Transient beam loading for reverse phase operation mode

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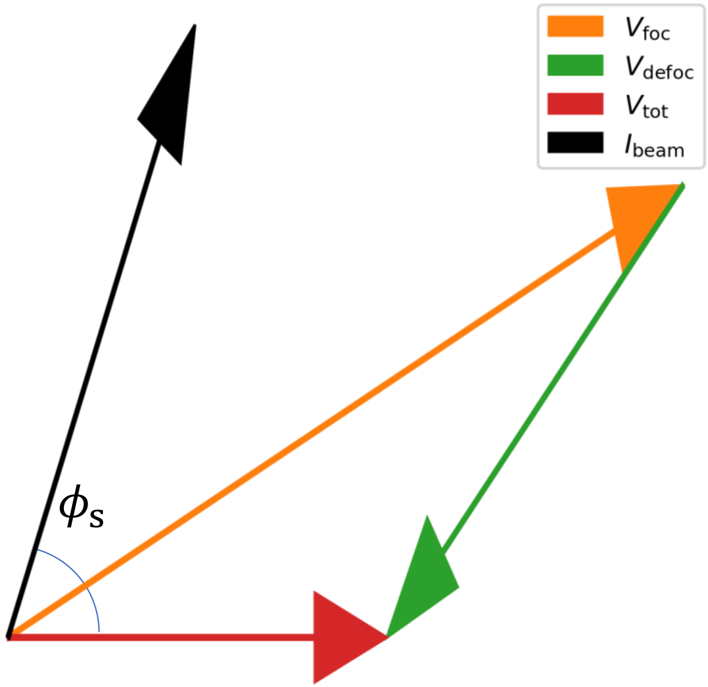
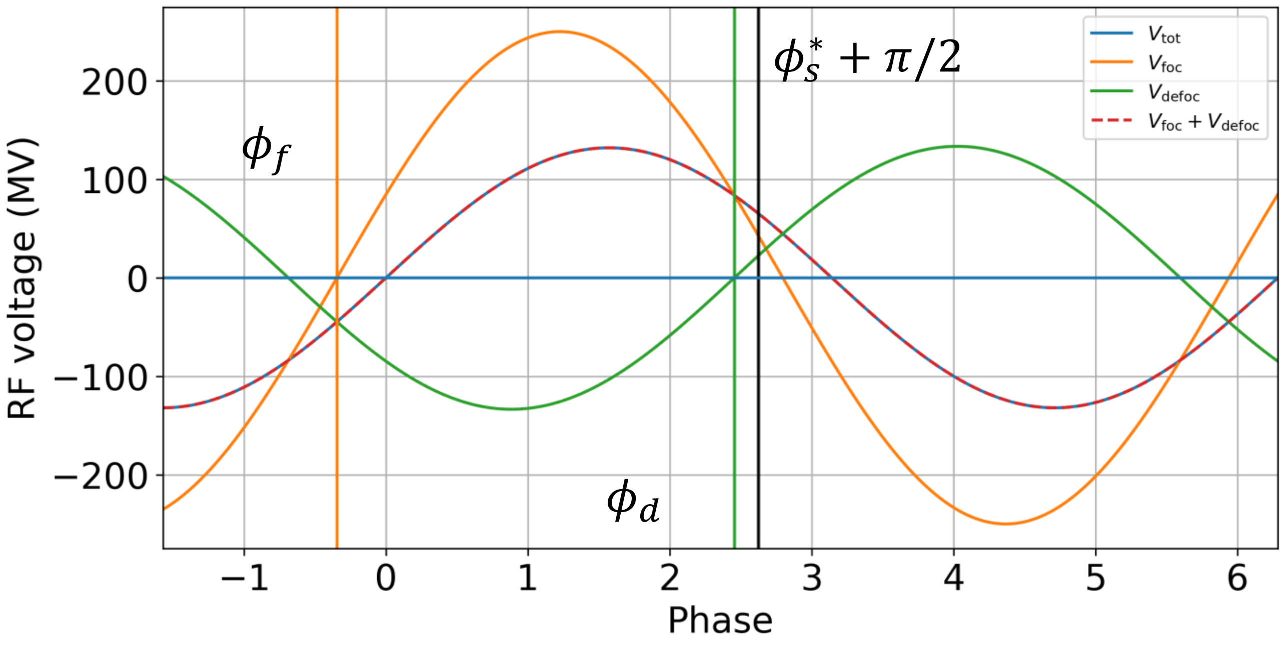
Reverse phase operation

Reverse phase operation (RPO) mode allows increasing RF cavity voltage *(Y. Morita et al., SRF, 2009)*
- Experimentally verified with high beam loading in KEKB *(Y. Morita et al., IPAC, 2010)*
- Baseline solution for EIC ESR *(e.g., J. Guo et al., IPAC, 2022)*

Phasors

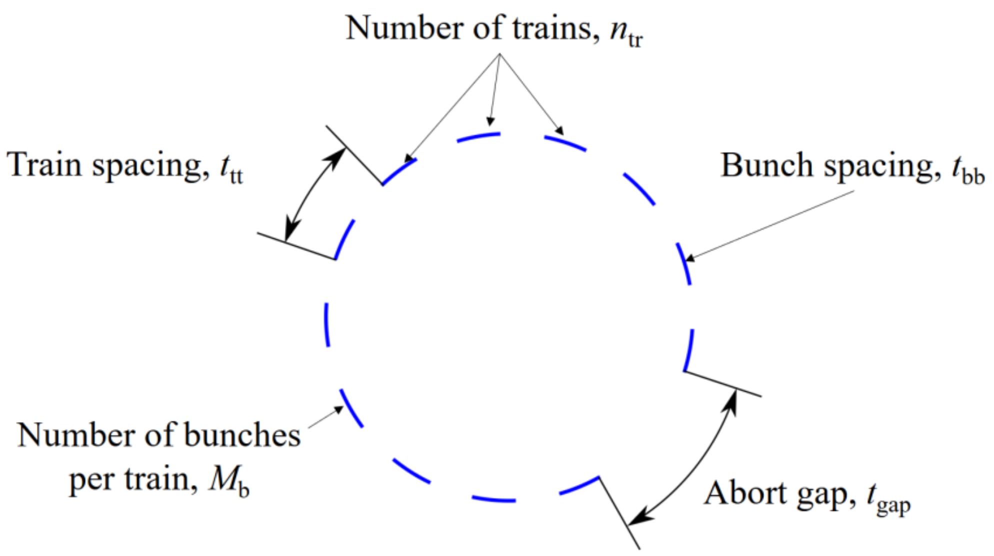


RF waves

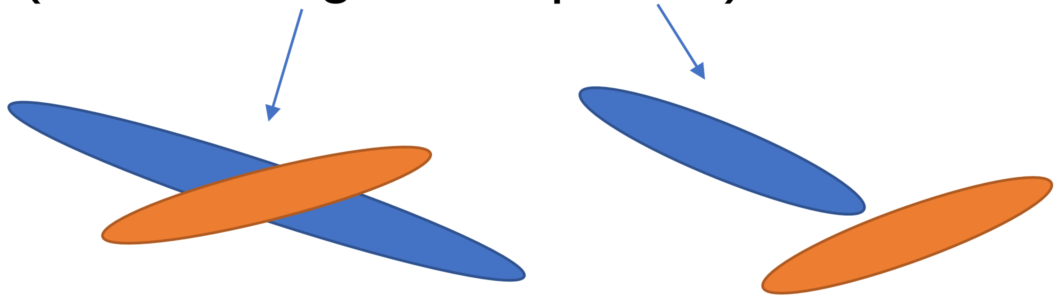


*Electron convention for synchronous phase: $\phi_{s,proton} = \frac{\pi}{2} + \phi_{s,electron}$

Transient beam loading



Gaps in machine filling will result in modulation beam parameters (bunch length and phase)



→ Modulations might impact luminosity and/or beam stability

Conventional approaches:

- Particle tracking simulations (difficult for 11200 bunches in FCC-ee Z)
- Steady-state time domain method *(J. Tückmantel, 2011)*
- Small-signal model in frequency domain *(F. Pedersen, 1992)*

→ Tracking simulation were applied for EIC (1160 bunches)
→ Small-signal model was adapted for the RPO case of FCC up to now

Reduced Pedersen model

General equations of beam-cavity interactions with reverse phase operation (RPO) mode (adaptation of formalism in *J. Tückmantel, 2011*):

$$I_{gf}(t) = \frac{V_f(t)}{2(R/Q)}\left(\frac{1}{Q_L} - 2i\frac{\Delta\omega_f}{\omega_{\mathrm{rf}}}\right) + \frac{I_{\mathrm{b,rf}}(t)}{2} + \frac{dV_f(t)}{dt}\frac{1}{\omega_{\mathrm{rf}}(R/Q)}$$

$$I_{gd}(t) = \frac{V_d(t)}{2(R/Q)}\left(\frac{1}{Q_L} - 2i\frac{\Delta\omega_d}{\omega_{\mathrm{rf}}}\right) + \frac{I_{\mathrm{b,rf}}(t)}{2} + \frac{dV_d(t)}{dt}\frac{1}{\omega_{\mathrm{rf}}(R/Q)}$$

Energy balance $\quad V_{\mathrm{tot}}\cos\phi_s = N_f A_f \cos\left(\phi_s - \phi_b + \phi_{cf} + \phi_f\right) + N_d A_d \cos(\phi_s - \phi_b + \phi_{cd} + \phi_d)$

To calculate beam-induced modulation we assume:

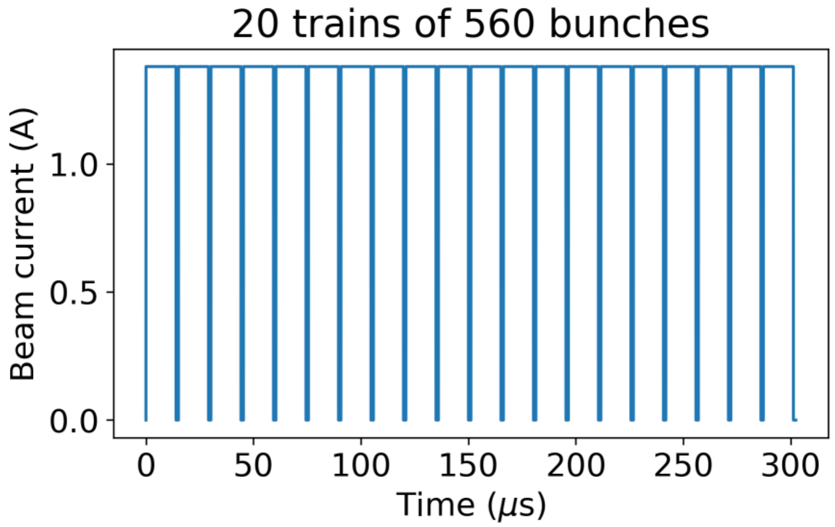
- $I_{gf,d}(t)$ = constant – no beam loading compensation
- $V_{f,d}(t) = A_{f,d}(t)e^{i\phi_f(t)+i\phi_{cf,d}}$, $I_{b,\mathrm{rf}}(t) = A_b(t)e^{-i\phi_s+i\phi_b(t)}$

