INVISIBLES25

Sufficient and Necessary Conditions for Collective Neutrino Instabilities

Based on Dasgupta & **Mukherjee**
2505.03886

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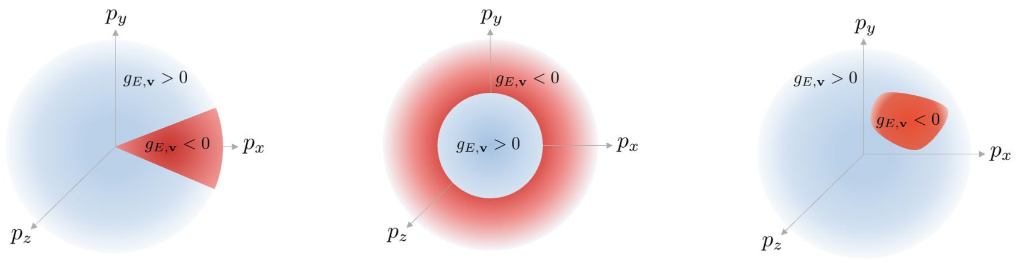
Crossings in distribution function

- Neutrinos evolve via the QKE,

$$i\,(\partial_t + \mathrm{v}.\,\partial_\mathrm{x})\rho_\mathrm{p} = [\mathsf{H}_\mathrm{p}, \rho_\mathrm{p}]$$

- Off diagonal components,

$$S_\mathbf{p} \sim \tilde{S}_\mathbf{p} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}$$



Fast Crossing | Slow Crossing | Mixed Crossing

B Dasgupta, **DM** (2025)

- Dispersion relation (for some eigenvector):

$$\mathcal{D}(\omega, \mathbf{k}) \equiv 1 + \int d\Gamma \frac{g_\Gamma}{\omega - \mathbf{k}\cdot\mathbf{v} - w_E} = 0$$

$$w_E = \frac{\Delta m^2}{2E}$$

$$g_\Gamma = g(w_E, \mathbf{v}) = \begin{cases} f_{\nu_e}(\mathbf{p}) - f_{\nu_\mu}(\mathbf{p}) & \text{for } w_E > 0 \\ f_{\bar{\nu}_\mu}(\mathbf{p}) - f_{\bar{\nu}_e}(\mathbf{p}) & \text{for } w_E < 0 \end{cases}$$

- Singularity at $h_\Gamma = \mathrm{k}\cdot\mathrm{v} + w_E$

$$\mathcal{D}(\omega, \mathbf{k}) \equiv 1 + \mathcal{I}_{\mathrm{PV}}(\omega) + i\,\mathcal{I}_\delta(\omega)$$

The Conditions and Proof

A. Statement of Sufficiency Condition

The dispersion relation $\mathcal{D}(\omega, \mathbf{k}) = 0$, admits a complex root ω with $\text{Im}\,\omega > 0$ at some wave-vector $\mathbf{k}$, for a time-like eigenvector and the inverted mass ordering, *if*

1a: the distribution function g_Γ has a zero-crossing at *some* point $\Gamma_0 = (w_{E_0}, \mathbf{v}_0)$, i.e., $g(w_{E_0}, \mathbf{v}_0) = 0$ and g_Γ takes both signs in the neighborhood,

1b: where the gradients of g_Γ and $h_\Gamma = \mathbf{k}\cdot\mathbf{v} + w_E$ have a positive dot-product, i.e., $(\nabla_\Gamma g_\Gamma)_0 \cdot (\nabla_\Gamma h_\Gamma)_0 > 0$, and

2a: the principal value $\mathcal{I}_{\text{PV}}(\omega_0) < -1$ at a frequency $\omega_0 = h_{\Gamma_0}$, where $\text{Im}\,\mathcal{I}(\omega_0) = 0$,

2b: while $\mathcal{I}_{\text{PV}}(\omega_i) > -1$, with $i = 1, 2, \ldots$, for any other frequencies ω_i where $\text{Im}\,\mathcal{I}(\omega_i) = 0$.

