime: $e^{iS} \to e^{-S}$ suitable weight for Monte Carlo integration
- Evaluate gauge integral by Monte Carlo integration $\Rightarrow$ gauge ensembles of N configuration, statistical error $\propto N^{-\frac{1}{2}}$:

$$\int \left[\prod_{x} \mathrm{d}U(x) \right] \det\{\mathcal{D}[U]\} e^{-S[U]} \, \mathcal{O}[q, \bar{q}, U] \to \sum_{U \sim P(U)}^{\text{ensemble}} \mathcal{O}[U],$$

Lattice QCD

Reduce spacetime $\mathbb{R}^4$ to finite lattice with spacing a, extensions $L^3 \times T$:

- put fermions on the grid points
- replace derivatives by symmetric differential quotient and integrals by sums
- restore gauge invariance (gauge links $U_\mu(x) \sim e^{iaA_\mu(x)}$)
- add pure gauge part, the plaquette, $\beta = 3g^{-2}$



$$S[q, \bar{q}, U] = \int \mathrm{d}^4x \ \bar{q}(x) \ \mathcal{D} \ q(x)$$

$$\mathcal{D} = i\gamma_\mu \partial^\mu - m\mathbb{1}$$

Lattice QCD

Reduce spacetime $\mathbb{R}^4$ to finite lattice with spacing a, extensions $L^3 \times T$:

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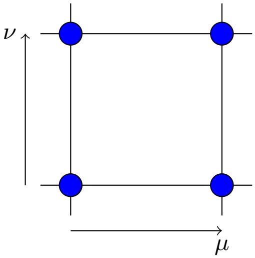

$$S[q, \bar{q}, U] = a^4 \sum_{x,y \in \Lambda_4} \bar{q}(x)\ \mathcal{D}(x|y)\ q(y)$$

$$\mathcal{D}(x|y) = \gamma_\mu \frac{\delta_{x+\hat{\mu},y} - \delta_{x-\hat{\mu},y}}{2a} - m\delta_{x,y}$$

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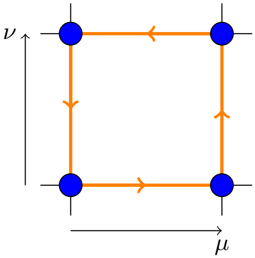

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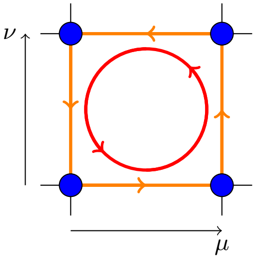

$$S[q, \bar{q}, U] = a^4 \sum_{x,y\in\Lambda_4} \bar{q}(x)\, \mathcal{D}(x|y)\, q(y) + \frac{\beta}{3} \sum_{x\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}\{\mathbb{1} - U_{\mu\nu}(x)\}$$

$$\mathcal{D}(x|y) = \gamma_\mu \frac{U_\mu(x)\, \delta_{x+\hat{\mu},y} - U^\dagger_\mu(x-\hat{\mu})\, \delta_{x-\hat{\mu},y}}{2a} - m\delta_{x,y}$$

Two-current matrix elements on the lattice

On the lattice, we have Euclidean time $t = it_M$:

$$\langle \mathcal{O}(t)\ J(\tau)\ \mathcal{O}^\dagger(0)\rangle = \sum_{i,j} Z_{\mathcal{O},i} Z^*_{\mathcal{O},j} \langle h_i|\ J\ |h_j\rangle\ e^{-(t-\tau)E_i} e^{-\tau E_j} \xrightarrow{t \gg \tau \gg 0} |Z_{\mathcal{O},0}|^2 \langle h_0|\ J\ |h_0\rangle\ e^{-tE_0}$$

$$\langle 0|\ \mathcal{O}\ |h_i\rangle = Z_{\mathcal{O},i}$$

where $|h_i\rangle$ are the states with energy E_i having overlap with the interpolating operators $\mathcal{O}$.

Two-current matrix element for an unpolarized proton:

$$\frac{1}{2}\sum_\lambda \langle p,\lambda|\mathcal{O}_i(y)\mathcal{O}_j(0)|p,\lambda\rangle\ |_{y^0=0} = 2V\sqrt{m^2+\vec{p}^2}\ \frac{C^{\vec{p},ij}_{\mathrm{4pt}}(t,\tau,\vec{y})}{C^{\vec{p}}_{\mathrm{2pt}}(t)}\Bigg|_{0\ll\tau\ll t}$$

with 4-point / 2-point function ($P_+ = \frac{1}{2}(\mathbb{1}+\gamma_4)$):

$$C^{\vec{p},ij}_{\mathrm{4pt}}(t,\tau,\vec{y}) = \sum_{\vec{z}\vec{z}'} e^{-i\vec{p}(\vec{z}'-\vec{z})}\ \left\langle \mathrm{tr}\left\{ P_+ \mathcal{P}^{\vec{p}}(\vec{z}',t)\ \mathcal{O}_i(\vec{0},\tau)\ \mathcal{O}_j(\vec{y},\tau)\ \overline{\mathcal{P}}^{\vec{p}}(\vec{z},0)\right\}\right\rangle$$

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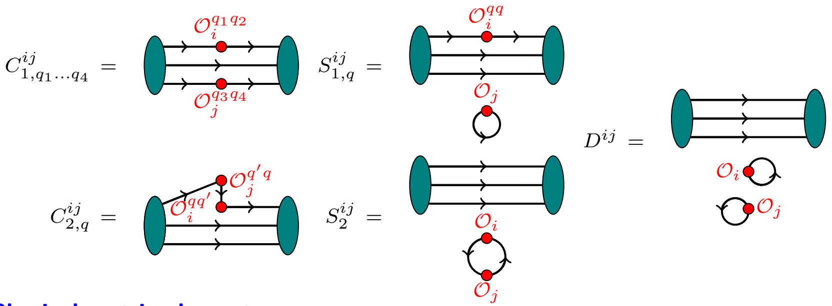
$\overline{\mathcal{P}}^{\vec{p}}(\vec{z}',t)$, $\mathcal{P}^{\vec{p}}(\vec{z}',t)$: interpolating operators for the proton