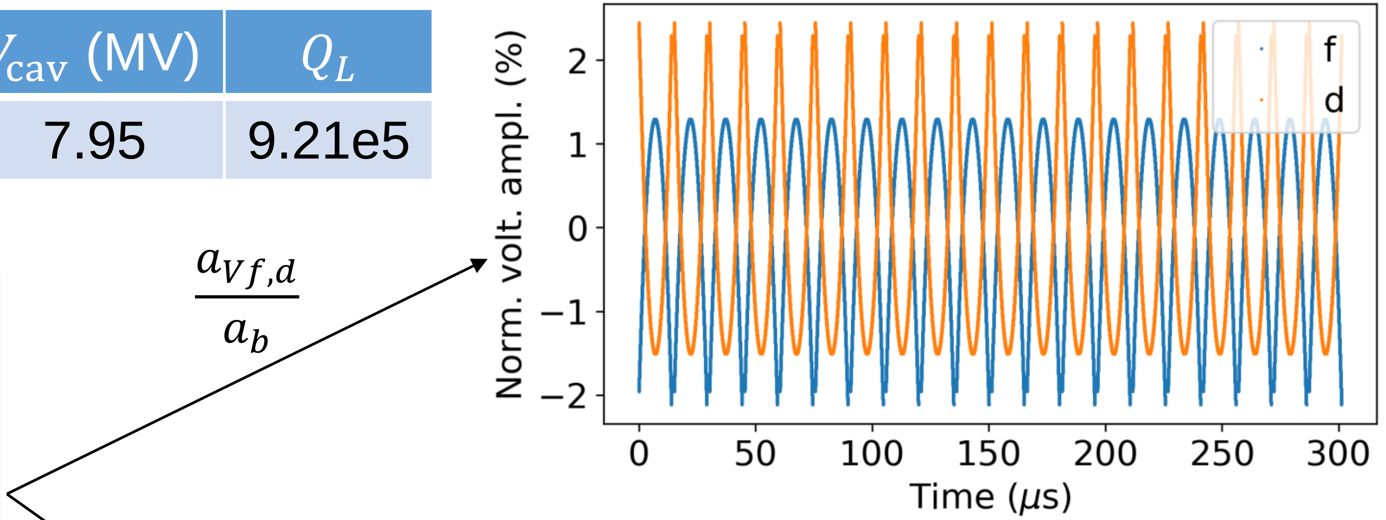
Then, system of equations is linearized to obtain transfer functions: $\frac{a_{Vf,d}}{a_b}, \frac{\phi_{f,d}}{a_b}, \frac{\phi_b}{a_b}$

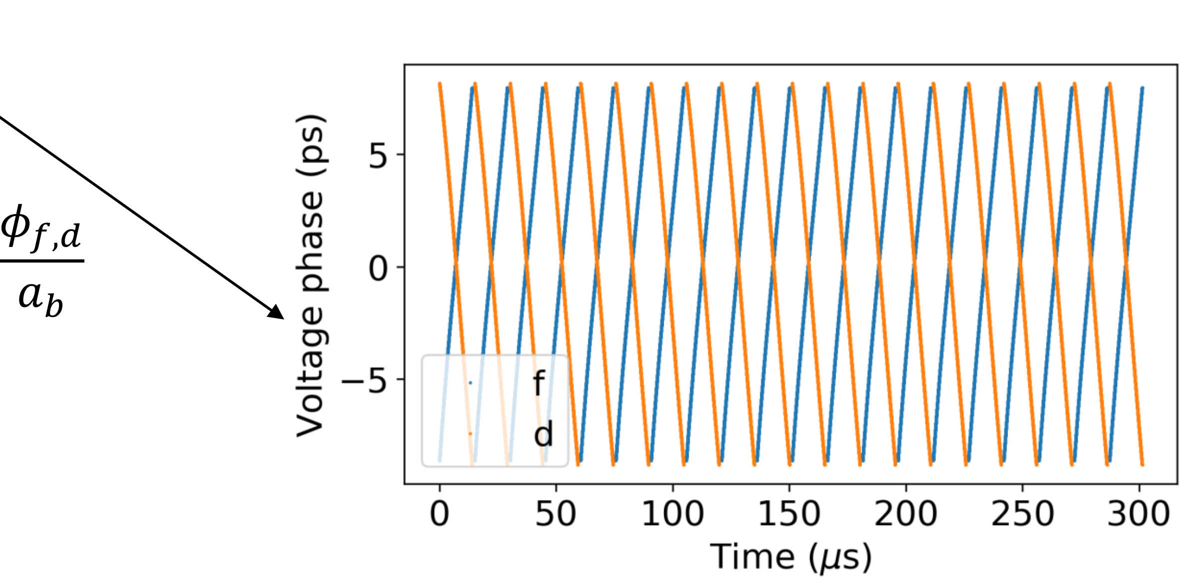
$A_{f,d} = V_{\mathrm{cav}}\left(1 + a_{Vf,d}\right), A_b(t) = |F_b| I_{b,\mathrm{dc}}\,(1 + a_b)$

Bunch-by-bunch spread of cavity parameters

	N_f	N_d	V_{tot} **Z** (MV)	V_{cav} (MV)	Q_L
Current	71	61	88	7.95	9.21e5



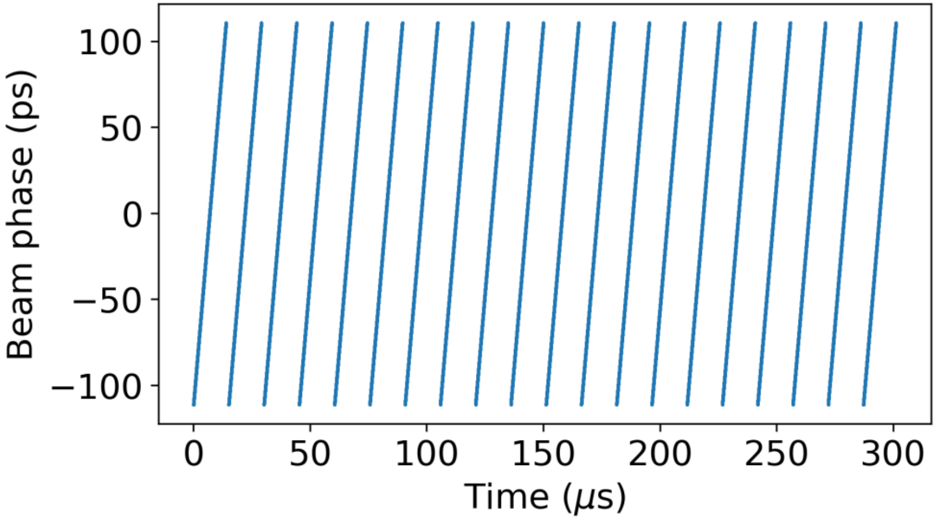




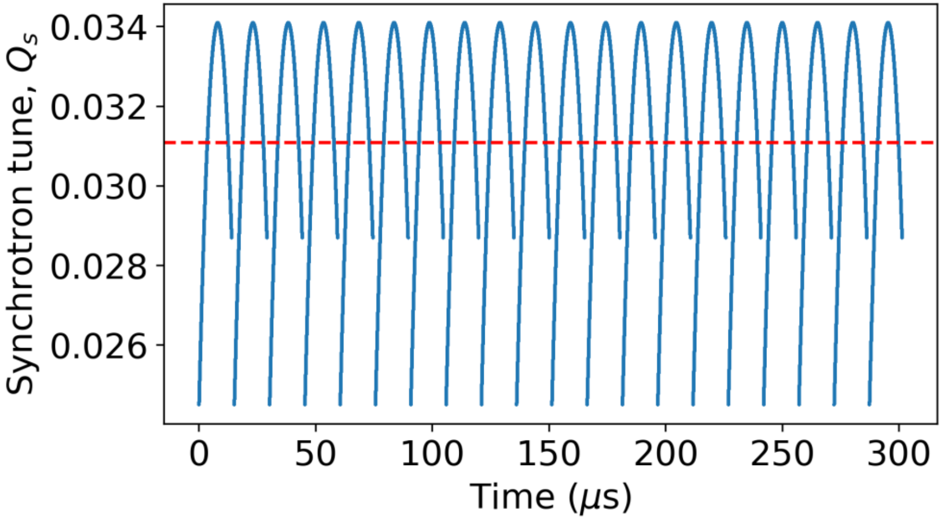
Note, the designed rms bunch length is 50 ps (with beamstrahlung)

Bunch-by-bunch spread of beam parameters

$\frac{\phi_b}{a_b}$



-21/+9 %

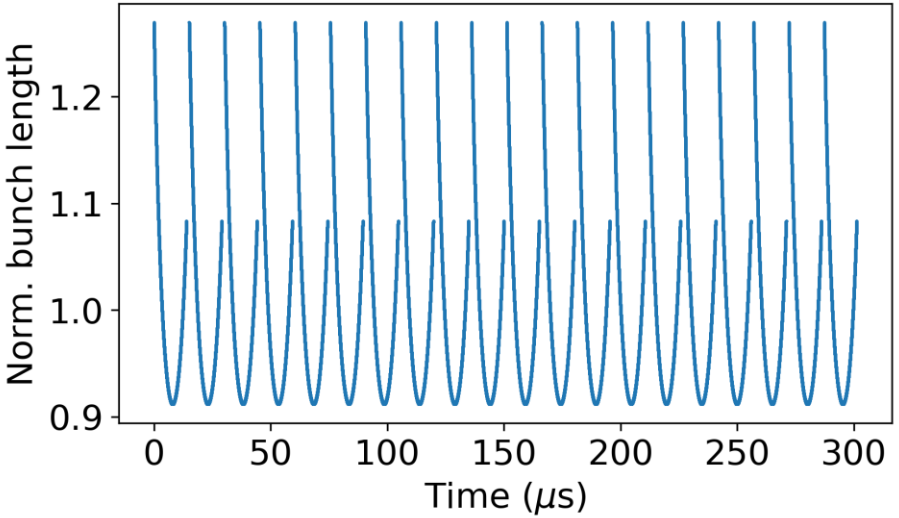


For identical rings, transients can be compensated by matching abort gaps (e.g., in PEPII, LHC,…)

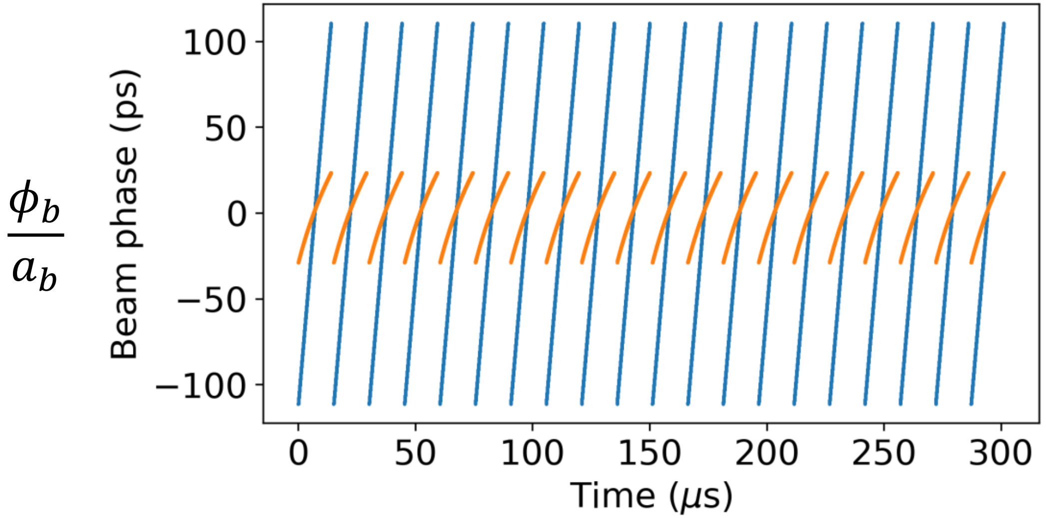
Imbalance of charge results in different detuning for electron and positron beams
→ Slightly different transients (most critical during filling)

Peak-to-peak spread of ~30% in synchrotron tune and bunch length can have a significant impact on beam stability

