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Theory Seminar

Resummation of Next-to-Leading Non-Global Logarithms

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Predictions in particle physics

Particles & interactions described in terms of quantum fields

$$\mathcal{L} = \bar{\psi}\,(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sqrt{\alpha_s}\,\bar{\psi}\,A\!\!\!/\,\psi$$

Predictions in particle physics

Particles & interactions described in terms of quantum fields

$$\mathcal{L} = \bar{\psi}\,(\mathrm{i}\,\partial\!\!\!/ - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \sqrt{\alpha_s}\;\bar{\psi}\,A\!\!\!/\,\psi$$

QFTs are typically not exactly solvable, perturbative approach necessary

$\Rightarrow$

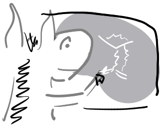
expansion in the strong coupling $\alpha_s \sim 0.1$

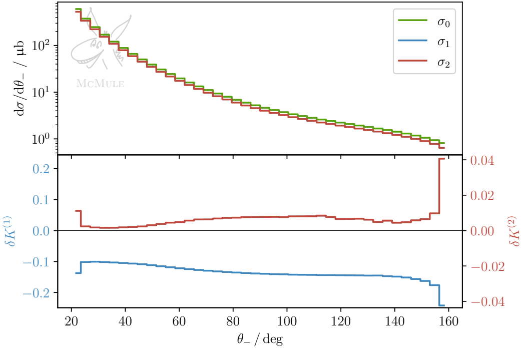
$$\sigma = \int \mathrm{d}\Phi_{\mathrm{LIPS}}\; |\mathcal{A}|^2\; \Theta_{\mathrm{cuts}}$$

$$= \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s\,\sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2\,\sigma_2}_{\text{NNLO}} + \mathcal{O}(\alpha_s^3)$$

Perturbative calculations: example

- NNLO Bhabha scattering $e^+e^- \to e^+e^-$

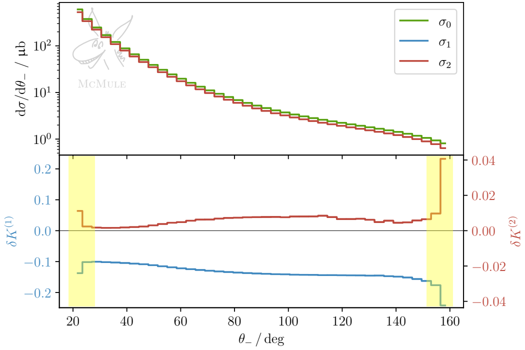




[Banerjee, Engel, NS, Signer, Ulrich, 2106.07469]

Perturbative calculations: breakdown

- Enhanced contributions in highlighted regions



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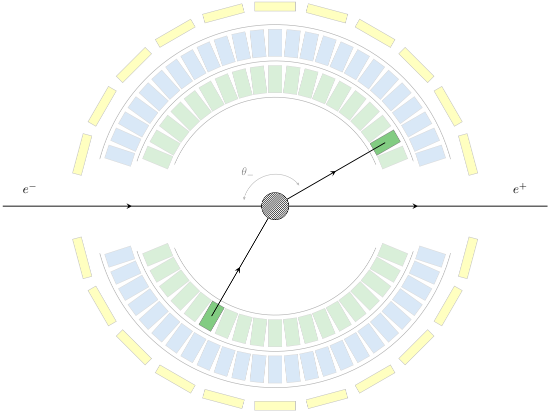
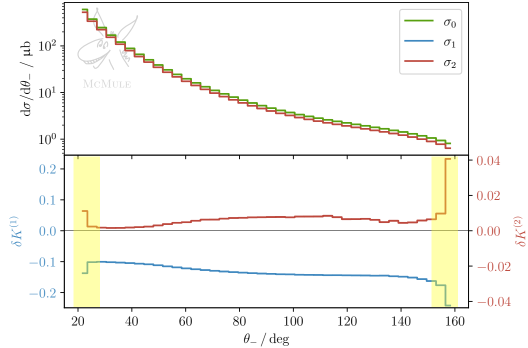
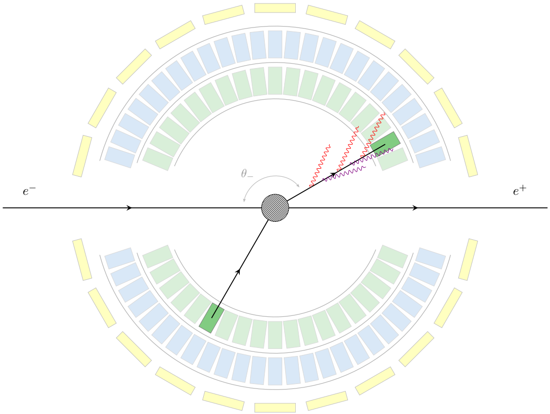
Perturbative calculations: breakdown

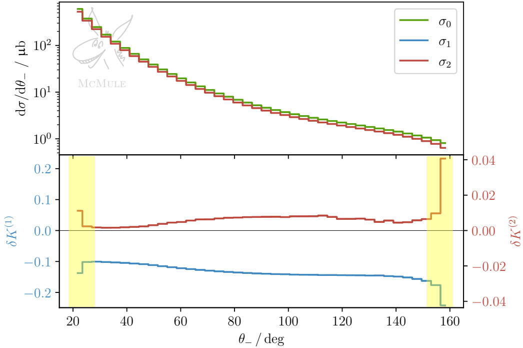




[Banerjee, Engel, NS, Signer, Ulrich, 2106.07469]

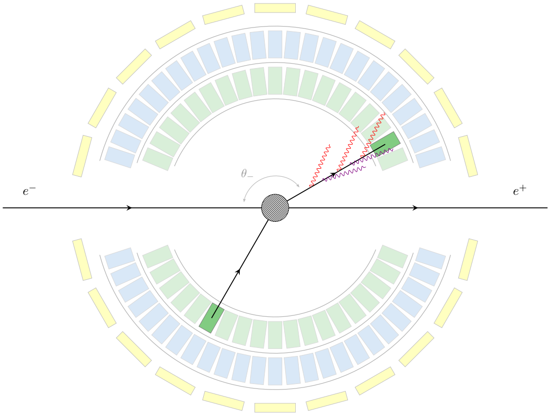
Contributions beyond fixed order





[Banerjee, Engel, NS, Signer, Ulrich, 2106.07469]

Contributions beyond fixed order



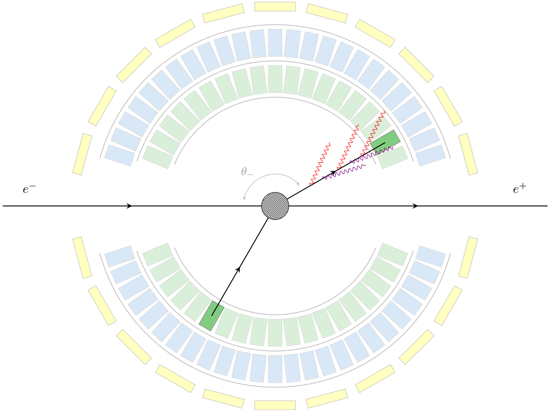
- Soft and collinear emissions (not captured by the detector) introduce additional scales

$$\mu_h \sim Q$$

$$\mu_s \sim Q_0$$

$$\mu_c \sim m_e$$

Contributions beyond fixed order



- For processes involving large scale hierarchies

$$Q_0 \ll Q$$
$$m_e \ll Q$$

fixed order perturbation theory breaks down due to logarithmically enhanced corrections

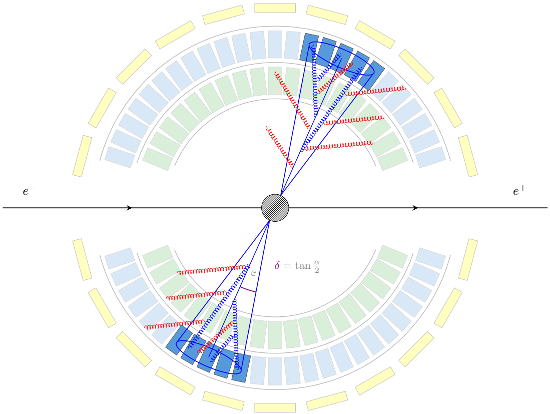
$$\alpha^n \, L^m \quad \text{with} \quad L = \log\left(\frac{Q}{Q_0}\right)$$

$$\alpha^n \, \ell^m \quad \text{with} \quad \ell = \log\left(\frac{Q}{m_e}\right)$$

in the cross section

$$\sigma = \sigma_{\mathrm{LO}} + \alpha\,\sigma_{\mathrm{NLO}} + \alpha^2\,\sigma_{\mathrm{NNLO}} + \cdots$$

