Hilbert series and applications
… or how to count stuff .

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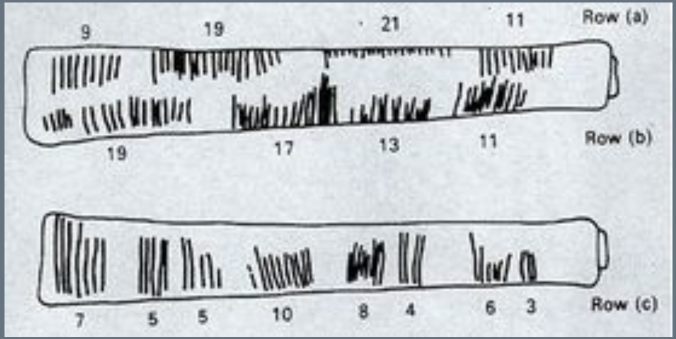
Based on: JHEP 06 (2024) 154 - 2312.13349 [hep-ph]
JHEP 03 (2025) 172 - 2412.05359 [hep-ph]
JHEP 07 (2025) 259 - 2412.16285 [hep-ph]

in collaboration with B. Grinstein, X. Lu and C. Miró

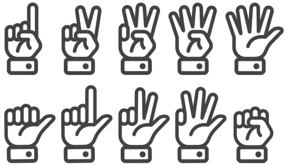
Ishango Bone

- One of the earliest evidence of human counting.
- 33.000-8.000 years old



Boyer, C. B., Merzbach, U. C. (3rd Ed.) (2011). A history of mathematics. New Jersey: John Wiley & Sons, Inc.

How to count stuff?



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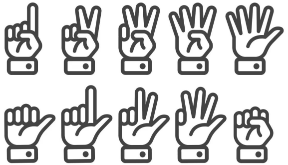

- ➔ Systematically
- ➔ Objects: operators, physical parameters, tensors

  Invariants/covariants

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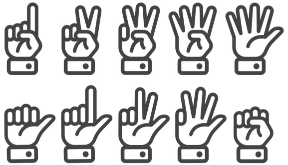

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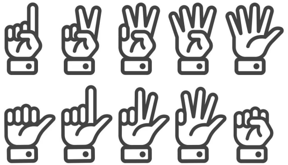

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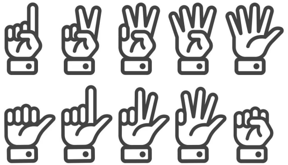
HILBERT SERIES

and tools in INVARIANT THEORY

Outline



1. Motivation: SMEFT and Accidental symmetries
2. Hilbert series
   - 2.1. Invariants
   - 2.2. Covariants
3. SMEFT with Hilbert Series
4. Accidental symmetries with Hilbert series

Based on **2412.05359 - Pablo Quílez**

SMEFT

$$\mathcal{L}_{\text{EFT}}\left(\Phi, \partial_\mu\right) = \mathcal{L}_{\text{EFT}}^{(0)} + \frac{1}{\Lambda}\mathcal{L}_{\text{EFT}}^{(1)} + \frac{1}{\Lambda^2}\mathcal{L}_{\text{EFT}}^{(2)} + \cdots + \frac{1}{\Lambda^k}\mathcal{L}_{\text{EFT}}^{(k)} + \cdots$$

- ➔ Field content $\Phi(x)$ $\quad \Phi \in \left\{H, Q, u, d, L, e, G^a_{\mu v}, W^a_{\mu v}, B_{\mu v}\right\}$
- ➔ Power counting $\frac{1}{\Lambda}$
- ➔ Symmetries H $\quad \begin{cases} \text{Lorentz invariance} \\ SU(3)_c \times SU(2)_W \times U(1)_Y \end{cases}$

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  - ◆ Integration by parts

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Finding an Operator basis can be tricky

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➔ NOT imposed but arise from Lorentz + Gauge Sym. + Matter Content

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- ➔ Ex: Dim-4 SM

$$\mathcal{L}_{\rm SM} = |DH|^2 + \sum_{\psi=q,u,d,l,e} \bar{\psi} i D\psi - \frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$
$$- \lambda\left(|H|^2 - \frac{1}{2}v^2\right)^2 - \left(\bar{q}Y_u\tilde{H}u + \bar{q}Y_dHd + \bar{l}Y_eHe + \text{ h.c. }\right)$$

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- Respects Baryon number $U(1)_{BN}$ and Lepton number $U(1)_{LN}$!
  - LN broken at dim-5 $\mathcal{L}^{d=5} = \frac{1}{2} \bar{\ell}_{L\alpha} \widetilde{\Phi} \frac{C^{\alpha\beta}_{\nu}}{\Lambda_{LN}} \widetilde{\Phi}^T \ell^c_{L\beta} + \text{h.c.}$
  - BN broken at dim-6 →

[Grzadkowski et al, 10]

	B-violating
Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d^\alpha_p)^T C u^\beta_r\right]\left[(q^{\gamma j}_s)^T C l^k_t\right]$
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Motivation: Custodial sym. $O(4) \simeq SU(2)_L \times SU(2)_R$

➔ Custodial Symmetry in **Higgs Sector of SMEFT** $\mathcal{L}_{\text{SMEFT}}\left(H, H^\dagger, \partial_\mu\right)$

$SU(2)_L \times U(1)_Y$

$$
\begin{aligned}
&\text{dim-}2: \quad H^\dagger H \\
&\text{dim-}4: \quad \left(H^\dagger H\right)^2, \left(\partial_\mu H^\dagger\right)\left(\partial^\mu H\right) \\
&\text{dim-}6: \quad \left(H^\dagger H\right)^3, \left(\partial_\mu |H|^2\right)\left(\partial^\mu |H|^2\right), \left(H^\dagger \partial_\mu H\right)^* \left(H^\dagger \partial^\mu H\right) \\
&\text{dim-}8: \quad \left(H^\dagger H\right)^4, \cdots
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$\cancel{O(4)}$

$$|H|^2 = H^\dagger H = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right)$$

Automatically $O(4)$-invariant !

$$SU(2)_L \times U(1)_Y$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

➔ $SU(2)_L \times U(1)_Y \Rightarrow O(4)$ for $\mathcal{L}_{\text{SMEFT}}\left(H, H^\dagger, \partial_\mu\right)$ up to mass dim 4

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$\text{dim-}2: \quad H^\dagger H$

Accidental symmetries in BSM: high quality axions, DM stability, hierarchy problem…

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★ How do we **verify an accidental symmetry**?

★ Can they hold to **all-orders**?

★ How do we **systematically find all** the accidental symmetries?

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★ ...

★ How ... **all** the accidental symmetries?

