Scalar scattering amplitudes from geometry

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based on 2308.00017 + work to appear
with Joe Davighi & Mohammad Alminawi

Swiss National Science Foundation

COMETA

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The key question

consider **a theory of N scalar fields**

$$\phi_i(x) \qquad i = 1 \dots N$$

under field redefinitions*

$$\phi_1 \mapsto \phi_1 + \phi_2^2 \qquad \phi_1 \mapsto \phi_1 e^{i\phi_2} \qquad \phi_2 \mapsto \frac{\phi_1 - \phi_2}{\sqrt{2}} \qquad \dots$$

- $\mathscr{L}$ **changes**
- Feynman rules **change**
- Feynman diagrams for a certain process **change**
- cross-sections and on-shell amplitudes **stay the same**

can we find a manifestly invariant parameterization of on-shell amplitudes?

*that preserve the mass spectrum

Phenomenological motivation: SMEFT vs HEFT

the two EFTs differ in how the 4 scalar fields of the SM are packaged in $SU(2) \times U(1)$ representations

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_2 + i\,\pi_1 \\ \phi - i\,\pi_3 \end{pmatrix} \qquad \text{vs.} \qquad \mathbf{U} = \exp\left[\frac{i\sigma^i \pi_i}{v}\right], \quad \phi$$

SMEFT **HEFT**

☞ fundamental differences are obscured by the field-redefinition relation between the EFTs

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SMEFT **HEFT**

☞ fundamental differences are obscured by the field-redefinition relation between the EFTs

tool of choice: **geometrical methods**

- formalism originally introduced in the 60–80s. has applications e.g. in quantum gravity
  Meetz 1969, (Ecker),Honerkamp 1971,1972, Tataru 1975, Alvarez-Gaumé+ 1981, Vilkovisky 1984, Gaillard 1986. . .
- first considered ∼10 yrs ago for a "universal" formulation of SMEFT/HEFT
  that can be studied independently of field representation Alonso,Jenkins,Manohar 1511.00724, 1605.03602
- in the last 5 years it has found several applications in the context of EFTs
  Pilaftis+ 2006.05831,2406.13594 Cohen+ 2008.08597,2108.03240,2202.06965,2312.06748, 2410.21378,2504.12371 Cheung+ 2111.03045,2202.06972
  Alonso+ 2109.13290,2207.02050,2307.14301 Helset+ 2210.08000,2212.03253 Craig+ 2305.09722 Assi+ 2307.03187,2504.18537
  Jenkins+ 2310.19883,2308.06315 Derda+ 2403.12142 Craig,Lee 2307.15742 IB+ 2308.00017 Li+ 2411.04173 Aigner+ 2503.09785 . . .

Geometrical description for scattering amplitudes

interactions with **2 derivatives** define a **metric** on the manifold of field configurations (field space)

$$\mathscr{L} = \frac{1}{2}\,\partial_\mu\phi^i\,\partial^\mu\phi^j\,g_{ij}(\phi) + \dots \qquad \longrightarrow \qquad g = g_{ij}(u)\,du^i du^j$$

☞ ϕ^i treated as coordinates on field space. theory is characterized by $R_{ijkl}, R_{ij}, R\dots$

On-shell amplitudes. for massless/soft scalars Cheung+ 2111.03045, Helset+ 2210.08000

$$\begin{aligned}
\mathcal{A}_{ijk} &= 0\\
\mathcal{A}_{ijkl} &= s_{ij}\,R_{ikjl} + s_{ik}\,R_{ijkl}\\
\mathcal{A}_{ijklm} &= s_{ij}\,\nabla_l R_{ikjm} + s_{ik}\,\nabla_l R_{ijkm} + s_{il}\nabla_k R_{ijlm} + s_{kl}\,\nabla_i R_{mkjl} + (s_{jl}+s_{il})\,\nabla_i R_{mjkl}
\end{aligned}$$

$R, \nabla R\dots$ are evaluated at the vacuum $\phi^i \equiv 0$, $\quad s_{ij} = (p_i + p_j)^2$

example:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1\left[1 + a_2\frac{\phi_2}{\Lambda}\right] + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2\left[1 + b_{11}\frac{\phi_1^2}{2\Lambda^2} + b_{111}\frac{\phi_1^3}{3!\Lambda^3}\right]$$

$$R_{1212} = \left(a_2{}^2 - 2b_{11}\right)/4\Lambda^2 \qquad \nabla_1 R_{1212} = -b_{111}/2\Lambda^3 \qquad \nabla_2 R_{1212} = -a_2{}^3/2\Lambda^3$$

What do we learn?