Large logarithms in dijet processes



- We also find a twofold pattern of logarithmic enhancements in dijet production

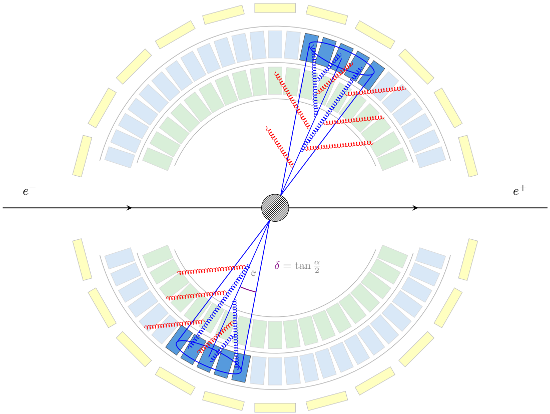
$$e^+ e^- \to \gamma^* \to 2\,\text{jets}$$

when (hard) radiation is restricted to be within the jets, since only (soft) radiation below Q_0 is allowed outside of the jets.

$$\sigma \sim 1 + \frac{\alpha_s}{\pi} C_F \left(3 \log \delta - 4 \log \delta \, \log \frac{Q}{Q_0} + \text{const.} \right)$$

[Sterman and Weinberg, 1977]

Large logarithms in dijet processes



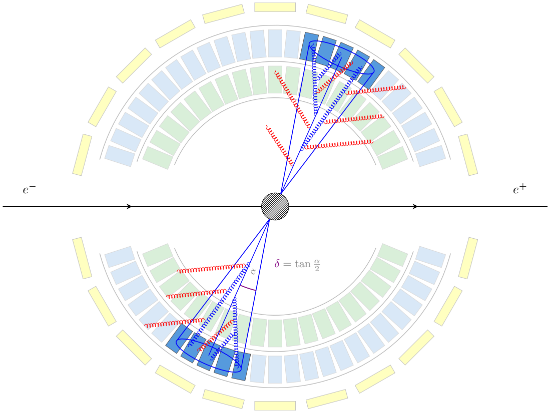
- Similarly, we find both soft and collinear logarithmic enhancements

We assume for the rest of this talk

$$\log \delta \sim 1$$

$$\log\left(\frac{Q}{Q_0}\right) \gg 1$$

Large logarithms in dijet processes



- Example:

$$Q_0 \sim 5\ \text{GeV}$$
$$Q \sim 1\ \text{TeV}$$

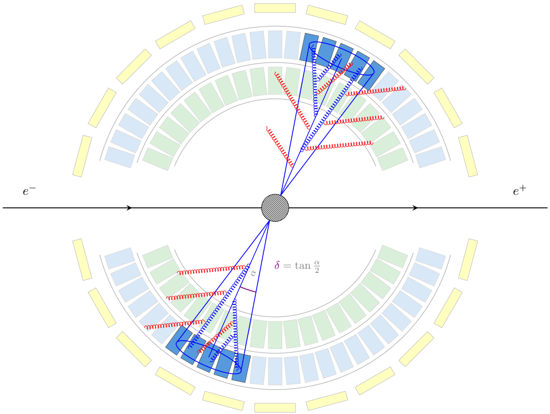
then the product; $\alpha_s \sim 0.1$ and $L = \log\frac{Q}{Q_0}$

$$\alpha_s\, L \sim \mathcal{O}(1)$$
$$\left(\alpha_s\, L\right)^2 \sim \mathcal{O}(1)$$
$$\left(\alpha_s\, L\right)^n \sim \mathcal{O}(1)$$

spoils the perturbative expansion!!!

$$\sigma_{\text{LO}} + \left(\alpha_s\, L\right)\sigma_{\text{NLO}} + \left(\alpha_s\, L\right)^2\sigma_{\text{NNLO}} + \cdots$$

Large logarithms in dijet processes



- Fixed order perturbation theory breaks down due to logarithmically enhanced corrections

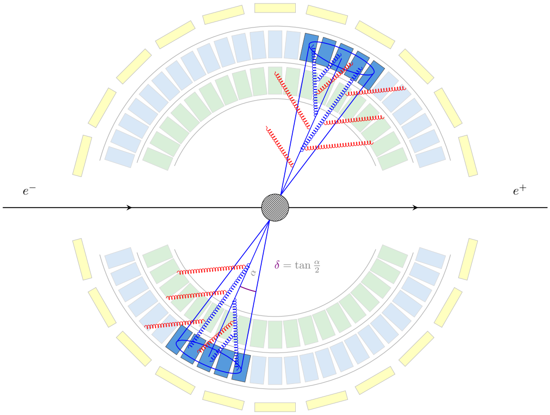
$$\alpha_s^n \, L^m \quad \text{with} \quad L = \log\left(\frac{Q}{Q_0}\right)$$

Identify

1. $(\alpha_s L)^n$ Leading Logarithms (LL)
2. $\alpha_s\,(\alpha_s L)^n$ Next-to-Leading Logarithms

$\Rightarrow$ (Re)assign LL, NLL, ... $\longrightarrow$ LO, NLO,...

Large logarithms in dijet processes



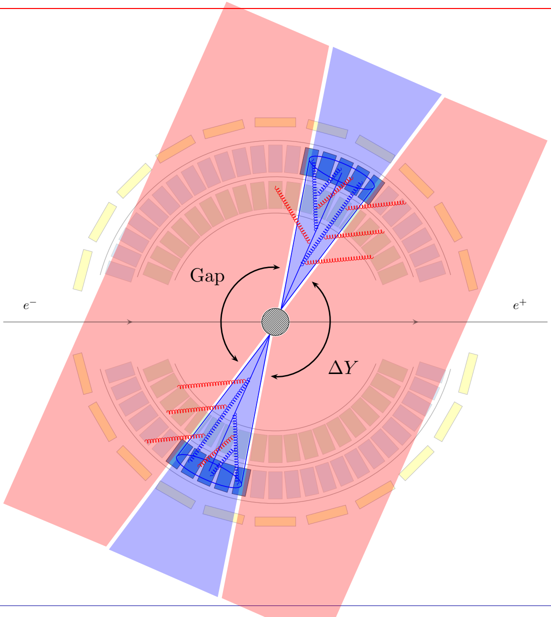
- Fixed order perturbation theory breaks down due to logarithmically enhanced corrections

$$\alpha_s^n \, L^m \quad \text{with} \quad L = \log\left(\frac{Q}{Q_0}\right)$$

To obtain reliable predictions across disparate scales, it is neccessary to capture the entire tower of logarithms

$\Rightarrow$ *Resummation*

Higher-order effects: Non-Global Logarithms



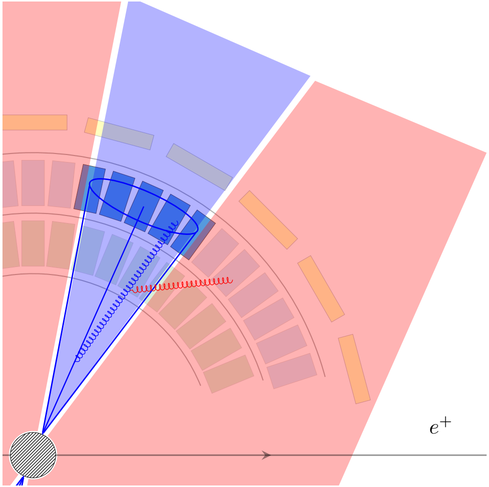
Jet cross sections involve angular cuts which constrain radiation within a corner of the phase space. As a consequence, logarithmically enhanced higher-order corrections known as

Non-Global Logarithms (NGLs)

arise.

[Dasgupta and Salam, hep-ph/0104277]

Higher-order effects: Non-Global Logarithms



The leading NGLs start at two loops

$$\sigma \sim 1 + \frac{\alpha_s}{\pi} C_F \left(3 \log \delta - 4 \log \delta \, \log \frac{Q}{Q_0} + \text{const.}\right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \Big[C_F^2 \, B_F(Q, Q_0, \delta)$$

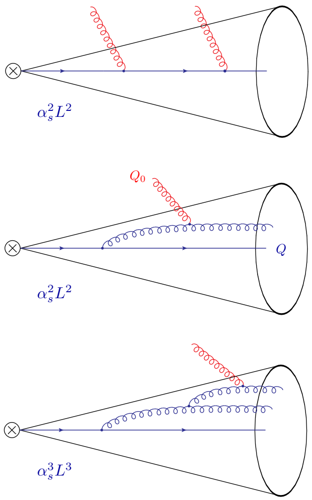
$$+ C_F \, T_F \, n_F \, B_{n_F}(Q, Q_0, \delta) + C_F \, C_A \; B_A(Q, Q_0, \delta) \Big]$$

$$\Big[-\zeta_2 + \text{Li}_2\left(e^{-2\Delta Y}\right) \Big] \log^2 \frac{Q}{Q_0} + \cdots$$

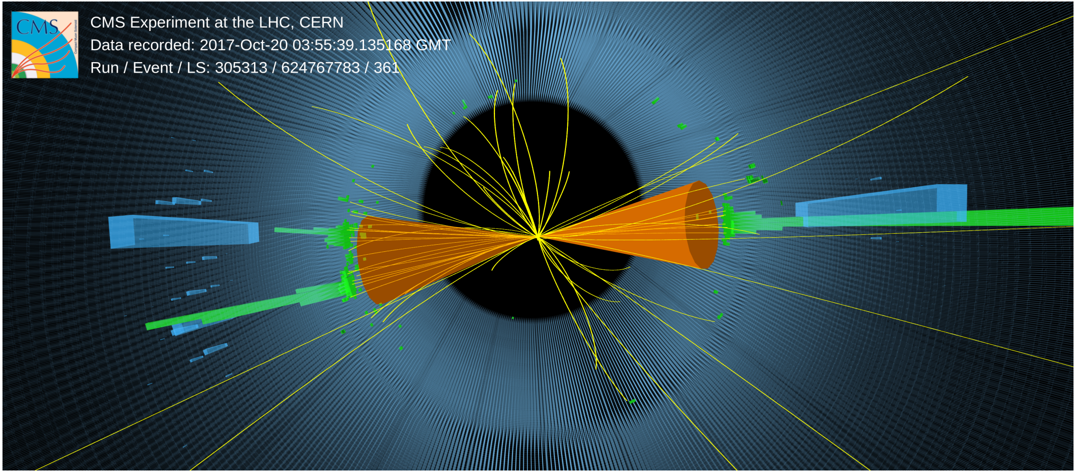
[Dasgupta and Salam, hep-ph/0203009]

Global or Non-Global?