HILBERT SERIES and tools in INVARIANT THEORY

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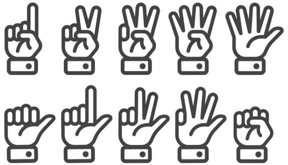
Demystifying

Hilbert series (for invariants)



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and a single complex scalar field $\{\phi_1,\, \phi_1^*\}$ charged $(+1, -1)$

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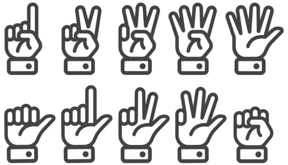

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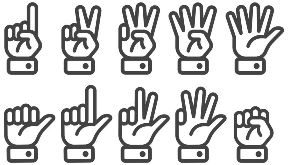

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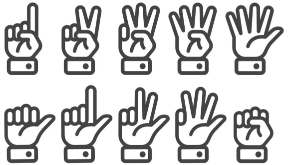

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$$\mathcal{H}_{\text{Inv}}^{U(1),\,(+1,-1)}(q) = 1 + q^2 + q^4 + q^6 + \cdots = \frac{1}{1-q^2}\,, \qquad |q| < 1$$

Based on **2412.05359 - Pablo Quílez**

Demystifying

Hilbert series II (for invariants)

- $G = U(1) \qquad \Phi = \left\{\phi_1,\, \phi_1^*,\, \phi_2,\, \phi_2^*\right\} \qquad Q = \{+1, -1, +1, -1\}$
- 4 Basic invariants:

$$I_1 \equiv \phi_1 \phi_1^* \,, \qquad I_2 \equiv \phi_2 \phi_2^* \,, \qquad I_3 \equiv \phi_1 \phi_2^* \,, \qquad I_4 \equiv \phi_2 \phi_1^* \,.$$

Based on **2312.13349**

Demystifying

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➔ Naively:

$$(1 \oplus I_1 \oplus I_1^2 \oplus \cdots)(1 \oplus I_2 \oplus I_2^2 \oplus \cdots)(1 \oplus I_3 \oplus I_3^2 \oplus \cdots)(1 \oplus I_4 \oplus I_4^2 \oplus \cdots) \implies \mathcal{H}_{\text{Inv}}\Big|_{\text{naive}} = \frac{1}{(1-q^2)^4}$$

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- ➔ True HS:

$$\mathcal{H}_{\text{Inv}} = \frac{1-q^4}{(1-q^2)^4} = \frac{1+q^2}{(1-q^2)^3}$$

Based on **2312.13349**

Demystifying

Hilbert series: primary and sec. invariants

- $G = U(1)$ $\Phi = \{\phi_1,\, \phi_1^*,\, \phi_2,\, \phi_2^*\}$ $Q = \{+1, -1, +1, -1\}$
- $$\mathcal{H}_{\rm Inv} = \frac{1+q^2}{(1-q^2)^3} = \left(1 + q^2 + q^4 + q^6 + \cdots\right)^3 \left(1+q^2\right)$$

$$\left\{P_1^{k_1}\right\} \otimes \left\{P_2^{k_2}\right\} \otimes \left\{P_3^{k_3}\right\} \otimes (1 \oplus S)\,,$$

3 Primary invariants **1 Secondary invariant**

$$P_1 = \phi_1\phi_1^*\,, \quad P_2 = \phi_2\phi_2^*\,, \quad P_3 = \phi_1\phi_2^* + \phi_2\phi_1^*\,, \quad S = \phi_1\phi_2^* - \phi_2\phi_1^*\,,$$

- Secondary only arises linearly since:

$$I_1 I_2 = I_3 I_4 \implies S^2 = P_3^2 - 4P_1P_2$$

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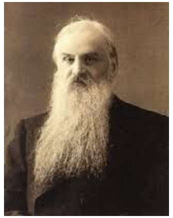
How to compute Hilbert series?

$$H(q) = \sum_{n=0}^{\infty} n_{\text{Inv}}(n)\, q^n$$



$$\Phi = \left\{\phi_1,\, \phi_2,\, \cdots,\, \phi_m\right\}, \qquad R_\Phi = \bigoplus_i R_{\phi_i}\,.$$

$$R_{\Phi^k} = \text{sym}\Big(\underbrace{R_\Phi \otimes R_\Phi \otimes \cdots \otimes R_\Phi}_{k}\Big) = n_{\text{Inv}}(k)\,\mathbf{Inv} \oplus \text{ other irreps}$$

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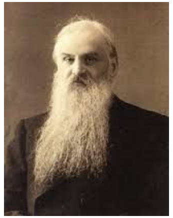

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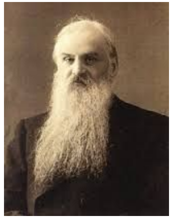

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$$R_1 = \text{Inv and } R_2 = R_{\Phi^k}$$

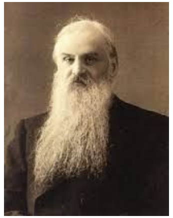
$$n_{\text{Inv}}(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_{\text{Inv}}(x)\, \chi_{R_{\Phi^k}}(x)$$

Based on **2412.05359 - Pablo Quílez**

How to compute Hilbert series?

$$H(q) = \sum_{n=0}^{\infty} n_{\mathrm{Inv}}(n)\, q^n$$

$$\Phi = \left\{\phi_1,\, \phi_2,\, \cdots,\, \phi_m\right\}, \qquad R_\Phi = \bigoplus_i R_{\phi_i}\,.$$



Molien formula to compute HS

$$\mathcal{H}^{G,R_\Phi}_{\mathrm{Inv}}(q) = \sum_{k=0}^{\infty} \int \mathrm{d}\mu_G(x)\, \chi_{R_{\Phi^k}}(x)\, q^k = \int \mathrm{d}\mu_G(x) \frac{1}{\det\left[1 - q g_{R_\Phi}(x)\right]}$$

$R_1 = \mathrm{Inv}$ and $R_2 = R_{\Phi^k}$

$$n_{\mathrm{Inv}}(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_{\mathrm{Inv}}(x)\, \chi_{R_{\Phi^k}}(x)$$

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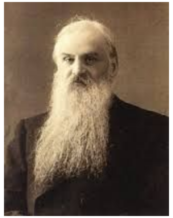

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It can be computed with our code: HilbCalc [Grinstein+Lu+Miró+PQ, 24]

$R_1 = \text{Inv}$ and $R_2 = R_{\Phi^k}$

$$n_{\text{Inv}}(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_{\text{Inv}}(x)\, \chi_{R_{\Phi^k}}(x)$$

Hilbert Series for flavor invariants

[Jenkins+Manohar, 09]
[Hanany et al, 10]
[Broer, 94]

$$\mathcal{L}_{\text{Yukawa}} = -\overline{Q}_L Y_u \widetilde{\Phi} u_R - \overline{Q}_L Y_d \Phi d_R + \text{ h.c.}$$

$$h_u \equiv Y_u Y_u^\dagger \quad h_d \equiv Y_d Y_d^\dagger$$

- ➔ Group: $G_F = U(3)_{Q_L} \times U(3)_{u_R} \times U(3)_{d_R}$
- ➔ Building blocks: $Y_u \sim (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}) \qquad Y_d \sim (\mathbf{3}, \mathbf{1}, \overline{\mathbf{3}})$
- ➔ Hilbert series: $\mathcal{H}_{\text{Inv}}(q) = \dfrac{1+q^{12}}{(1-q^2)^2 (1-q^4)^3 (1-q^6)^4 (1-q^8)}$