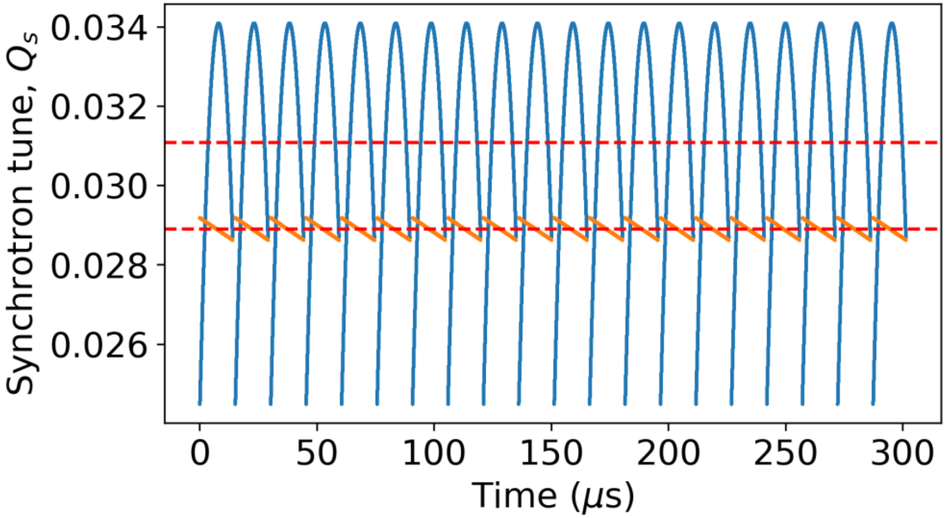
-9/+27 %



Bunch-by-bunch spread of beam parameters



-21/+9 %

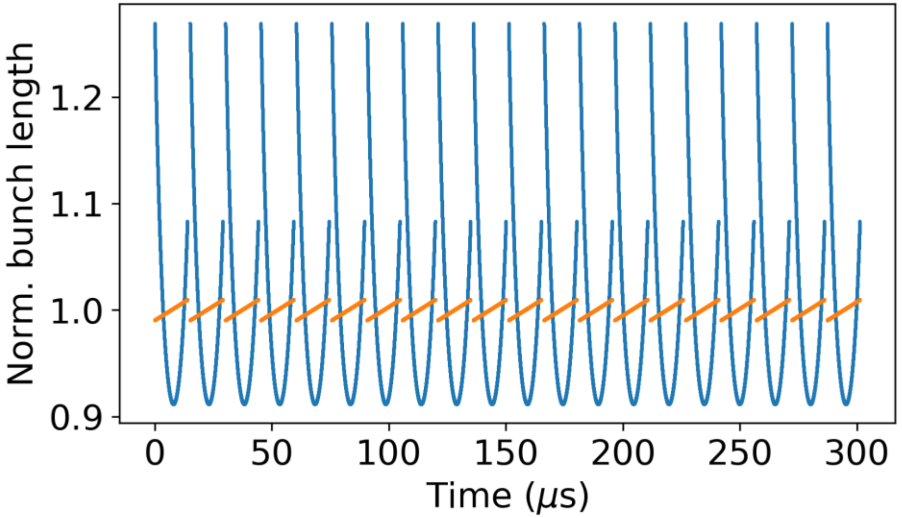


For identical rings, transients can be compensated by matching abort gaps (e.g., in PEPII, LHC,…)

Imbalance of charge results in different detuning for electron and positron beams
→ Slightly different transients (most critical during filling)

Peak-to-peak spread of ~30% in synchrotron tune and bunch length can have a significant impact on beam stability
→ We lose a factor of 15 wrt to 1-cell RF system

-9/+27 %

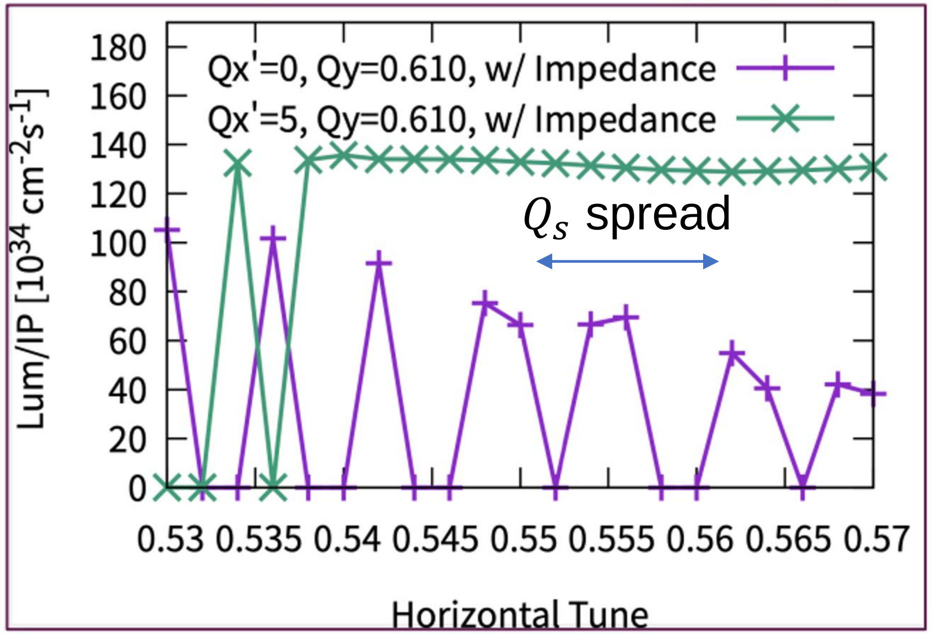


Critical impact of spread

Interplay between beam-beam and coupling impedance

A positive chromaticity has a beneficial effect on the beam-beam. Self-consistent simulations show a luminosity per IP close to the nominal value of 141 x 10^{34} $cm^{-2}s^{-1}$ by properly choosing the collider working point.

Y. Zhang, IHEP, Beijing, China



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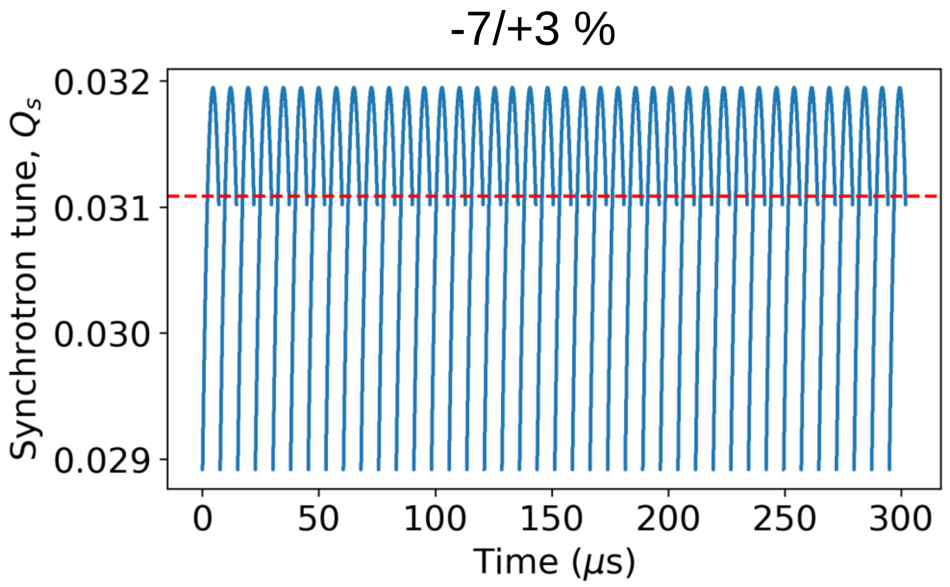
M. Migliorati, FCC Week 2024

→ No stable region for a horizontal tune can be found in presence of large Q_s spread. Possible mitigations need to be studied

Possible scenarios

Peak-to-peak beam phase spread ∝ $\Delta\omega_{\mathrm{opt}}\tau_{\mathrm{gap}}N_{\mathrm{tot}}/(N_f - N_d)$

1. New filling scheme (e.g., 40 trains of 280 bunches)
   → Spread is reduced by a factor of ~3
   → Gaps become twice shorted (~600 ns) - most likely unfeasible for the extraction system (1 us kicker rise time)



2. Higher total RF voltage for Z?

Optimal quality factor $$Q_{L,\mathrm{opt}} = \frac{V_{\mathrm{cav}}^2 N_{\mathrm{tot}}}{2P_{\mathrm{SR}}(R/Q)}$$

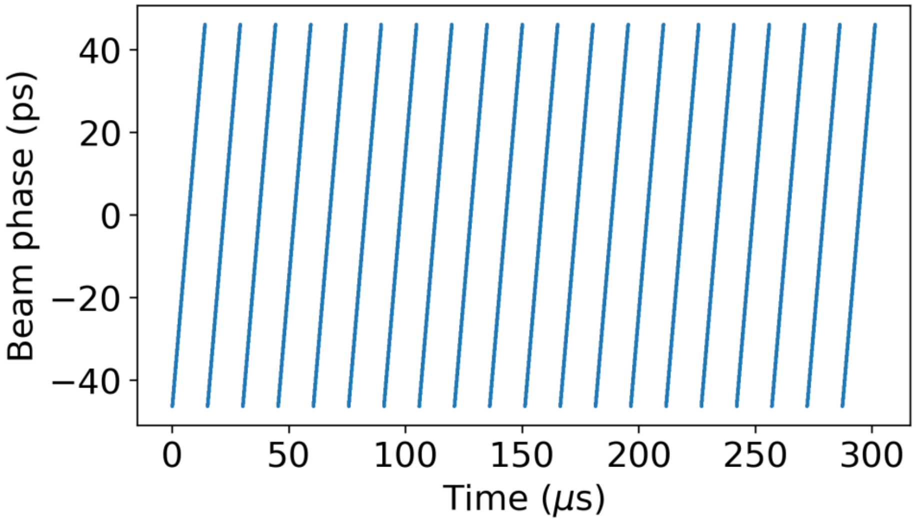
Since $Q_{L,\mathrm{opt}}$ should be the same for Z, W, and ZH, V_{cav} cannot be changed

→ Optimal detuning is also unchanged $$\Delta\omega_{\mathrm{opt}} = -\frac{\omega_{\mathrm{rf}}(R/Q)|F_b|I_{b,dc}}{2V_{\mathrm{cav}}}\sqrt{1 - \frac{U_0^2}{e^2V_{\mathrm{cav}}^2N_{\mathrm{tot}}^2}}$$

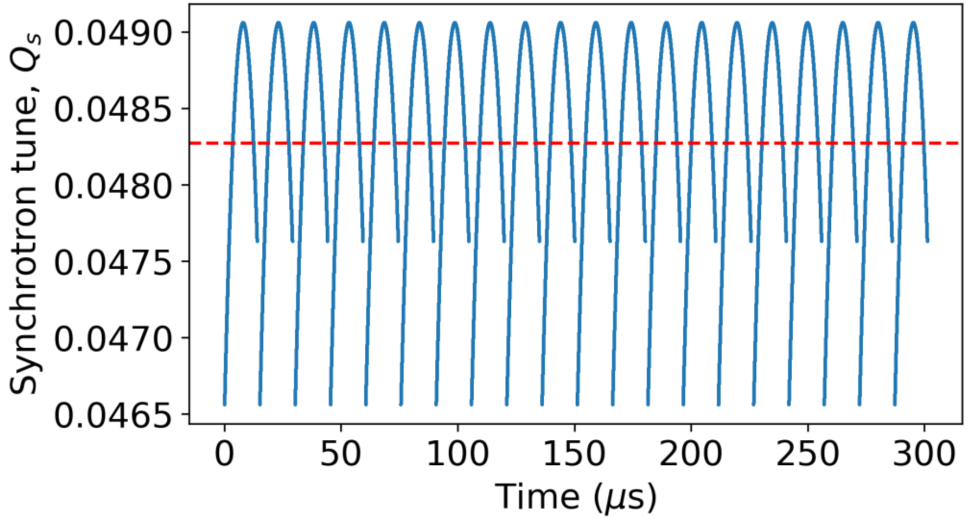
The only knob is to change $N_f - N_d$ by changing V_{tot}: $$V_{\mathrm{cav}} = \frac{V_{\mathrm{tot}}}{N_{\mathrm{tot}}}\sqrt{\frac{U_0^2}{e^2V_{\mathrm{tot}}^2} + \left(1 - \frac{U_0^2}{e^2V_{\mathrm{tot}}^2}\right)\frac{N_{\mathrm{tot}}^2}{(N_f - N_d)^2}}$$

Higher RF voltage

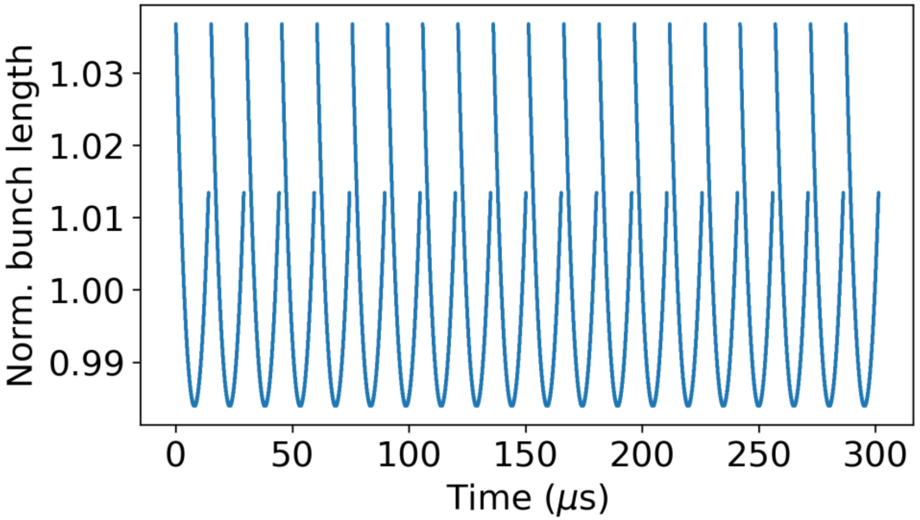
	N_f	N_d	V_{tot} **Z** (MV)	V_{cav} (MV)	Q_L
Current	71	61	88	7.95	9.21e5
Option 2	78	54	195	7.95	9.21e5



-3.6/+1.6 %



-1.6/+3.7 %



Higher RF voltage reduces parameter spread to ~5%
Peak-to-peak Qs spread 2.5e-3

Impact on parameters (oversimplified scaling)

FCC-ee collider parameters for the GHC lattice as of Aug. 2, 2024.