- $\mathcal{A}$ are **tensors** under diffeomorphism field redefinitions — Cohen+ 2202.06965,2312.06748, Alminawi wip

  true even with masses, up to terms that vanish at the vacuum ($\frac{\partial\Gamma}{\partial\phi}$) or for on-shell external legs ($\frac{\partial^2\Gamma}{\partial\phi\partial\phi}$)

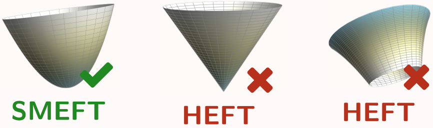
- if ϕ^i are mass eigenstates, $\mathcal{A}_{i_1\dots i_n}$ with fixed $i_1 \dots i_n$ indices is **invariant**

  if they are not, we can move to a mass eigenstate basis $\phi^i \to \phi'^a$
  by LSZ, a physical on-shell amplitude is $\mathcal{A}'_{a_1\dots a_n} = U^{i_1}{}_{a_1} \dots U^{i_n}{}_{a_n} \mathcal{A}_{i_1\dots i_n}$ with $U^i{}_a = \delta\phi^i/\delta\phi'^a$

- $R, \nabla R \dots$ at $\phi^i = 0$ are **invariant, measurable** quantities

- the expressions of $\mathcal{A}_{i_1\dots i_n}$ in terms of Riemanns are **universal**: computed once for all theories

- a theory can be characterized by the geometry of its field space



Cohen,Craig,Lu,Sutherland 2008.08597

Limitations & Challenges

- interactions with $\partial^{\geqslant 4}$ or $\partial^0 \equiv V(\phi)$ are not assigned a geometric meaning

  ☞ some attempts with "functional geometry" Helset+ 2210.08000,2202.06972, Cohen+ 2202.06965,2312.06748,2410.21378
  Lagrange spaces Craig+ 2305.09722 and jet bundles Craig,Lee 2307.15742, Alminawi,IB,Davighi 2308.00017

- invariance under **derivative field redefinitions** is not captured, e.g. $\phi_1 \to \phi_1 + \Box\phi_2/\Lambda^2$

- computing higher-point amplitudes is highly non trivial!

  Feynman rules are *not* covariant, e.g.

$$\mathcal{R}_{ijk} = \frac{i}{2}\left[p_i^2\,\Gamma_{ijk} + p_j^2\,\Gamma_{jki} + p_k^2\,\Gamma_{kij}\right]$$

  recovering covariant expressions requires recombining dozens of non-tensor terms into tensors, e.g.

$$g_{in}\left(\partial_k\Gamma^n_{lj} - \partial_l\Gamma^n_{kj} + \Gamma^n_{km}\Gamma^m_{lj} - \Gamma^n_{lm}\Gamma^m_{kj}\right) = R_{ijkl}$$

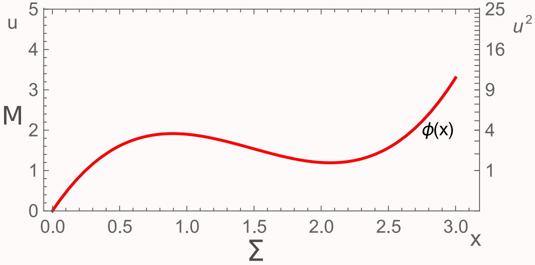
  some simplification can be achieved with normal coordinates, where $\Gamma_{ijk} = 0$.
  however, there is no basis in which all non-covariant terms vanish

Fibre bundle picture

main aim: include derivatives $\partial_\mu \phi(x) \rightarrow$ keep ϕ dependence on x manifest!

natural structure: **fibre bundle**

Alminawi,IB,Davighi 2308.00017



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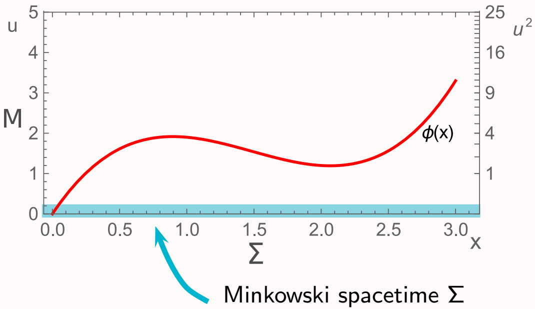

Minkowski spacetime Σ

w/ coord x^μ, metric η

$\equiv$ **base space**

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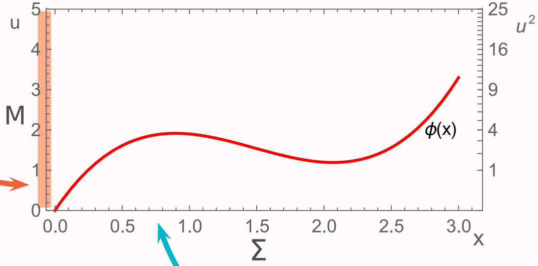
field config. space M

w/ coord u^i

$\equiv$ **fibre**



Minkowski spacetime Σ

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Fibre bundle picture

main aim: include derivatives $\partial_\mu \phi(x) \to$ keep ϕ dependence on x manifest!

