Modular Invariance and the Strong CP problem

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in collaboration with:

Antonio Marrone, Alessandro Strumia and Arsenii Titov, 2505.20395

[Alessandro Strumia and Arsenii Titov, 2305.08908

Matteo Parricciatu,Alessandro Strumia and Arsenii Titov, 2406.01689

Robert Ziegler, 2411.08101]

the strong CP problem

- $$\mathcal{L}_{QCD} = \bar{q}(i\not{D} - m)q - \frac{1}{4g_3^2}\mathcal{G}^a_{\mu\nu}\mathcal{G}^{a\mu\nu} + \frac{\theta_{QCD}}{32\pi^2}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{a\mu\nu}$$

$$\bar{\theta} = \theta_{QCD} + \arg\det m$$

$$d_n \approx 1.2 \times 10^{-16}\,\bar{\theta}\; e \cdot cm$$

→ $|\bar{\theta}| \lesssim 10^{-10}$ & $\delta_{CKM} \approx \mathcal{O}(1)$

- Axion solution

$\bar{\theta}$ promoted to a field, the axion, pseudoGB of a global, anomalous $U(1)_{PQ}$ symmetry

VEV dynamically relaxed to zero by QCD dynamics

axion virtues

1. $\bar{\theta} = 0$ whatever is the source of $\arg\det m$
2. link to Dark Matter
3. a wealth of accessible lab tests
4. plenty of candidates from string theory → talk by Nicole Righi

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axion quality problem

e.g. from string theory

$$V(a) = V_{QCD}(a) + M^4 e^{-\mathcal{S}} \cos(\frac{a}{f_a} + \delta)$$

$$\begin{matrix} M = M_P \\ \delta = \mathcal{O}(1) \end{matrix} \rightarrow \mathcal{S} \geq 200$$

- strong CP solved by an ALP dominated by Planck dynamics ?
  an unappealing possibility, since 1., 2., 3. are no more guaranteed
- can we exclude it?
- if allowed, what is its most natural realization?

here: a superstring-inspired model

1. supersymmetry in the UV

2. modular invariance

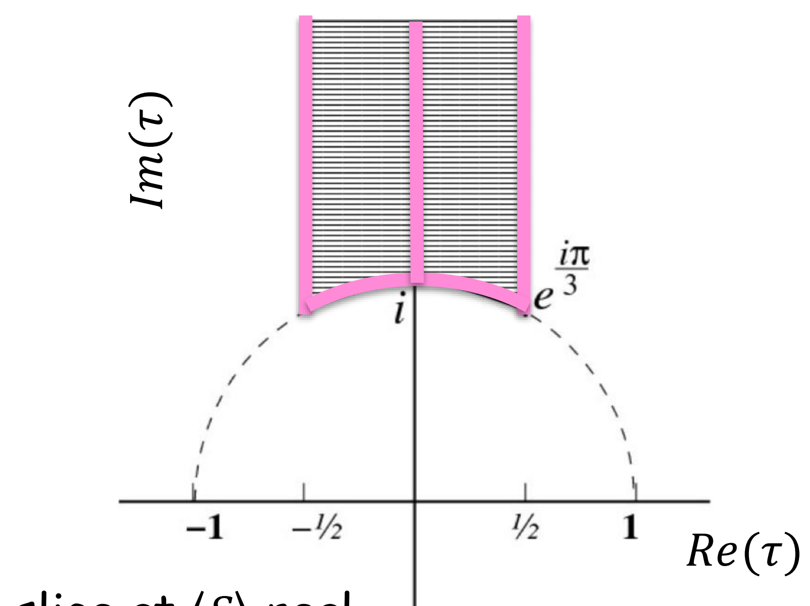
$\mathcal{M} = \{\tau \in \mathbb{C}, Im(\tau) > 0\}$ moduli space

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

vacua related by this are equivalent

3. CP spontaneously broken

CP violation depends on the specific vacuum chosen by the theory



slice at $\langle S \rangle$ real

supersymmetry

- rigid SUSY

gauge kinetic function

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\, K + \left[\int d^2\theta\, w + \frac{1}{16} \int d^2\theta\, f\, WW + h.c \right]$$

kinetic terms

Yukawa couplings $\mathcal{Y}$

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{QCD}}{8\pi^2}$$

not zero, field-dependent

real VEVs for $H_{u,d}$

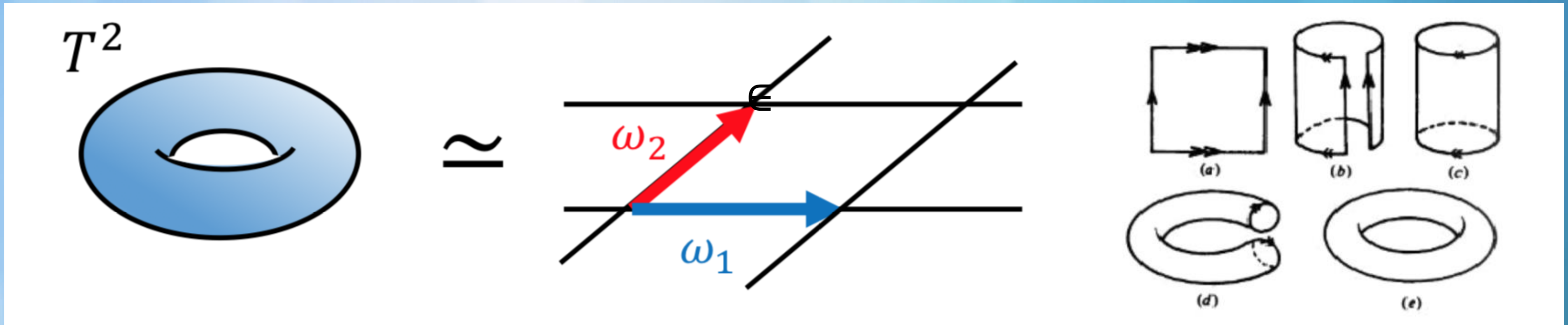
no dependence on K

- $$\bar{\theta} = \arg\left[e^{-8\pi^2 f_3} \det Y_q\right]$$

G. Hiller, M. Schmaltz, 'Solving the Strong CP Problem with Supersymmetry', Phys.Lett.B 514 (2001) 263 [arXiv:hep-ph/0105254].