Two-current matrix elements on the lattice

Wick contractions



Physical matrix elements

$$\langle p|\,\mathcal{O}_i^{uu}(\vec{y})\mathcal{O}_j^{dd}(\vec{0})\,|p\rangle = C^{ij,\vec{p}}_{1,uudd}(\vec{y}) + S^{ij,\vec{p}}_{1,u}(\vec{y}) + S^{ji,\vec{p}}_{1,d}(-\vec{y}) + D^{ij,\vec{p}}(\vec{y})$$

$$\begin{aligned}\langle p|\,\mathcal{O}_i^{uu}(\vec{y})\mathcal{O}_j^{uu}(\vec{0})\,|p\rangle &= C^{ij,\vec{p}}_{1,uuuu}(\vec{y}) + C^{ij,\vec{p}}_{2,u}(\vec{y}) + C^{ji,\vec{p}}_{2,u}(-\vec{y}) + S^{ij,\vec{p}}_{1,u}(\vec{y}) + S^{ji,\vec{p}}_{1,u}(-\vec{y}) \\ &\quad + S^{ij,\vec{p}}_{2}(\vec{y}) + D^{ij,\vec{p}}(\vec{y})\end{aligned}$$

$$\begin{aligned}\langle p|\,\mathcal{O}_i^{dd}(\vec{y})\mathcal{O}_j^{dd}(\vec{0})\,|p\rangle &= C^{ij,\vec{p}}_{2,d}(\vec{y}) + C^{ji,\vec{p}}_{2,d}(-\vec{y}) + S^{ij,\vec{p}}_{1,d}(\vec{y}) + S^{ji,\vec{p}}_{1,d}(-\vec{y}) + S^{ij,\vec{p}}_{2}(\vec{y}) + D^{ij,\vec{p}}(\vec{y}) \\ &= C^{ij,\vec{p}}_{1,duud}(\vec{y}) + C^{ij,\vec{p}}_{2,d}(\vec{y}) + C^{ji,\vec{p}}_{2,u}(-\vec{y}) + S^{ij,\vec{p}}_{2}(\vec{y})\end{aligned}$$

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Content

Lattice studies of Mellin moments

CLS ensembles ($n_f = 2+1$, Wilson fermions, order-a improved *[arXiv:1411.3982]*):

id	β	$a[\mathrm{fm}]$	$L^3 \times T$	$m_\pi[\mathrm{MeV}]$	$m_K[\mathrm{MeV}]$	$m_\pi La$	$La[\mathrm{fm}]$	conf.
H102	3.4	0.0856	$32^3 \times 96$	355	441	4.9	2.7	990
B451	3.46	0.076	$32^3 \times 64$	418	590	5.1	2.4	750
B452	3.46	0.076	$32^3 \times 64$	354	555	4.4	2.4	750
S400	3.46	0.076	$32^3 \times 128$	354	442	4.4	2.4	1000
N203	3.55	0.0644	$48^3 \times 128$	345	441	5.4	3.1	300

- $t = t_{\mathrm{snk}} - t_{\mathrm{src}} = \begin{cases} 12a & \vec{p} = \vec{0} \\ 10a & \vec{p} \neq \vec{0} \end{cases}$, Insertion time: Fit to constant or fix: $\tau = t_{\mathrm{src}} + t/2$
- **Momenta** up to $|\vec{p}| = \sqrt{12}\frac{2\pi}{La}$ ($1.39\ \mathrm{GeV} \lesssim |\vec{p}|_{\max} \lesssim 1.79\ \mathrm{GeV}$)
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- **Renormalization** and **conversion** to $\overline{\mathrm{MS}}$ at $\mu = 2\mathrm{GeV}$ *[arXiv:2012.06284]*

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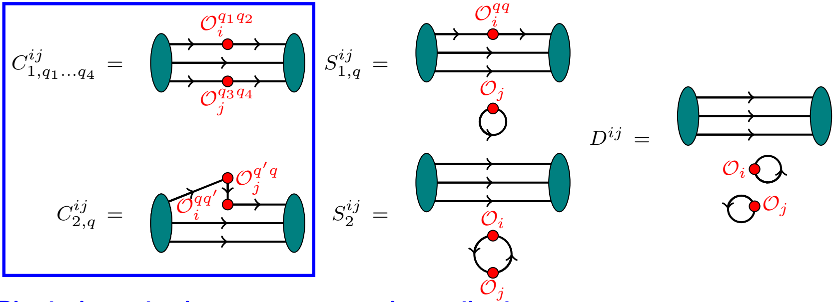
Two-current matrix elements on the lattice

Wick contractions



Physical matrix elements, connected contributions

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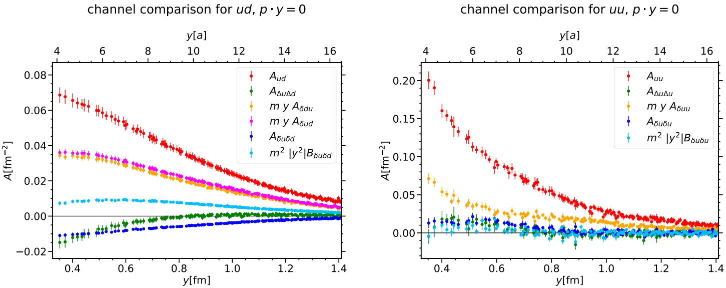
Results: Polarization dependence

Invariant functions $A(py = 0, y^2)$, connected graphs only (notation $y = \sqrt{-y^2}$, $y^2 = y^\mu y_\mu$):