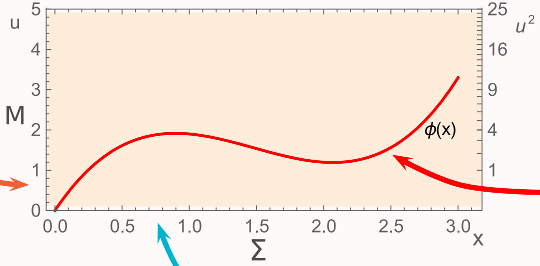
natural structure: **fibre bundle** (E, Σ, π) Alminawi,IB,Davighi 2308.00017

locally: $E_x = \Sigma_x \times M$

field config. space M

w/ coord u^i

$\equiv$ **fibre**



field $\phi^i(x) : \Sigma \to M$

$\equiv$ **section** of E over Σ

$\neq u^i$

Minkowski spacetime Σ

w/ coord x^μ, metric η

$\equiv$ **base space**

Fibre bundle metric $\to \partial^2$ scalar Lagrangian

the fibre bundle is a Riemannian manifold, on which we can build a **metric**

$$g = \begin{pmatrix} dx^\mu & du^i \end{pmatrix} \begin{pmatrix} g_{\mu\nu} & g_{\mu j} \\ g_{\nu i} & g_{ij} \end{pmatrix} \begin{pmatrix} dx^\nu \\ du^j \end{pmatrix} = g_{\mu\nu} dx^\mu dx^\nu + g_{ij} du^i du^j + 2 g_{\mu i} dx^\mu du^i$$

Poincaré invariance $\Rightarrow g_{IJ}$ depend on u^i but *not* on x^μ, $g_{\mu i} \equiv 0$

⚠ $g_{\mu\nu} \neq \eta_{\mu\nu}$

Metric to Lagrangian.

$$\mathscr{L} = \frac{\Lambda^4}{2} \eta^{\mu\nu} g_{\mu\nu}(\phi) + \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j$$

$\equiv -V(\phi)$

same as usual geo

geometric interpretation of the scalar potential!

Scalar on-shell amplitudes from fibre bundle geometry

2-point amplitudes = inverse propagators are always tensors:

Alminawi,IB,Davighi wip

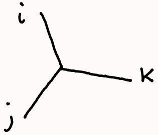
$$\Delta^{-1}_{ij}(p^2) = -i\left(p^2 g_{ij} + \Lambda^4 R^{\mu}{}_{i\mu j}/2\right) \quad \longrightarrow \quad \Delta^{ij}(p^2) = i\left(p^2 g_{ij} + R^{\mu}{}_{i\mu j}/2\right)^{-1}$$

where $\frac{\Lambda^4}{2} R^{\mu}{}_{i\mu j} = \frac{\Lambda^4}{2}\eta^{\mu\nu}\partial_i\partial_j g_{\mu\nu} = -\partial_i\partial_j V = -m^2_{ij}$

$g, R, \nabla R \ldots$ at vacuum

3-point on-shell amplitudes $\mathcal{A}_{ijk} = i\frac{\Lambda^4}{2}\nabla_i R^{\mu}{}_{j\mu k}$

= trilinear vertex computed with all external lines on-shell



example $\mathscr{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1\left[1 + a_2\frac{\phi_2}{\Lambda}\right] + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \left[\frac{m_1^2}{2}\phi_1^2 + \frac{m_2^2}{2}\phi_2^2 + \frac{v_{112}}{2}\Lambda\phi_1^2\phi_2\right]$

$\to \Lambda^4\ \nabla_2 R^{\mu}{}_{1\mu 1} = \frac{a_2}{4\Lambda}(2m_1^2 - m_2^2) - \frac{\Lambda v_{112}}{2}$

both kinetic and potential contributions are simultaneously accounted for! **bundle feature!**

Scalar on-shell amplitudes from fibre bundle geometry

all n-point on-shell amplitudes can be computed by

1. deriving "covariant Feynman Rules" $\mathcal{R}_{i_1\ldots i_n}$
2. gluing them together with Δ^{ij} propagators, in the usual diagrammatic way

Alminawi,IB,Davighi wip

☺ covariant FR can be predicted with a generalization of normal coordinates expansion of $g_{\mu\nu}$, g_{ij}

☺ non-tensor terms never need to be evaluated

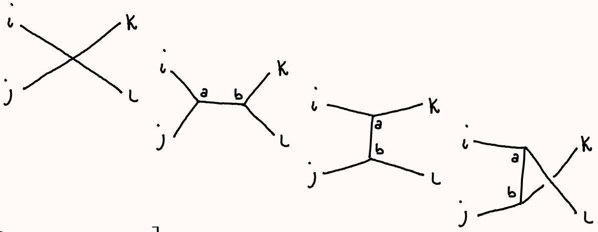
☞ non trivial result: field-redefinition-invariant scalar on-shell amplitudes can be built connecting **vertices** that behave as if all their lines were on shell, and are **individually invariant** under redefinitions all n-point on-shell amplitudes can be computed by

1. deriving "covariant Feynman Rules" $\mathcal{R}_{i_1\dots i_n}$
2. gluing them together with Δ^{ij} propagators, in the usual diagrammatic way

Alminawi,IB,Davighi wip

4-point on-shell amplitudes.