modular invariance

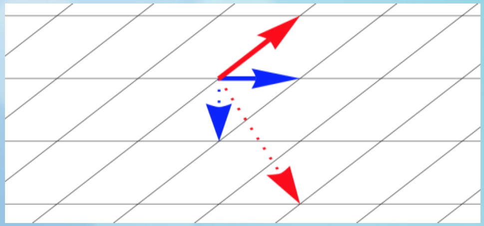
- a discrete gauge symmetry removing redundancy in parametrization of a torus



tori
parametrized by

$$\mathcal{M} = \left\{ \tau = \frac{\omega_2}{\omega_1} \quad Im(\tau) > 0 \right\}$$

lattice left invariant by modular transformations:



$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \in SL(2, Z)$$

a, b, c, d integers

$ad - bc = 1$

modular invariance

- modular functions: transform covariantly under $SL(2,Z)$

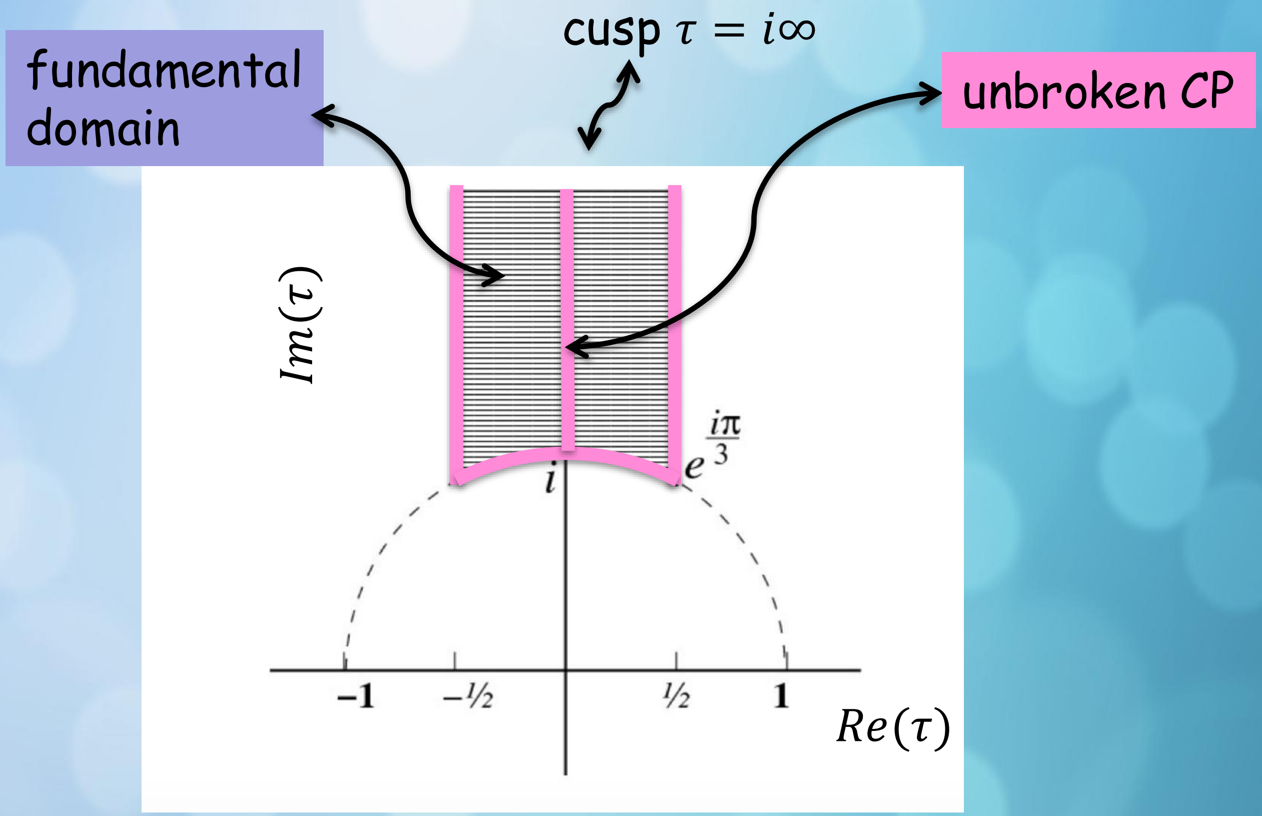
$$A(\gamma\tau) = (c\,\tau + d)^{\overset{\text{weight}}{k}} A(\tau) \qquad c\,\tau + d = \left(\frac{d\gamma\tau}{d\tau}\right)^{-1/2}$$

if holomorphic everywhere: modular forms

$$A(\gamma\tau) = A(\tau) \quad \Rightarrow \quad A(\tau) = \text{constant}$$

CP spontaneously broken

- τ promoted to a field. Through a gauge choice we can restrict τ to the fundamental domain



- CP

$$\tau \to -\tau^*$$

[up to modular transformations]

[Novichkov, Penedo, Petcov and Titov 1905.11970
Baur, Nilles, Trautner and Vaudrevange, 1901.03251]

Field content

moduli space parametrized by two gauge-singlets

$$\varphi \to (c\,\tau + d)^{-k_\varphi}\ \varphi$$

matter multiplets

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

$$V \to V$$

$$S \to S$$

$$a = Im\, S$$

our ALP

$\bar{\theta}$ becomes field-dependent

$$\bar{\theta} = \arg A(S,\tau)$$

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)}\, \det Y_q(\tau)$$

holomorphic

main idea:

holomorphic functions with too much symmetry are constants

$A(S,\tau)$ in modular invariant theories

1. Yukawa couplings are τ-dependent modular functions

modular function of weight k_{det}

$$\det Y_q(\gamma\tau) = (c\tau + d)^{k_{\det}} \quad \det Y_q(\tau)$$

$$k_{\det} \quad = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) + 3k_{H_u} + 3k_{H_d}$$

2 anomaly-free theory if

$$e^{-8\pi^2 f_a(S,\gamma\tau)} = (c\tau + d)^{k_{fa}} e^{-8\pi^2 f_a(S,\tau)}$$

$$k_{f_a} = -\sum_M 2T_a(M)k_M$$

modular function of weight k_{f_a}

$$k_{f_3} = -\sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right)$$

general result

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det Y_q(\tau)$$

$$A(S,\gamma\tau) = (c\tau + d)^{k_A} A(S,\tau)$$

$$k_A = 3(k_{H_u} + k_{H_d})$$

conditions for $\bar{\theta} = 0$

1. the sum of the weights in the Higgs sector vanishes,

$$k_{H_u} + k_{H_d} = 0$$

→ $A(S, \gamma\tau) = A(S, \tau)$

2. $A(S, \tau)$ is holomorphic

→ $A(S, \tau)$ is τ-independent

3. τ is the only source of CP-breaking

$$\langle Im\ S \rangle = 0$$

→ $A(S, \tau)$ is a real constant

we further assume it is positive

→ $\bar{\theta} = \arg A(S, \tau) = 0$

independently from the particular vacuum selected by the modulus τ

Example of $Y_q(\tau)$

Milne, S.C. (2001). Hankel Determinants of Eisenstein Series. In: Garvan, F.G., Ismail, M.E.H. (eds) Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics. Developments in Mathematics, vol 4. Springer, Boston, MA. https://doi.org/10.1007/978-1-4613-0257-5_10