Based on **2312.13349**

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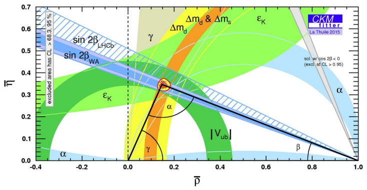

➔ Properties

- 10 prim. inv. = 10 phys. param.
- Polynomial invariants form a ring
- Positive coefs. in numerator
- Palindromic numerator
- Hironaka decomposition

10 Primary invariants

$$P_{2,0} = \text{Tr}[h_u], \quad P_{0,2} = \text{Tr}[h_d],$$
$$P_{4,0} = \text{Tr}[h_u^2], \quad P_{0,4} = \text{Tr}[h_d^2],$$
$$P_{2,2} = \text{Tr}[h_u h_d],$$
$$P_{6,0} = \text{Tr}[h_u^3], \quad P_{0,6} = \text{Tr}[h_d^3],$$
$$P_{4,2} = \text{Tr}[h_u^2 h_d], \quad P_{2,4} = \text{Tr}[h_u h_d^2],$$
$$P_{4,4} = \text{Tr}[h_u^2 h_d^2],$$

1 Secondary invariant

$$S = \text{Im Tr}\left[h_u h_d h_u^2 h_d^2\right] = -\frac{i}{2}\det\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right] \equiv \text{Jarlskog determinant}$$

$$J^2 = \text{poly}(P_1, \ldots, P_{10})$$

Based on **2312.13349**

Extension: Hilbert series for covariants

➔ Hilbert Series can also count rep-R covariants

$$R_{\Phi^k} = n_R(k)\, R \oplus \text{ other irreps}\,.$$

$$n_{\text{Inv}}(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_{\text{Inv}}(x)\, \chi_{R_{\Phi^k}}(x)$$

Based on **2312.13349**

Extension: Hilbert series for covariants

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$$\chi^*_{\text{Inv}}(x) = 1 \longrightarrow \chi^*_R(x)$$

$$n_R(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_R(x)\, \chi_{R_{\Phi^k}}(x)\,.$$

$$\mathcal{H}^{G,\,R_\Phi}_R(q) \equiv \sum_{k=0}^{\infty} n_R(k)\, q^k = \int \mathrm{d}\mu_G(x)\, \chi^*_R(x)\, \frac{1}{\det\left[1 - q\, g_{R_\Phi}(x)\right]}\,.$$

Based on **2312.13349**

Extension: Hilbert series for covariants

Hilbert series for covariants and their applications to Minimal Flavor Violation

Benjamín Grinstein,[a] Xiaochuan Lu,[a] Luca Merlo,[b] and Pablo Quílez[a]

[Grinstein+Lu+Merlo+PQ, 23]
[Grinstein+Lu+Miró+PQ, 24]

$$\mathcal{H}_R^{G,\,R_\Phi}(q) \equiv \sum_{k=0}^{\infty} n_R(k)\, q^k = \int \mathrm{d}\mu_G(x)\, \chi_R^*(x)\, \frac{1}{\det\left[1 - q\, g_{R_\Phi}(x)\right]}\,.$$

Based on **2312.13349**

Some Applications of Hilbert Series

- ➔ Supersymmetric gauge theories , general supersymmetric EFTs [Benvenuti et al, 07] [Feng et al, 07] [Gray et al, 08] [Delgado et al, 23]
- ➔ SMEFT, SMEFT with gravity [Lehman et al, 15] [Henning, et al, 15] [Lehman et al, 16] [Henning, et al, 17] [Marinissen et al, 20] [Kondo, et al, 23] [Ruhdorfer et al, 19]
- ➔ QCD Chiral Lagrangian, Higgs EFT, NRQED and NRQCD [Graf et al, 21] [Graf et al, 22] [Sun, et al, 22] [Kobach, et al, 17] [Kobach, et al, 18]
- ➔ EFTs for axion-like particles [Grojean et al, 23]
- ➔ Primary observables at colliders [Chang, et al, 22]
- ➔ Flavor invariants [Jenkins+Manohar, 09] [Hanany et al, 10]
- ➔ Covariants, MFV and saturation theorem [Grinstein+Lu+Merlo+PQ, 23] [Grinstein+Lu+Miró+PQ, 24]
- ➔ Higher-Spin Dark Matter Meets Hilbert Series

Based on **2412.05359 - Pablo Quílez**

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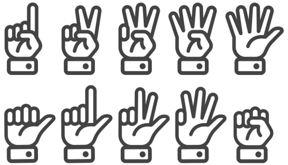
- ➔ **Higher-Spin Dark Matter Meets Hilbert Series** → Poster by Bruno

Based on **2412.05359 - Pablo Quílez**

SMEFT

[Henning, Xiaochuan Lu, Melia, Murayama, 1512.03433]



$$\mathcal{L}_{\mathrm{EFT}}\left(\Phi, \partial_\mu\right) = \mathcal{L}_{\mathrm{EFT}}^{(0)} + \frac{1}{\Lambda}\mathcal{L}_{\mathrm{EFT}}^{(1)} + \frac{1}{\Lambda^2}\mathcal{L}_{\mathrm{EFT}}^{(2)} + \cdots + \frac{1}{\Lambda^k}\mathcal{L}_{\mathrm{EFT}}^{(k)} + \cdots$$

Single particle Module (EOM)

$$R_\phi = \begin{pmatrix} \phi \\ \partial_\mu \phi \\ \partial_{\{\mu_1}\partial_{\mu_2\}}\phi \\ \vdots \end{pmatrix}.$$

Integration by parts

$$\mathcal{H}_K^{SO(4)} = \mathcal{H}_Q^{SO(4)} - \mathcal{H}_{f(\partial_\mu Q^\mu)}$$

$$\mathcal{H}_{f(\partial_\mu Q^\mu)} = \mathcal{H}_Q^{\text{co-exact 0-form}}$$

$$= q\left(\mathcal{H}_Q^{\text{1-form}} - \mathcal{H}_Q^{\text{cohomology 1-form}}\right) - q\mathcal{H}_Q^{\text{co-exact 1-form}}$$