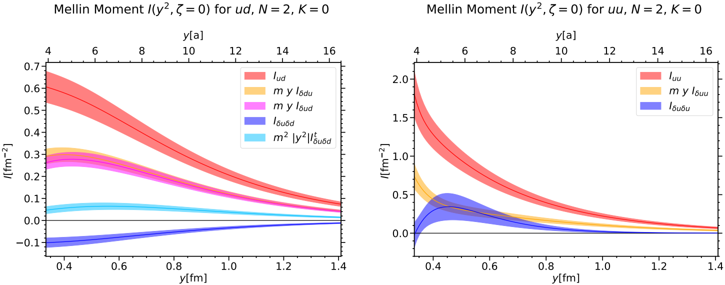


[arXiv:2106.03451]

- Signal of good quality for most channels
- *ud*: Clear contributions from all polarized channels (large for δud, δdu)
- *uu*: Polarization effects suppressed, but visible for δuu
- Moments: Similar conclusions

Results: Polarization dependence

DPD moments $I(\boldsymbol{y}^2)$ (notation $y = |\boldsymbol{y}|$):



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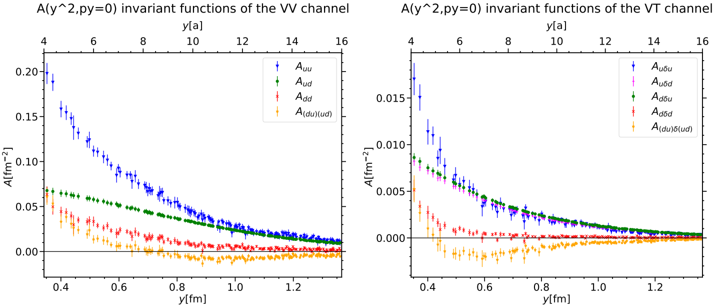
Results: Flavor dependence



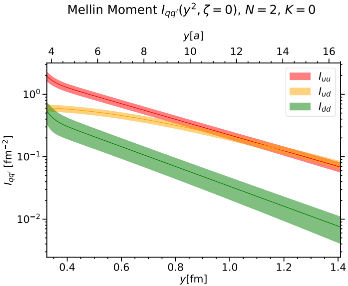
[arXiv:2106.03451,arXiv:2401.14855]

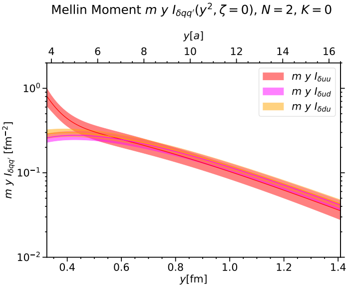
- Clear flavor dependence visible, behavior of *uu* and *dd* different from *du*
- Size of interference effects comparable to *dd*, sign change possible
- Reminder: Assumption for the pocket formula:

$$F_{ab}(x_1, x_2, \boldsymbol{y}) = f_a(x_1) f_b(x_2) T(\boldsymbol{y}) \quad \Rightarrow \quad I_{ab}(\boldsymbol{y}^2) = C_{ab} T(\boldsymbol{y}^2)$$

  with **unique** $T(\boldsymbol{y})$
- **Clearly not fulfilled**

Results: Flavor dependence





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- Size of interference effects comparable to *dd*, sign change possible
- Reminder: Assumption for the pocket formula:

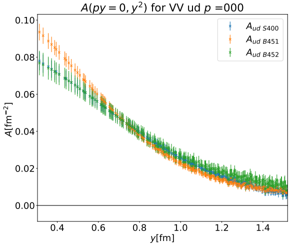
$$F_{ab}(x_1, x_2, \boldsymbol{y}) = f_a(x_1) f_b(x_2) T(\boldsymbol{y}) \quad \Rightarrow \quad I_{ab}(\boldsymbol{y}^2) = C_{ab} T(\boldsymbol{y}^2)$$

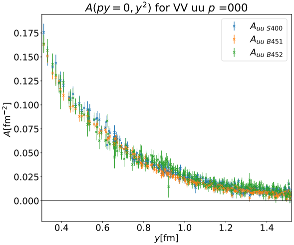
  with **unique** $T(\boldsymbol{y})$
- **Clearly not fulfilled**

Towards the physical point

PRELIMINARY!

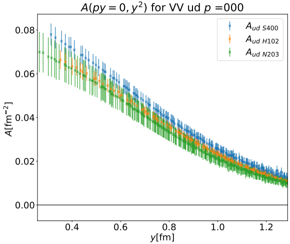
Dependence on the quark mass:

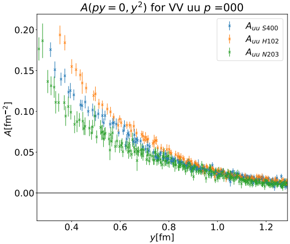




- B451: $m_\pi = 418$ MeV, $m_K = 590$ MeV
- B452: $m_\pi = 354$ MeV, $m_K = 555$ MeV
- S400: $m_\pi = 354$ MeV, $m_K = 442$ MeV

Dependence on the lattice spacing:





- H102: $a = 0.0856$ fm
- S400: $a = 0.076$ fm
- N203: $a = 0.0644$ fm

Comparison with $SU(6)$-model

$SU(6)$-symmetric spin-flavor part of the proton's wave-function :

$$|p^{\uparrow}\rangle = \frac{1}{3\sqrt{2}} \left[|u^{\uparrow}u^{\downarrow}d^{\uparrow}\rangle + |u^{\downarrow}u^{\uparrow}d^{\uparrow}\rangle - 2\,|u^{\uparrow}u^{\uparrow}d^{\downarrow}\rangle + |u^{\uparrow}d^{\uparrow}u^{\downarrow}\rangle + |u^{\downarrow}d^{\uparrow}u^{\uparrow}\rangle - 2\,|u^{\uparrow}d^{\downarrow}u^{\uparrow}\rangle + |d^{\uparrow}u^{\uparrow}u^{\downarrow}\rangle + |d^{\uparrow}u^{\downarrow}u^{\uparrow}\rangle - 2\,|d^{\downarrow}u^{\uparrow}u^{\uparrow}\rangle \right] .$$

$$\mathcal{O}_{(ud)} = \frac{1}{2} \left[(\bar{u}^{\uparrow}\gamma^{+}d^{\uparrow}) + (\bar{u}^{\downarrow}\gamma^{+}d^{\downarrow}) \right] \qquad \mathcal{O}_{\Delta(ud)} = \frac{1}{2} \left[(\bar{u}^{\uparrow}\gamma^{+}d^{\uparrow}) - (\bar{u}^{\downarrow}\gamma^{+}d^{\downarrow}) \right] \qquad f_{aa'} \propto \langle p^{\uparrow}|\, \mathcal{O}_a \mathcal{O}_{a'}\, |p^{\uparrow}\rangle$$