- $k_{Q_i} = k_{U_i^c} = k_{D_i^c} = (2,4,6)$ $\quad k_{H_u} + k_{H_d} = 0$

$$Y_{u,d}(\tau) = \begin{pmatrix} E_4 & E_6 & E_8 \\ E_6 & E_8 & E_{10} \\ E_8 & E_{10} & E_{12} \end{pmatrix}$$

$$E_{2k} \equiv \sum_{m\neq0,n\neq0} \frac{1}{(m+\tau n)^{2k}} \qquad (k>1)$$

$$\det Y_{u,d}(\tau) \propto \Delta(\tau)^2$$

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

discriminant form

$$q = e^{i\,2\pi\tau}$$

- strong CP solved by the choice

$$f_3(S,\tau) = k_3 S + \frac{1}{8\pi^2}\log\Delta(\tau)^4$$

[same structure as in string theory compactifications]

$$A(S,\gamma\tau) = A(S,\tau)$$

$$\propto e^{-8\pi^2 k_3 S}$$

$$Re\ S > 0 \quad \langle Im\ S\rangle = 0$$

$$\bar{\theta} = 0$$

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discriminant form

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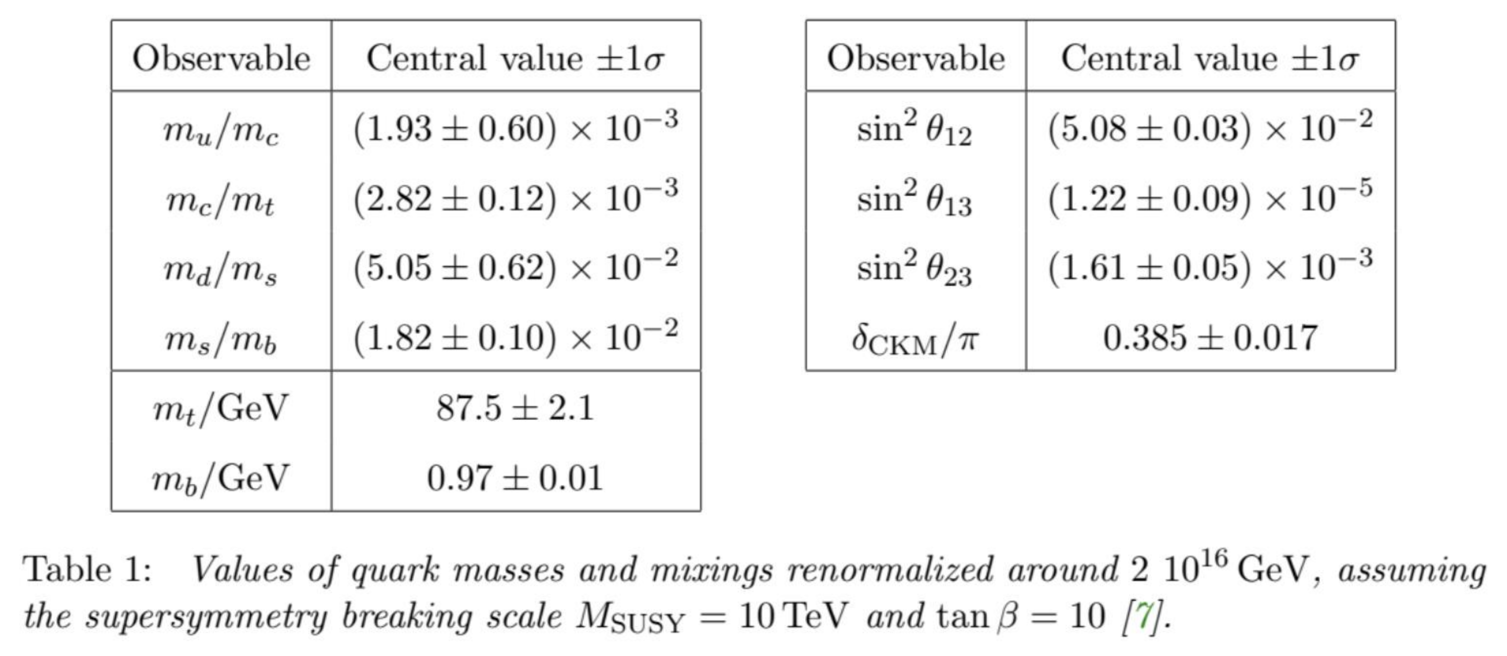
- in the basis where kinetic terms are canonical

$$Y^q_{\rm can} = \begin{pmatrix} c_{Q_1^c} c_{Q_1} y^2 E_4 & c_{Q_1^c} c_{Q_2} y^3 E_6 & c_{Q_1^c} c_{Q_3} y^4 E_8 \\ c_{Q_2^c} c_{Q_1} y^3 E_6 & c_{Q_2^c} c_{Q_2} y^4 E_8 & c_{Q_2^c} c_{Q_3} y^5 E_{10} \\ c_{Q_3^c} c_{Q_1} y^4 E_8 & c_{Q_3^c} c_{Q_2} y^5 E_{10} & c_{Q_3^c} c_{Q_3} y^6 E_{12} \end{pmatrix}$$

$$K = \sum_{i=1}^{3} \left[c_{Q_i}^{-2} y^{-k_{Q_i}} |Q_i|^2 + c_{U_i^c}^{-2} y^{-k_{U_i^c}} U_i^{c\,2} + c_{D_i^c}^{-2} y^{-k_{D_i^c}} D_i^{c\,2} \right]$$

$$y \equiv 2\, Im\, \tau$$



Observable	Central value $\pm1\sigma$
m_u/m_c	$(1.93 \pm 0.60) \times 10^{-3}$
m_c/m_t	$(2.82 \pm 0.12) \times 10^{-3}$
m_d/m_s	$(5.05 \pm 0.62) \times 10^{-2}$
m_s/m_b	$(1.82 \pm 0.10) \times 10^{-2}$
m_t/GeV	87.5 ± 2.1
m_b/GeV	0.97 ± 0.01