$$
\begin{aligned}
\mathcal{A}_{ijkl} =& \mathcal{R}_{ijkl} \\
&+ \mathcal{R}_{ija}\,\Delta^{ab}(s)\,\mathcal{R}_{bkl} \\
&+ \mathcal{R}_{ika}\,\Delta^{ab}(t)\,\mathcal{R}_{bjl} \\
&+ \mathcal{R}_{ila}\,\Delta^{ab}(u)\,\mathcal{R}_{bkl} \\
=& \mathcal{R}_{ijkl} + \frac{1}{4!}\left[3\,\mathcal{R}_{ija}\,\Delta^{ab}(s_{ij})\,\mathcal{R}_{bkl} + \text{perm}_{ijkl}\right]
\end{aligned}
$$



where

$$
\mathcal{R}_{ijk} = \mathcal{A}_{ijk} = i\frac{\Lambda^4}{2}\nabla_i R^{\mu}{}_{j\mu k} \qquad \mathcal{R}_{ijkl} = \frac{i}{4!}\left[\frac{\Lambda^4}{2}\nabla_i\nabla_j R^{\mu}{}_{k\mu l} - 2\Lambda^4 R^{\mu}{}_{i\nu j} R^{\nu}{}_{k\mu l} - 2s_{ij}\,R_{ilkj} + \text{perm}_{ijkl}\right]
$$

all n-point on-shell amplitudes can be computed by

1. deriving "covariant Feynman Rules" $\mathcal{R}_{i_1 \dots i_n}$
2. gluing them together with Δ^{ij} propagators, in the usual diagrammatic way

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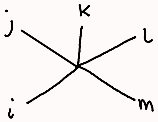
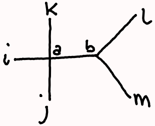
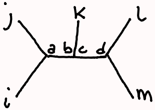
5-point on-shell amplitudes.







$$
\begin{aligned}
\mathcal{A}_{ijklm} = \; & \mathcal{R}_{ijklm} \\
& + \frac{1}{5!}\Big[10\, \mathcal{R}_{ijka}\, \Delta^{ab}(s_{45})\, \mathcal{R}_{blm} + \\
& \quad + 15\, \mathcal{R}_{ija}\, \Delta^{ab}(s_{ij})\, \mathcal{R}_{kbc}\, \Delta^{cd}(s_{lm})\, \mathcal{R}_{dlm} \\
& \quad + \text{perm}_{ijklm} \Big]
\end{aligned}
$$

where

$$
\mathcal{R}_{ijklm} = \frac{i}{5!}\left[\frac{\Lambda^4}{2} \nabla_i \nabla_j \nabla_k R^{\mu}{}_{l\mu m} - 7\Lambda^4 R^{\mu}{}_{i\nu j} \nabla_k R^{\nu}{}_{l\mu m} - 5 s_{ij} \nabla_m R_{ilkj} + \text{perm}_{ijklm} \right]
$$

6-point on-shell amplitudes.

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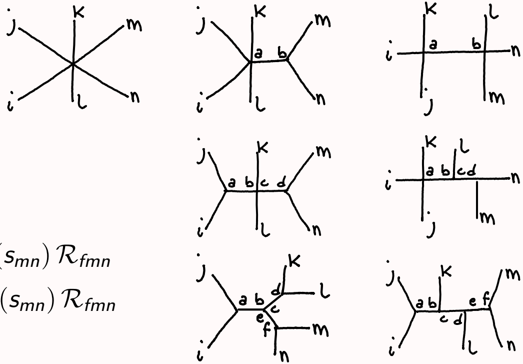
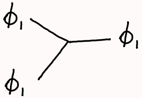
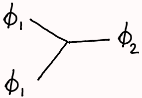
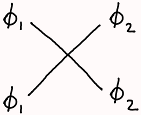
$$\mathcal{A}_{ijklmn} = \mathcal{R}_{ijklmn}$$
$$+\frac{1}{6!}\Big[15\,\mathcal{R}_{ijkla}\,\Delta^{ab}(s_{mn})\,\mathcal{R}_{bmn}$$
$$+10\,\mathcal{R}_{ijka}\,\Delta^{ab}(s_{ijk})\,\mathcal{R}_{blmn}$$
$$+45\,\mathcal{R}_{ijab}\,\Delta^{ac}(s_{kl})\,\mathcal{R}_{ckl}\,\Delta^{bd}(s_{56})\,\mathcal{R}_{dmn}$$
$$+60\,\mathcal{R}_{ijka}\,\Delta^{ab}(s_{ijkl})\,\mathcal{R}_{bck}\,\Delta^{cd}(s_{mn})\,\mathcal{R}_{dmn}$$
$$+15\,\mathcal{R}_{ija}\,\Delta^{ab}(s_{ij})\,\mathcal{R}_{bcd}\,\Delta^{ce}(s_{kl})\,\mathcal{R}_{ekl}\,\Delta^{df}(s_{mn})\,\mathcal{R}_{fmn}$$
$$+90\,\mathcal{R}_{ija}\,\Delta^{ab}(s_{ij})\,\mathcal{R}_{bck}\,\Delta^{cd}(s_{ijk})\,\mathcal{R}_{del}\,\Delta^{ef}(s_{mn})\,\mathcal{R}_{fmn}$$
$$+\text{perm}_{ijklmn}\Big]$$



where
$$\mathcal{R}_{ijklmn} = \frac{i}{6!}\Big[\frac{\Lambda^4}{2}\nabla_i\nabla_j\nabla_k\nabla_l R^{\mu}{}_{m\mu n} - 11\Lambda^4 R^{\mu}{}_{i\nu j}\nabla_k\nabla_l R^{\nu}{}_{m\mu n} - 7\Lambda^4\nabla_i R^{\mu}{}_{j\nu k}\nabla_l R^{\mu}{}_{m\nu n}$$
$$+ 8\Lambda^4 R^{\mu}{}_{i\nu j} R^{\nu}{}_{k\rho l} R^{\rho}{}_{m\mu n} - 9 s_{ij}\nabla_m\nabla_n R_{ilkj} - 8 s_{in} R_{ijk}{}^{q} R_{qlmn} + \text{perm}_{ijklmn}\Big]$$