Beam energy	[GeV]	45.6	80	120	182.5
Layout		PA31-3.0			
# of IPs		4			
Circumference	[km]	90.658728			
Bend. radius of arc dipole	[km]	10.021			
Energy loss / turn	[GeV]	0.0390	0.369	1.86	9.94
SR power / beam	[MW]	50			
Beam current	[mA]	1283	135	26.8	5.0
Colliding bunches / beam		11200	1852	300	64
Colliding bunch population	$[10^{11}]$	2.16	1.38	1.69	1.48
Hor. emittance at collision ε_x	[nm]	0.70	2.16	0.66	1.51
Ver. emittance at collision ε_y	[pm]	1.9	2.0	1.0	1.36
Lattice ver. emittance $\varepsilon_{y,\mathrm{lattice}}$	[pm]	0.87	1.20	0.57	0.94
Arc cell		Long 90/90		90/90	
Momentum compaction α_p	$[10^{-6}]$	28.67		7.52	
Arc sext families		75		146	
$\beta^*_{x/y}$	[mm]	110 / 0.7	220 / 1	240 / 1	[illegible]
Transverse tunes $Q_{x/y}$		218.158 / 222.220	218.185 / 222.2[illegible]	[illegible]	[illegible]
Chromaticities $Q'_{x/y}$		0 / +5	0 / +5	[illegible]	[illegible]
Energy spread (SR/BS) σ_δ	[%]	0.039 / 0.110	0.069 / 0.105	0.102 / 0.176	[illegible]
Bunch length (SR/BS) σ_z	[mm]	5.57 / 15.6	3.46 / 5.28	[illegible]	1.91 / 2.32
RF voltage 400/800 MHz	[GV]	0.079 / 0	1.00 / 0	[illegible]	2.1 / 9.20
Harm. number for 400 MHz		[illegible]			
RF frequency (400 MHz)	MHz	[illegible]			
Synchrotron tune Q_s		0.0289	0.0809	[illegible]	0.0881
Long. damping time	[turns]	1171	218	[illegible]	19.4
RF acceptance	[%]	1.06	3.32	[illegible]	3.06
Energy acceptance (DA)	[%]	±1.0	±1.0	[illegible]	-2.8/+2.5
Beam crossing angle at IP θ_x	[mrad]	±15			
Crab waist ratio	[%]	70	55	[illegible]	[illegible]
Beam-beam ξ_x/ξ_y[a]		0.0022 / 0.0977	0.013 / 0.12[illegible]	[illegible]	[illegible]
Piwinski angle $(\theta_x \sigma_{z,\mathrm{BS}})/\sigma^*_x$		26.6	3.6	[illegible]	[illegible]
Lifetime (q + BS + lattice)	[sec]	11800	4500	6000	[illegible]
Lifetime (lum)[b]	[sec]	1330	960	600	[illegible]
Luminosity / IP	$[10^{34}/\mathrm{cm}^2\mathrm{s}]$	143	20	7.5	[illegible]

[a] incl. hourglass.
[b] only the energy acceptance is taken into account for the cross section, no beam size effect.

3.3/9.4

$\sigma_z \propto \frac{1}{\sqrt{V_{tot}}}$

0.195/0

0.0483

2.35

xx/0.162

$\xi_y \propto \frac{N_p}{\sigma_z}$

?

$L \propto \xi_y$

K. Oide, 2024

X-Z instability
(K. Ohmi, 2016**)**

Higher Q_s
→ stronger low order resonance but more space available between them

Bunches are ~50% shorter (assuming the same σ_δ)
→ stronger beamstrahlung (ξ_y increase is smaller)
→ stronger impact of longitudinal impedance?

Resonant depolarization
Figure of merit SMI $=\nu_s \sigma_\delta / Q_s$ ~ 1.3-1.4 for baseline
→ is reduced to ~0.85 (SMI<1 is preferred)

Many more aspects to be re-analyzed…

Summary

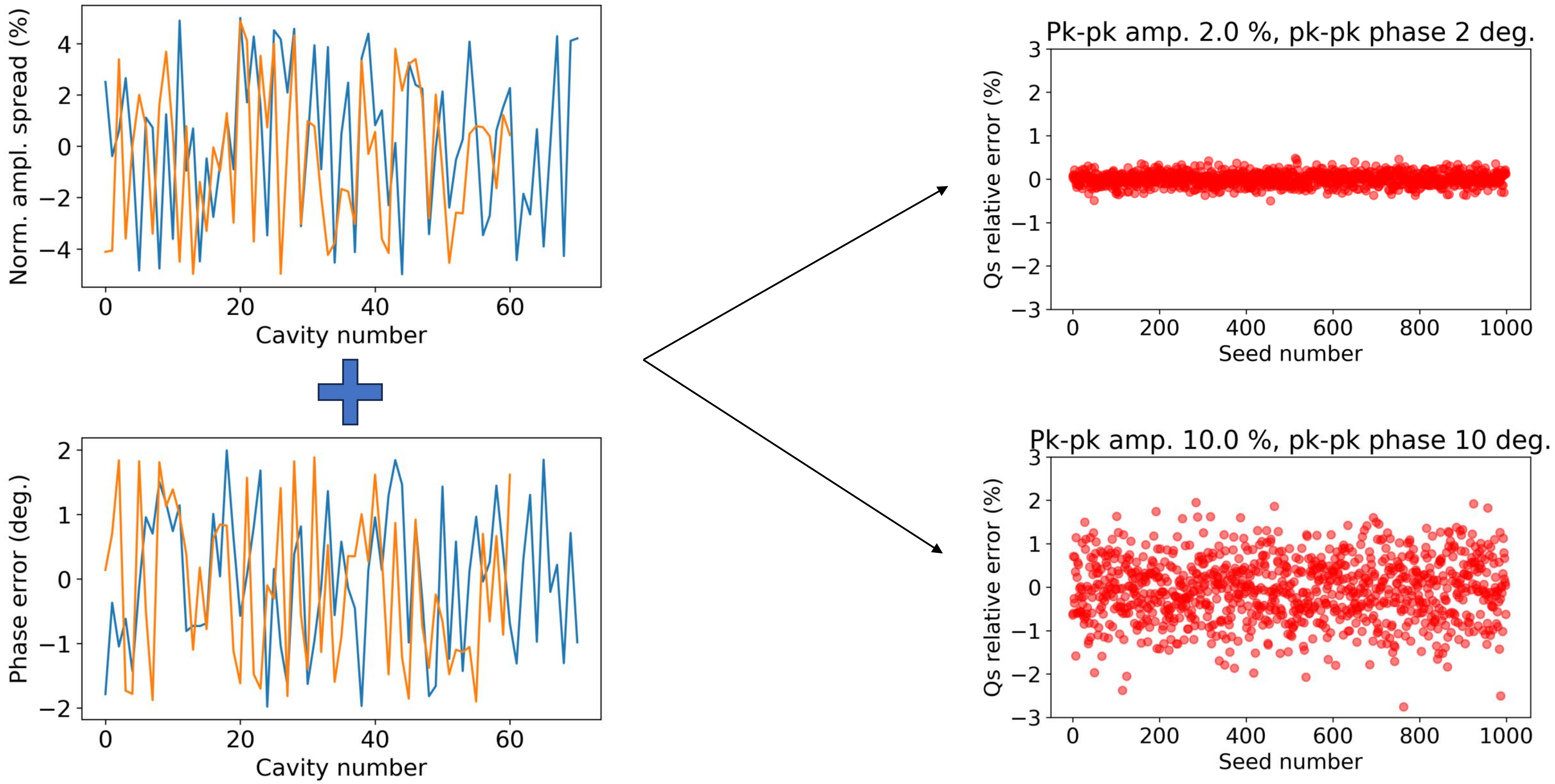
Reverse Phase Operation (RPO) mode aims to avoid hardware modification of RF system between Z, W, and ZH modes

- Synchrotron frequency and bunch length spread due to transient beam loading could be a potential showstopper.
- Possible mitigations are reduction of gap length or increase of total RF voltage
- Would it be possible to find a new parameter set before end Feasibility Study?

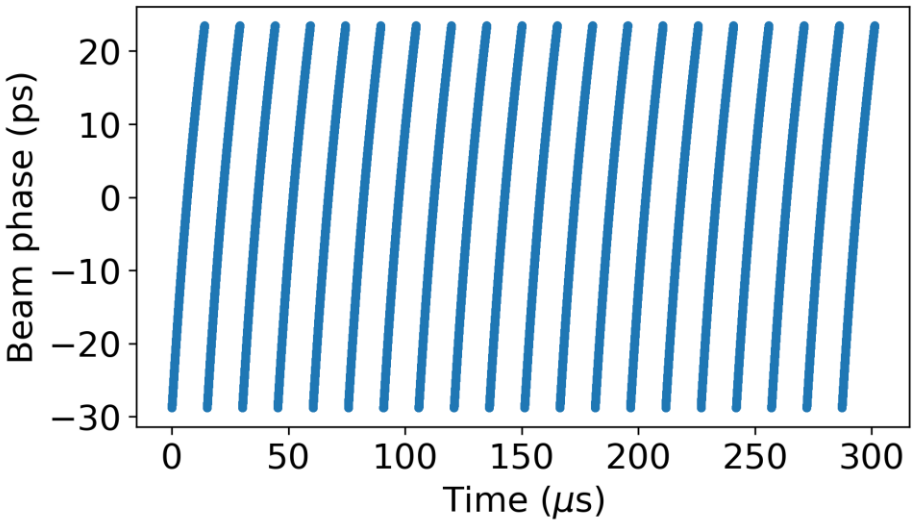
Thank you for your attention!