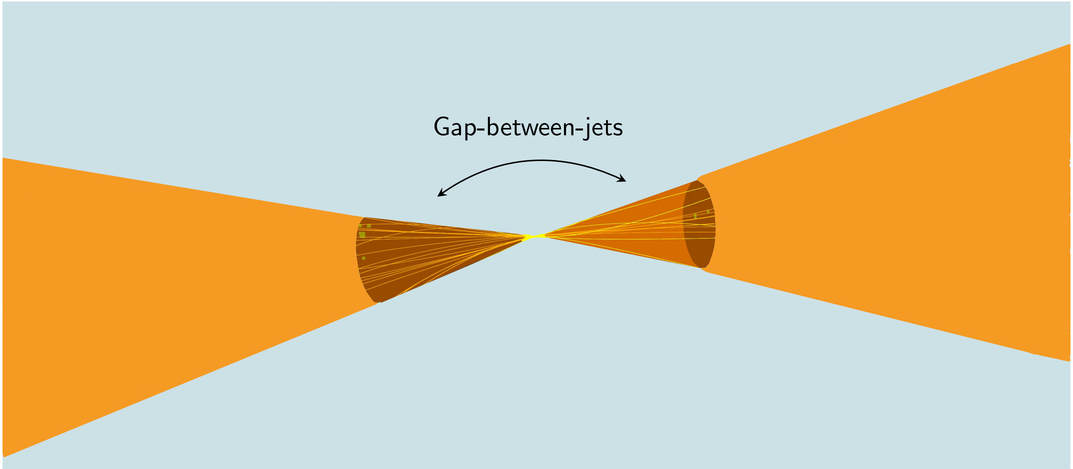
- NGLs arise due to secondary soft gluon emissions inside jets
- Not captured by standard resummation methods
  $\Rightarrow$ even leading NGLs $(\alpha_s L)^n$ do not simply exponentiate
- At large N_c leading NGLs can be obtained with a parton shower [Dasgupta and Salam, hep-ph/0104277] or by solving the non-linear BMS integral equation [Banfi et. al., hep-ph/0206076]



Non-Global logarithms are ubiquitous



Non-Global logarithms are ubiquitous



Non-global logarithms: Recent advances

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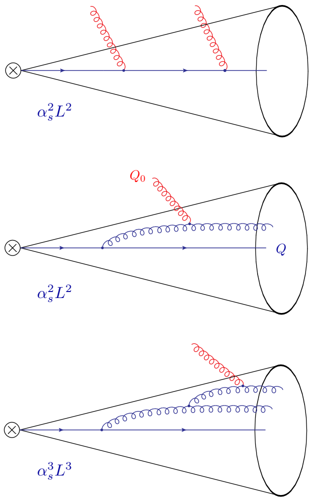
- LL at large N_c with general-purpose shower
  - PanScales [2002.11114, 2207.09467]
  - ALARIC [2208.06057, 2404.14360]

- Finite-N_c results for leading NGLs [Weigert, hep-ph/0312050], [Hatta, Ueda, Hagiwara; 1304.6930, 1507.07641, 2011.04154], [De Angelis, Forshaw, Plätzer; 2007.09648]

- First NLL numerical results in the large-N_c limit
  - Extension of BMS framework to NLL [Monni et. al., 2104.06416] and numerical implementation in MC code GNOLE [Monni et. al., 2111.02413]
  - Double-soft effects implemented in the PanGlobal family of showers, numerical results available [PanScales, 2307.11142]
  - Ingredients for resummation of subleading effects of NGLs using modern EFT techniques [Becher et. al., 1605.02737, 1901.09038, 2112.02108]



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MARZILI

(Monte-cArlo for the RenormaliZation group Improved calculation of non-global LogarIthms)

soon to appear on `gitlab`

Outline

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Parton shower framework

- RG methods
- $\Gamma^{(2)}$ anomalous dimension

Comparison against

- Gnole
- PanScales

for $e^+e^- \to 2\,\text{jets}$

Gap-between-jets at hadron collider

Factorisation theorem

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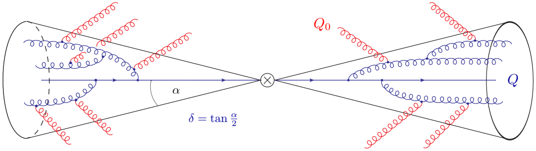
- Cross section for jet production in e^+e^- collisions with veto on radiation factorises into hard $\mathcal{H}_m$ and soft $\mathcal{S}_m$ functions [Becher et. al., 1508.06645]

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$



- Factorisation separates contributions from scales Q and Q_0
  $\Rightarrow$ natural way to perform resummation via RGEs

- Hard functions fulfill RG equations

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu} \mathcal{H}_m(Q, \mu) = -\sum_{l=2}^{m} \mathcal{H}_l(Q, \mu)\, \Gamma_{lm}(Q, \mu)$$

Factorisation theorem

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- Factorisation separates contributions from scales Q and Q_0 $\Rightarrow$ natural way to perform resummation via RGEs

- Hard functions fulfill RG equations

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\mathcal{H}_m(Q,\mu) = -\sum_{l=2}^{m} \mathcal{H}_l(Q,\mu)\,\Gamma_{lm}(Q,\mu)$$

- $\mathcal{H}_m(\{\underline{n}\},Q,\mu) \sim |\mathcal{M}_m\rangle\langle\mathcal{M}_m|$ describes m hard partons along fixed directions $\{n_1,\cdots,n_m\}$

$$\mathcal{H}_m(\{\underline{n}\},Q,\epsilon) = \frac{1}{2Q^2}\sum_{\text{spins}}\prod_{i=1}^{m}\int \frac{dE_i\,E_i^{d-3}}{\tilde{c}^\epsilon\,(2\pi)^2}\,|\mathcal{M}_m(\{\underline{p}\})\rangle\langle\mathcal{M}_m(\{\underline{p}\})| \times (2\pi)^d\,\delta\Big(Q-\sum_{i=1}^{m}E_i\Big)\,\delta^{(d-1)}(\vec{p}_{\text{tot}})\,\Theta_{\text{in}}(\{\underline{n}\})$$

- $\mathcal{S}_m(\{\underline{n}\},Q_0,\mu)$ is the squared amplitude with m Wilson lines along fixed directions $\{n_1,\cdots,n_m\}$

$$\mathcal{S}_m(\{\underline{n}\},Q_0,\epsilon) = \sum\!\!\!\!\!\!\!\int_{X_s} \langle 0|\,\boldsymbol{S}_1^\dagger(n_1)\ldots\boldsymbol{S}_m^\dagger(n_m)\,|X_s\rangle\langle X_s|\,\boldsymbol{S}_1(n_1)\ldots \times \ldots\boldsymbol{S}_m(n_m)\,|0\rangle\,\theta(Q_0-E_{\text{out}})$$

Resummation by RG evolution

- Cross section for jet production in e^+e^- collisions with veto on radiation factorises into hard $\mathcal{H}_m$ and soft $\mathcal{S}_m$ functions [Becher et. al., 1508.06645]

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

- Factorisation separates contributions from scales Q and Q_0

  $\Rightarrow$ natural way to perform resummation via RGEs

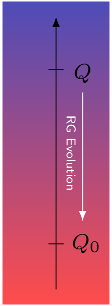
- Hard functions fulfill RG equations

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu} \mathcal{H}_m(Q, \mu) = -\sum_{l=2}^{m} \mathcal{H}_l(Q, \mu)\, \Gamma_{lm}(Q, \mu)$$

Procedure to solve the RGEs

1. Compute $\mathcal{H}_m$ at hard scale $\mu_h = Q$
2. Evolve $\mathcal{H}_m$ to soft scale $\mu_s = Q_0$
3. Evaluate $\mathcal{S}_m$ at soft scale $\mu_s = Q_0$