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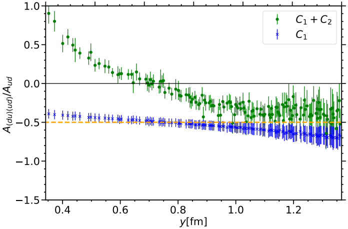
$f_{(du)(ud)}/f_{ud} \overset{!}{=} 1/2$



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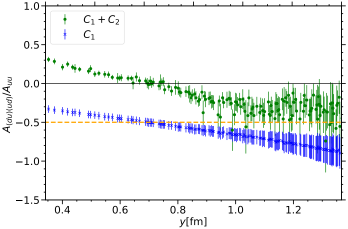

[arXiv:2401.14855]

- C_1 data for unpolarized quarks roughly coincides with $SU(6)$ prediction (orange line)
- Large deviations in particular for small y when considering all leading contractions
- No agreement for polarized channels

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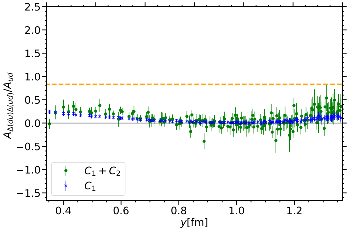
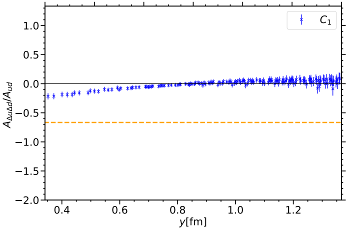
$f_{\Delta(du)\Delta(ud)}/f_{ud} \overset{!}{=} 5/6$



$f_{\Delta u \Delta d}/f_{ud} \overset{!}{=} -2/3$



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Factorization tests: $A(py, y^2)$

Factorization in terms of impact parameter distributions $f_q(x, \boldsymbol{b})$:

$$F_{qq'}(x_1, x_2, \boldsymbol{y}) \approx \int \mathrm{d}^2\boldsymbol{b}\; f_q(x_1, \boldsymbol{b} + \boldsymbol{y})\; f_{q'}(x_2, \boldsymbol{b})$$

Factorization tests: $A(py, y^2)$

On the level of invariant functions $A(py, y^2)$ ($t(\zeta, \boldsymbol{r}^2) = -(\zeta^2 m^2 + \boldsymbol{r}^2)/(1-\zeta)$):

$$A_{qq'}(py = 0, y^2) \approx \frac{1}{2\pi^2} \int \mathrm{d}\zeta \frac{1 - \frac{\zeta}{2}}{1 - \zeta} \int \mathrm{d}r \; r \; J_0(yr) \left[\left(1 - \frac{\zeta^2}{(2-\zeta)^2} \right) F_1^q(t) F_1^{q'}(t) + \cdots \right]$$

$\Rightarrow$ Obtain form factors F_1, F_2 from the lattice *[RQCD '20]*

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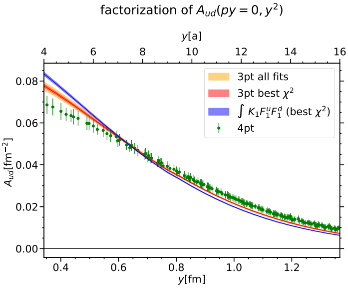
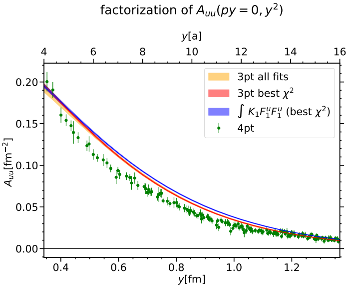
Results for A_{ud} and A_{uu}:





[arXiv:2106.03451]

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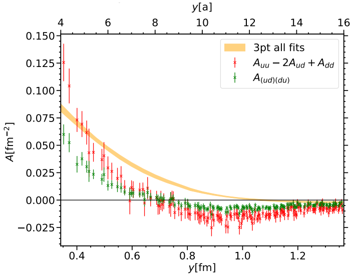
Results for $A_{(ud)(du)}$:



[arXiv:2401.14855]

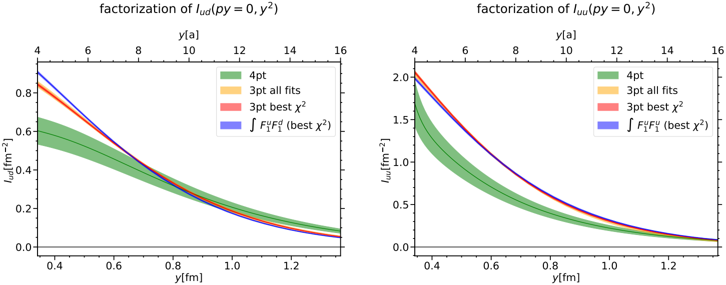
Factorization tests: $I(y^2)$

For the Mellin moments

$$I_{qq'}(\boldsymbol{y}) \approx \int \frac{\mathrm{d}r}{2\pi}\, rJ_0(ry) \left[F_1^q(-\boldsymbol{r}^2) F_1^{q'}(-\boldsymbol{r}^2) + \frac{\boldsymbol{r}^2}{4m^2} F_2^q(-\boldsymbol{r}^2) F_2^{q'}(-\boldsymbol{r}^2) \right]$$