Backup slides

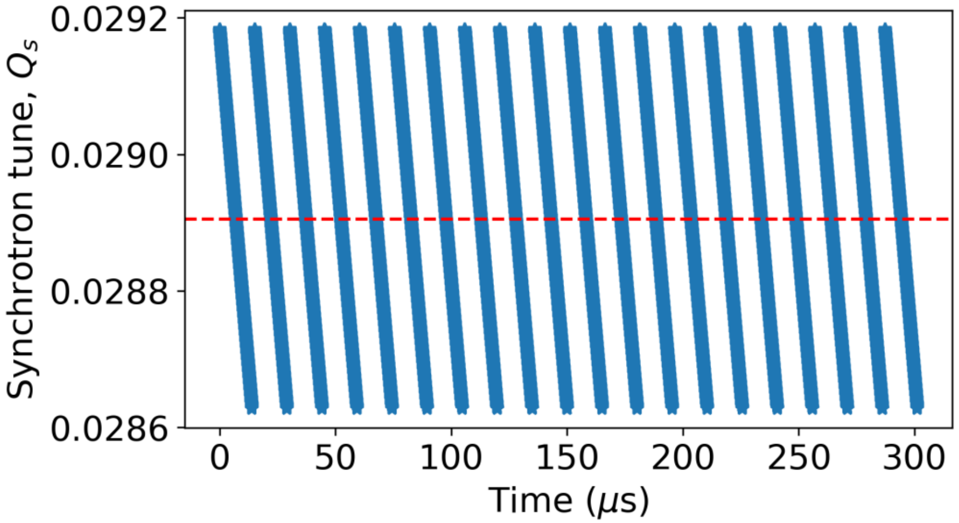
Parameter sensitivity of RPO



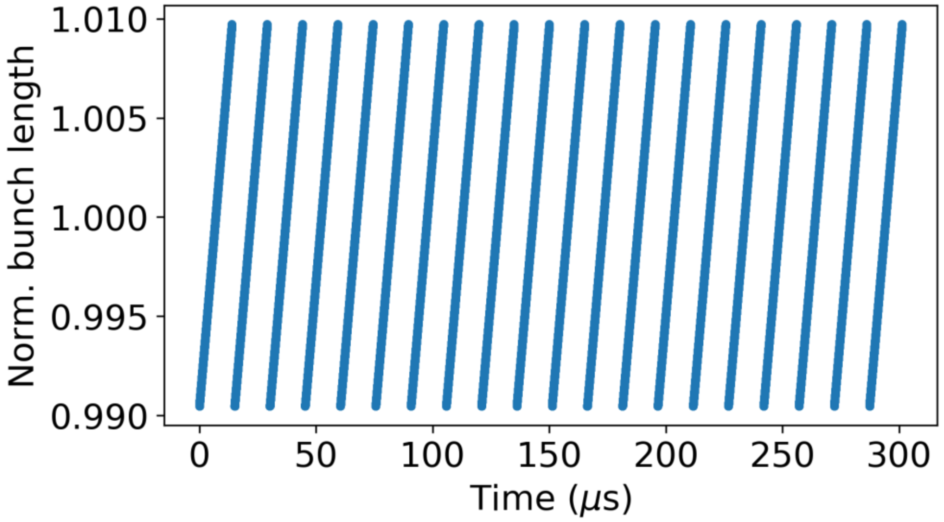
Transients for baseline scenario (56 1-cell cavities)



+1 – 1%



+1 – 1%



Q_s spread ~3e4

Almost negligible spread of bunch-by-bunch parameters

Global parameters (Déjà vu)

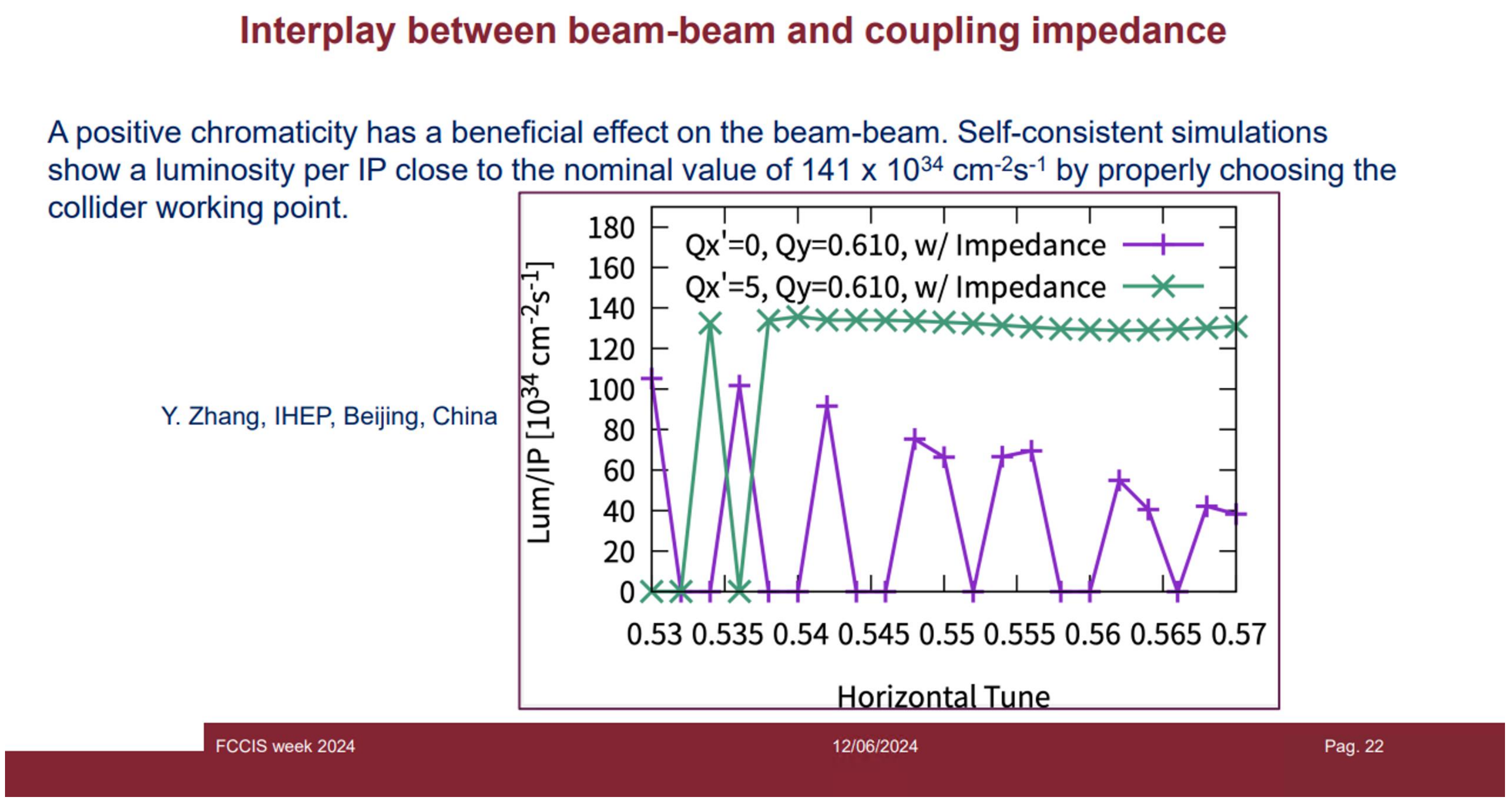
August 2024

Beam energy	[GeV]	45.6	
Layout			
# of IPs			
Circumference	[km]		
Bend. radius of arc dipole	[km]		
Energy loss / turn	[GeV]	0.0390	
SR power / beam	[MW]		
Beam current	[mA]	1283	
Colliding bunches / beam		11200	
Colliding bunch population	[10^{11}]	2.16	
Hor. emittance at collision ε_x	[nm]	0.70	
Ver. emittance at collision ε_y	[pm]	1.9	
Lattice ver. emittance $\varepsilon_{y,\text{lattice}}$	[pm]	0.87	
Arc cell		Long 90/90	
Momentum compaction α_p	[10^{-6}]	28.67	
Arc sext families		75	
$\beta^*_{x/y}$	[mm]	110 / 0.7	
Transverse tunes $Q_{x/y}$		218.158 / 222.220	218.1
Chromaticities $Q'_{x/y}$		0 / +5	
Energy spread (SR/BS) σ_δ	[%]	0.039 / 0.110	0.
Bunch length (SR/BS) σ_z	[mm]	5.57 / 15.6	3
RF voltage 400/800 MHz	[GV]	0.079 / 0	
Harm. number for 400 MHz			
RF frequency (400 MHz)	MHz		
Synchrotron tune Q_s		0.0289	
Long. damping time	[turns]	1171	
RF acceptance	[%]	1.06	
Energy acceptance (DA)	[%]	±1.0	
Beam crossing angle at IP θ_x	[mrad]		
Crab waist ratio	[%]	70	
Beam-beam ξ_x/ξ_y[a]		0.0022 / 0.0977	0.
Piwinski angle $(\theta_x \sigma_{z,\text{BS}})/\sigma^*_x$		26.6	
Lifetime (q + BS + lattice)	[sec]	11800	
Lifetime (lum)[b]	[sec]	1330	
Luminosity / IP	[10^{34}/cm^{2}s]	143	

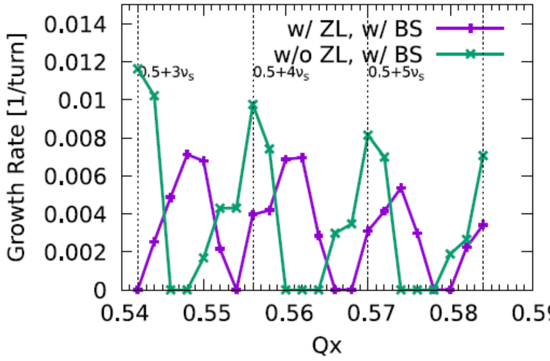
March 2022 $\beta^*_x = 10\,\text{cm}$ @Z

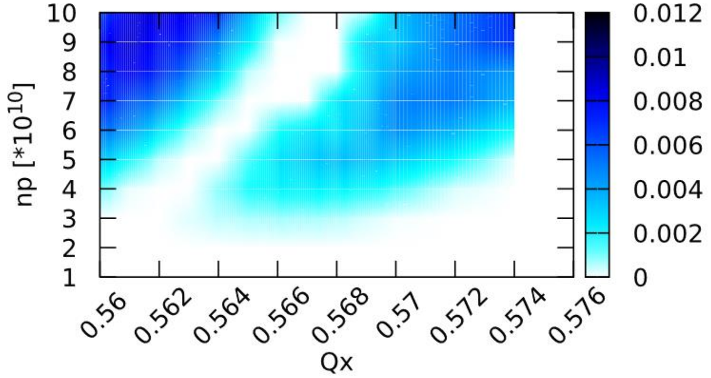
Beam energy	[GeV]	45.6	80
Layout		PA31	
# of IPs		4	
Circumference	[km]	91.174117	
Bending radius of arc dipole	[km]	9.9	
Energy loss / turn	[GeV]	0.0391	0.370
SR power / beam	[MW]	5	
Beam current	[mA]	1280	135
Bunches / beam		10000	880
Bunch population	[10^{11}]	2.43	2.91
Horizontal emittance ε_x	[nm]	0.71	2.16
Vertical emittance ε_y	[pm]	1.42	4.32
Arc cell		Long 90/90	
Momentum compaction α_p	[10^{-6}]	28.5	
Arc sextupole families		75	
$\beta^*_{x/y}$	[mm]	100 / 0.8	200 / 1.0
Transverse tunes/IP $Q_{x/y}$		53.563 / 53.600	
Energy spread (SR/BS) σ_δ	[%]	0.038 / 0.132	0.069 / 0.154
Bunch length (SR/BS) σ_z	[mm]	4.38 / 15.4	3.55 / 8.01
RF voltage 400/800 MHz	[GV]	0.120 / 0	1.0 / 0
Harmonic number for 400 MHz		121	
RF freuqeuncy (400 MHz)	MHz	399.994581	
Synchrotron tune Q_s		0.0370	0.0801
Long. damping time	[turns]	1168	217
RF acceptance	[%]	1.6	3.4
Energy acceptance (DA)	[%]	±1.3	±1.3
Beam-beam ξ_x/ξ_y[a]		0.0023 / 0.135	0.011 / 0.125
Luminosity / IP	[10^{34}/cm^{2}s]	182	19.4
Lifetime (q + BS)	[sec]	–	
Lifetime (lum)	[sec]	1129	1070

Impact on beam stability (first thoughts)



CEPC CDR





X-Z instability

Higher Q_s → lower order resonance (stronger)

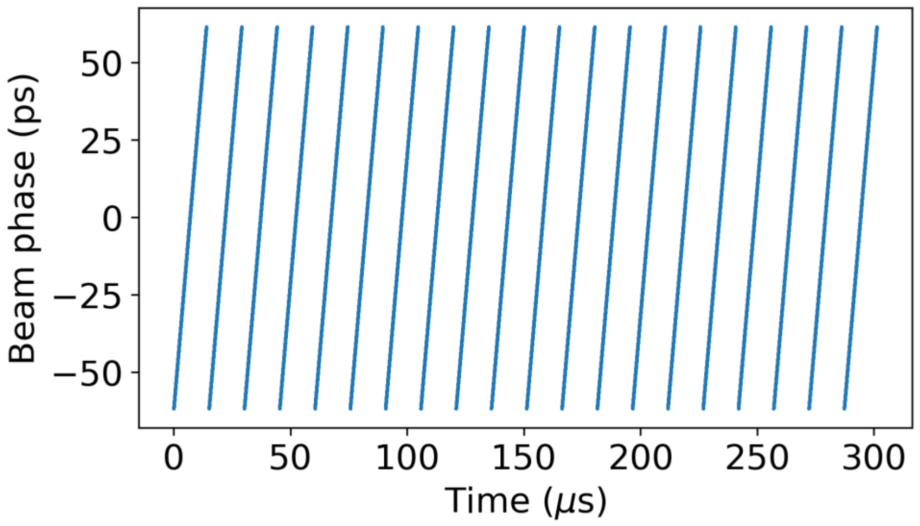
Higher V_{tot} → ~50% shorter bunches (assuming the same dp/p) → stronger beamstrahlung (ξ_y increase)