$\Rightarrow$ Resums large logarithms $\log \frac{Q_0}{Q}$



Resummation by RG evolution

Clear prescription how to perform resummation at any given accuracy

- LL
  $\mathcal{H}_2(\mu_h) = \sigma_0 \mathbb{1}$
  $\mathcal{H}_m(\mu_h) = 0$ for $m > 2$
  $\mathcal{S}_m(\mu_s) = \mathbb{1}$
  $\Gamma_{lm}^{(1)}$ one-loop anomalous dimension

- NLL
  $\mathcal{H}_2(\mu_h) = \sigma_0 |C_V|^2 \mathbb{1}$ one-loop virtual
  $\mathcal{H}_3(\mu_h)$ hard real emission corrections
  $\mathcal{S}_m^{(1)}(\mu_s)$ one-loop soft corrections
  $\Gamma_{lm}^{(2)}$ two-loop anomalous dimension

Extraction of anomalous dimension

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- Anomalous dimension $\mathbf{\Gamma}$ arises from soft singularities of hard functions

$$\mathcal{H}_m(Q,\mu) = \sum_{l=2}^{m} \mathcal{H}_l^{\text{bare}}(Q,\mu)\ \left(\mathbf{Z}^{-1}\right)_{lm}(Q,\mu)$$

$$\left(\mathbf{Z}^{-1}\right) = \mathbb{1} + \frac{\alpha_s}{4\pi}\frac{1}{2\epsilon}\mathbf{\Gamma}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{8\epsilon^2}\mathbf{\Gamma}^{(1)}\otimes\mathbf{\Gamma}^{(1)} - \frac{\beta_0}{4\epsilon^2}\mathbf{\Gamma}^{(1)} + \frac{1}{4\epsilon}\mathbf{\Gamma}^{(2)}\right]$$

$\mathbf{\Gamma}^{(1)}$ is $(-2)\times$ soft divergence of the one-loop hard function

Renormalisation $Z_\alpha = 1 - \frac{\beta_0}{\epsilon}\frac{\alpha_s}{4\pi}$

$\mathbf{\Gamma}^{(2)}$ is (-4) times single pole of the two-loop hard function

$\Gamma^{(1)}$: real

- To obtain the real contribution take the soft limit of $\mathcal{H}_{m+1}$

soft dipole W_{ij}^q

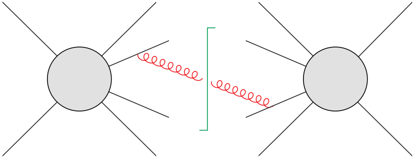
$$\mathcal{H}_{m+1}(\{\underline{n}, n_q\}, Q, \epsilon) = -g_s^2 \int_0^\Lambda \frac{dE_q\, E_q^{d-3}}{\tilde{c}^\epsilon\, (2\pi)^2}\, \theta_{\text{in}}(n_q) \sum_{(ij)} \frac{n_i \cdot n_j}{n_i \cdot n_q\; n_j \cdot n_q}\, \boldsymbol{T}_i^a\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon)\, \boldsymbol{T}_j^{\tilde{a}}$$

$$= \frac{2}{\epsilon} \frac{\alpha_s}{4\pi}\, \theta_{\text{in}}(n_q) \sum_{(ij)} W_{ij}^q\, \boldsymbol{T}_i^a\, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon)\, \boldsymbol{T}_j^{\tilde{a}}$$

- Directly obtain

$$\boldsymbol{R}_m = -4 \sum_{(ij)} \boldsymbol{T}_{i,L}^a \boldsymbol{T}_{j,R}^{\tilde{a}}\, W_{ij}^q\, \theta_{\text{in}}(n_q)$$

$\Rightarrow$ generates additional gluon inside jet



$\Gamma^{(1)}$: virtual

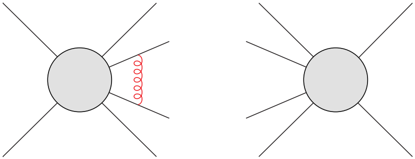
- To obtain the virtual contribution perform q^0 integral using residues

$$\mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) = \frac{g_s^2}{2} \sum_{(ij)} \int \frac{d^d q}{(2\pi)^d} \frac{-i}{q^2 + i0} \frac{n_i \cdot n_j}{[n_i \cdot q + i0]\,[-n_j \cdot q + i0]} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \, \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}$$

$$= -\frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \sum_{(ij)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \left[\int [\mathrm{d}\Omega_q] W_{ij}^q + i\pi \right] \mathcal{H}_m(\{\underline{n}\}, Q, \epsilon) + \text{h.c.}$$

- Directly obtain

$$\boldsymbol{V}_m = 2 \sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right) \int [\mathrm{d}\Omega_q] \, W_{ij}^q$$
$$- 2 \sum_{(ij)} \left[\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} - \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right] \times i\pi \, \Pi_{ij}$$



$\Gamma^{(1)}$ [Becher et. al., 1508.06645]

- One-loop anomalous dimension $\boldsymbol{\Gamma}^{(1)}$

$$\boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\boldsymbol{R}_m = -4 \sum_{(ij)} \boldsymbol{T}^a_{i,L} \boldsymbol{T}^{\tilde{a}}_{j,R} \, W^q_{ij} \, \theta_{\text{in}}(n_q)$$

$$\boldsymbol{V}_m = 2 \sum_{(ij)} (\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}) \int [\mathrm{d}\Omega_q] \, W^q_{ij}$$

$$- 2 \sum_{(ij)} \left[\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} - \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}\right] \times i\pi \, \Pi_{ij}$$

Glauber/Coulomb contribution

! $\boldsymbol{\Gamma}^{(1)}$ infinite dimensional matrix

- No Glauber contribution in e^+e^-
- Singular when soft emission is collinear to hard partons
  $\Rightarrow$ Collinear finite by combining real and virtual

$\Gamma^{(2)}$ [Becher et. al., 2112.02108; see also Caron-Huot, 1501.03754]

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- Two-loop anomalous dimension $\boldsymbol{\Gamma}^{(2)}$ has been calculated by considering (double) soft limits of hard functions

$$\boldsymbol{\Gamma}^{(2)} = \begin{pmatrix} \boldsymbol{v}_2 & \boldsymbol{r}_2 & \boldsymbol{d}_2 & 0 & \dots \\ 0 & \boldsymbol{v}_3 & \boldsymbol{r}_3 & \boldsymbol{d}_3 & \dots \\ 0 & 0 & \boldsymbol{v}_4 & \boldsymbol{r}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{v}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\boldsymbol{d}_m$: double real emission

$\boldsymbol{r}_m$: real-virtual correction

$\boldsymbol{v}_m$: double-virtual correction

$$\boldsymbol{d}_m = \sum_{(ij)} \sum_{k} i f^{abc} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,R} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,L} \right) K_{ijk;qr} \theta_{\mathsf{in}}(n_q) \theta_{\mathsf{in}}(n_r)$$

$$-2 \sum_{(ij)} \boldsymbol{T}^c_{i,L} \boldsymbol{T}^c_{j,R} K_{ij;qr} \theta_{\mathsf{in}}(n_q) \theta_{\mathsf{in}}(n_r)$$

$$\boldsymbol{r}_m = -2 \sum_{i} \sum_{(jk)} i f^{abc} (\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,R} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,L}) \int \left[d^2 \Omega_r \right] K_{ijk;qr} \theta_{\mathsf{in}}(n_q)$$

$$- \sum_{(ij)} \boldsymbol{T}^a_{i,L} \boldsymbol{T}^a_{j,R} \left\{ W^q_{ij} \left[4\beta_0 \ln(2W^q_{ij}) + \gamma_1^{\text{cusp}} \right] - 2 \int \left[d^2 \Omega_r \right] K_{ij;qr} \right\} \theta_{\mathsf{in}}(n_q)$$

$$+8 i\pi \sum_{i} \sum_{(jk)} i f^{abc} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,R} + \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,L} \right) W^q_{ij} \ln W^q_{jk} \, \theta_{\mathsf{in}}(n_q)$$

$$\boldsymbol{v}_m = \sum_{(ijk)} i f^{abc} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^b_{j,L} \boldsymbol{T}^c_{k,L} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^b_{j,R} \boldsymbol{T}^c_{k,R} \right) \int \left[d^2 \Omega_q \right] \int \left[d^2 \Omega_r \right] K_{ijk;qr}$$

$$+ \sum_{(ij)} \frac{1}{2} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^a_{j,L} + \boldsymbol{T}^a_{i,R} \boldsymbol{T}^a_{j,R} \right) \int \left[d^2 \Omega_q \right] W^q_{ij} \left[4\beta_0 \ln(2W^q_{ij}) + \gamma_1^{\text{cusp}} \right]$$

$$- i\pi \sum_{(ij)} \frac{1}{2} \left(\boldsymbol{T}^a_{i,L} \boldsymbol{T}^a_{j,L} - \boldsymbol{T}^a_{i,R} \boldsymbol{T}^a_{j,R} \right) \Pi_{ij} \, \gamma_1^{\mathsf{cusp}}$$

+ additional terms from converting to angular integrals in $d = 4$

Towards a solution of the RGEs

- Started with a factorisation theorem for jet production with veto on radiation which provides natural way to perform resummation via RG evolution of hard functions $\mathcal{H}_m$

$$\sigma(Q, Q_0) = \sum_{m=m_0}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, Q, \mu) \otimes \mathcal{S}_m(\{\underline{n}\}, Q_0, \mu) \rangle$$

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu} \mathcal{H}_m(Q, \mu) = -\sum_{l=2}^{m} \mathcal{H}_l(Q, \mu)\, \Gamma_{lm}(Q, \mu)$$

The RG evolution is governed by the anomalous dimension which has been extracted up to two-loops

$$\mathbf{\Gamma} = \frac{\alpha_s}{4\pi}\mathbf{\Gamma}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \mathbf{\Gamma}^{(2)} + \cdots \qquad \Rightarrow \quad \text{NLL accuracy}$$

? How to solve complicated RGEs ?