Example: 4-point on-shell amplitude from fibre bundle geometry

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1\left[1 + a_2\frac{\phi_2}{\Lambda}\right] + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \left[\frac{m_1^2}{2}\phi_1^2 + \frac{m_2^2}{2}\phi_2^2 + \frac{v_{111}}{3!}\Lambda\phi_1^3 + \frac{v_{1122}}{4}\phi_1^2\phi_2^2\right]$$

Usual FR		**Covariant FR**
$-i\Lambda\, v_{111}$		$-i\Lambda v_{111}$
$-i\dfrac{a_2}{2\Lambda}\left[s_{11} - 2m_1^2\right]$		$-i\dfrac{a_2}{2\Lambda}\left[2m_1^2 - m_2^2\right]$
$-i\,v_{1122}$		$-i\,v_{1122} - i\dfrac{{a_2}^2}{2\Lambda^2}\left[4m_1^2 - m_2^2 - \dfrac{s_{11}}{2}\right]$

Example: 4-point on-shell amplitude from fibre bundle geometry

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1\left[1 + a_2\frac{\phi_2}{\Lambda}\right] + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \left[\frac{m_1^2}{2}\phi_1^2 + \frac{m_2^2}{2}\phi_2^2 + \frac{v_{111}}{3!}\Lambda\phi_1^3 + \frac{v_{1122}}{4}\phi_1^2\phi_2^2\right]$$

$$\begin{aligned}\mathcal{A}_{1122} = \\ & - v_{1122} \\ & + \frac{{a_2}^2}{4\Lambda^2}\left(s - 8m_1^2 + 2m_2^2\right) \\ & - \frac{{a_2}^2}{\Lambda^2}\left(m_1^2 - \frac{m_2^2}{2}\right)^2\left[\frac{1}{t - m_1^2} + \frac{1}{u - m_1^2}\right]\end{aligned}$$

Example: 4-point on-shell amplitude from fibre bundle geometry

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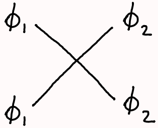
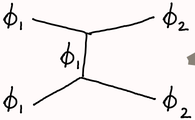
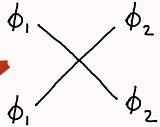
Usual diagrams

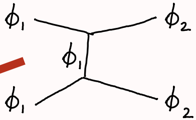




$$\mathcal{A}_{1122} = - v_{1122} + \frac{{a_2}^2}{4\Lambda^2}\left(s - 8m_1^2 + 2m_2^2\right) - \frac{{a_2}^2}{\Lambda^2}\left(m_1^2 - \frac{m_2^2}{2}\right)^2\left[\frac{1}{t - m_1^2} + \frac{1}{u - m_1^2}\right]$$

Covariant diagrams





Loops

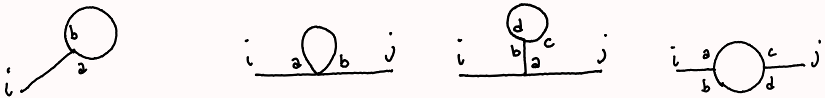
- on-shell amplitudes of **any loop order** can be written as **covariant tensors**

Cohen+ 2312.06748
Alonso,West 2207.02050

- vacuum and on-shell conditions become $\frac{\delta \Gamma^{(L)}}{\delta \phi_i} = 0$, $\frac{\delta^2 \Gamma^{(L)}}{\delta \phi^i \delta \phi^j} = 0$
- verified that covariant FR technology works for 1- and 2-point functions at 1-loop

$$\mathcal{A}_i^{(1)} = \int \frac{d^d k}{(2\pi)^d} \mathcal{R}_{iab} \Delta^{ab}(k^2)$$

$$\mathcal{A}_{ij}^{(1)} = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \mathcal{R}_{ijab} \Delta^{ab}(k^2) + \mathcal{R}_{ija} \Delta^{ab}(0) \mathcal{R}_{bcd} \Delta^{cd}(k^2) + \mathcal{R}_{iab} \Delta^{ac}(k^2) \Delta^{bd}((p+k)^2) \mathcal{R}_{cdj}$$



- in principle should generalize for any n-point [in progress]

Summary & Outlook

- **geometrical methods** long used to describe scalar theory in a **“universal” way**
  - ☞ tensors at the vacuum are field-redefinition-invariant, measureable quantities
  - ☞ theories can be characterized by geometric properties
- we proposed a novel approach that accounts for spacetime dependence of fields switching from a field space to a **fibre bundle** picture
- identified general prescription to get **on-shell amplitudes** in scalar theories with up to 2 ∂
  - ☞ includes geometrical description of scalar potential
  - ☞ based on covariant Feynman Rules, assembled in the usual approach
  - ☞ avoids heaviest computational hurdles
- a lot to be done!
  - solidify application to higher loops, renormalization
  - including higher derivatives $\rightarrow$ **jet bundles geometry**
  - gauging, consider extension to fermions
  - ...