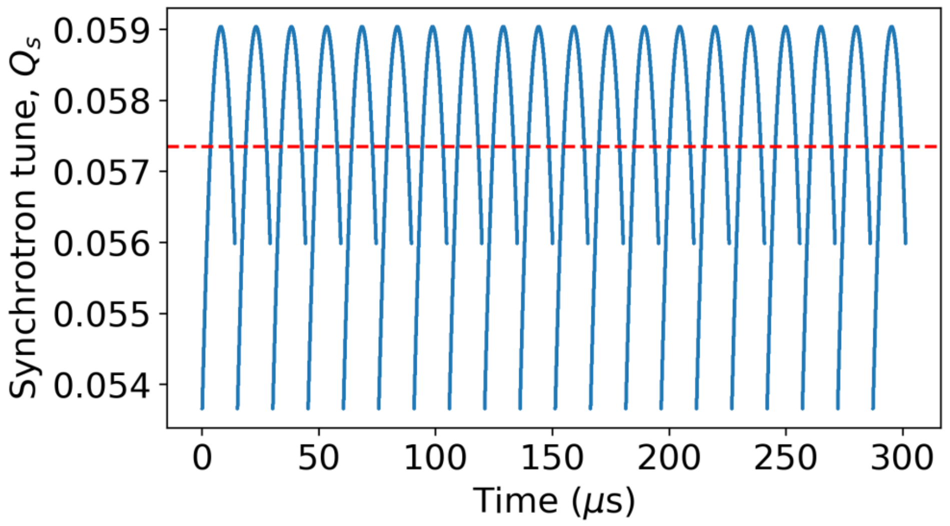
Stabilizing role of chromaticity is not fully understood

Resonant depolarization

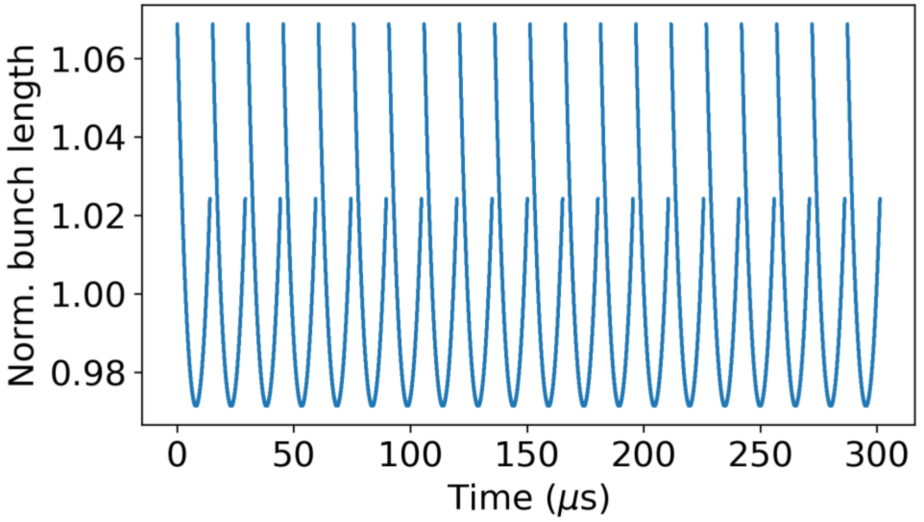
Figure of merit SMI = nus * (sigma_E/E) / Qs ~ 1.3-1.4 for baseline → reduces ~0.85 (SMI<1 is preferred)

Changing optics

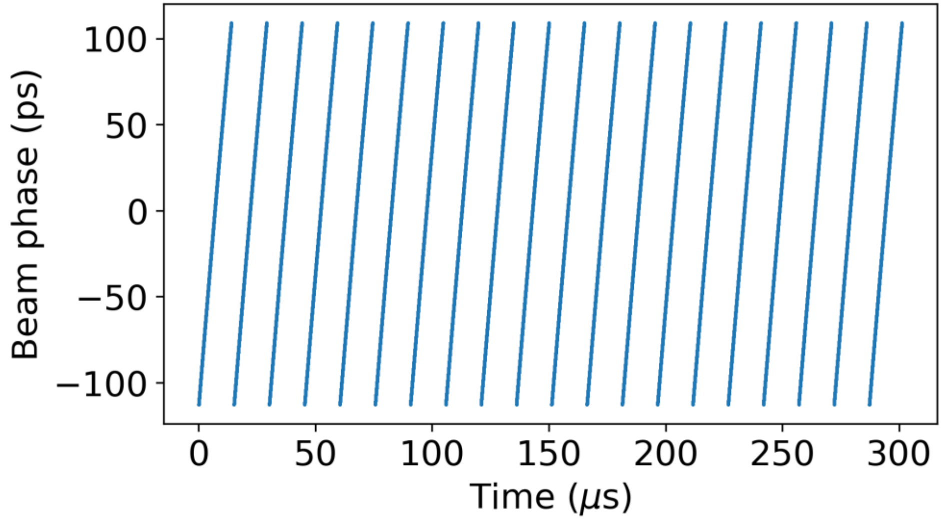




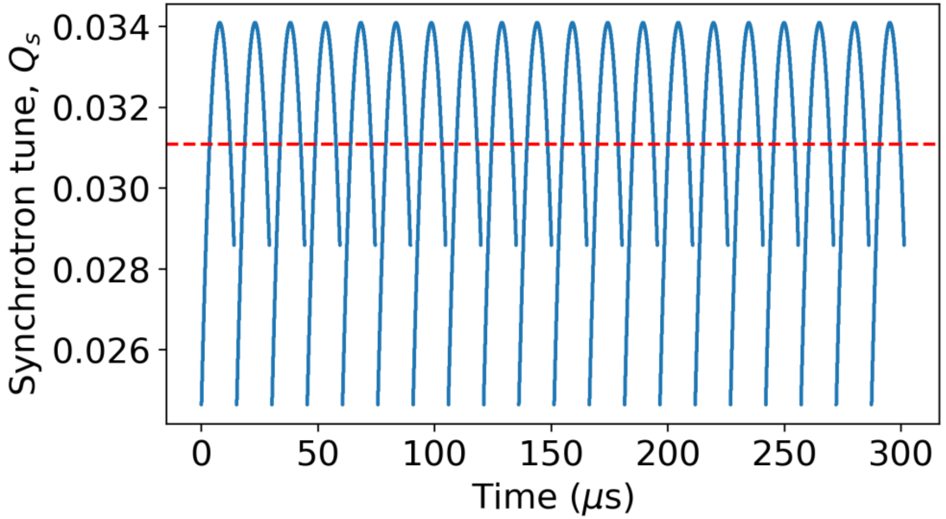
New $\alpha_p = 53.7\text{e-}6$



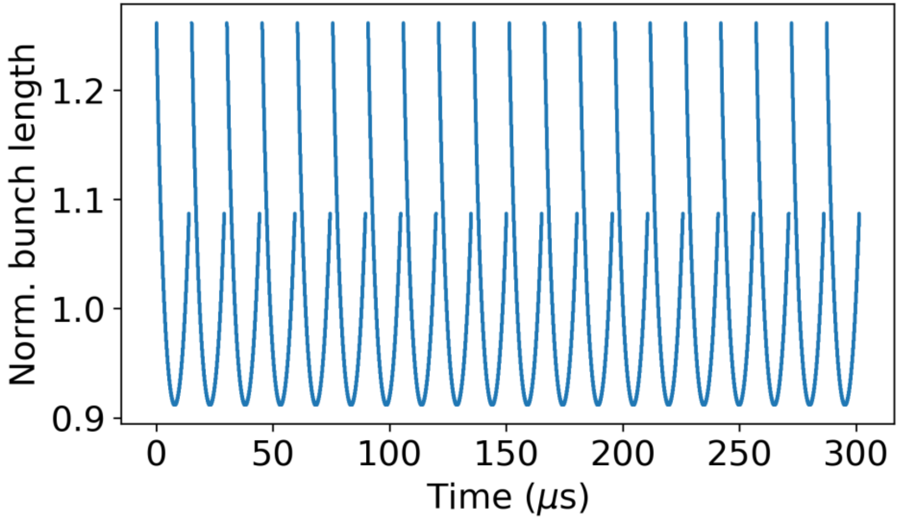
Reduced Q_L=1.5e5 for same V_{tot}



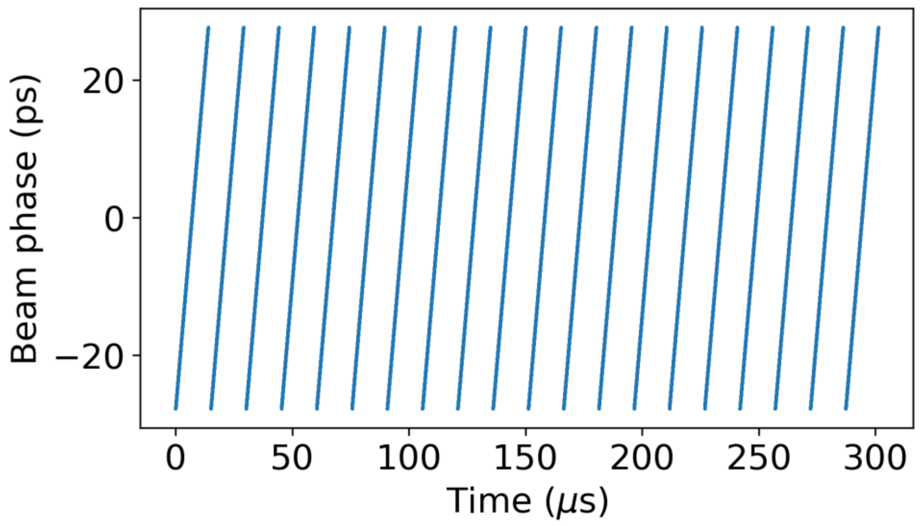
-21/+10 %

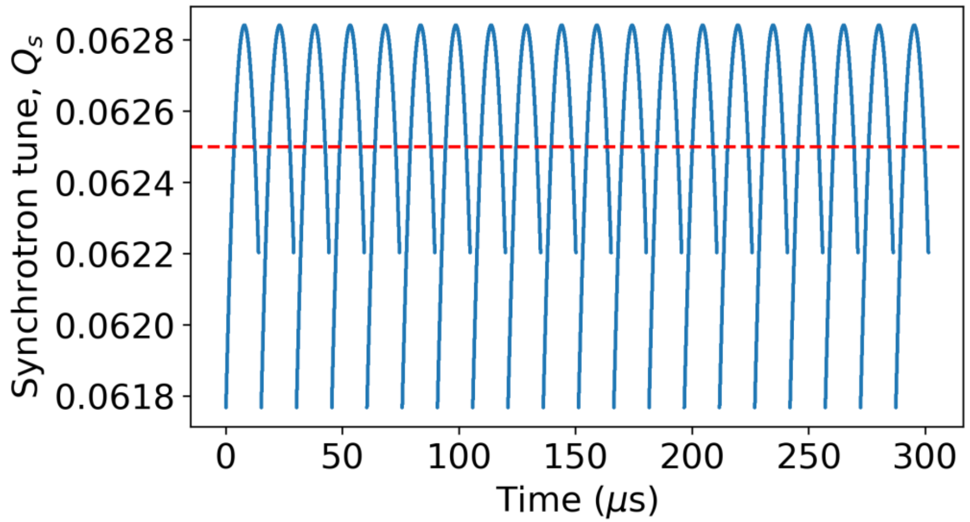


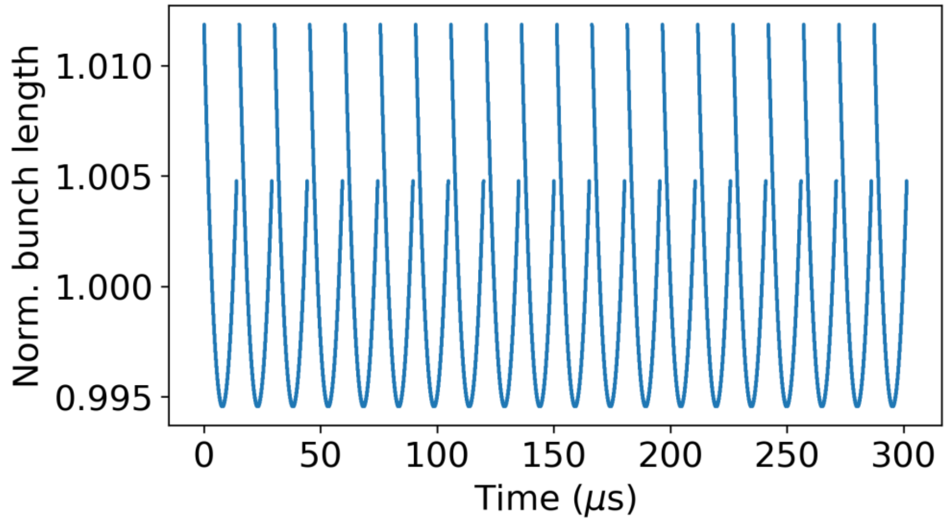
-9/+26 %



320 MV


Beam phase (ps)
20
0
−20
0
50
100
150
200
250
300
Time (μs)