⇒ Obtain form factors F_1, F_2 from the lattice *[RQCD '20]*

Results for I_{ud} and I_{uu}:



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Comparable size but deviations are visible

Content

Summary

- DPDs play a crucial role in the description of DPS, however, they are hard to determine in theory and experiment.
- Ab-initio calculations of DPDs or related quantities can be performed in lattice QCD:
  - Two-current matrix elements of local operators: Mellin moments
  - LaMET approach, quasi-DPDs: also access x-dependence of DPDs
- Current status and results:
  - Calculations have been performed for the first Mellin moment of color-singlet quark-quark DPDs for the pion and the nucleon.
  - Considered different quark-polarization and -flavors including flavor interference.
  - Checked factorization assumptions.
  - Compared with simple $SU(6)$ quark-model.
  - Steps towards the physical point: Dependence on the quark mass and the lattice spacing (preliminary!).
- Outlook:
  - Improve extractions of the Mellin moments (use higher proton momentum, employ better models for fits, use more sophisticated method to tackle inverse problem).
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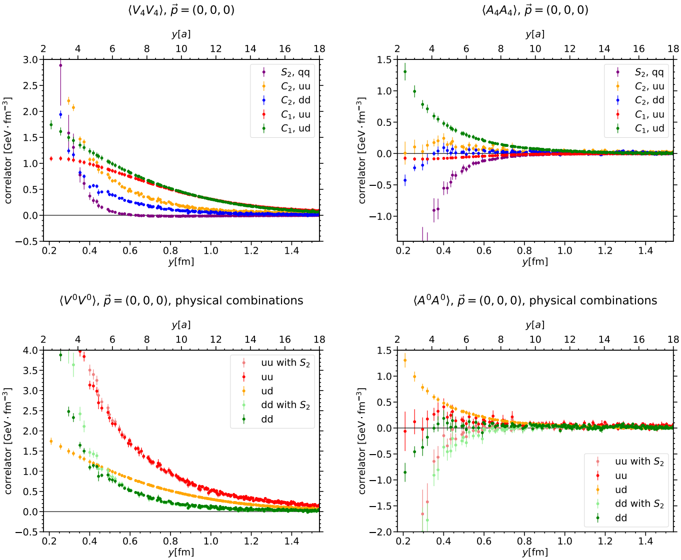
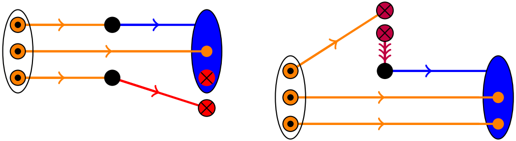
Thank you for your attention!

Backup Slides

Results for Two-current Matrix Elements



Technical Details



point source / propagator

stochastic source / propagator / with HPE

sequential source / propagator with constituents

- APE smearing *[Nucl. Phys. B251 (1985)]*
- Boosted sources (momentum smearing) *[arXiv:1602.05525]*
- Sequential source technique *[Nucl. Phys. B316 (1989)]*
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- Remove trivial terms from stoch. propagators by applying hopping parameter expansion:
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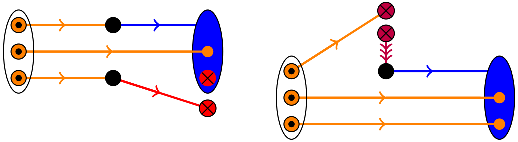

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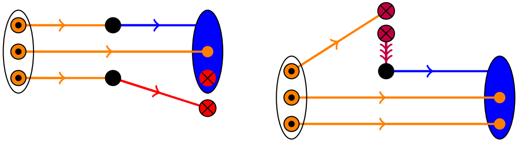

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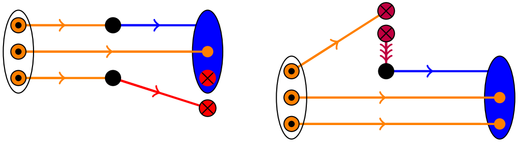

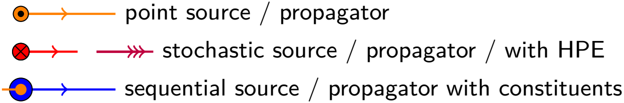

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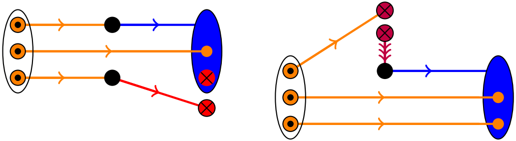

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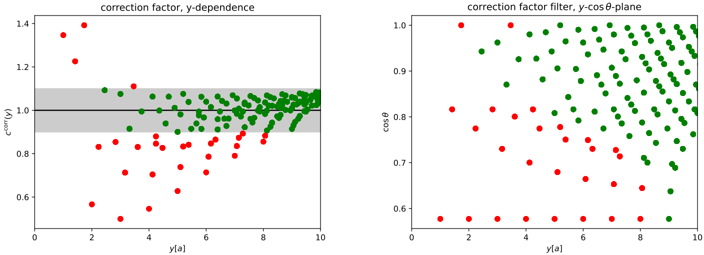
Propagator Anisotropy



Reduce effects caused by lattice propagator anisotropy (following the idea of *[arXiv:1807.06671]*):