• • •

Cohomology operators

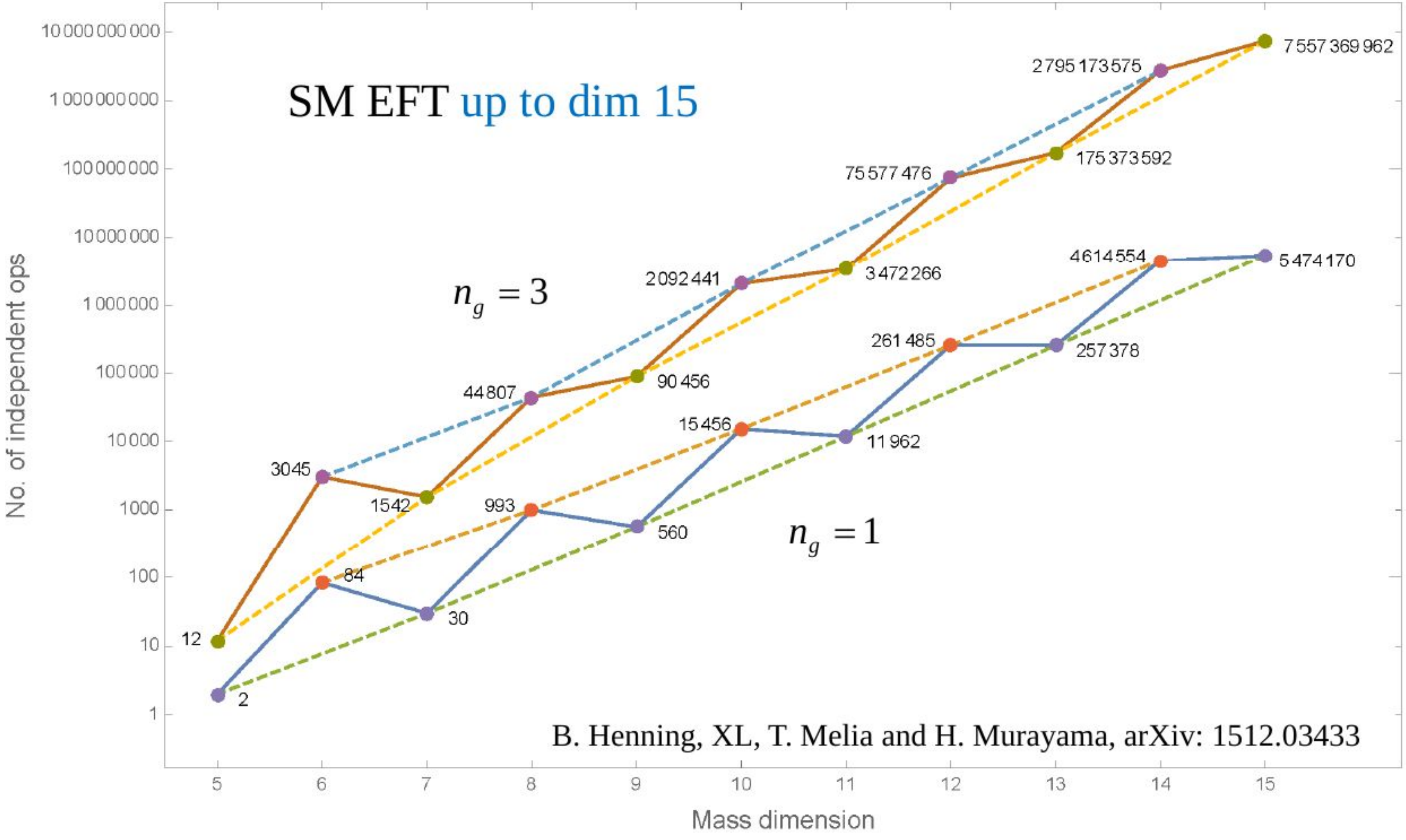
$$\Delta\mathcal{H}_K^{SO(4)}(q)$$

History of Enumerating SMEFT Operators

Slide from Xiaochuan Lu

- dim 6, $n_g = 1$ — 1986 — Buchmuller and Wyler, Nucl. Phys. B 268 (1986) 621 — ~~80~~ operators
  - 2010 — Grzadkowski, Iskrzynski, Misiak, and Rosiek, arXiv: 1008.4884 — 59 + ~~5~~
- dim 6, general n_g — 2013 - 14 — Alonso, Chang, Jenkins, Manohar, and Shotwell, arXiv: 1405.0486 — 59 + 4
  - Alonso, Jenkins, Manohar, and Trott, arXiv: 1312.2014
    - $n_g = 1$: 76 + 8 = 84
    - $n_g = 3$: 2499 + 546 = 3045
- $\begin{cases} \text{dim 7, general } n_g \\ \text{dim 8, } n_g = 1 \end{cases}$ — 2014 - 15 — Lehman and Martin, arXiv: 1410.4193, 1503.07537, 1510.00372 — ~~931~~
  - Henning, **XL**, Melia, and Murayama, arXiv: 1512.03433 — 993

Based on **2412.05359 - Pablo Quílez**



SM EFT up to dim 15
$n_g = 3$
$n_g = 1$
No. of independent ops
Mass dimension
10 000 000 000
1 000 000 000
100 000 000
10 000 000
1 000 000
100 000
10 000
1000
100
10
1
5
6
7
8
9
10
11
12
13
14
15
12
3045
1542
44 807
90 456
2 092 441
3 472 266
75 577 476
175 373 592
2 795 173 575
7 557 369 962
2
84
30
993
560
15 456
11 962
261 485
257 378
4 614 554
5 474 170
B. Henning, XL, T. Melia and H. Murayama, arXiv: 1512.03433

General criterion for Accidental Sym,

➔ All-order accidental symmetries of the potential

$$H \Rightarrow G \text{ for } V(\Phi) \text{ to all orders} \quad \Longleftrightarrow \quad \mathcal{H}^{H,\Phi}_{\text{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\text{Inv}}(q)$$

➔ Accidental symmetries of the potential until order k

$$H \Rightarrow G \text{ for } V(\Phi) \text{ up to order } k \geq 0 \quad \Longleftrightarrow \quad \mathcal{H}^{H,\Phi}_{\text{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\text{Inv}}(q) + \mathcal{O}(q^{k+1})$$

➔ Accidental symmetries of the Lagrangian until order k

$$H \Rightarrow G \text{ for } \mathcal{L}_{\text{EFT}} \text{ up to } k \geq 0 \quad \Longleftrightarrow \quad \mathcal{H}^{\text{Lorentz } \times H, \mathcal{M}_\Phi}_{\text{Inv, IBP}}(q) = \mathcal{H}^{\text{Lorentz } \times G, \mathcal{M}_\Phi}_{\text{Inv, IBP}}(q) + \mathcal{O}\left(q^{k+1}\right)$$

[Henning+Lu+Melia+Murayama,17]

Accidental symmetries with HS

1. Verifying Accidental Symmetries with **Hilbert series**
2. Three Classes of **All-Order Accidental Symmetries**

   – $SU(N) \Rightarrow SU(N) \times U(1)$
   – $SU(N) \times U(1) \Rightarrow SO(2N)$
   – $SU(2N-1) \Rightarrow SU(2N)$

   $H \Rightarrow G$ for $V(\Phi)$
   $\mathcal{L}_{\mathrm{EFT}} = V(\Phi) + \mathcal{O}(\partial_\mu)$

   ➔ Impact of derivatives
3. Novel Approach using **Friendship Relation**

   ➔ All-order accidental sym. : *friends*
   ➔ Finite-order accidental sym. : *friends ma non troppo*

Based on **2412.05359 - Pablo Quílez**

All-order Acc. Sym. : Class I

Class I	$G = SU(N) \times U(1)$	$H = SU(N)$
$\Phi(x)$	$\phi \sim (\square\,, +1)$ $\phi^* \sim (\bar{\square}\,, -1)$	$\varphi \sim \square$ $\varphi^* \sim \bar{\square}$

$$H \Rightarrow G \text{ for } V(\Phi) \text{ to all orders}$$

$$\mathcal{H}^{H,\Phi}_{\text{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\text{Inv}}(q)$$