Synchrotron tune, Q_s
0.0628
0.0626
0.0624
0.0622
0.0620
0.0618
0
50
100
150
200
250
300
Time (μs)


Norm. bunch length
1.010
1.005
1.000
0.995
0
50
100
150
200
250
300
Time (μs)

Motivation

RF power for SRF cavities with circulators is minimized for optimal parameters:

Optimal detuning $\Delta\omega_{\text{opt}} = -\dfrac{\omega_{\text{rf}}(R/Q)|F_b|I_{b,\text{dc}}\sin\phi_s}{2V_{\text{cav}}}$

$$F_b = 2\frac{\mathcal{F}[\lambda(t)]_{\omega=\omega_{\text{rf}}}}{\mathcal{F}[\lambda(t)]_{\omega=0}}$$

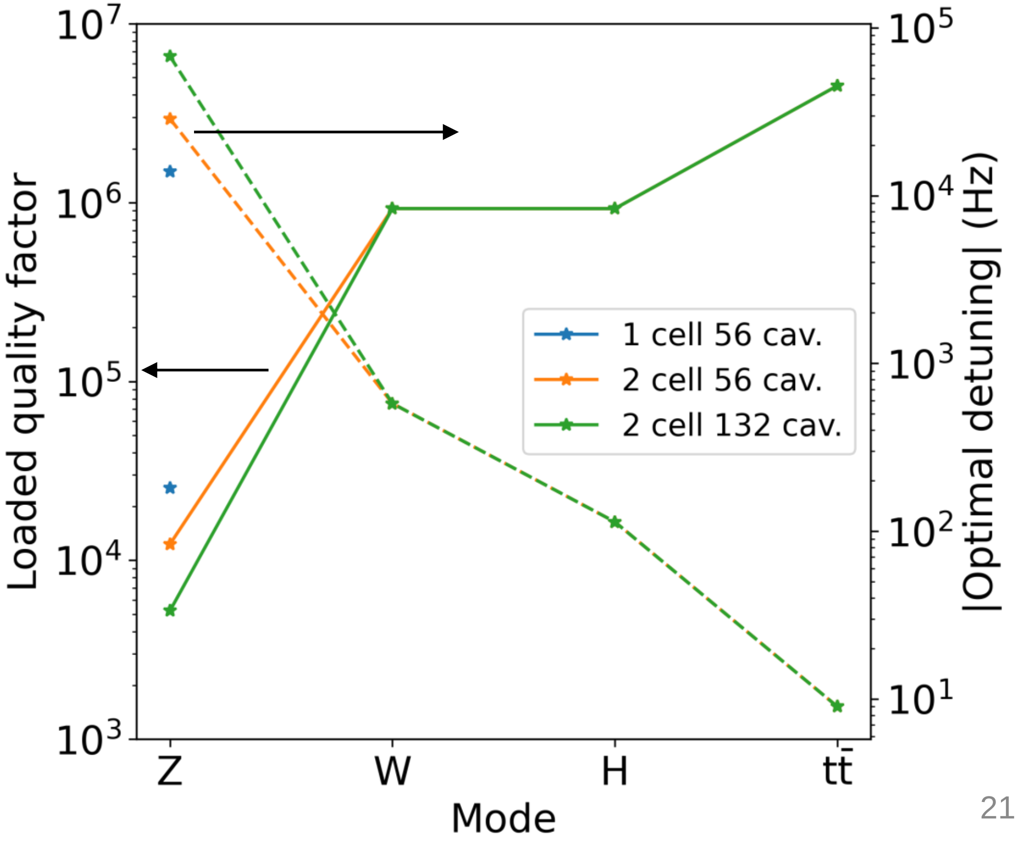
Optimal quality factor $Q_{\text{ext,opt}} = \dfrac{V_{\text{cav}}}{|F_b|(R/Q)I_{b,\text{dc}}\cos\phi_s}$

Keeping 2-cell cavities for Z, W, H, (and $t\bar{t}$):

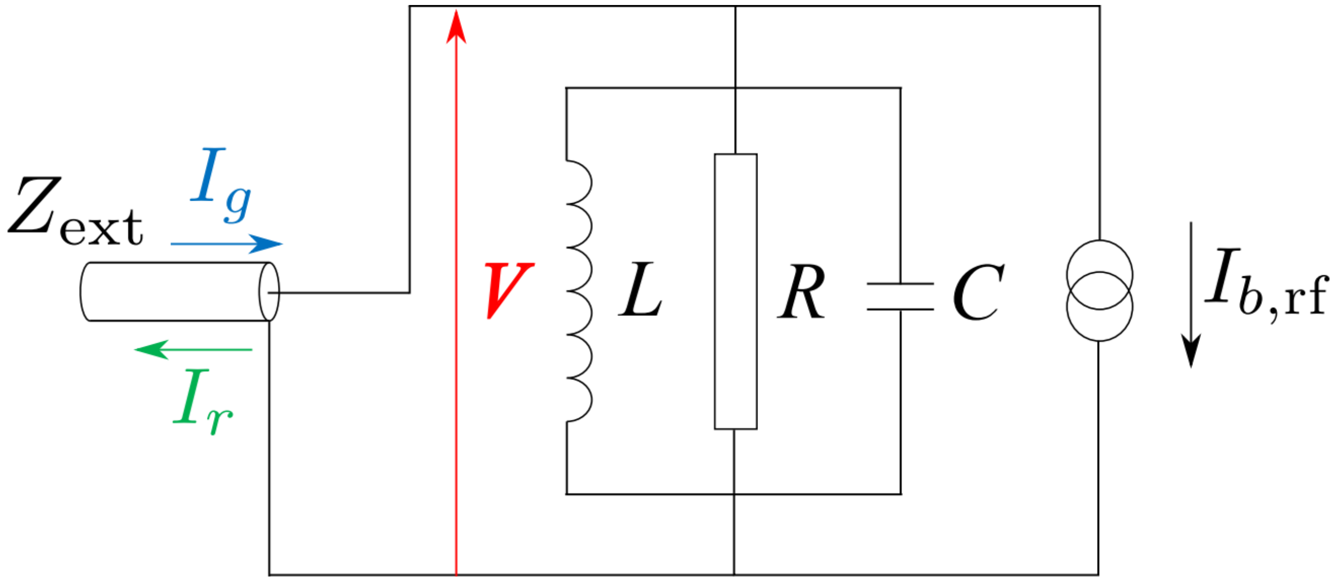
→ Large range for $Q_{\text{ext,opt}}$ adjustment (a factor of ~75-600) starting from $\sim 5 \times 10^3$: possible FPC solutions was studied *(S. Gorgi Zadeh and E. Montesinos, CERN SRF, 2024; see also slides of F. Gerigk, FCC Week 2024)*
→ Incresed detuning enhances instability due to fundamental mode

Can the voltage per cavity be increased for Z mode?

Optimal parameters for different scenarios



Beam loading model: main equation



Generator current $$I_g = \frac{V}{2(R/Q)}\left(\frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}} - 2i\frac{\Delta\omega}{\omega_{\text{rf}}}\right) + \frac{I_{b,\text{rf}}}{2}$$

Generator power $$P_g = \frac{1}{2} Z_{\text{ext}} |I_g|^2 = \frac{1}{2}(R/Q) Q_{\text{ext}} |I_g|^2$$

Fixed parameters are V, (R/Q), Q_0, ω_{rf}, $I_{b,\text{rf}}$, while V, $\Delta\omega$, and Q_{ext} can be adjusted

See, e.g., J. Tückmantel, CERN Report No. CERN-ATS-Note-2011- 002 TECH, 2011

RF power requirements

Constraints:
- The same $Q_{\text{ext,opt}}$ for all cavities to avoid a movable fundamental power coupler design
- The same $P_{g,\text{opt}}$ to have the identical power sources and uniform power distribution (role of variations is under study)

$$Q_{\text{ext,opt}} = \frac{|V_{\text{cav}}|}{|F_b|(R/Q)I_{b,\text{dc}}\cos(\phi_s + \phi_c)}$$

$$P_{g,\text{opt}} = \frac{|V_{\text{cav}}||F_b|I_{b,\text{dc}}\cos(\phi_s + \phi_c)}{2}$$

→ Cavity voltage must be the same for all cavities: $\cos(\phi_s + \phi_{\text{foc}}) = \cos(\phi_s + \phi_{\text{defoc}}) \rightarrow \phi_{\text{foc}} = -2\phi_s - \phi_{\text{defoc}}$

Starting with energy gain per turn

$$N_{\text{foc}}|V_{\text{cav}}|\cos(\phi_s + \phi_{\text{foc}}) + N_{\text{defoc}}|V_{\text{cav}}|\cos(\phi_s + \phi_{\text{defoc}}) = V_{\text{tot}}\cos\phi_s$$

$$\times \frac{|F_b|I_{b,\text{dc}}}{2}$$

$$N_{\text{foc}}\frac{|V_{\text{cav}}||F_b|I_{b,\text{dc}}\cos(\phi_s + \phi_{\text{foc}})}{2} + N_{\text{defoc}}\frac{|V_{\text{cav}}||F_b|I_{b,\text{dc}}\cos(\phi_s + \phi_{\text{defoc}})}{2} = \frac{|F_b|I_{b,\text{dc}}}{2}V_{\text{tot}}\cos\phi_s$$

$$\cos\phi_s = \frac{U_0}{V_{\text{tot}}} \qquad |F_b| \approx 2$$

$$N_{\text{foc}}P_{g,\text{foc}} + N_{\text{defoc}}P_{g,\text{defoc}} = I_{b,\text{dc}}U_0 = P_{\text{SR}}$$

$$P_{g,\text{foc}} = P_{g,\text{defoc}} = P_{g,\text{opt}}$$

$$N_{\text{foc}} + N_{\text{defoc}} = N_{\text{tot}}$$

$$P_{g,\text{opt}} = \frac{P_{SR}}{N_{\text{tot}}}$$

→ No RF power overshoot is needed for RPO if optimal detuning and optimal quality factor are used

Reverse phasing mode equations

Preservation of energy gain

$$N_{\text{foc}}|V_{\text{cav}}|\cos(\phi_s+\phi_{\text{foc}})+N_{\text{defoc}}|V_{\text{cav}}|\cos(\phi_s+\phi_{\text{defoc}})=V_{\text{tot}}\cos\phi_s$$

Preservation of synchrotron tune

$$N_{\text{foc}}|V_{\text{cav}}|\sin(\phi_s+\phi_{\text{foc}})+N_{\text{defoc}}|V_{\text{cav}}|\sin(\phi_s+\phi_{\text{defoc}})=V_{\text{tot}}\sin\phi_s$$