Observable	Central value $\pm1\sigma$
$\sin^2\theta_{12}$	$(5.08 \pm 0.03) \times 10^{-2}$
$\sin^2\theta_{13}$	$(1.22 \pm 0.09) \times 10^{-5}$
$\sin^2\theta_{23}$	$(1.61 \pm 0.05) \times 10^{-3}$
$\delta_{\mathrm{CKM}}/\pi$	0.385 ± 0.017

Table 1: *Values of quark masses and mixings renormalized around* $2\ 10^{16}$ GeV, *assuming the supersymmetry breaking scale* $M_{\mathrm{SUSY}} = 10\,\mathrm{TeV}$ *and* $\tan\beta = 10$ *[7].*

best fit values

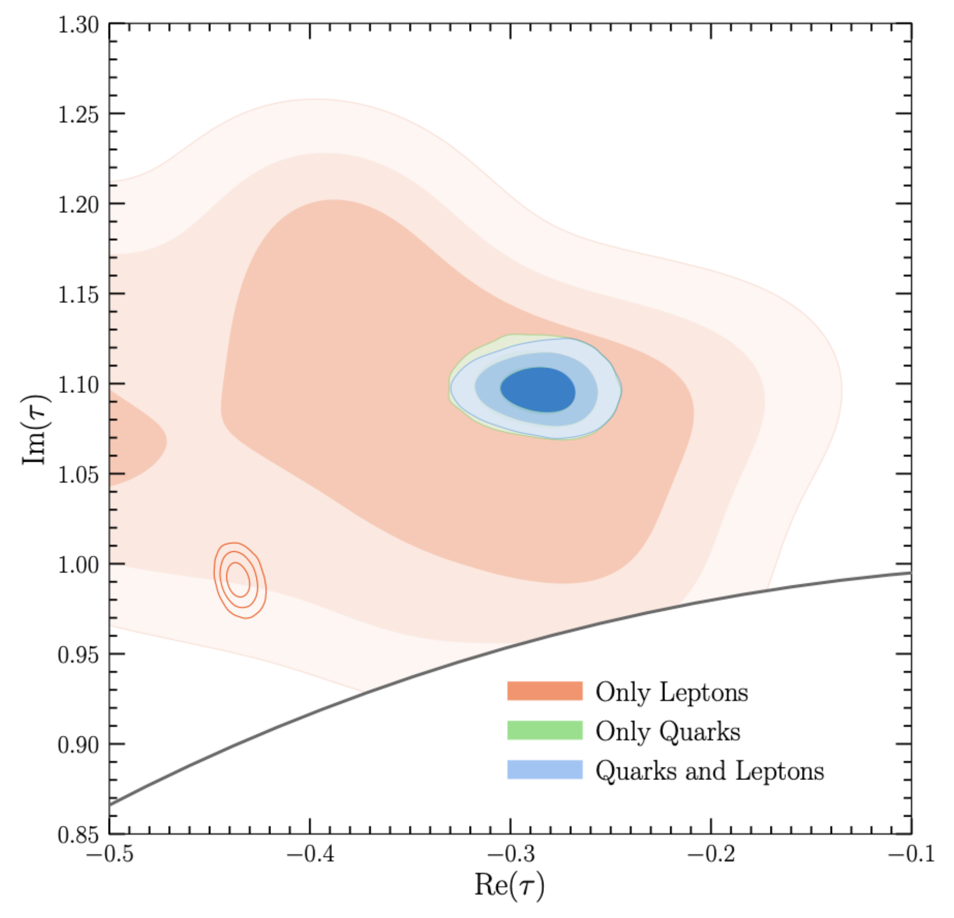
8 parameters + τ

$$\tau = -0.286 + 1.096i$$

$$q_{13} = 0.037\,,\ \ q_{23} = 0.075\,,\ \ u_{13} = 0.035\,,\ \ u_{23} = 19.98\,,\ \ d_{13} = 3.44\,,\ \ d_{23} = 0.203$$

$$c_{U_3^c} c_{Q_3} = 4.15\ 10^{-4} \qquad \text{and} \qquad c_{D_3^c} c_{Q_3} = 4.76\ 10^{-4}\,.$$

$q_{i3} \equiv C_{Q_i}/C_{Q_3}$ $\qquad$ $u_{i3} \equiv C_{U_i^c}/C_{U_3^c}$ $\qquad$ $d_{i3} \equiv C_{D_i^c}/C_{D_3^c}$ $\qquad$ $i = 1,2$



leptons require 6+2=8 more parameters

normal ordering

$$m_1 \approx 10\ \text{meV}\,, \qquad m_{\beta\beta} \approx 6\ \text{meV}\,, \qquad \sum_{i=1}^{3} m_i \approx 74\ \text{meV}\,,$$

$$\delta_{\text{PMNS}} \approx 0.94\,\pi\,, \qquad \alpha_{21} \approx 1.29\,\pi\,, \qquad \alpha_{31} \approx 0.14\,\pi\,.$$

deviations from $\bar{\theta} = 0$

SUSY unbroken

no corrections from K
no corrections from nonrenormalizable operators: $SL(2, \mathbb{Z})$

SUSY breaking corrections

potentially big if soft terms violate flavour in a generic way

minimized if $\Lambda_{CP} \gg \Lambda_{SUSY}$ (as e.g. in gauge mediation)

and soft breaking terms respect the flavour structure of the SM

$$\bar{\theta} \lesssim \frac{M_t^4 M_b^4 M_c^2 M_s^2}{v^{12}} J_{\mathrm{CP}} \tan^6 \beta \sim 10^{-28} \tan^6 \beta.$$

SM corrections

negligible: $\bar{\theta} \leq 10^{-18}$ at four loops

J.R. Ellis, M.K. Gaillard, 'Strong and Weak CP Violation', Nucl.Phys.B 150 (1979) 141.