➔ H-invariants $\varphi^\dagger\varphi = \phi^\dagger\phi$ are automatically G-invariants

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Hilbert series:
$$\begin{cases} \mathcal{H}^{G,\Phi}_{\text{Inv}} = \frac{1}{1-\phi\phi^*} \to \frac{1}{1-q^2} \\ \mathcal{H}^{H,\Phi}_{\text{Inv}} = \frac{1}{1-\varphi\varphi^*} \to \frac{1}{1-q^2} \end{cases}$$

$$\mathcal{H}^{SU(N)\times U(1),\,(\phi,\phi^*)}_{\text{Inv}}(q) = \mathcal{H}^{SU(N),\,(\varphi,\varphi^*)}_{\text{Inv}}(q)$$

$$\mathcal{H}^{G}_{\text{Inv}} = \oint \frac{dx}{2\pi i x} \prod_{i=1}^{r} \frac{dx_i}{2\pi i x_i} \prod_{1\le i<j\le N} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=1}^{N} \frac{1}{(1-\phi\, x_i x)(1-\phi^*\, x_i^{-1} x^{-1})},$$

Based on **2412.05359 - Pablo Quílez**

All-order Acc. Sym. : Class I, k flavors

Class I, Multiple Flavors (k_1, k_2)	$G = SU(N) \times U(1)$	$H = SU(N)$
$\Phi(x)$	$\phi_i \sim (\square\,,\ +1)$ $\phi_j^* \sim (\bar{\square}\,,\ -1)$	$\varphi_i \sim \square$ $\varphi_j^* \sim \bar{\square}$

$H \Rightarrow G$ for $V(\Phi)$ to all orders

$$\mathcal{H}_{\text{Inv}}^{H,\Phi}(q) = \mathcal{H}_{\text{Inv}}^{G,\Phi}(q)$$

➔ H-invariants $\varphi_j^\dagger \varphi_i = \phi_j^\dagger \phi_i$ are automatically G-invariants

$$k_1, k_2 < N : \quad \begin{cases} \mathcal{H}_{\text{Inv}}^{G,\Phi} = \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{1}{1-\phi_i \phi_j^*} & \to \frac{1}{(1-q^2)^{k_1 k_2}} \\ \mathcal{H}_{\text{Inv}}^{H,\Phi} = \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{1}{1-\varphi_i \varphi_j^*} & \to \frac{1}{(1-q^2)^{k_1 k_2}} \end{cases}$$

SAME Hilbert series

Based on **2412.05359 - Pablo Quílez**

All-order Acc. Sym. : Class I, k flavors

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$H \Rightarrow G$ for $V(\Phi)$ to all orders

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SAME Hilbert series

$$k_1 = k_2 = N : \quad \mathcal{H}_{\text{Inv}}^{H,\Phi} = \frac{1 - \varphi_1 \varphi_1^* \cdots \varphi_N \varphi_N^*}{(1 - \varphi_1 \cdots \varphi_N)(1 - \varphi_1^* \cdots \varphi_N^*)} \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{1 - \varphi_i \varphi_j^*} \to \frac{1 + q^N}{(1 - q^N)} \frac{1}{(1 - q^2)^{N^2}}$$

Disagree for $k_{1,2}$=N

Based on **2412.05359 - Pablo Quílez**

All-order Acc. Sym. : Class I, k flavors

Class I, Multiple Flavors (k_1, k_2)	$G = SU(N) \times U(1)$	$H = SU(N)$
$\Phi(x)$	$\phi_i \sim (\square, +1)$ $\phi_j^* \sim (\bar{\square}, -1)$	$\varphi_i \sim \square$ $\varphi_j^* \sim \bar{\square}$

$H \Rightarrow G$ for $V(\Phi)$ to all orders

$$\mathcal{H}_{\text{Inv}}^{H,\Phi}(q) = \mathcal{H}_{\text{Inv}}^{G,\Phi}(q)$$

➔ H-invariants $\varphi_j^\dagger \varphi_i = \phi_j^\dagger \phi_i$ are automatically G-invariants

$$k_1, k_2 < N : \begin{cases} \mathcal{H}_{\text{Inv}}^{G,\Phi} = \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{1}{1-\phi_i \phi_j^*} \rightarrow \frac{1}{(1-q^2)^{k_1 k_2}} \\ \mathcal{H}_{\text{Inv}}^{H,\Phi} = \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} \frac{1}{1-\varphi_i \varphi_j^*} \rightarrow \frac{1}{(1-q^2)^{k_1 k_2}} \end{cases}$$

SAME Hilbert series

$k_1 = k_2 = N$ Accidental up to order N-1 $\mathcal{H}_{\text{Inv}}^{H,\Phi}(q) = \mathcal{H}_{\text{Inv}}^{G,\Phi}(q) + \mathcal{O}(q^N) \rightarrow \frac{1+q^N}{(1-q^N)} \frac{1}{(1-q^2)^{N^2}}$

Disagree for $k_{1,2}$=N

All-order Acc. Sym. : Class II

Class II	$G = SO(2N)$	$H = SU(N) \times U(1)$
$\Phi(x)$	$\phi \sim \square$	$\varphi \sim (\square\,, +1)$ $\varphi^* \sim (\bar{\square}\,, -1)$

$$\phi = \begin{pmatrix} \mathrm{Re}\,\varphi \\ \mathrm{Im}\,\varphi \end{pmatrix}$$

$H \Rightarrow G$ for $V(\Phi)$ to all orders

$$\mathcal{H}^{H,\Phi}_{\mathrm{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\mathrm{Inv}}(q)$$

➔ H-invariants $\varphi^\dagger \varphi = \phi^T \phi$ are automatically G-invariants

Based on **2412.05359 - Pablo Quílez**

All-order Acc. Sym. : Class II

Class II	$G = SO(2N)$	$H = SU(N) \times U(1)$
$\Phi(x)$	$\phi \sim \square$	$\varphi \sim (\square\,,\, +1)$ $\varphi^* \sim (\bar{\square}\,,\, -1)$

$$\phi = \begin{pmatrix} \mathrm{Re}\,\varphi \\ \mathrm{Im}\,\varphi \end{pmatrix}$$

$H \Rightarrow G$ for $V(\Phi)$ to all orders

$$\mathcal{H}^{H,\Phi}_{\mathrm{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\mathrm{Inv}}(q)$$

➔ H-invariants $\varphi^\dagger \varphi = \phi^T \phi$ are automatically G-invariants

$$\text{Hilbert series:} \begin{cases} \mathcal{H}^{G,\Phi}_{\mathrm{Inv}} = \frac{1}{1-\phi\phi^*} \to \frac{1}{1-q^2} \\ \mathcal{H}^{H,\Phi}_{\mathrm{Inv}} = \frac{1}{1-\varphi\varphi^*} \to \frac{1}{1-q^2} \end{cases}$$