→ Cavity voltage

$$|V_{\text{cav}}|=\frac{V_{\text{tot}}}{N_{\text{tot}}}\sqrt{\frac{U_0^2}{V_{\text{tot}}^2}+\left(1-\frac{U_0^2}{V_{\text{tot}}^2}\right)\frac{N_{\text{tot}}^2}{(N_{\text{foc}}-N_{\text{defoc}})^2}}$$

Optimal detuning

$$\Delta\omega_{\text{opt}}=-\frac{\omega_{\text{rf}}(R/Q)|F_b|I_{b,\text{dc}}}{2V_{\text{cav}}}\sqrt{1-\frac{U_0^2}{V_{\text{cav}}^2N_{\text{tot}}^2}}$$

See, also A. Blednykh et al, EIC-ADD-TN-33, 2022

Phases

$$\phi_{\text{foc}}=-\phi_s+\arccos\left(\frac{V_{\text{tot}}\cos\phi_s}{N_{\text{tot}}V_{\text{cav}}}\right)\qquad\phi_{\text{defoc}}=-\phi_s-\arccos\left(\frac{V_{\text{tot}}\cos\phi_s}{N_{\text{tot}}V_{\text{cav}}}\right)$$

The aim is to keep V_{cav}, $P_{g,\text{opt}}$, and $Q_{\text{ext,opt}}$ for Z, W, and ZH modes
→ Cavity voltage can be change in discrete steps of $N_{\text{foc}}-N_{\text{defoc}}=2,4,\ldots$

Derivations for arbitrary cavity phase (1/2)

Generator current $$I_g = \frac{V}{2(R/Q)}\left(\frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}} - 2i\frac{\Delta\omega}{\omega_{\text{rf}}}\right) + \frac{I_{b,\text{rf}}}{2}$$

0

$$F_b = 2\frac{\mathcal{F}[\lambda(t)]_{\omega=\omega_{\text{rf}}}}{\mathcal{F}[\lambda(t)]_{\omega=0}}$$

Complex quantities: I_g, V, and $I_{b,\text{rf}} \rightarrow$ $I_g = |I_g|e^{i\phi_L}$, $V = |V_{\text{cav}}|e^{i\phi_c}$, $I_{b,\text{rf}} = |F_b|I_{b,\text{dc}}e^{-i\phi_s}$

$$|I_g|e^{i\phi_L} = \frac{|V_{\text{cav}}|e^{i\phi_c}}{2(R/Q)}\left(\frac{1}{Q_{\text{ext}}} - 2i\frac{\Delta\omega}{\omega_{\text{rf}}}\right) + \frac{|F_b|I_{b,\text{dc}}e^{-i\phi_s}}{2} \quad \Big| \times e^{-i\phi_c}$$

$$|I_g|e^{i\phi_L - i\phi_c} = \frac{|V_{\text{cav}}|}{2(R/Q)}\left(\frac{1}{Q_{\text{ext}}} - 2i\frac{\Delta\omega}{\omega_{\text{rf}}}\right) + \frac{|F_b|I_{b,\text{dc}}e^{-i\phi_s - i\phi_c}}{2}$$

Then splitting in real and imaginary parts:

Derivations for arbitrary cavity phase (2/2)

$$|I_g|e^{i\phi_L - i\phi_c} = \frac{|V_{\text{cav}}|}{2(R/Q)Q_{\text{ext}}} + \frac{|F_b|I_{b,\text{dc}}\cos(\phi_s + \phi_c)}{2} - i\left[\frac{|V_{\text{cav}}|}{(R/Q)}\frac{\Delta\omega}{\omega_{\text{rf}}} + \frac{|F_b|I_{b,\text{dc}}\sin(\phi_s + \phi_c)}{2}\right]$$

$$P_g = \frac{1}{2}(R/Q)Q_{\text{ext}}|I_g|^2$$

$$= \frac{1}{2}(R/Q)Q_{\text{ext}}\left[\frac{|V_{\text{cav}}|}{2(R/Q)Q_{\text{ext}}} + \frac{|F_b|I_{b,\text{dc}}\cos(\phi_s + \phi_c)}{2}\right]^2$$

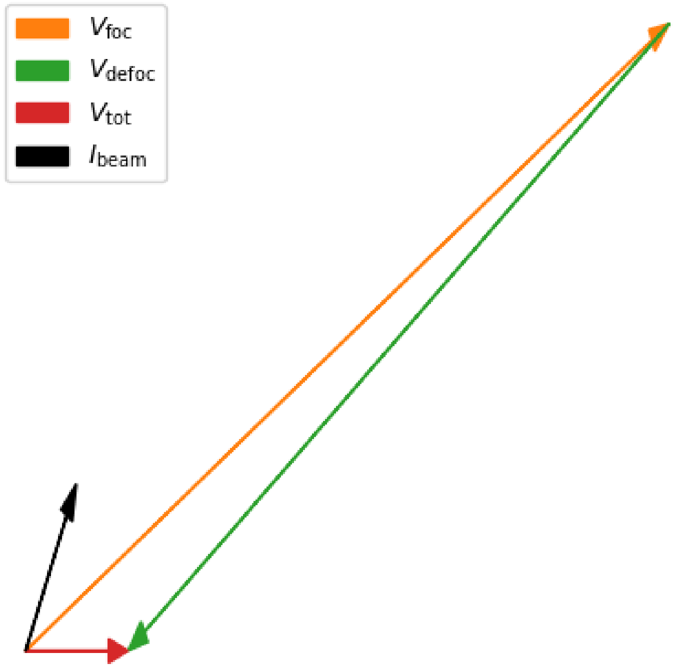
→ Minimized for $Q_{\text{ext,opt}} = \frac{|V_{\text{cav}}|}{|F_b|(R/Q)I_{b,\text{dc}}\cos(\phi_s+\phi_c)}$

$$+\frac{1}{2}(R/Q)Q_{\text{ext}}\left[\frac{|V_{\text{cav}}|}{(R/Q)}\frac{\Delta\omega}{\omega_{\text{rf}}} + \frac{|F_b|I_{b,\text{dc}}\sin(\phi_s + \phi_c)}{2}\right]^2$$

→ = 0 for $\Delta\omega_{\text{opt}} = -\frac{\omega_{\text{rf}}(R/Q)|F_b|I_{b,\text{dc}}\sin(\phi_s+\phi_c)}{2|V_{\text{cav}}|}$

Setting $\phi_c = 0$ recovers classical equations for optimal parameters
Adjusting ϕ_c, $Q_{\text{ext,opt}}$ can be modified to meet certain constraints

The minimum power $P_{g,\text{opt}} = \frac{|V_{\text{cav}}||F_b|I_{b,\text{dc}}\cos(\phi_s + \phi_c)}{2}$



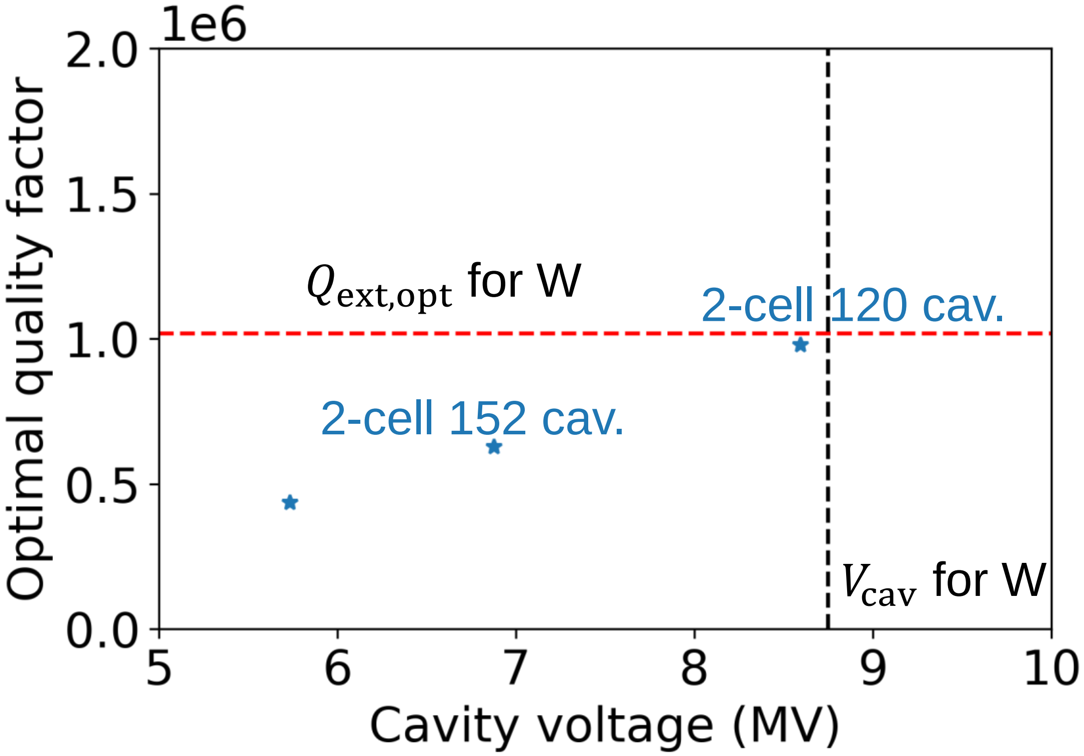
V_{foc}
V_{defoc}
V_{tot}
I_{beam}

Preliminary results

$$|V_{\mathrm{cav}}| = \frac{V_{\mathrm{tot}}}{N_{\mathrm{tot}}}\sqrt{\frac{U_0^2}{V_{\mathrm{tot}}^2} + \left(1 - \frac{U_0^2}{V_{\mathrm{tot}}^2}\right)\frac{N_{\mathrm{tot}}^2}{(N_{\mathrm{foc}} - N_{\mathrm{defoc}})^2}}$$

Optimal quality factor with RPO

$$Q_{\mathrm{ext,opt}} = \frac{V_{\mathrm{cav}}^2 N_{\mathrm{tot}}}{V_{\mathrm{tot}}(R/Q)|F_b| I_{b,\mathrm{dc}} \cos\phi_s}$$



→ RPO is under evaluation potentially allowing for the same optimal quality factor for Z, W, and H modes

Reverse phasing mode equations

Constraints: $|V_{\text{cav}}|$ and $P_{g,\text{opt}}$ are the same for focusing and defocusing cavities

$\rightarrow \cos(\phi_s + \phi_{\text{foc}}) = \cos(\phi_s + \phi_{\text{defoc}}) \rightarrow \phi_{\text{foc}} = -2\phi_s - \phi_{\text{defoc}}$

$$P_{g,\text{opt}} = \frac{|V_{\text{cav}}||F_b|I_{b,\text{dc}}\cos(\phi_s + \phi_c)}{2}$$

Preservation of energy gain

$$N_{\text{foc}}|V_{\text{cav}}|\cos(\phi_s + \phi_{\text{foc}}) + N_{\text{defoc}}|V_{\text{cav}}|\cos(\phi_s + \phi_{\text{defoc}}) = V_{\text{tot}}\cos\phi_s$$

Preservation of synchrotron tune

$$N_{\text{foc}}|V_{\text{cav}}|\sin(\phi_s + \phi_{\text{foc}}) + N_{\text{defoc}}|V_{\text{cav}}|\sin(\phi_s + \phi_{\text{defoc}}) = V_{\text{tot}}\sin\phi_s$$

	RPO	Classical
Optimal quality factor	$Q_{\text{ext,opt}} = \dfrac{V_{\text{cav}}^2 N_{\text{tot}}}{V_{\text{tot}}(R/Q)\lvert F_b\rvert I_{b,\text{dc}}\cos\phi_s}$	$Q_{\text{ext,opt}} = \dfrac{V_{\text{cav}}}{\lvert F_b\rvert(R/Q)I_{b,\text{dc}}\cos\phi_s}$
Optimal detuning	$\Delta\omega_{\text{opt}} = -\dfrac{\omega_{\text{rf}}(R/Q)\lvert F_b\rvert I_{b,\text{dc}}}{2V_{\text{cav}}}\sqrt{1 - \dfrac{\cos^2\phi_s V_{\text{tot}}^2}{V_{\text{cav}}^2 N_{\text{tot}}^2}}$	$\Delta\omega_{\text{opt}} = -\dfrac{\omega_{\text{rf}}(R/Q)\lvert F_b\rvert I_{b,\text{dc}}\sin\phi_s}{2V_{\text{cav}}}$

Reduced Pedersen model