B. Dasgupta and **D. Mukherjee**

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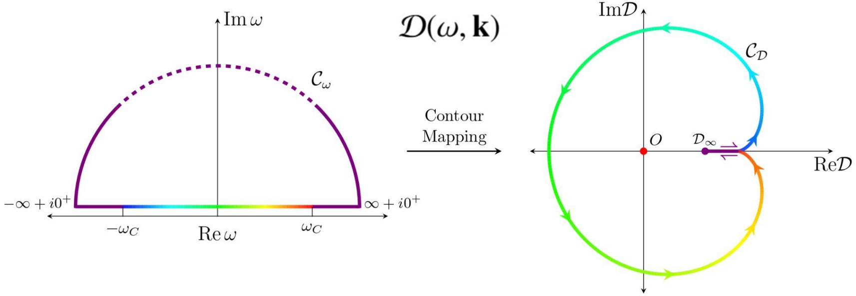


- Sign-flip and Localisation

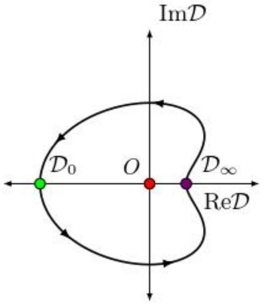
  Near terms dominate

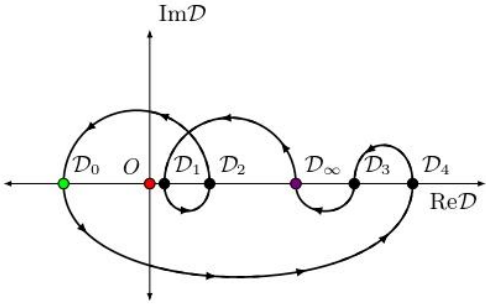
- Contour Mapping, Winding and Roots

  A loop in ω around the upper half-plane maps to D.
  D encircles the origin if the conditions are met.
  A nonzero winding guarantees a root of DR in UHP

Essence of the Conditions

- Local Conditions:
  - Cond 1a. (crossing) requires g to change sign at some point.
  - Cond 1b. (grad-positivity) ensures D-contour goes from above-to-below (at ω_0).
- Global Conditions:
  - Cond 2a. (Negative PV) $\mathcal{I}_{PV}\,(\omega_0) < -1$, ensures D-0 is on the left of origin.
  - Cond 2b. (Positive PV) $\mathcal{I}_{PV}\,(\omega_i) > -1$, ensures all other points on the right of origin.

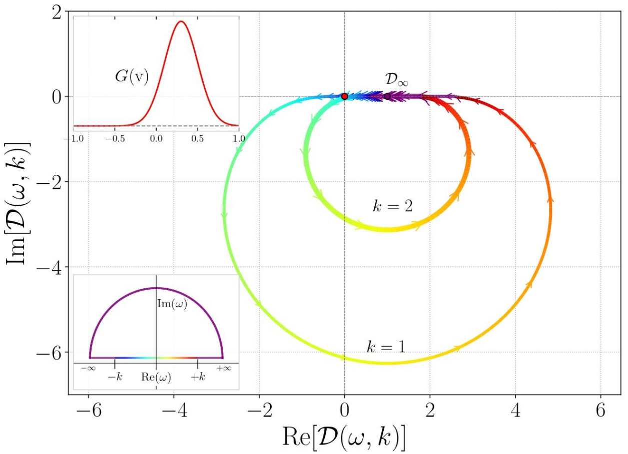




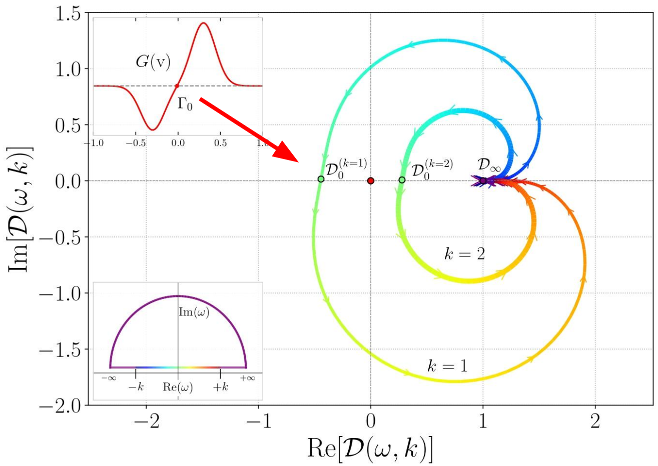
Example: Fast Crossing

- Crossing only in velocity:

$$\mathcal{D}_{\rm fast}(\omega, \mathbf{k}) = 1 + \int d\mathbf{v} \frac{G_{\mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}} = 0$$



No *crossing* implies No *winding*.