$N = 2 \quad \to \quad$ Custodial symmetry in Higgs sector of SMEFT

$$SU(2)_L \times U(1)_Y : \begin{cases} H \sim (\mathbf{2}, +\frac{1}{2}) \\ H^* \sim (\mathbf{2}, -\frac{1}{2}) \end{cases}$$

SAME Hilbert series

Based on **2412.05359 - Pablo Quílez**

All-order Acc. Sym. : Class II

Class II	$G = SO(2N)$	$H = SU(N) \times U(1)$
$\Phi(x)$	$\phi \sim \square$	$\varphi \sim (\square\,, +1)$ $\varphi^* \sim (\bar{\square}\,, -1)$

$$\phi = \begin{pmatrix} \mathrm{Re}\,\varphi \\ \mathrm{Im}\,\varphi \end{pmatrix}$$

$H \Rightarrow G$ for $V(\Phi)$ to all orders

$$\mathcal{H}^{H,\Phi}_{\mathrm{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\mathrm{Inv}}(q)$$

➔ H-invariants $\varphi^\dagger \varphi = \phi^T \phi$ are automatically G-invariants

$$k < N : \quad \begin{cases} \mathcal{H}^{G,\Phi}_{\mathrm{Inv}} = \prod_{1 \le i \le j \le k} \frac{1}{1-\phi_i \phi_j} \quad \to \quad \frac{1}{(1-q^2)^{k(k+1)/2}} \\ \mathcal{H}^{H,\Phi}_{\mathrm{Inv}} = \prod_{i,j=1}^{k} \frac{1}{1-\varphi_i \varphi_j^*} \quad \to \quad \frac{1}{(1-q^2)^{k^2}} \end{cases}$$

disagree when $k > 1$

Based on **2412.05359 - Pablo Quílez**

Impact of Derivative Interactions

➔ All-order accidental symmetries of the potential

$$H \Rightarrow G \text{ for } V(\Phi) \text{ to all orders} \quad \Longleftrightarrow \quad \mathcal{H}_{\text{Inv}}^{H,\Phi}(q) = \mathcal{H}_{\text{Inv}}^{G,\Phi}(q)$$

➔ Accidental symmetries of the Lagrangian until order k

$$H \Rightarrow G \text{ for } \mathcal{L}_{\text{EFT}} \text{ up to } k \geq 0 \quad \Longleftrightarrow \quad \mathcal{H}_{\text{Inv, IBP}}^{\text{Lorentz } \times H, \mathcal{M}_\Phi}(q) = \mathcal{H}_{\text{Inv, IBP}}^{\text{Lorentz } \times G, \mathcal{M}_\Phi}(q) + \mathcal{O}\left(q^{k+1}\right)$$

➔ Custodial Symmetry in **Higgs Sector of SMEFT** $\mathcal{L}_{\text{SMEFT}}\left(H, H^\dagger, \partial_\mu\right)$

$$\begin{aligned}
\text{dim-}2: &\quad H^\dagger H \\
\text{dim-}4: &\quad \left(H^\dagger H\right)^2, \left(\partial_\mu H^\dagger\right)\left(\partial^\mu H\right) \\
\text{dim-}6: &\quad \left(H^\dagger H\right)^3, \left(\partial_\mu |H|^2\right)\left(\partial^\mu |H|^2\right), \boxed{\left(H^\dagger \partial_\mu H\right)^* \left(H^\dagger \partial^\mu H\right)} \\
\text{dim-}8: &\quad \left(H^\dagger H\right)^4, \cdots
\end{aligned}$$

~~$O(4)$~~

Based on **2412.05359 - Pablo Quílez**

Impact of Derivative Interactions: Class I

➔ Class I all-order accidental symm. $H = SU(4) \Rightarrow G = SU(4)\times U(1)$

$$\begin{aligned}\mathcal{H}^{\text{Lorentz}\times G,\,\mathcal{M}_\Phi}_{\text{Inv, IBP}} &= 1+\phi\phi^* + \left[(\phi\phi^*)^2 + \phi\phi^*\partial^2\right] + \left[(\phi\phi^*)^3 + 2(\phi\phi^*)^2\partial^2\right] + \left[(\phi\phi^*)^4 + 2(\phi\phi^*)^3\partial^2 + 3(\phi\phi^*)^2\partial^4\right] \\ &\quad + \left[(\phi\phi^*)^5 + 2(\phi\phi^*)^4\partial^2 + 11(\phi\phi^*)^3\partial^4 + 4(\phi\phi^*)^2\partial^6\right] + \mathcal{O}(\text{dim-11}) \\ &= 1+q^2+2q^4+3q^6+6q^8+18q^{10}+\mathcal{O}(q^{11})\end{aligned}$$

$$\begin{aligned}\mathcal{H}^{\text{Lorentz}\times H,\,\mathcal{M}_\Phi}_{\text{Inv, IBP}} &= 1+\varphi\varphi^* + \left[(\varphi\varphi^*)^2 + \varphi\varphi^*\partial^2\right] + \left[(\varphi\varphi^*)^3 + 2(\varphi\varphi^*)^2\partial^2\right] + \left[(\varphi\varphi^*)^4 + 2(\varphi\varphi^*)^3\partial^2 + 3(\varphi\varphi^*)^2\partial^4\right] \\ &\quad + \left[\begin{matrix}(\varphi\varphi^*)^5 + 2(\varphi\varphi^*)^4\partial^2 + 11(\varphi\varphi^*)^3\partial^4 + 4(\varphi\varphi^*)^2\partial^6 \\ +(\varphi^4+\varphi^{*4})(\varphi\varphi^*)\partial^4 + (\varphi^4+\varphi^{*4})\partial^6\end{matrix}\right] + \mathcal{O}(\text{dim-11}) \\ &= 1+q^2+2q^4+3q^6+6q^8+(18+4)q^{10}+\mathcal{O}(q^{11})\end{aligned}$$

Impact of Derivative Interactions: Class I

➔ Class I all-order accidental symm. $H = SU(4) \;\Rightarrow\; G = SU(4)\times U(1)$

$$\mathcal{H}^{\text{Lorentz}\times G,\,\mathcal{M}_\Phi}_{\text{Inv, IBP}} = 1 + \phi\phi^* + \left[(\phi\phi^*)^2 + \phi\phi^*\partial^2\right] + \left[(\phi\phi^*)^3 + 2(\phi\phi^*)^2\partial^2\right] + \left[(\phi\phi^*)^4 + 2(\phi\phi^*)^3\partial^2 + 3(\phi\phi^*)^2\partial^4\right]$$

$$SU(4) \implies SU(4)\times U(1) \text{ for } V(\Phi) \text{ to all orders}$$

$$SU(4) \implies SU(4)\times U(1) \text{ for } \mathcal{L}_{EFT} \text{ up to mass dim 9}$$