I.B. Khriplovich, 'Quark Electric Dipole Moment and Induced θ Term in the Kobayashi-Maskawa Model', Phys.Lett.B 173 (1986) 193.

to recap

- in a SUSY & CP & modular-invariant theory:
  τ can generate a large CKM phase without contributing to $\bar{\theta}$

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to recap

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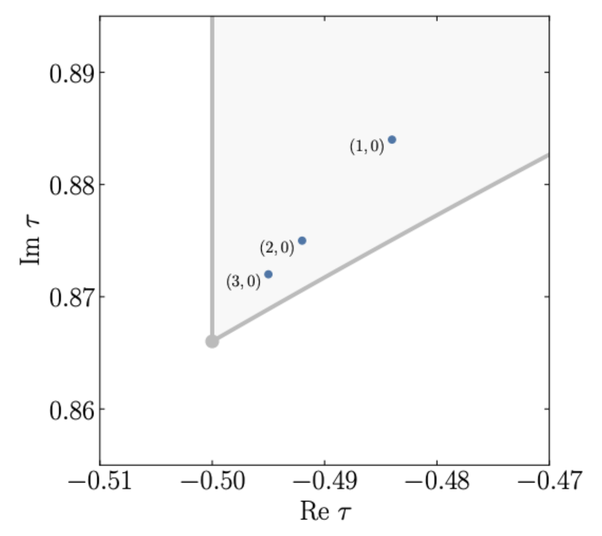
- evidence for modular-invariant potentials where τ spontaneously breaks CP

  [Novichkov, Penedo, Petcov 2201.02020
  Leedom, Righi, Westphal 2212.03876]



to recap

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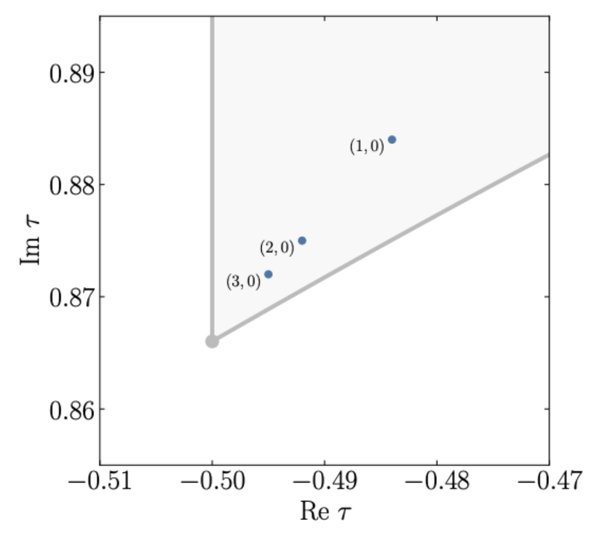


- options for $\langle Im\ S \rangle = 0$

  - dynamics dominated by QCD → axion

  - S stabilized at a CP-conserving minimum by Planck-scale dynamics
    if so, $\bar{\theta} = 0$ not altered by flavour physics.

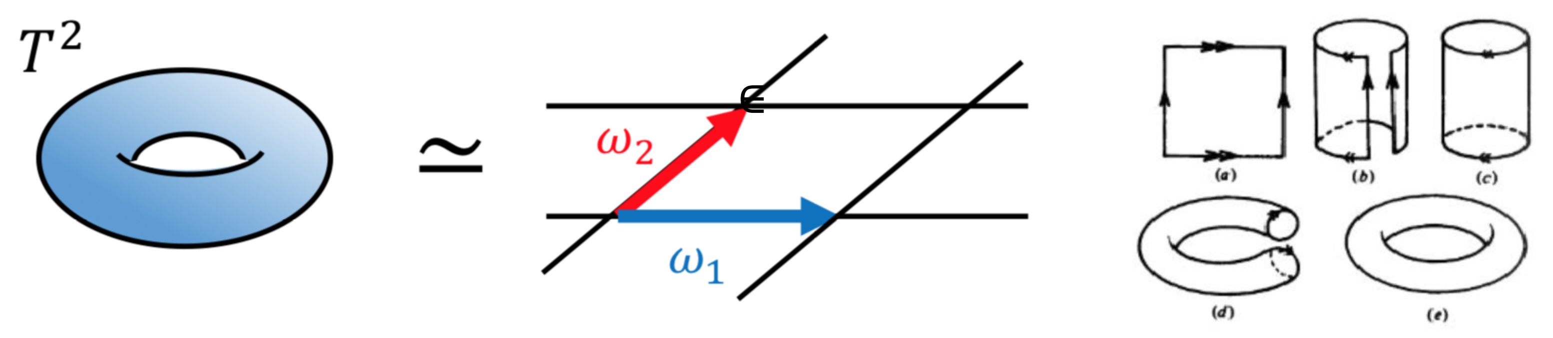
    [Higaki, Kobayashi, Nasu and Otsuka 2405.18813]


THANK
YOU!

back-up slides

modular invariance

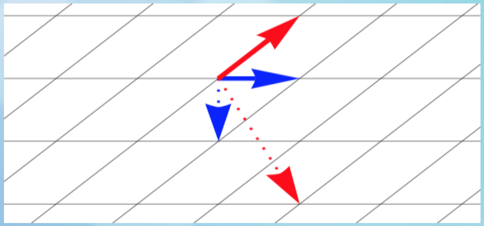
- a discrete gauge symmetry removing redundancy in parametrization of a torus



tori parametrized by

$$\mathcal{M} = \left\{ \tau = \frac{\omega_2}{\omega_1} \quad Im(\tau) > 0 \right\}$$

lattice left invariant by modular transformations:



$$\tau \to \frac{a\tau + b}{c\tau + d} \in SL(2, Z)$$

a, b, c, d integers
$ad - bc = 1$

singularities

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det Y_q(\tau)$$

cannot be both holomorphic everywhere [have opposite weight]

- $e^{-8\pi^2 f_3(S,\tau)}$ expected to be singular at $\tau = i\infty$ [Gonzalo, Ibanez, Uranga, 1812.06520]

Distance Conjecture: [Ooguri, Vafa 0605264]

$\tau = i\infty$ is infinitely far away from any point in D

explicit computation in string theory compactifications

$$f_3(S,\tau) = k_3 S - \frac{k_{f_3}}{8\pi^2} \log \eta(\tau) + \cdots$$

- $\det Y_q(\tau)$ exhibits a zero at $\tau = i\infty$ if

discriminant form

$$k_{\det} = 12\, m > 0$$

➡

$$\det Y_q(\tau) \propto \Delta(\tau)^m$$

$$q \equiv e^{i\, 2\pi\tau}$$

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

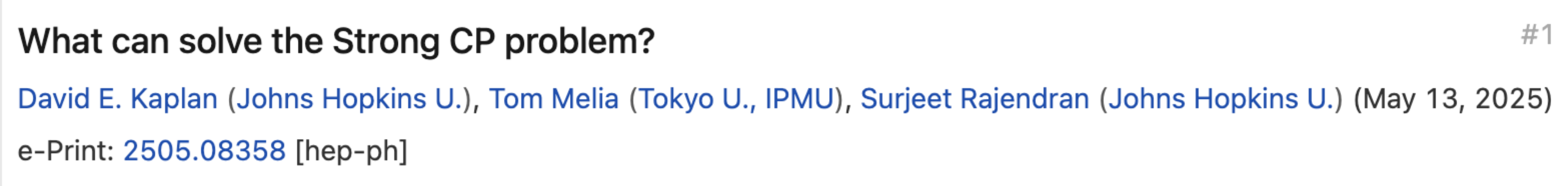
transformation of $f_a(S,\tau)$

$$f_a(S,\tau) \to f_a(S,\gamma\tau) - \frac{1}{8\pi^2}\sum_M 2T_a(M)k_M \log(c\tau + d)$$

intrinsic
τ-dependence

anomaly

[Konishi, Shizuya 1985;
Ferrara, Kounnas, Lust, Zwirner, 1991;
Dixon, Kaplunovsky, Louis, 1991;
N. Arkani-Hamed, H. Murayama 9707133]