...stay tuned!

Backup slides

Covariant Feynman rules

$$\mathcal{R}_{ijk} = i\frac{\Lambda^4}{2}\nabla_i R^{\mu}{}_{j\mu k}$$

$$\mathcal{R}_{ijkl} = \frac{i}{4!}\left[\frac{\Lambda^4}{2}\nabla_i\nabla_j R^{\mu}{}_{k\mu l} - 2\Lambda^4 R^{\mu}{}_{i\nu j} R^{\nu}{}_{k\mu l} - 2s_{ij}\, R_{ilkj} + \text{perm}\right]$$

$$\mathcal{R}_{ijklm} = \frac{i}{5!}\left[\frac{\Lambda^4}{2}\nabla_i\nabla_j\nabla_k R^{\mu}{}_{l\mu m} - 7\Lambda^4 R^{\mu}{}_{i\nu j}\nabla_k R^{\nu}{}_{l\mu m} - 5s_{ij}\,\nabla_m R_{ilkj} + \text{perm}\right]$$

$$\mathcal{R}_{ijklmn} = \frac{i}{6!}\Bigg[\frac{\Lambda^4}{2}\nabla_i\nabla_j\nabla_k\nabla_l R^{\mu}{}_{m\mu n} - 11\Lambda^4 R^{\mu}{}_{i\nu j}\nabla_k\nabla_l R^{\nu}{}_{m\mu n} - 7\Lambda^4\nabla_i R^{\mu}{}_{j\nu k}\nabla_l R^{\mu}{}_{m\nu n}$$
$$+ 8\Lambda^4 R^{\mu}{}_{i\nu j} R^{\nu}{}_{k\rho l} R^{\rho}{}_{m\mu n} - 9s_{ij}\,\nabla_m\nabla_n R_{ilkj} - 8s_{in}\, R_{ijk}{}^{q} R_{qlmn} + \text{perm}\Bigg]$$

$$\mathcal{R}_{ijklmno} = \frac{i}{7!}\Bigg[\frac{\Lambda^4}{2}\nabla_i\nabla_j\nabla_k\nabla_l\nabla_m R^{\mu}{}_{n\mu o} - 16\Lambda^4 R^{\mu}{}_{i\nu j}\nabla_k\nabla_l\nabla_m R^{\nu}{}_{n\mu o} - 25\Lambda^4\nabla_i R^{\mu}{}_{j\nu k}\nabla_l\nabla_m R^{\nu}{}_{n\mu o}$$
$$+ 64\Lambda^4 R^{\mu}{}_{i\nu j} R^{\nu}{}_{k\rho l}\nabla_m R^{\rho}{}_{n\mu o} - 14s_{ij}\,\nabla_m\nabla_n\nabla_o R_{ilkj} - 28s_{in}\, R_{ijk}{}^{q}\nabla_l R_{qmno} + \text{perm}\Bigg]$$

Fibre bundle geometry

Alminawi,IB,Davighi in progress

$\bar{X} \equiv X\big|_{\mathrm{vacuum}} \qquad A_{,b} \equiv \partial_b A$

Christoffel symbols

$$\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{ij} = \Gamma^{i}_{j\mu} = 0$$

$$\Gamma^{i}_{\mu\nu} = -\frac{g^{im}}{2} g_{\mu\nu,m}$$

$$\Gamma^{\mu}_{i\nu} = \frac{g^{\mu\rho}}{2} g_{\rho\nu,i}$$

$$\Gamma^{i}_{jk} = \frac{g^{im}}{2}\left[g_{jm,k} + g_{km,j} - g_{jk,m}\right]$$

evaluating at the **vacuum** of the theory: $\overline{g_{\mu\nu,i}} = -\eta_{\mu\nu}\overline{V_{,i}}/2 \equiv 0 \quad \forall i$

$$\bar{\Gamma}^{i}_{jk} = \frac{\bar{g}^{im}}{2}\left[\overline{g_{jm,k}} + \overline{g_{km,j}} - \overline{g_{jk,m}}\right]$$

all others $= 0$

Fibre bundle geometry

Alminawi,IB,Davighi in progress

$\bar{X} \equiv X\big|_{\text{vacuum}} \qquad A_{,b} \equiv \partial_b A$

Riemann tensors

$$R^i_{\mu\nu\rho} = R^\mu_{i\nu\rho} = R^\mu_{\nu i\rho} = R^\mu_{\nu\rho i} = R^i_{jk\mu} = R^i_{j\mu k} = R^i_{\mu jk} = R^\mu_{ijk} \equiv 0$$

$$R^\mu_{\nu\rho\sigma} = \Gamma^\mu_{\rho m}\Gamma^m_{\nu\sigma} - \Gamma^\mu_{\sigma m}\Gamma^m_{\nu\rho}$$
$$R^\mu_{ij\nu} = \Gamma^\mu_{i\nu,j} + \Gamma^\mu_{j\rho}\Gamma^\rho_{i\nu} - \Gamma^\mu_{m\nu}\Gamma^m_{ij}$$
$$R^i_{j\mu\nu} = \Gamma^i_{\mu\rho}\Gamma^\rho_{\nu j} - \Gamma^i_{\nu\rho}\Gamma^\rho_{\mu j}$$