$\Rightarrow$ Monte Carlo Methods

Numerical solution of RGEs

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RGEs not yet in a suitable form for implementation in a MC framework

- Change variables from $\mu \to t = \frac{\alpha_s}{4\pi} \log \frac{\mu_h}{\mu_s} + \mathcal{O}(\alpha_s^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\langle \mathcal{H}(t) \big| = \big\langle \mathcal{H}(t) \big| \,\hat{\Gamma}(t) \qquad \to \text{formal solution} \qquad \big\langle \mathcal{H}(t) \big| = \big\langle \mathcal{H}(0) \big| \,\mathbb{P} \exp \left[\int_0^t \mathrm{d}t' \,\hat{\Gamma}(t') \right]$$

- Expand anomalous dimension perturbatively $\hat{\Gamma}(t) = \Gamma^{(1)}(t) + \frac{\alpha_s}{4\pi} \Delta\Gamma(t) + \mathcal{O}(\alpha_s^2) \to$ Interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\langle \mathcal{H}^I(t) \big| = \big\langle \mathcal{H}^I(t) \big| \, e^{t\,\Gamma^{(1)}} \left[\frac{\alpha_s}{4\pi} \Delta\Gamma(t) \right] e^{-t\,\Gamma^{(1)}}$$

- Solve RG evolution iteratively including subleading contributions due to $\Delta\Gamma$

$$\sigma \sim \big\langle \mathcal{H}(t) \big| \mathcal{S}(t) \big\rangle = \big\langle \mathcal{H}(0) \big| \left[e^{t\,\Gamma^{(1)}} + \int_0^t \mathrm{d}t' \, e^{t'\,\Gamma^{(1)}} \left[\frac{\alpha_s}{4\pi} \Delta\Gamma(t') \right] e^{(t-t')\,\Gamma^{(1)}} \right] \big| \mathcal{S}(t) \big\rangle \quad \to \quad \text{suitable for MC}$$

Parton Shower at LL accuracy

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- In practice coupled RGEs for hard functions $\mathcal{H}_m$, however these simplify at LL due to the form of $\mathbf{\Gamma}^{(1)}$

$$\mathbf{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

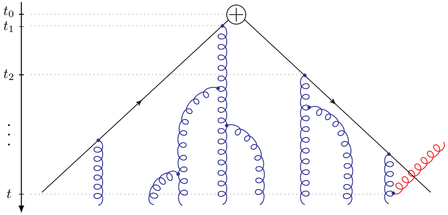
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_m(t) = \mathcal{H}_m(t)\boldsymbol{V}_m + \mathcal{H}_{m-1}(t)\boldsymbol{R}_{m-1}$$

$$\mathcal{H}_m(t) = \mathcal{H}_m(t_0)e^{(t-t_0)\boldsymbol{V}_m} + \int_{t_0}^{t}\mathrm{d}t'\,\mathcal{H}_{m-1}(t')\boldsymbol{R}_{m-1}e^{(t-t')\boldsymbol{V}_m}$$

- Introduce shower-time t

$$\mu \to t = \frac{\alpha_s}{4\pi}\log\frac{\mu_h}{\mu_s} + \mathcal{O}(\alpha_s^2)$$

[Balsiger et. al., 1803.07045]

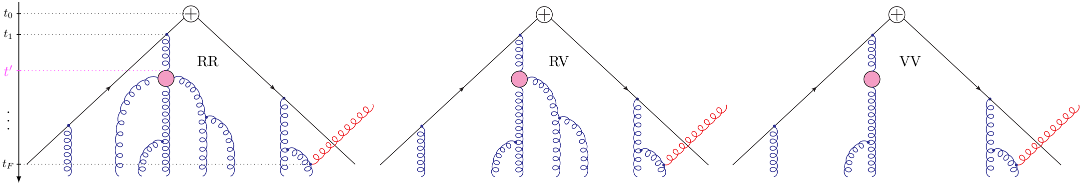


Parton Shower: NLL

- Include corrections of $\mathcal{H}_m$ due to $\Gamma^{(2)}$ $\Rightarrow$ NLL resummation

$$\Delta\mathcal{H}_m(t) = \mathcal{H}_k(t_0)\Delta\boldsymbol{U}_{km}(t,t_0)$$
$$= \mathcal{H}_k(t_0)\int_{t_0}^{t} \mathrm{d}t' \; \boldsymbol{U}_{kl}(t'-t_0) \cdot \frac{\alpha(t')}{4\pi}\left(\Gamma^{(2)}_{ll'} - \frac{\beta_1}{\beta_0}\Gamma^{(1)}_{ll'}\right) \cdot \boldsymbol{U}_{l'm}(t-t')$$

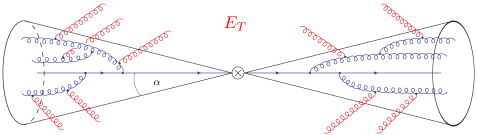
LL evolution	Insertion of $\Gamma^{(2)}$	LL evolution

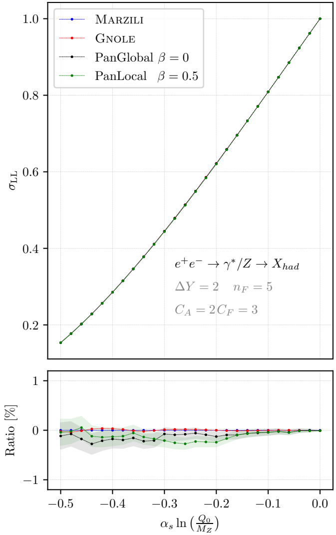


Validation: (N)LL resummation interjet energy flow

- Gap fraction: fraction of events with transverse energy E_T in gap below Q_0

$$R(Q_0) \equiv \frac{1}{\sigma_{\text{tot}}} \int_0^{Q_0} dE_T \, \frac{d\sigma}{dE_T}$$





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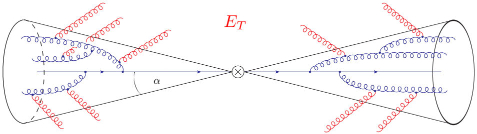

- Ongoing work with Pier Monni on detailed numerical comparison with
  - GNOLE [2111.02413]
  - PanScales [2307.11142]

- Delicate to isolate pure NLL correction
  - GNOLE and PanScales: extrapolation $\alpha_s \to 0$
  - Very small collinear cutoffs & high statistics

Detailed comparison with Gnole at NLL

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- We divide the NLL contribution into different pieces

$$\sigma_{\mathrm{NLL}} \quad \sim \quad \sigma_{\mathrm{hard}} \quad + \quad \sigma_{\mathrm{soft}} \quad + \quad \Delta\sigma_{\mathrm{run.}} \quad + \quad \sigma_{\Gamma^{(2)}}$$

Detailed comparison with Gnole at NLL

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- We divide the NLL contribution into different pieces

$\overline{\mathrm{MS}}$ – Scheme

$$\sigma_{\mathrm{NLL}} \sim \sigma_{\mathrm{hard}} + \sigma_{\mathrm{soft}} + \Delta\sigma_{\mathrm{run.}} + \sigma_{\Gamma(2)}$$

σ_{NLL}	σ_{hard}	σ_{soft}	$\Delta\sigma_{\mathrm{run.}}$	$\sigma_{\Gamma(2)}$
↕ ✓	↕ ✓	↕ ✗	↕ ✗	↕ ✗

$$\sigma_{\mathrm{NLL}}^{\textsc{Gnole}} \sim \sigma_{\mathrm{hard}} + \sigma_{\mathrm{soft}} + \Delta\sigma_{\mathrm{run.}} + \sigma_{\Gamma(2)}$$

Detailed comparison with Gnole at NLL

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- We add a piece proportional to ϵ to the anomalous dimension

$$\widetilde{\mathbf{\Gamma}}^{(1)} = \mathbf{\Gamma}^{(1)} + \epsilon \Delta \mathbf{\Gamma}^{(1)}$$

$\widetilde{\overline{\mathrm{MS}}}$ – Scheme [Caron-Huot, 1501.03754]

! Affects NLL contributions through renormalisation

Detailed comparison with Gnole at NLL

◦ We divide the NLL contribution into different pieces

$\widetilde{\overline{\text{MS}}}$ – Scheme

$$\sigma_{\text{NLL}} \sim \sigma_{\text{hard}} + \sigma_{\text{soft}} + \Delta\sigma_{\text{run.}} + \sigma_{\Gamma(2)}$$