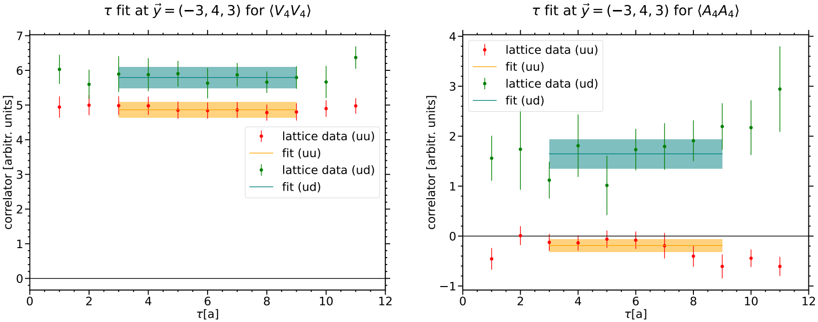
- Remove chiral odd contribution: Take current combination $(VV + AA)/2$
- Consider correction factor for the chiral even part of the lattice propagator:

$$c^{\text{corr}}(y) = \left(\text{tr}\left\{\not{y} M^{\text{free}}(y) \frac{y^2 \pi^2}{2}\right\}\right)^{-1} \frac{-m^2 y^2}{2} K_2\left(m\sqrt{-y^2}\right)$$

- Drop all data points with $|c^{\text{corr}} - 1| > 0.1$.
- Multiply remaining points by c^{corr}.

Time dependence

Looking for hints to excited states:



- Data very flat and clear for $\langle V_4 V_4 \rangle$ w.r.t. τ.
- No **significant** curvature for $\langle V_4 V_4 \rangle$ and $\langle A_4 A_4 \rangle$.

Parametrization of DPDs

Decomposition in terms of rotational invariant functions $f(x_1, x_2, \boldsymbol{y}^2)$:

$$
\begin{aligned}
F_{qq'}(x_1, x_2, \boldsymbol{y}) &= f_{qq'}(x_1, x_2, y^2) \\
F_{\Delta q \Delta q}(x_1, x_2, \boldsymbol{y}) &= f_{\Delta q \Delta q'}(x_1, x_2, y^2) \\
F_{q \Delta q'}(x_1, x_2, \boldsymbol{y}) &= F_{\Delta q q'}(x_1, x_2, \boldsymbol{y}) = 0 \\
F^{j}_{q \delta q'}(x_1, x_2, \boldsymbol{y}) &= \epsilon^{j\ell} y^{\ell} m \; f_{q \delta q'}(x_1, x_2, y^2) \\
F_{\Delta q \delta q'}(x_1, x_2, \boldsymbol{y}) &= F_{\delta q \Delta q'}(x_1, x_2, \boldsymbol{y}) = 0 \\
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F^{jk}_{\delta q \delta q'}(x_1, x_2, \boldsymbol{y}) &= \delta^{jk} f_{\delta q \delta q'}(x_1, x_2, y^2) + \left(2 y^j y^k - \delta^{jk} \boldsymbol{y}^2\right) m^2 f^{t}_{\delta q \delta q'}(x_1, x_2, y^2)
\end{aligned}
$$

Parametrization of DPDs

Decomposition in terms of rotational invariant functions $I(\boldsymbol{y}^2)$:

$$
\begin{aligned}
M_{qq'}(\boldsymbol{y}) &= I_{qq'}(\boldsymbol{y}^2) \\
M_{\Delta q \Delta q}(\boldsymbol{y}) &= I_{\Delta q \Delta q'}(\boldsymbol{y}^2) \\
M_{q \Delta q'}(\boldsymbol{y}) &= M_{\Delta q q'}(\boldsymbol{y}) = 0 \\
M^j_{q \delta q'}(\boldsymbol{y}) &= \epsilon^{j\ell} y^\ell m \; I_{q \delta q'}(\boldsymbol{y}^2) \\
M_{\Delta q \delta q'}(\boldsymbol{y}) &= M_{\delta q \Delta q'}(\boldsymbol{y}) = 0 \\
M^j_{\delta q q'}(\boldsymbol{y}) &= \epsilon^{j\ell} y^\ell m \; I_{\delta q q'}(\boldsymbol{y}^2) \\
M^{jk}_{\delta q \delta q'}(\boldsymbol{y}) &= \delta^{jk} I_{\delta q \delta q'}(\boldsymbol{y}^2) + \left(2 y^j y^k - \delta^{jk} \boldsymbol{y}^2\right) m^2 I^t_{\delta q \delta q'}(\boldsymbol{y}^2)
\end{aligned}
$$

Parametrization of two-current matrix elements

Decomposition in terms of Lorentz invariant functions $A(py, y^2), B(py, y^2), \ldots$, blue: twist-2 contributions:

$$\langle p|\, V_q^{\{\mu}(0) V_{q'}^{\nu\}}(y)\, |p\rangle =$$

$$= (2p^\mu p^\nu - \tfrac{m^2}{2} g^{\mu\nu}) A_{q'q}(py, y^2) + (2p^{\{\mu} y^{\nu\}} - \tfrac{py}{2} g^{\mu\nu}) m^2 B_{q'q}(py, y^2)$$

$$+ (2y^\mu y^\nu - \tfrac{y^2}{2} g^{\mu\nu}) m^4 C_{q'q}(py, y^2) + g^{\mu\nu} D_{q'q}(py, y^2)$$

$$\langle p|\, A_q^{\{\mu}(0) A_{q'}^{\nu\}}(y)\, |p\rangle =$$

$$= (2p^\mu p^\nu - \tfrac{m^2}{2} g^{\mu\nu}) A_{\Delta q'\Delta q}(py, y^2) + (2p^{\{\mu} y^{\nu\}} - \tfrac{py}{2} g^{\mu\nu}) m^2 B_{\Delta q'\Delta q}(py, y^2)$$

$$+ (2y^\mu y^\nu - \tfrac{y^2}{2} g^{\mu\nu}) m^4 C_{\Delta q'\Delta q}(py, y^2) + g^{\mu\nu} D_{\Delta q'\Delta q}(py, y^2)$$

$$\langle p|\, T_q^{\mu\nu}(0) V_{q'}^{\rho}(y)\, |p\rangle + \tfrac{2}{3} g_{\lambda\sigma} g^{\rho[\mu} \langle p|\, T_q^{\nu]\lambda}(0) V_{q'}^{\sigma}(y)\, |p\rangle =$$