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk}$$
$$R^\mu_{\nu ij} = \Gamma^\mu_{\nu j,i} - \Gamma^\mu_{\nu i,j} + \Gamma^\mu_{i\rho}\Gamma^\rho_{\nu j} - \Gamma^\mu_{j\rho}\Gamma^\rho_{\nu i}$$
$$R^i_{\mu j\nu} = \Gamma^i_{\mu\nu,j} - \Gamma^i_{\nu\rho}\Gamma^\rho_{\mu j} + \Gamma^i_{jm}\Gamma^m_{\mu\nu}$$

evaluating at the **vacuum** of the theory: $\overline{g_{\mu\nu,i}} = -\eta_{\mu\nu}\overline{V_{,i}}/2 \equiv 0 \quad \forall i$

$$\bar{R}^\mu_{\nu\rho\sigma} = \bar{R}^\mu_{\nu ij} = \bar{R}^i_{j\mu\nu} = 0$$
$$\bar{R}^\mu_{ij\nu} = \overline{\Gamma^\mu_{i\nu,j}}$$

$$\bar{R}^i_{jkl} = \overline{\Gamma^i_{jl,k}} - \overline{\Gamma^i_{jk,l}} + \bar{\Gamma}^i_{km}\bar{\Gamma}^m_{jl} - \bar{\Gamma}^i_{lm}\bar{\Gamma}^m_{jk}$$
$$\bar{R}^i_{\mu j\nu} = \overline{\Gamma^i_{\mu\nu,j}}$$

Fibre bundle geometry

Alminawi,IB,Davighi in progress

$\bar{X} \equiv X\big|_{\text{vacuum}} \qquad A_{,b} \equiv \partial_b A$

Covariant derivatives of the Riemann tensors.

The non-vanishing options are

$$\nabla_\alpha R^i_{\mu\nu\rho} \quad \nabla_\alpha R^\mu_{i\nu\rho} \quad \nabla_\alpha R^\mu_{\nu i\rho} \quad \nabla_a R^\mu_{\nu\rho\sigma} \quad \nabla_a R^\mu_{ij\nu} \quad \nabla_a R^\mu_{\nu ij}$$

$$\nabla_\alpha R^i_{jk\mu} \quad \nabla_\alpha R^i_{\mu jk} \quad \nabla_\alpha R^\mu_{ijk} \quad \nabla_a R^i_{j\mu\nu} \quad \nabla_a R^i_{\mu j\nu} \quad \nabla_a R^i_{jkl}$$

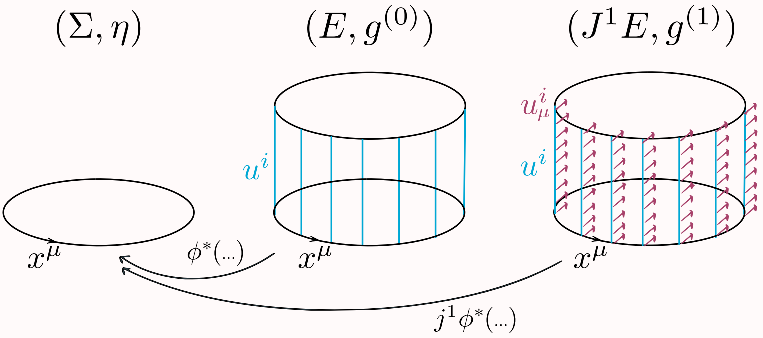
evaluating at the **vacuum** of the theory: $\overline{g_{\mu\nu,i}} = -\eta_{\mu\nu}\overline{V_{,i}}/2 \equiv 0 \quad \forall i$

the only surviving objects are

$$\overline{\nabla_a R^i_{jkl}} \qquad \overline{\nabla_a R^i_{\mu j\nu}} \qquad \overline{\nabla_a R^\mu_{ij\nu}}$$

Jet bundles

Saunders 1989. see also Craig,Lee 2307.15742



$j^i_x\phi$ = r-jet of ϕ at x = equivalence class containing sections identical up to r-th derivative

J^rE = r-jet bundle = $\{j^r_x\phi | x \in \Sigma, \phi \in \Gamma_x(\pi)\}$ is a differentiable manifold.   we use only J^1E

1-jet bundle metric $\to \partial^4$ scalar Lagrangian

the 1-jet bundle is a Riemannian manifold, on which we can build a **metric**

$$g^{(1)} = \begin{pmatrix} dx^\mu & du^i & du^i_\mu \end{pmatrix} \begin{pmatrix} g_{\mu\nu} & g_{\mu j} & g^{\nu}_{\mu j} \\ g_{\nu i} & g_{ij} & g^{\nu}_{ij} \\ g^{\mu}_{\nu i} & g^{\mu}_{ij} & g^{\mu\nu}_{ij} \end{pmatrix} \begin{pmatrix} dx^\nu \\ du^j \\ du^j_\nu \end{pmatrix}$$

$$= g_{\mu\nu} dx^\mu dx^\nu + 2 g_{\mu i} dx^\mu du^i + 2 g^{\nu}_{\mu j}\, dx^\mu du^j_\nu + g_{ij}\, du^i du^j + 2 g^{\nu}_{ij}\, du^i du^j_\nu + g^{\mu\nu}_{ij}\, du^i_\mu du^j_\nu$$

Poincaré invariance $\Rightarrow g_{IJ}$ depend on u^i, u^i_μ but *not* on x^μ

Metric to Lagrangian.