What can solve the Strong CP problem? #1

David E. Kaplan (Johns Hopkins U.), Tom Melia (Tokyo U., IPMU), Surjeet Rajendran (Johns Hopkins U.) (May 13, 2025)

e-Print: 2505.08358 [hep-ph]

- θ_{QCD} is not a Lagrangian parameter as a mass, a coupling, …
  it is a variable that labels a vacuum

  the Hilbert space of physical states is the whole $\mathcal{H} = \sum_{\theta} \mathcal{H}_{\theta}$

- strong CP problem = why do we live in $\mathcal{H}_0$ if the universe had access to the whole $\mathcal{H}$?
  cannot be solved by enforcing P/CP: the vacuum can violate CP, even in a CP-invariant theory

- a problem with this picture: each $\mathcal{H}_\theta$ is a superselection sector

$\langle\theta|A|\theta'\rangle = 0$ for any observable A

$|\psi_1\rangle = |\theta\rangle + |-\theta\rangle$ $\quad$ $|\psi_2\rangle = |\theta\rangle - |-\theta\rangle$ represent one and the same state

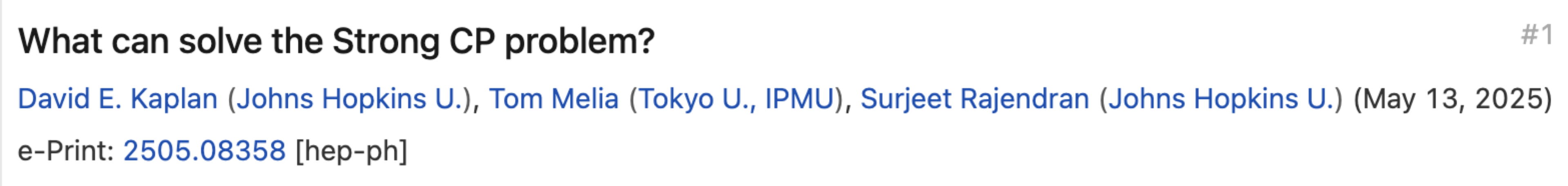
but $\langle\psi_1|P|\psi_1\rangle = +1$

$\langle\psi_2|P|\psi_2\rangle = -1$

➡ P cannot be defined in $\mathcal{H}$

- to define P/CP in QCD, we need to take $\mathcal{H} = \mathcal{H}_0$ or $\mathcal{H} = \mathcal{H}_\pi$

the unique vacuum is CP-invariant



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e-Print: 2505.08358 [hep-ph]

- in our framework CP is conserved but $\bar{\theta}$ is a dynamical variable
i.e. we do not set $\theta_{QCD} = 0$ by CP invariance

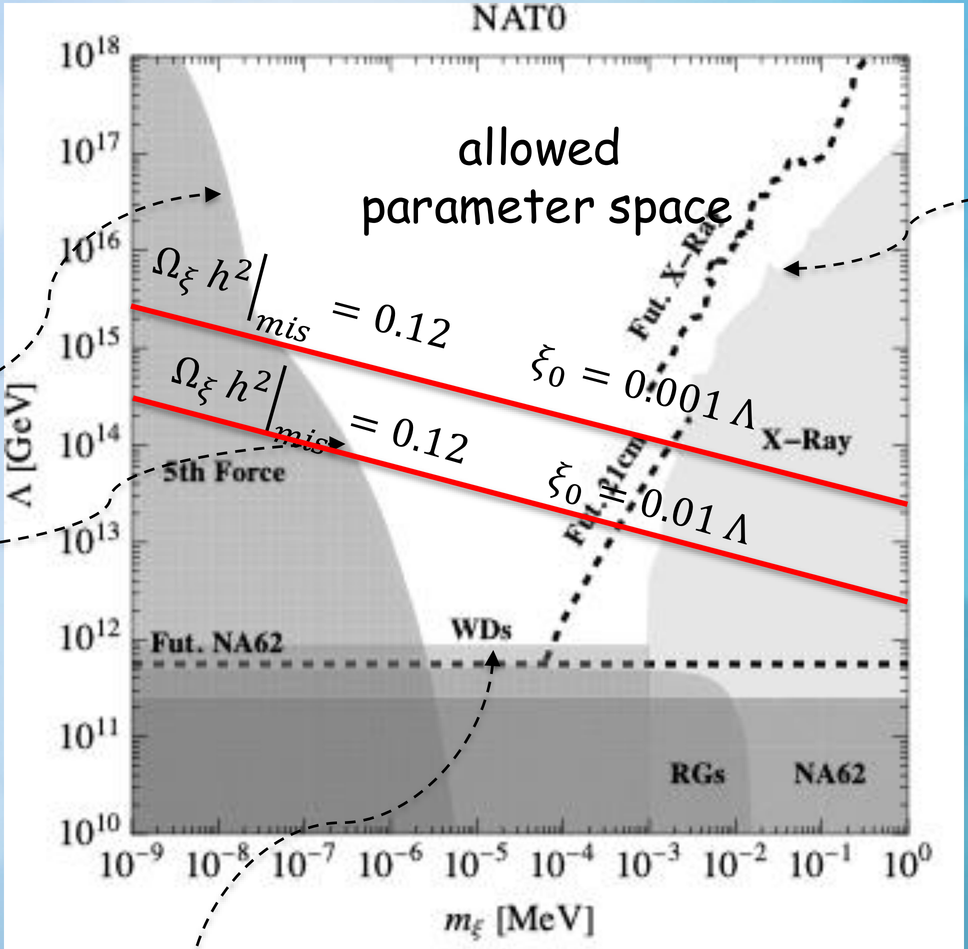
$$\bar{\theta} = \arg[e^{-8\pi^2 f_3(S,\tau)} \det Y_q(\tau)]$$

CP invariance → $f_3(S^*, \tau^*) = f_3^*(S, \tau)$

θ_{QCD} is inside S → $f_3(S, \tau) = S + \cdots$

a light spin-0 component in τ ?