$$= (4y^{[\mu} p^{\nu]} p^\rho + \tfrac{4m^2}{3} g^{\rho[\mu} y^{\nu]} - \tfrac{4py}{3} g^{\rho[\mu} p^{\nu]}) m\, A_{q'\delta q}(py, y^2)$$

$$+ (4y^{[\mu} p^{\nu]} y^\rho + \tfrac{4py}{3} g^{\rho[\mu} y^{\nu]} - \tfrac{4y^2}{3} g^{\rho[\mu} p^{\nu]}) m^3 B_{q'\delta q}(py, y^2)$$

$$\tfrac{1}{2} \langle p|\, T_q^{\mu\nu}(0) T_{q'}^{\rho\sigma}(y)\, |p\rangle + \tfrac{1}{2} \langle p|\, T_q^{\rho\sigma}(0) T_{q'}^{\mu\nu}(y)\, |p\rangle =$$

$$= -8p^{[\nu} g^{\mu][\rho} p^{\sigma]} A_{\delta q'\delta q}(py, y^2) - (16y^{[\mu} p^{\nu]} y^{[\rho} p^{\sigma]} - 8y^2\ p^{[\nu} g^{\mu][\rho} p^{\sigma]}) m^2 B_{\delta q'\delta q}(py, y^2)$$

$$- (4p^{[\nu} g^{\mu][\rho} y^{\sigma]} + 4y^{[\nu} g^{\mu][\rho} p^{\sigma]}) m^2 C_{\delta q'\delta q}(py, y^2) - 8y^{[\nu} g^{\mu][\rho} y^{\sigma]} m^4 D_{\delta q'\delta q}(py, y^2)$$

$$+ 2g^{\mu[\rho} g^{\sigma]\nu} m^2 E_{\delta q'\delta q}(py, y^2)$$

Fit ansatz for invariant functions

Skewed DPDs (additional phase with skewness ζ):

$$F_{ab}(x_1, x_2, \zeta, \boldsymbol{y}) := 2p^+ \int \mathrm{d}y^- \, e^{-i\zeta p^+ y^-} \left[\prod_{i=1,2} \frac{\mathrm{d}z_i^-}{2\pi} e^{ix_i p^+ z_i} \right] \langle p | \, \mathcal{O}_a(y, z_1) \mathcal{O}_b(0, z_2) \, | p \rangle$$

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Symmetries and support region:

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Ansatz:

$$I(\zeta, \boldsymbol{y}^2) \propto \sum_n \zeta^{2n} \Theta(1 - \zeta^2) \quad \Rightarrow \quad A(py, y^2) = A(0, y^2) \sum_n a_n(y^2) h_n(py)$$

$$h_n(x) := \frac{1}{2} \int_{-1}^{1} \mathrm{d}\zeta e^{ix\zeta} \zeta^{2n} = \sin(x) s_n(x) + \cos(x) c_n(x)$$

$$s_n(x) := \sum_{m=0}^{n} \frac{(2n)!(-1)^m}{(2n-2m)! x^{1+2m}} \qquad c_n(x) := \sum_{m=0}^{n-1} \frac{(2n)!(-1)^m}{(2n-2m-1)! x^{2+2m}}$$

Fit ansatz for invariant functions

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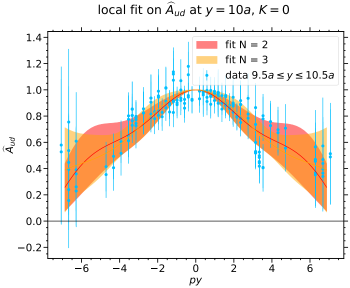

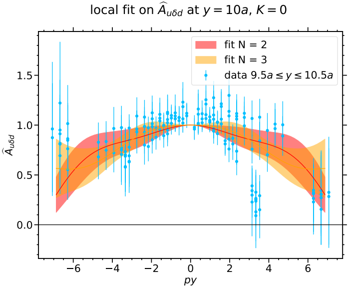

Fit ansatz for invariant functions

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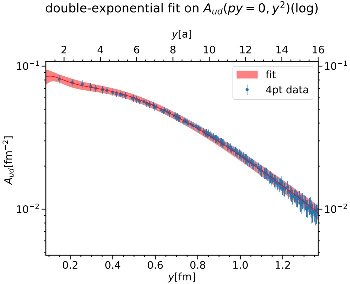
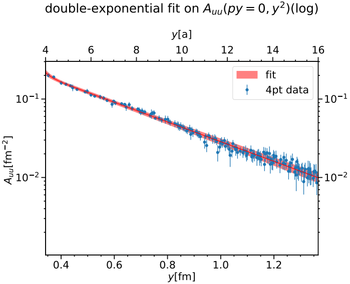
y^2-dependence:

$$A(0, y^2) = \sum_{i=1,2} A_i (\eta_i y)^\delta e^{-\eta_i (y - y_0)}$$





Fit ansatz for invariant functions

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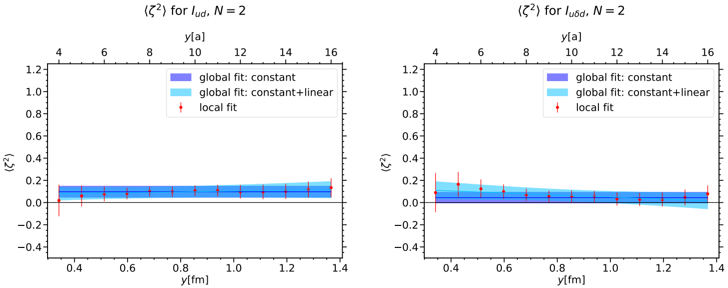
Ansatz $a_m(y^2)$:

$$a_m(y^2) = \sum_k c'_{mk} \sqrt{-y^2}^k \qquad (Tc')_{nk} = c_{nk} \qquad \left. \frac{\partial^{2n} A(py, y^2)}{\partial (py)^{2n}} \right|_{py=0} = A(0, y^2) \sum_k c_{nk} \sqrt{-y^2}^k$$