Inv, IBP

$$+\begin{bmatrix}(\varphi\varphi^*)^5 + 2(\varphi\varphi^*)^4\partial^2 + 11(\varphi\varphi^*)^3\partial^4 + 4(\varphi\varphi^*)^2\partial^6 \\ +(\varphi^4+\varphi^{*4})(\varphi\varphi^*)\partial^4 + (\varphi^4+\varphi^{*4})\partial^6\end{bmatrix} + \mathcal{O}(\text{dim-11})$$

$$= 1 + q^2 + 2q^4 + 3q^6 + 6q^8 + (18+4)q^{10} + \mathcal{O}(q^{11})$$

Based on **2412.05359 - Pablo Quílez**

Accidental symmetries with HS

1. Verifying Accidental Symmetries with **Hilbert series**

2. Three Classes of **All-Order Accidental Symmetries**

   – $SU(N) \Rightarrow SU(N) \times U(1)$
   – $SU(N) \times U(1) \Rightarrow SO(2N)$
   – $SU(2N-1) \Rightarrow SU(2N)$

   $H \Rightarrow G$ for $V(\Phi)$
   $\mathcal{L}_{\text{EFT}} = V(\Phi) + \mathcal{O}(\partial_\mu)$

   ➔ Impact of derivatives

3. Novel Approach using **Friendship Relation**

   ➔ All-order accidental sym. : *friends*
   ➔ Finite-order accidental sym. : *friends ma non troppo*

Based on **2412.05359 - Pablo Quílez**

Conclusions

- **Hilbert series** are an ideal systematic tool to count stuff.

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Conclusions

- **Hilbert series** are an ideal systematic tool to count stuff.
- They are way better than markings in bones.
- **So you should add them to your phenomenologist toolkit!**

Thank you

Back up slides

Brion Theorem

6.2. Expand

$$\frac{\mathscr{H}(B(M))}{\mathscr{H}(B)} = r(M) - s(M)(1-t) + O(1-t)^2.$$

The so-defined constants $r(M)$ and $s(M)$ are closely related to the B-module structure. The integer $r(M) = P(B(M);1)/P(B;1)$ is just the rank of $B(M)$ as a B-module. If the generic orbit in V is closed, isomorphic to G/H, then $r(M) = \dim M^H$ (See **[Sch1]** or **[Bri]**).

All-order Acc. Sym. : Class III

$$H \Rightarrow G \text{ for } V(\Phi) \text{ to all orders}$$

$$\mathcal{H}^{H,\Phi}_{\text{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\text{Inv}}(q)$$

Class III	$G = SU(2N)$	$H = SU(2N-1) \times U(1)$
$\Phi(x)$	$\phi \sim \boxminus$	$\varphi_1 \sim (\boxminus, +2)$ $\varphi_2 \sim (\square, -2N+2)$
Primary Invariants	$\phi^{[12} \cdots \phi^{(2N-1)(2N)]}$	$\varphi_1^{[12} \cdots \varphi_1^{(2N-3)(2N-2)} \varphi_2^{2N-1]}$

Based on **2412.05359 - Pablo Quílez**

Accidental sym. from Invariant Theory

➔ Branching rule $G \longrightarrow H$

$$\text{Irrep}_G = \bigoplus_{\text{Irrep}\ _H} \text{Irrep}_H = \dim\left(\text{Irrep}_G^H\right) \text{Inv}_H \oplus \text{non-invariant}\ \ \text{Irrep}_H$$

Number of H-invariants in Irrep_G

Accidental sym. from Invariant Theory

➔ Branching rule $G \longrightarrow H$

$$\text{Irrep}_G = \bigoplus_{\text{Irrep}_H} \text{Irrep}_H = \dim\left(\text{Irrep}_G^H\right) \text{Inv}_H \oplus \text{non-invariant Irrep}_H$$

Number of H-invariants in Irrep_G

Ex. $SU(4) \rightarrow SU(3)$

$$\mathbf{4} \rightarrow \mathbf{3} + \mathbf{1}$$

$$\mathbf{6} \rightarrow \mathbf{3} + \bar{\mathbf{3}}$$

Based on **2412.05359 - Pablo Quílez**

Accidental sym. from Invariant Theory

➔ Branching rule $G \longrightarrow H$

$$\text{Irrep}_G = \bigoplus_{\text{Irrep}_H} \text{Irrep}_H = \dim\left(\text{Irrep}_G^H\right) \text{Inv}_H \oplus \text{non-invariant Irrep}_H$$

Number of H-invariants in Irrep_G

Ex. $SU(4) \to SU(3)$

$$\mathbf{4} \to \mathbf{3} + \mathbf{1}$$
$$\mathbf{6} \to \mathbf{3} + \bar{\mathbf{3}}$$

- If I can build with $\{\Phi\}$ an irrep* R_G that contains H-invariants, then those H-invariants break G, and G is NOT an accidental sym.

$$\phi_1 \phi_2 \ldots \phi_N \sim R_G \to \text{Inv}_H \oplus \text{non-invariant Irrep}_H$$

*other than the identity

Accidental sym. from Invariant Theory

➔ Branching rule $G \longrightarrow H$

Ir

For G to be **all-order accidental**,

the field content must **forbid the construction**

of any* irrep of G that contains H-invariant components.

*other than the identity

- If I ca … then those H-invariants break G, and G is NOT an accidental sym.

$$\phi_1\phi_2\ldots\phi_N \sim R_G \to \mathrm{Inv}_H \oplus \text{non-invariant } \mathrm{Irrep}_H$$

[2312.13349-Grinstein+Lu+Merlo+**PQ**, 23]

Hilbert series for covariants: Brion Th.

- ➔ How do we know if an Irrep_G is forbidden by the field content?
  - ◆ Hilbert series for covariants!

Hilbert series for covariants and their applications to Minimal Flavor Violation

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Benjamín Grinstein,[a] Xiaochuan Lu,[a] Luca Merlo,[b] and Pablo Quílez[a]

Hilbert series for covariants: Brion Th.

➔ How do we know if an Irrep_G is forbidden by the field content?

- Hilbert series for covariants!
- Brion Theorem:

Only scalars

Hilbert series for covariants and their applications to Minimal Flavor Violation

Benjamín Grinstein,[a] Xiaochuan Lu,[a] Luca Merlo,[b] and Pablo Quílez[a]

$$G \xrightarrow{\langle\Phi\rangle} \boxed{H_\Phi} \subset G$$

→ unbroken group under a generic vev

*stabilizer subgroup, little group or isotropy group

$$\text{rank}\left({}_{\text{r}_{\text{Inv}}}\mathcal{M}_R^{G,R_\Phi}\right) = \dim\left(R^{H_\Phi}\right)$$

Only irreps R_G that contain H_Φ-invariants can be constructed with Φ

Hilbert series for covariants: Brion Th.

➔ How do we know if an Irrep$_G$ is forbidden by the field content?

◆ H ... applications

Only scalars

◆ B ... nd Pablo Quílez

... y group

For G to be **all-order accidental**,

no **irrep* of G can contain** simultaneously

H-invariant(s) and H$_\Phi$-invariant(s).