FF, Robert Ziegler, 2411.0810]1



limits from
Inverse
Square
Law
of gravity

X-rays diffuse
emissions
from DM decay
in galaxy clusters

stellar energy loss
in Red Giants

axion solution

- 🟩 $\bar{\theta}$ dynamically relaxed to zero by the axion, would-be GB of a global, anomalous $U(1)_{PQ}$ symmetry

- 🟩 provides a candidate for DM
- 🟩 many axion candidates in e.g. superstring theories

- 🟥 axion quality problem

  minimum of $V(a)$ should be at $a = 0$

$$V(a) = V_{QCD}(a) - M^4 e^{-S} \cos(\frac{a}{f_a} + \delta)$$

$$M = M_P$$
$$\delta = \mathcal{O}(1)$$

⇒ $S \geq 200$

- 🟥 axion undetected, so far

Nelson-Barr solution

- CP ia a symmetry of the UV,
  SB to get $\bar{\theta} = 0$ & $\delta_{CKM} = \mathcal{O}(1)$

- CP $\Rightarrow$ $\theta_{QCD} = 0$

- heavy vector-like quark sector

$$m = \begin{array}{c|c} Q & q \\ \hline \mu & \lambda_a \eta_a \\ 0 & y\,v \end{array}$$

- CP spontaneously broken
  by $\langle \eta_a \rangle$ complex

  [one is not enough]

- $\mu \approx \lambda_a \eta_a$ [tuning]

our solution

no extra matter

CP spontaneously broken
by τ alone

no tuning

$\mathcal{N} = 1$ supergravity

- K and w no more independent

$$\mathcal{G} = \frac{K}{M_{Pl}^2} + \log\left|\frac{w}{M_{Pl}^3}\right|^2$$

$$K = -k_W M_{Pl}^2 \log(-i\tau + i\tau^+) + \cdots$$

$$w(\tau) \to (c\tau + d)^{-k_W}\, w(\tau)$$

- $\bar{\theta} = \arg A \qquad \text{where now} \qquad A = e^{-8\pi^2 f_3} W^{-C_3} \det Y_u \det Y_d\,.$

$$[\arg M_3 = -\arg W]$$

- $k_{f_a} + \sum_M 2T_a(M)k_M + k_W\left[C_a - \sum_M T_a(M)\right] = 0.$

$$A(S, \gamma\tau) = (c\tau + d)^{k_A}\, A(S, \tau)$$

$$k_A = 3(k_{H_u} + k_{H_d})$$

gauge coupling unification

$$f_a = \frac{1}{g_a^2} - i\,\frac{\theta_{QCD}}{8\pi^2}$$

holomorphic coupling ≠ physical

$$K_S = -\bar{M}_{\rm Pl}^2 \ln(S+\bar{S})\,, \qquad f_a = \kappa_a S - \frac{k_{f_a}}{8\pi^2}\ln\eta^2(\tau)$$

$$\frac{16\pi^2}{g_a^{c2}(\mu)} = 16\pi^2\kappa_a \mathrm{Re}\,S + 2C_a\ln\frac{\kappa_a}{2} + b_a\ln\frac{\Lambda^2}{2\mathrm{Re}\,S\,\mu^2} + \Delta_a(\tau,\bar{\tau})$$

unification condition

1-loop running

threshold corrections

$$\Delta_a(\tau,\bar{\tau}) \equiv -k_{f_a}\ln y|\eta(\tau)|^4$$

dependence on: SUSY-breaking scale, sparticle spectrum, k_a levels, …

modular forms

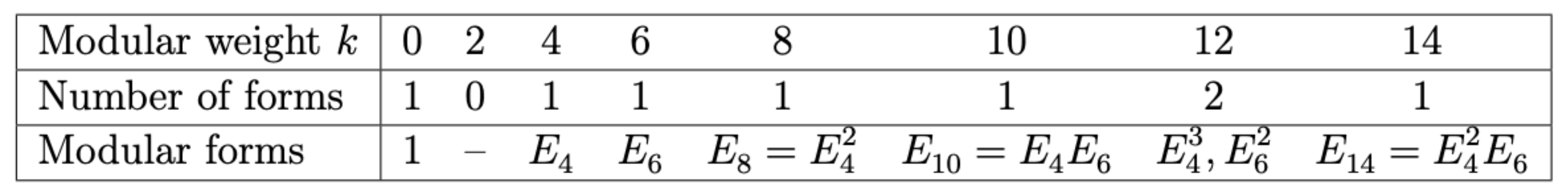
$$f(\gamma\tau) = (c\tau + d)^k \, f(\tau)$$

& $f(\tau)$ holomorphic everywhere
included at $\tau = i\,\infty$

$k < 0$: no modular forms

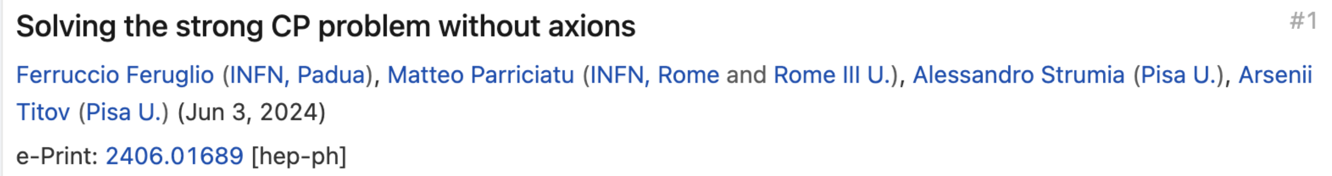
$k = 0$: modular forms are constants

$k > 0$: modular forms polynomials in $E_4(\tau), E_6(\tau)$



Modular weight k	0	2	4	6	8	10	12	14
Number of forms	1	0	1	1	1	1	2	1
Modular forms	1	–	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	E_4^3, E_6^2	$E_{14} = E_4^2 E_6$

variants



Solving the strong CP problem without axions #1

Ferruccio Feruglio (INFN, Padua), Matteo Parriciatu (INFN, Rome and Rome III U.), Alessandro Strumia (Pisa U.), Arsenii Titov (Pisa U.) (Jun 3, 2024)

e-Print: 2406.01689 [hep-ph]