Fit ansatz for invariant functions

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Total ansatz (red: fit parameters)

$$A(py, y^2) = \sum_{i=1,2} A_i e^{-\eta_i (y - y_0)} \sum_{n,m=0}^{N} \sum_{k=0}^{K} c'_{nk} \, \sqrt{-y^2}^{\,k+\delta} \, {\eta_i}^{\delta} \, h_n(py)$$

$$I(\boldsymbol{y}^2) = \pi \sum_{i=1,2} A_i e^{-\eta_i (y - y_0)} \sum_{k=0}^{K} c'_{0k} \sqrt{-y^2}^{\,k+\delta} {\eta_i}^{\delta}$$

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- $c'_{nk} = T^{-1}_{nm} c_{mk}$
- Notice: $c_{00} \equiv 1$ and $c_{01} \equiv 0$ (only influence $A(0, y^2)$)
- In this work: $(N, K) = (2, 0), (2, 1), (3, 0)$

Fit ansatz for invariant functions

Skewed DPDs (additional phase with skewness ζ):

$$F_{ab}(x_1, x_2, \zeta, \boldsymbol{y}) := 2p^+ \int \mathrm{d}y^- \, e^{-i\zeta p^+ y^-} \left[\prod_{i=1,2} \frac{\mathrm{d}z_i^-}{2\pi} e^{i x_i p^+ z_i} \right] \langle p | \, \mathcal{O}_a(y, z_1) \mathcal{O}_b(0, z_2) \, | p \rangle$$

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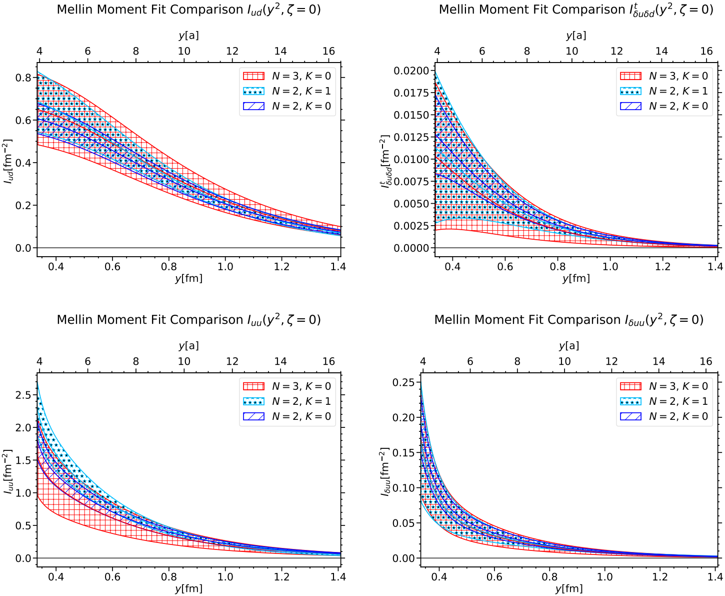
Caution: Preliminary ansatz! We are currently exploring more sophisticated models based on parton splitting at small $\boldsymbol{y}$

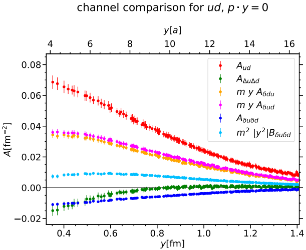
Mellin moments: Fit ansatz dependence

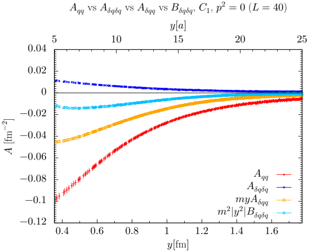


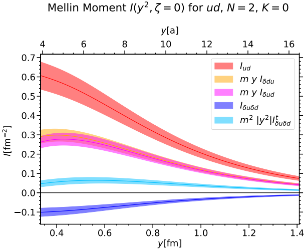
Results for the pion

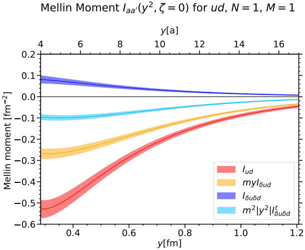
Comparison of A_{ab} and I_{ab} for ud:

proton (p)	pion π^+









Cross check: The number sum rule

For $x_1 > 0$ (otherwise $F_{qq'}(x_1, ...) \to -F_{\bar{q}q'}(-x_1, ...)$)

The number sum rule *[Gaunt, Stirling '10; Diehl, Plößl, Schäfer '19]*

$$\int_{-1}^{1} \mathrm{d}x_2 \int_{b_0/\mu} \mathrm{d}^2\boldsymbol{y} F_{qq'}(x_1, x_2, \boldsymbol{y}; \mu) =$$
$$= (N_{q'} + \delta_{q\bar{q}'} - \delta_{qq'}) f_q(x_1; \mu) + \mathcal{O}(\alpha_s(\mu)) + \mathcal{O}((b_0\Lambda/\mu)^2)$$

with $b_0 = 2e^{-\gamma}$ and $\mu = 2\ \mathrm{GeV}$ ($\gamma \approx 0.577$, splitting singularity $\sim \alpha_s/\boldsymbol{y}^2$)

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Implies for I_{ud}

$$\int_{b_0/\mu} \mathrm{d}^2\boldsymbol{y} I_{ud}(\boldsymbol{y}^2) = 2 + \mathcal{O}(\alpha_s^2(\mu)) + \mathcal{O}((b_0\Lambda/\mu)^2)$$

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From our data:

N	K	χ^2/dof	integral
2	0	0.47	1.93(23)
3	0	0.46	2.07(51)
2	1	0.46	1.98(24)