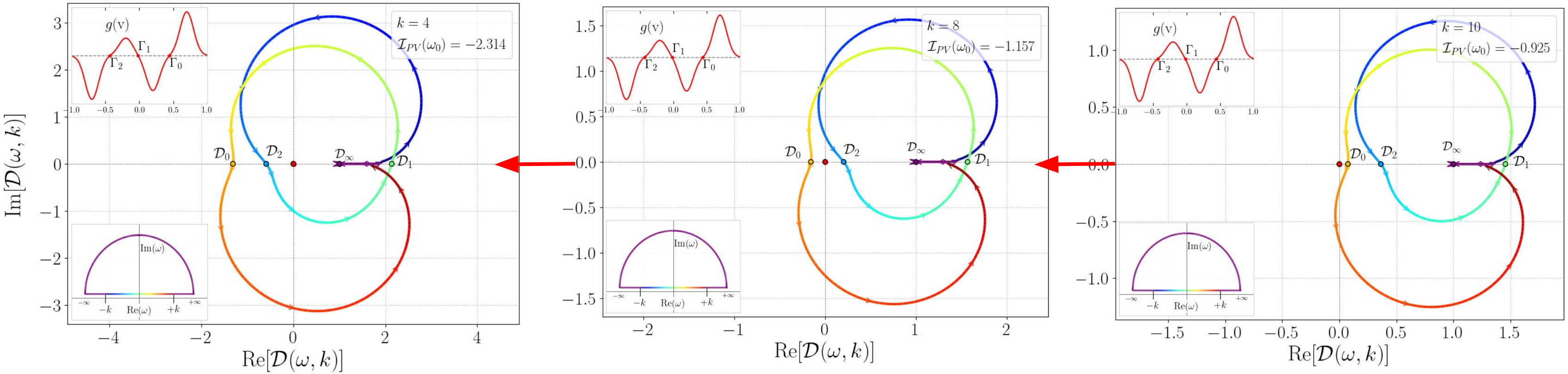
Single *crossing* implies *winding*.

- Condition 1a. (Crossing)

Example: Fast Crossing

- Multiple crossings: Scale $\mathbf{k} \to \lambda \mathbf{k}$ to get $\mathcal{I}_{\mathrm{PV}}(\omega_i) \to \frac{1}{\lambda} \mathcal{I}_{\mathrm{PV}}(\omega_i)$
- Need PV<0 for one of the crossing; can be satisfied for

$$\max(0, -\kappa_1) \equiv \lambda_{\min} < \lambda < \lambda_{\max} \equiv -\min(0, \kappa_0)$$

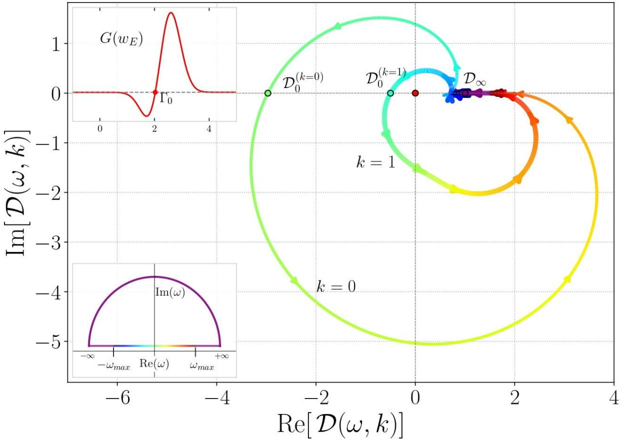


- Condition 2a. and 2b. (Negative PV and Scaling)