*other than the identity

$$\text{rank}\left({}_{r_{\text{Inv}}}\mathcal{M}_R^{G,H_\Phi}\right) = \dim\left(R^{H_\Phi}\right)$$

Only irreps R$_G$ that contain H$_\Phi$-invariants can be constructed with Φ

Friendship relation

Let H_1 and H_2 be two subgroups of G. We say that H_1 and H_2 are *friends* of each other when the only G-irrep that contains both Inv_{H_1} and Inv_{H_2} is Inv_G:

$$\dim\left(\mathrm{Irrep}_G^{H_1}\right)\dim\left(\mathrm{Irrep}_G^{H_2}\right) = 0 \qquad \forall \qquad \mathrm{Irrep}_G \neq \mathrm{Inv}_G\,. \tag{4.18}$$

Irrep_G	$\dim\left(\mathrm{Irrep}_G^{H_1}\right) \neq 0$	$\dim\left(\mathrm{Irrep}_G^{H_2}\right) \neq 0$
type-1	✓	✓
type-2	✓	✗
type-3	✗	✓
type-4	✗	✗

Based on **2412.05359 - Pablo Quílez**

Friendship relation

Let H_1 and H_2 be two subgroups of G. We say that H_1 and H_2 are *friends* of each other when the only G-irrep that contains both Inv_{H_1} and Inv_{H_2} is Inv_G:

$$\dim\left(\mathrm{Irrep}_G^{H_1}\right)\dim\left(\mathrm{Irrep}_G^{H_2}\right) = 0 \qquad \forall \qquad \mathrm{Irrep}_G \neq \mathrm{Inv}_G\,. \qquad (4.18)$$

$H_1 = G$: a friend of any $H_2 \subset G$

$H_1 = \{\mathbf{e}\}$: only friend is $H_2 = G$

$G = H_1 \times H_2$: H_1 and H_2 are friends

$$\left.\begin{array}{r} F \text{ is a friend of } H_1 \\ F \subset H_2 \subset G \end{array}\right\} \Rightarrow H_2 \text{ is a friend of } H_1$$

Friendship relation

$$G \xrightarrow{\langle \Phi \rangle} \boxed{H_\Phi} \subset G$$

$$H \Rightarrow G \text{ for } V(\Phi) \text{ to all orders}$$

$$\Longleftrightarrow$$

$$\mathcal{H}^{H,\Phi}_{\text{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\text{Inv}}(q)$$

$$\Longleftrightarrow$$

$$H \text{ and } H_\Phi \text{ are friends}$$

Friendship relation

$$G \xrightarrow{\langle \Phi \rangle} \boxed{H_\Phi} \subset G$$

$H \Rightarrow G$ for $V(\Phi)$ to all orders
$\Longleftrightarrow$
$\mathcal{H}^{H,\Phi}_{\rm Inv}(q) = \mathcal{H}^{G,\Phi}_{\rm Inv}(q)$
$\Longleftrightarrow$
H and H_Φ are friends

➔ All-order accidental symmetry iff H and H_Φ are friends.

Class I: $\quad G = SU(N)\times U(1) \quad , \quad H = SU(N)$

$$\Phi = \begin{cases} k_1 \times (\square, +1) \\ k_2 \times (\bar{\square}, -1) \end{cases} \longrightarrow H_\Phi = U(N-k)\times \mathbb{Z}_k \quad , \quad k = \max(k_1, k_2) < N$$

Class II: $\quad G = SO(2N) \quad , \quad H = SU(N)\times U(1)$

$$\Phi = \text{vector} \longrightarrow H_\Phi = SO(2N-1)$$

Class III: $\quad G = SU(2N) \quad , \quad H = SU(2N-1)$

$$\Phi = \begin{smallmatrix}\square\\\square\end{smallmatrix} \longrightarrow H_\Phi = Sp(2N)$$

Based on **2412.05359 - Pablo Quílez**

Friendship relation

$$G \xrightarrow{\langle\Phi\rangle} \boxed{H_\Phi} \subset G$$

$H \Rightarrow G$ for $V(\Phi)$ to all orders
$\Longleftrightarrow$
$\mathcal{H}^{H,\Phi}_{\text{Inv}}(q) = \mathcal{H}^{G,\Phi}_{\text{Inv}}(q)$
$\Longleftrightarrow$
H and H_Φ are friends

➔ All-order accidental symmetry iff H and H_Φ are friends.

Class I: $G = SU(N)\times U(1)$, $\Phi = \begin{cases} k_1 \times (\square, +1) \\ k_2 \times (\bar{\square}, -1) \end{cases}$ → $H = SU(N)$, $H_\Phi = U(N-k)\times \mathbb{Z}_k$, $k = \max(k_1, k_2) < N$ **FRIENDS**

Class II: $G = SO(2N)$, $\Phi =$ vector → $H = SU(N)\times U(1)$, $H_\Phi = SO(2N-1)$ **FRIENDS**

Class III: $G = SU(2N)$, $\Phi = \begin{smallmatrix}\square\\\square\end{smallmatrix}$ → $H = SU(2N-1)$, $H_\Phi = Sp(2N)$ **FRIENDS**

Based on **2412.05359 - Pablo Quílez**

Friends and friends *ma non troppo*

Class III , $N = 4$

$G = SU(8)$ Irrep Dynkin label	Dimension (name)	$H = SU(7)$ Invariants	$H_\Phi = Sp(8)$ Invariants
(1000000)	**8**	1	0
(0100000)	**28**	0	1
(2000000)	**36**	1	0
(0010000)	**56**	0	0
(1000001)	**63**	1	0
(0001000)	**70**	0	1
(3000000)	**120**	1	0
(1100000)	**168**	0	0
(0100001)	**216**	0	0
(2000001)	**280**	1	0
(4000000)	**330**	1	0
(0200000)	**336**	0	1
(1010000)	**378**	0	0
(0010001)	**420**	0	0
(0001001)	**504**	0	0
(2100000)	**630**	0	0
(0100010)	**720**	0	1
(0000005)	**792**	1	0
(3000001)	**924**	1	0
(2000010)	**945**	0	0
(0000110)	**1008**	0	0
(0000200)	**1176**	0	0
(2000002)	**1232**	1	0

$G = SU(6)$ Irrep Dynkin label	Dimension (name)	$H = SU(3) \times SU(2)$ Invariants	$H_\Phi = SU(5)$ Invariants
(10000)	**6**	0	1
(01000)	**15**	0	0
(00100)	**20**	0	0
(20000)	**21**	0	1
(10001)	**35**	0	1
(30000)	**56**	0	1
(11000)	**70**	0	0
(01001)	**84**	0	0
(00101)	**105**	0	0
(00020)	**105′**	0	0
(20001)	**120**	0	1
(00004)	**126**	0	1
(00200)	**175**	0	0
(01010)	**189**	1	0
(00110)	**210**	0	0
(00012)	**210′**	0	0
(00005)	**252**	0	1
(20010)	**280**	0	0
(30001)	**315**	0	1
(00102)	**336**	0	0
(11001)	**384**	0	0
(20002)	**405**	1	1
(00021)	**420**	0	0
(60000)	**462**	0	1
(03000)	**490**	1	0
(00013)	**504**	0	0
(10101)	**540**	0	0
(10020)	**560**	0	0
(40001)	**700**	0	1

Based on **2412.05359 - Pablo Quílez**