$\updownarrow$ ✓ $\quad$ $\updownarrow$ ✓ $\quad$ $\updownarrow$ ✓ $\quad$ $\updownarrow$ ✓ $\quad$ $\updownarrow$ ✓

$$\sigma^{\text{GNOLE}}_{\text{NLL}} \sim \sigma_{\text{hard}} + \sigma_{\text{soft}} + \Delta\sigma_{\text{run.}} + \sigma_{\Gamma(2)}$$

Detailed comparison with Gnole at NLL

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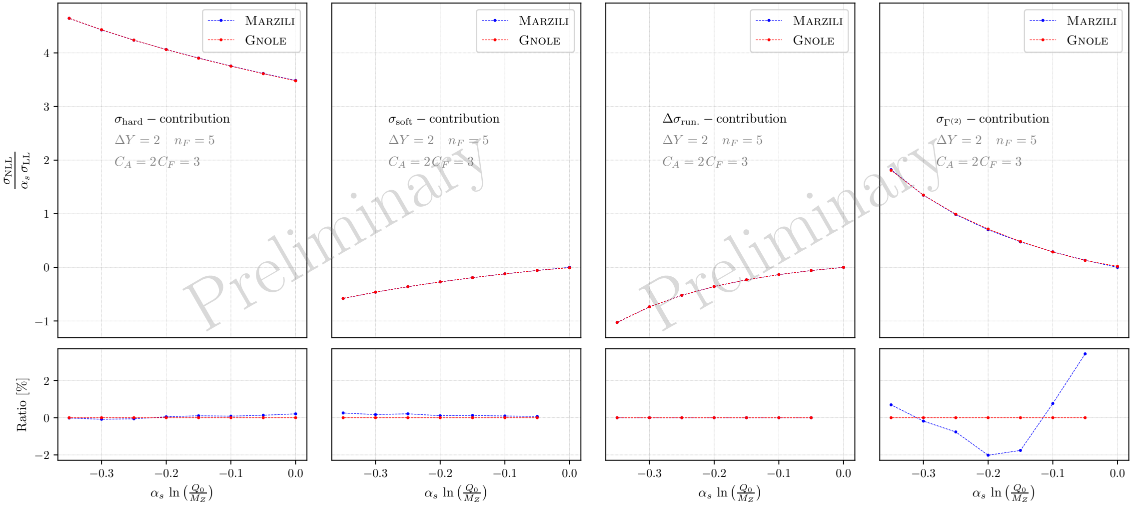
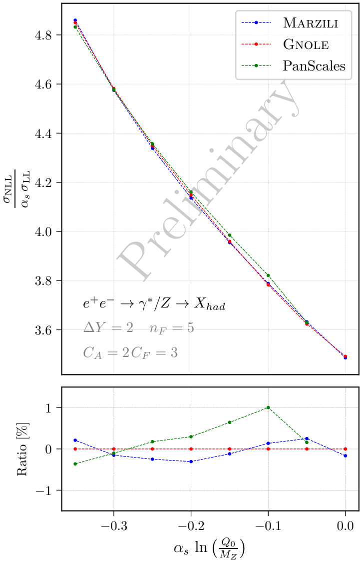


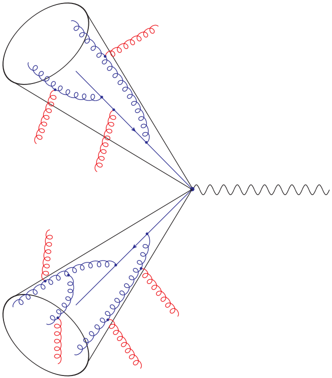
Comparison with PanScales at NLL



Good agreement between the frameworks

- Marzili [2307.02283]
- Gnole [2111.02413]
- PanScales [2307.11142]

Subleading NGLs at Hadron Colliders



Simplest observable: gap fraction in Z – production

- Gap is defined by vetoing hadronic radiation at central rapidities in interval ΔY around the beam
- Cross section factorises in the large-N_c limit exactly in the same way as for e^+e^-
- Same observables used by PanScales collaboration
- First relevant process in a long list

Observable

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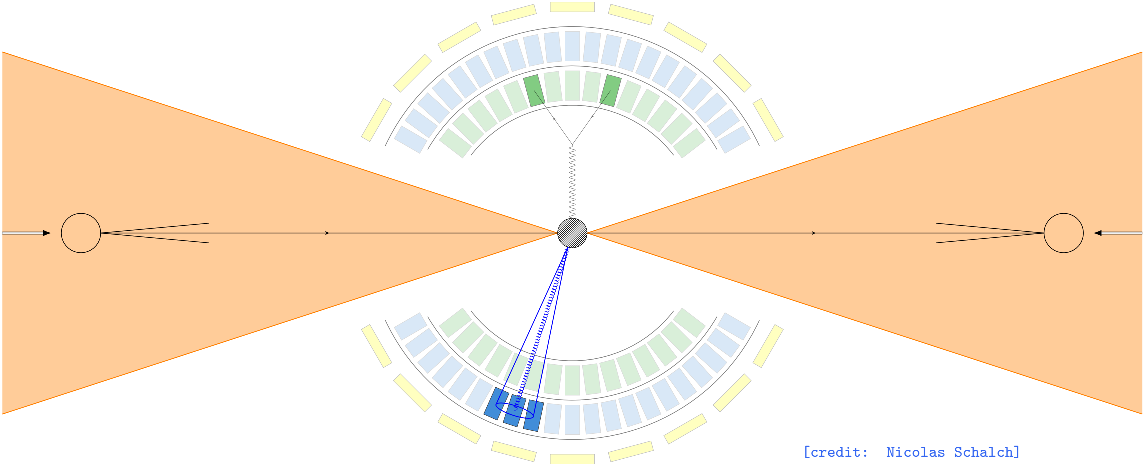
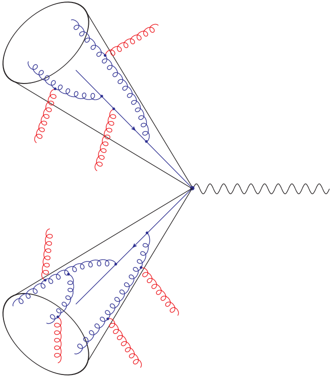


[credit: Nicolas Schalch]

Symmetric gap in restframe of Z

Subleading NGLs at Hadron Colliders: Ingredients



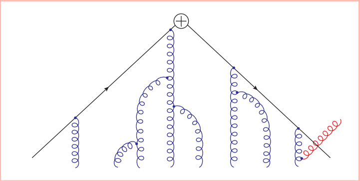
- Required ingredients to resum subleading NGLs
  - One-loop virtual $\mathcal{H}_2(\mu_h) = \sigma_0 |C_V|^2 \, \mathbb{1}$ (✓)
  - Hard real emission corrections $\mathcal{H}_3^{(1)}(\mu_h)$ ✗
  - One-loop soft corrections $\mathcal{S}_m^{(1)}(\mu_s)$ (✓)
  - Two-loop anomalous dimension $\Gamma_{lm}^{(2)}$ (✓)

Extraction of $\mathcal{H}_2^{(1)}$ and $\mathcal{H}_3^{(1)}$

- The virtual corrections due to $\mathcal{H}_2^{(1)}$ factorise such that we obtain these by multiplying the standard dijet hard function H_2 to the LL result $\mathcal{S}_2$

$$\frac{\alpha_s(\mu_h)}{4\pi}\sum_{m=2}^{\infty}\langle\mathcal{H}_2^{(1)}(Q,\mu_h)\otimes\boldsymbol{U}_{2m}(\mu_s,\mu_h)\,\hat{\otimes}\,1\rangle=\sigma_0\;H_2(Q,\mu_h)\;\langle\mathcal{S}_2(\{\bar{n},n\},Q_0,\mu_h)\rangle$$

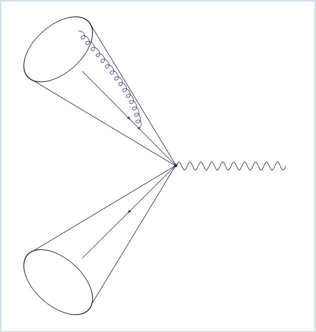
$$C_F\left[-8\ln^2\frac{\mu}{Q}-12\ln\frac{\mu}{Q}-16+\frac{7}{3}\pi^2\right]$$