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu} g_{\mu\nu} + g_{\mu i}\partial^\mu\phi^i + g^{\nu}_{\mu j}\partial^\mu\partial^\nu\phi^j + \frac{1}{2} g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + g^{\nu}_{ij}\partial_\rho\phi^i\partial^\rho\partial_\nu\phi^j + \frac{1}{2} g^{\mu\nu}_{ij}\partial_\rho\partial_\mu\phi^i\partial^\rho\partial_\nu\phi^j$$

a redundant basis of operators with up to 4 derivatives

Scalar Lagrangian from 1-jet bundle metric: 1 scalar case

retaining only terms leading to operators with **up to 4 derivatives**

$$\frac{g_{\mu\nu}}{\Lambda^4} = -\frac{\eta_{\mu\nu}}{2}V(u) + \left[\frac{u_\mu u_\nu}{\Lambda^4} + \frac{\eta_{\mu\nu}}{4}\frac{u_\rho u^\rho}{\Lambda^4}\right]\frac{J(u)}{2} + \left[\frac{u_\mu u_\nu}{\Lambda^4} + \frac{\eta_{\mu\nu}}{4}\frac{u_\rho u^\rho}{\Lambda^4}\right]\frac{u_\sigma u^\sigma}{\Lambda^4}\frac{K(u)}{2}$$

$$\frac{g_{\mu u}}{\Lambda^2} = \frac{u_\mu}{\Lambda^2}G(u) + \frac{u_\mu u_\rho u^\rho}{\Lambda^6}H(u)$$

$$g^{\nu}_{\mu u} = \delta^\nu_\mu E(u) + \frac{u^\nu u_\mu}{\Lambda^4}F_1(u) + \delta^\nu_\mu \frac{u_\rho u^\rho}{\Lambda^4}F_2(u)$$

$$g_{uu} = C(u) + \frac{u_\rho u^\rho}{\Lambda^4}D(u) \qquad \Lambda g^\mu_{uu} = \frac{u^\mu}{\Lambda}B(u) \qquad \Lambda^2 g^{\mu\nu}_{uu} = \eta^{\mu\nu}A(u)$$

pulls back to

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\,(C + 2G + J) - \Lambda(\Box\phi)E - \Lambda^4 V$$
$$+ \frac{\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi}{\Lambda^2}\frac{A}{2} + \frac{\partial_\mu\partial_\nu\phi\partial^\mu\phi\partial^\nu\phi}{\Lambda^3}(B + F_1) + \frac{(\Box\phi)(\partial_\mu\phi\partial^\mu\phi)}{\Lambda^3}F_2 + \frac{(\partial_\mu\phi\partial^\mu\phi)^2}{\Lambda^4}\frac{D + 2H + K}{2}$$

Extension to higher derivatives

metric $g^{(r)}$ of a r-jet bundle $\longrightarrow$ **redundant** basis of operators with up to $2(r+1)$ deriv.

r-jet bundle has coordinates $y^I = (x^\mu, u^i, u^i_{\mu_1}, u^i_{\mu_1\mu_2}, \ldots, u^i_{\mu_1\ldots\mu_r})$

$$g^{(r)} = \begin{pmatrix} dx^\mu & du^i & du^i_{\mu_1} & \cdots & du^i_{\mu_1\ldots\mu_r} \end{pmatrix} \begin{pmatrix} g_{\mu\nu} & g_{\mu j} & g^{\nu_1}_{\mu j} & \cdots & g^{\nu_1\ldots\nu_r}_{\mu j} \\ g_{\nu i} & g_{ij} & g^{\nu_1}_{ij} & \cdots & g^{\nu_1\ldots\nu_r}_{ij} \\ g^{\mu_1}_{\nu i} & g^{\mu_1}_{ij} & g^{\mu_1\nu_1}_{ij} & \cdots & g^{\mu_1\nu_1\ldots\nu_r}_{ij} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g^{\mu_1\ldots\mu_r}_{\nu i} & g^{\mu_1\ldots\mu_r}_{ij} & g^{\mu_1\ldots\mu_r\nu_1}_{ij} & \cdots & g^{\mu_1\ldots\mu_r\nu_1\ldots\nu_r}_{ij} \end{pmatrix} \begin{pmatrix} dx^\nu \\ du^j \\ du^j_{\nu_1} \\ \cdots \\ du^j_{\nu_1\ldots\nu_r} \end{pmatrix}$$

- arbitrary internal symmetries (or absence thereof) can always be implemented
- many redundancies! different metric entries mapping to same operators, IBP, EOM, diffeos...