- higher levels, smaller weight

modular forms associated with subgroups of $SL(2,Z)$

$$k_{Q_i} = k_{U_i^c} = k_{D_i^c} = (-1,0+1)\ or\ (-2,0+2)$$

perhaps easier to occur in string theory

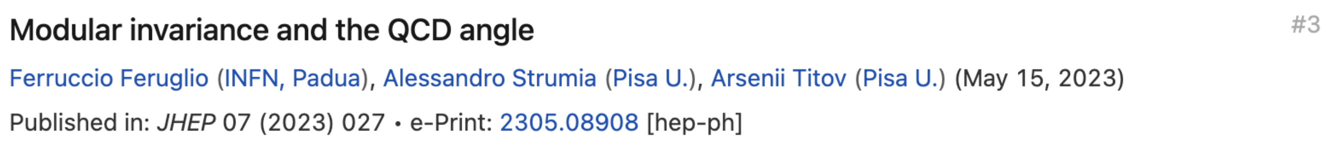
- With heavy vector-like quarks

anomaly of IR theory canceled by a nontrivial gauge kinetic function

$$f_{IR} = f_{UV} - \frac{1}{8\pi^2}\log\det Y_{Heavy}(\tau)$$

many more viable patterns of quark mass matrices

- can be extended to supergravity



Modular invariance and the QCD angle #3

Ferruccio Feruglio (INFN, Padua), Alessandro Strumia (Pisa U.), Arsenii Titov (Pisa U.) (May 15, 2023)

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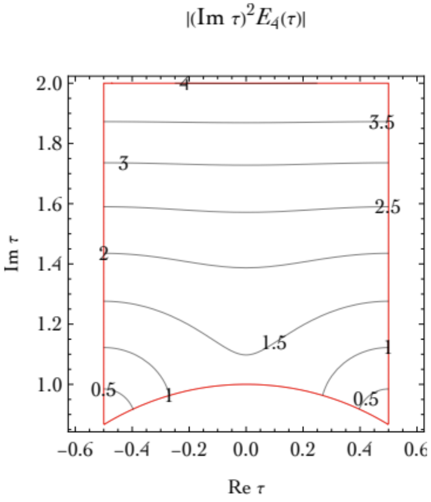
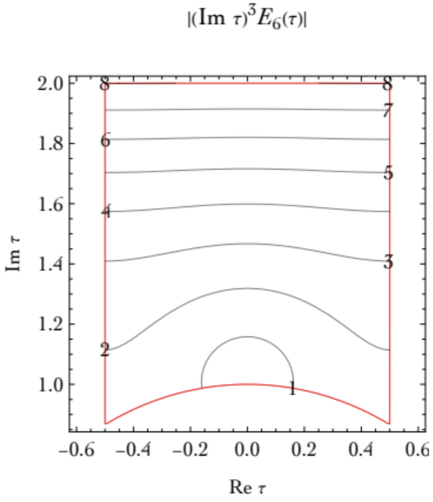
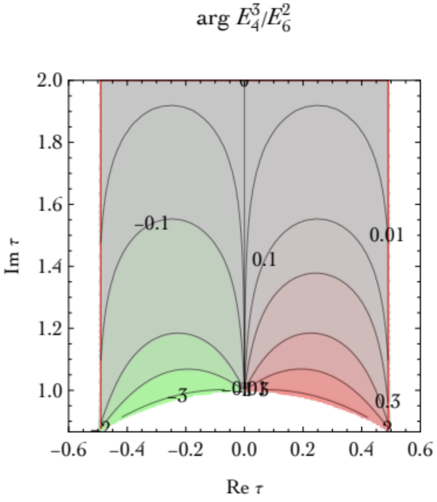


$|(\mathrm{Im}\ \tau)^2 E_4(\tau)|$



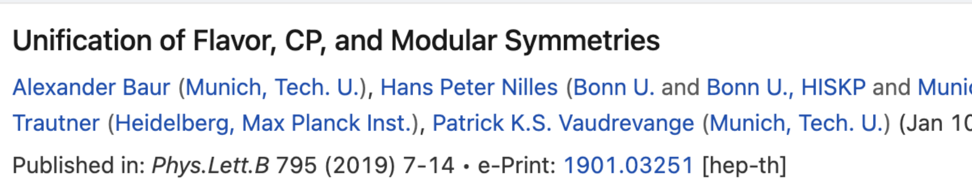
$|(\mathrm{Im}\ \tau)^3 E_6(\tau)|$



$\arg E_4^3/E_6^2$

Ingredients

1. CP in the UV
2. Yukawa couplings are field-dependent quantities
3. the vacuum has a redundant description: vacua related by $SL(2,\mathbb{Z})$ are equivalent
4. CP and $SL(2,\mathbb{Z})$ are unified in a gauge flavour symmetry
5. absence of anomalies
6. singularities in the EFT

	Ingredients	String Theory
1.	CP in the UV	the four-dimensional CP symmetry is a gauge symmetry in most string theory compactifications.
2.	Yukawa couplings are field-dependent quantities	string theory has no free parameters and couplings are set by moduli VEVs
3.	the vacuum has a redundant description: vacua related by $SL(2,\mathbb{Z})$ are equivalent	modular invariance is a key ingredient of string theory compactifications
4.	CP and $SL(2,\mathbb{Z})$ are unified in a gauge flavour symmetry	 **Unification of Flavor, CP, and Modular Symmetries** Alexander Baur (Munich, Tech. U.), Hans Peter Nilles (Bonn U. and Bonn U., HISKP and Muni Trautner (Heidelberg, Max Planck Inst.), Patrick K.S. Vaudrevange (Munich, Tech. U.) (Jan 1 Published in: *Phys.Lett.B* 795 (2019) 7-14 • e-Print: 1901.03251 [hep-th]
5.	absence of anomalies	mandatory in string theory
6.	singularities in the EFT	emergence of singularities at infinite distances in moduli space.

to recap

- in a SUSY & CP & modular-invariant theory:
  τ can generate a large CKM phase without contributing to $\bar{\theta}$

e.g. $f_3(S,\tau) = k_3 S - \frac{k_{f_3}}{8\pi^2}\log\eta(\tau) + \cdots$

$$\frac{k_3}{16}\int d^2\theta\, S\, W_3 W_3 - \underbrace{\frac{k_{f_3}}{16}\int d^2\theta\, \frac{\log\eta(\tau)}{8\pi^2} W_3 W_3 + \int d^2\theta\, w(\tau) + h.c}_{\boxed{\delta\bar{\theta} = 0}}$$

modular invariance

- a discrete gauge symmetry removing redundancy in parametrization of a torus

tori parametrized by $\mathcal{M} = \{\tau \in \mathbb{C}, Im(\tau) > 0\}$

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$$

- modular functions: transform covariantly under $SL(2, Z)$

weight

$$A(\gamma\tau) = (c\,\tau + d)^k A(\tau) \qquad c\,\tau + d = \left(\frac{d\gamma\tau}{d\tau}\right)^{-1/2}$$

if holomorphic everywhere: modular forms

$$A(\gamma\tau) = A(\tau) \quad \Rightarrow \quad A(\tau) = \text{constant}$$