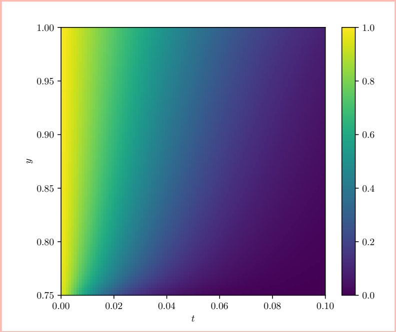


Extraction of $\mathcal{H}_2^{(1)}$ and $\mathcal{H}_3^{(1)}$

- To calculate the contribution due to $\mathcal{H}_3^{(1)}$, which depends on $\theta_{qg}(y)$, we convolute the real corrections of $q\,\bar{q} \to Z$ with the three particle soft function which is obtained via a LL RG evolution

$$\frac{\alpha_s(\mu_h)}{4\pi} \sum_{m=3}^{\infty} \langle\, \mathcal{H}_3^{(1)}(\{n_1, n_2, n_3\}, Q, \mu_h) \otimes \boldsymbol{U}_{3m}(\{\underline{n}\}, \mu_s, \mu_h)\, \hat{\otimes}\, 1 \,\rangle$$

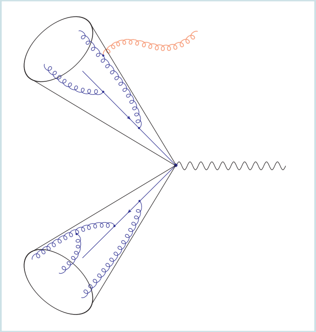




Calculation of $\mathcal{S}_m^{(1)}$ and ΔU_{2m}

◦ The one-loop corrections to the soft function $\mathcal{S}_m^{(1)}$, which represents the emission of a soft particle into the gap, is directly obtained from our shower

$$\frac{\alpha_s(\mu_h)}{4\pi}\sum_{m=2}^{\infty}\left\langle \mathcal{H}_2^{(0)}(Q,\mu_h)\otimes \boldsymbol{U}_{2m}(\mu_s,\mu_h)\;\hat{\otimes}\;\mathcal{S}_m^{(1)}(Q_0,\mu_s)\right\rangle$$

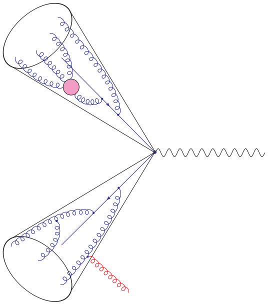


$$\int \frac{d^d k}{(2\pi)^{d-1}}\,\frac{W_{ij}^k}{E_k^2}\,\delta(k^2)\,\theta(k^0)\,\theta(Q_0-k_T)\,\Theta_{\text{out}}(n_k)$$

Calculation of $\mathcal{S}_m^{(1)}$ and ΔU_{2m}

- The contributions due to the insertion of the two-loop anomalous dimension is obtained within our parton shower framework; in practice, we start a LL shower prior to the insertion and then restart a LL shower

$$\sum_{m=2}^{\infty} \langle \mathcal{H}_2^{(0)}(Q,\mu_h) \otimes \Delta \boldsymbol{U}_{2m}(\mu_s,\mu_h) \,\hat{\otimes}\, 1 \rangle$$



$$\boldsymbol{d}_m = + N_c \left(K_{ij;qr} + K_{ji;qr} \right) \theta_{\text{in}}(n_q)\, \theta_{\text{in}}(n_r) - 8 N_c^2 M_{ij;qr}\, \theta_{\text{in}}(n_q)\, \theta_{\text{in}}(n_r)$$

$$\boldsymbol{r}_m = - N_c \int [d^2\Omega_r] \left(K_{ij;qr} + K_{ji;qr} \right) \theta_{\text{in}}(n_q) + 8 N_c^2 \int [d^2\Omega_r]\, M_{ij;qr}\, \theta_{\text{in}}(n_q) - N_c \left(4\beta_0 X_{ij}^q - \gamma_1^{\text{cusp}} W_{ij}^q \right) \theta_{\text{in}}(n_q)$$

$$\boldsymbol{v}_m = + N_c \int [d^2\Omega_q] \left(4\beta_0 X_{ij}^q - \gamma_1^{\text{cusp}} W_{ij}^q \right)$$

Resummed gap fraction for $pp \to Z \to \ell^+\ell^- + X_{had}$

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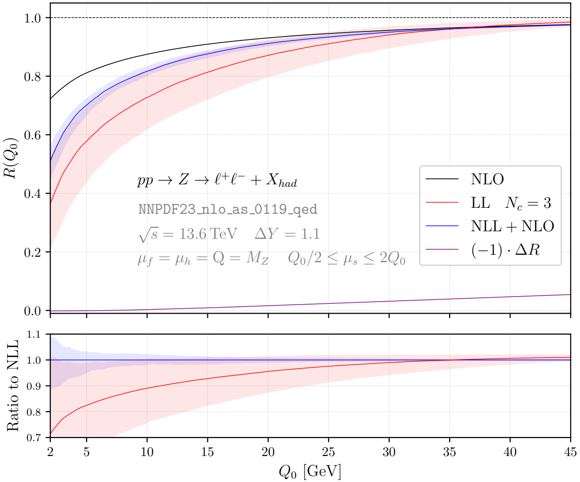
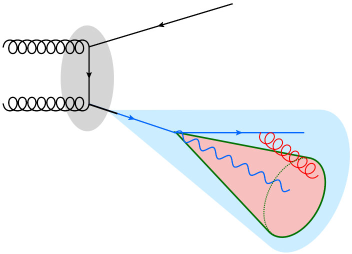


- Many ingredients the same as for e^+e^- case
- $N_c = 3$ LL obtained from [Hatta and Ueda; 1304.6930]

Glauber phases neglected, but superleading logarithms turn out to be small for $q\bar{q} \to Z$

Outlook: Photon Isolation at the LHC

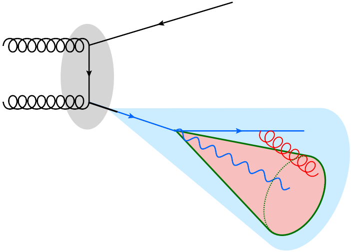


- Distinguish experimentally between photons produced in
  - hard scattering processes
  - other sources (i.e. energetic hadronic decays)

Isolate photons γ from hadronic background radiation

$\Rightarrow$ Large logarithms $\log R$ and $\log \frac{Q_0}{Q}$

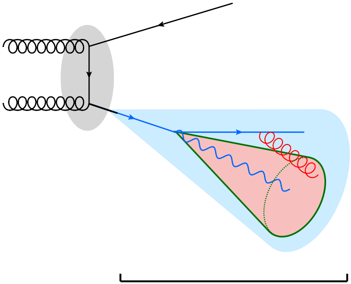
Outlook: Photon Isolation at the LHC



- LL have been calculated in [Favrod paper; 2208.01554]
- Interesting application of our formalism

Include running of $\mathbf{\Gamma}^{(2)}$ as well as matching corrections to increase accuracy to NLL

Factorisation in photon isolation



$\mathcal{F}_{q\to\gamma}$

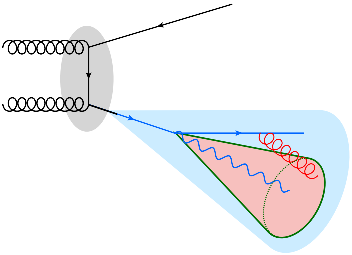
- For small R all isolation effects can be factorised into a cone fragmentation function $\mathcal{F}_{i\to\gamma}$

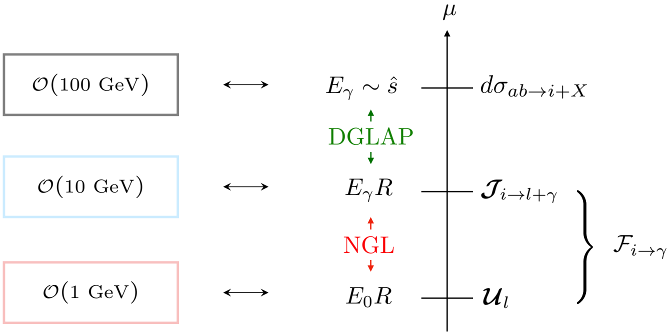
$$\frac{\mathrm{d}\sigma(E_0,R)}{\mathrm{d}E_\gamma} = \frac{\mathrm{d}\sigma^{\mathrm{dir}}_{\gamma+X}}{\mathrm{d}E_\gamma} + \sum_{i=q,\bar{q},g} \int dz \frac{\mathrm{d}\sigma_{i+X}}{\mathrm{d}E_i} \mathcal{F}_{i\to\gamma}(z,E_\gamma,E_0,R) + \mathcal{O}(R)$$

capturing all perturbative effect associated with the isolation.