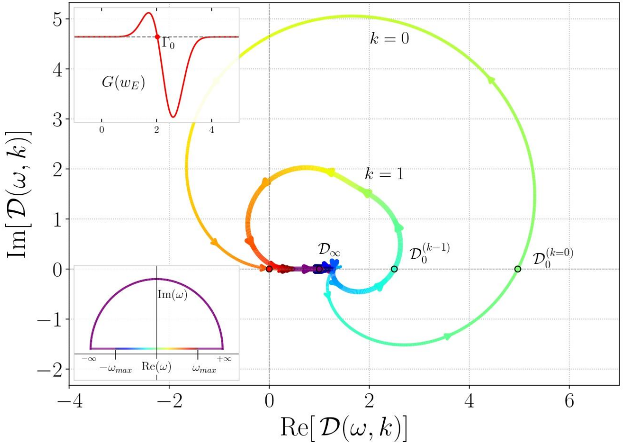
Example: Slow Crossing

- Crossing only in energy:

$$\mathcal{D}_{\text{slow}}(\omega, \mathbf{k}) = 1 + \int d\Gamma \frac{g_{w_E}}{\omega - \mathbf{k}\cdot\mathbf{v} - w_E} = 0$$



Correct *slope* implies *winding.*

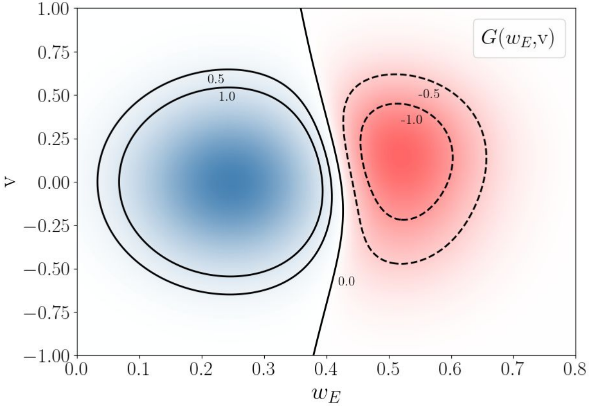


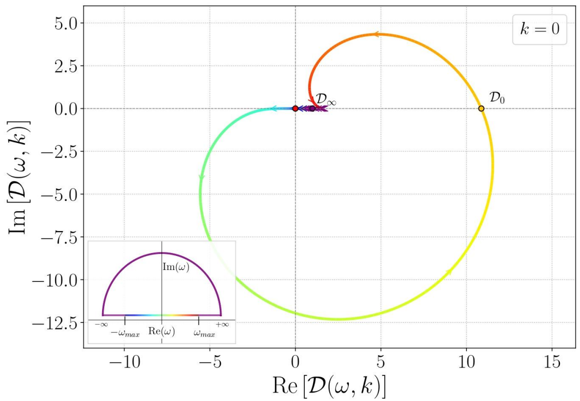
Wrong *slope* implies No *winding.*

- Condition 1b. (Gradient-positivity)

Example: Mixed Crossing

- Non-trivial crossings in both energy and velocity.
- From Cond 1b, $\left(\mathbf{k} - (\mathbf{k}\cdot\mathbf{v}_0)\mathbf{v}_0\right)\cdot\left(\nabla_{\mathbf{v}} g_{\mathbf{\Gamma}}\right)_0 > -w_{E_0}^{\,2}\left(\frac{\partial g_{\mathbf{\Gamma}}}{\partial w_E}\right)_0$, for some k that satisfies Cond. 2a & 2b.





Wrong slope (dg/dw<0) no encirclement (for k=0).

Summary and Remarks

- Our proof :
    - Existence proof that predicts complex roots in UHP.
    - Conditions are proven for IH (time-like eigenvectors), but can be adapted for NH with appropriate changes.
    - Conditions 1a, 1b, and 2a are also ***necessary*.**

- Some caveats:
    - One-to-One connection between crossing and instability is not established.
    - Not a prescription for determining all unstable modes.
    - Condition 2b is not strictly necessary.

Thank You!!

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