[Favrod paper; 2208.01554]

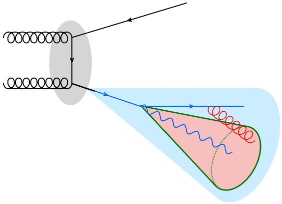
(Re-)Factorisation of $\mathcal{F}_{i\to\gamma}$

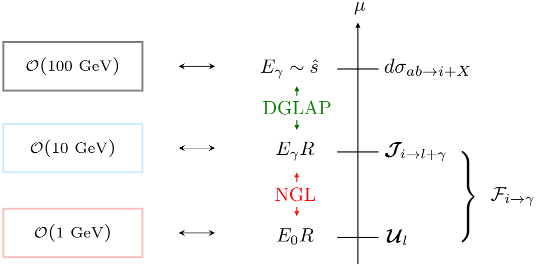




[Favrod paper; 2208.01554]

(Re-)Factorisation of $\mathcal{F}_{i\to\gamma}$





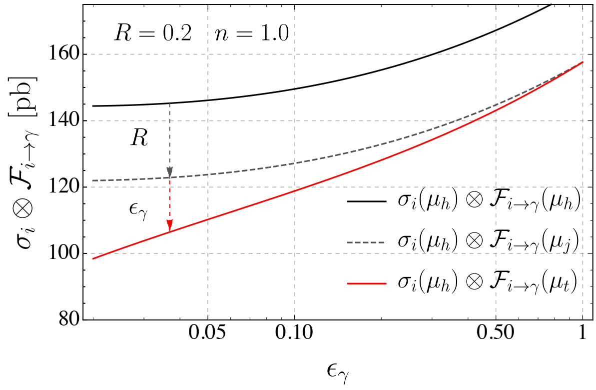
$$\mathcal{F}_{i\to\gamma}(z, R\,E_\gamma, R\,E_0, \mu) = \sum_{l=1}^{\infty} \left\langle \mathcal{J}_{i\to\gamma+l}(\{\underline{n}\}, R\,E_\gamma, z, \mu) \otimes \mathcal{U}_l\,(\{\underline{n}\},\, R\,E_0, \mu) \right\rangle$$

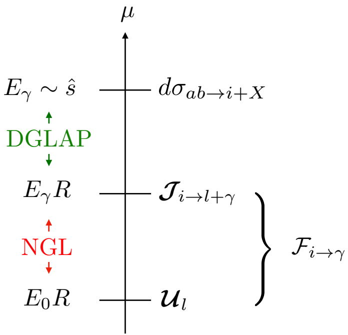
energetic collinear radiation outside cone

soft radiation inside cone

[Favrod paper; 2208.01554]

Resummation of $\log R$ and $\log \epsilon_\gamma$





[Favrod paper; 2208.01554]

Conclusion

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Summary

- Fiducial cuts lead to phase-space constraints

  ⇒ Intricate pattern of logs: NGLs (& SLL)

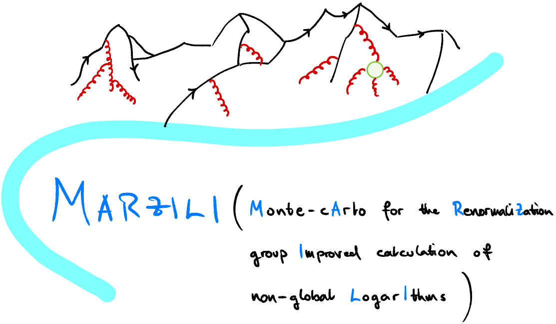
- Implemented two-loop anomalous dimension inside parton shower framework MARZILI

  ⇒ NLL resummation for gap fraction

  ⇒ First results for Hadron Collider process

Outlook

- Photon isolation at the LHC



BACKUP

Predictions in particle physics

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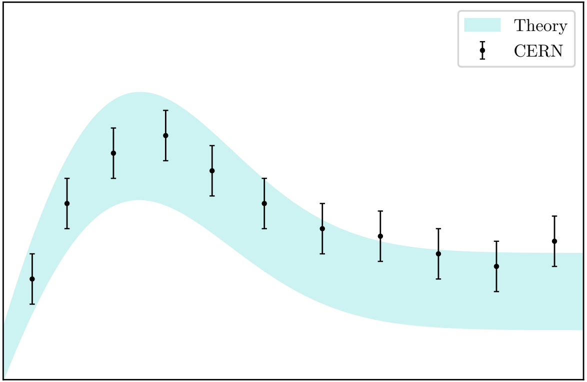
- Particles & interactions described in terms of quantum fields

- QFTs are typically not exactly solvable, perturbative approach necessary

  ⇒ expansion in terms of the strong coupling α_s

$$\sigma = \int \mathrm{d}\Phi_{\mathrm{LIPS}}\, |\mathcal{A}|^2$$

$$= \underbrace{\sigma_0}_{\text{LO}} + \mathcal{O}(\alpha_s)$$



[Figure inspired by S. Jones]

Predictions in particle physics

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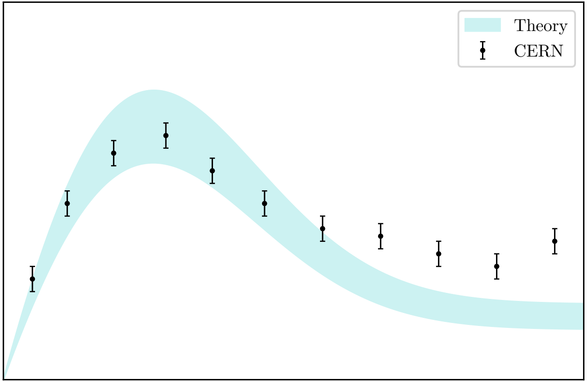
- QFTs are typically not exactly solvable, perturbative approach necessary

  ⇒ expansion in terms of the strong coupling α_s

$$\sigma = \int \mathrm{d}\Phi_{\text{LIPS}}\, |\mathcal{A}|^2 = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s\,\sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2\,\sigma_2}_{\text{NNLO}} + \mathcal{O}(\alpha_s^3)$$



[Figure inspired by S. Jones]

Extraction of anomalous dimension

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- Anomalous dimension $\mathbf{\Gamma}$ arises from soft singularities of hard functions

$$\mathcal{H}_m(Q,\mu) = \sum_{l=2}^{m} \mathcal{H}_l^{\text{bare}}(Q,\mu)\ \left(\mathbf{Z}^{-1}\right)_{lm}(Q,\mu)$$

$$\left(\mathbf{Z}^{-1}\right) = \mathbb{1} + \frac{\alpha_s}{4\pi}\frac{1}{2\epsilon}\mathbf{\Gamma}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{8\epsilon^2}\mathbf{\Gamma}^{(1)}\otimes\mathbf{\Gamma}^{(1)} - \frac{\beta_0}{4\epsilon^2}\mathbf{\Gamma}^{(1)} + \frac{1}{4\epsilon}\mathbf{\Gamma}^{(2)}\right]$$

$\mathbf{\Gamma}^{(1)}$ is $(-2)\times$ soft divergence of the one-loop hard function

Renormalisation $Z_\alpha = 1 - \frac{\beta_0}{\epsilon}\frac{\alpha_S}{4\pi}$

$\mathbf{\Gamma}^{(2)}$ is (-4) times single pole of the two-loop hard function

- At $\mathcal{O}(\alpha_s)$ soft singularities arise when either a real or a virtual gluon becomes soft

Angular functions

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- Three-leg correlations combine with ϵ-terms

$$K_{ijk;qr} = 8\left(W^q_{ik}W^r_{jk} - W^q_{ik}W^r_{jq} - W^q_{ir}W^r_{jk} + W^q_{ij}W^r_{jq}\right)\ln\left(\frac{n_{kq}}{n_{kr}}\right)$$

$$M_{ij;qr} = \left(W^q_{ik}W^r_{jk} - W^q_{ik}W^r_{jq} - W^q_{ir}W^r_{jk} + W^q_{ij}W^r_{jq}\right)\ \ln\left(\frac{s^2_{\phi_{qr}}}{s^2_{\phi_{qx}}}\right)$$

- Two-leg correlations (diverges for $q \parallel r$)

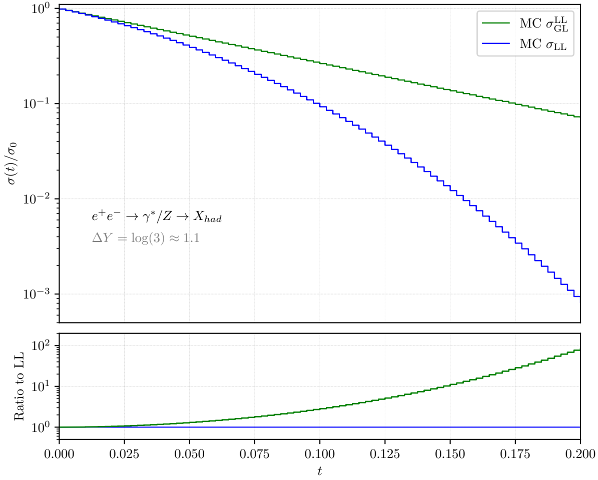
$$K_{ij;qr} = C_A K^{(a)}_{ij;qr} + \left[n_F T_F - 2C_A\right] K^{(b)}_{ij;qr} + \left[C_A - 2n_F T_F + n_S T_S\right] K^{(c)}_{ij;qr}$$

$$K^{(a)}_{ij;qr} = \frac{4n_{ij}}{n_{iq}n_{qr}n_{jr}}\left[1 + \frac{n_{ij}n_{qr}}{n_{iq}n_{jr} - n_{ir}n_{jq}}\right]\ln\frac{n_{iq}n_{jr}}{n_{ir}n_{jq}}$$

$$K^{(b)}_{ij;qr} = \frac{8n_{ij}}{n_{qr}(n_{iq}n_{jr} - n_{ir}n_{jq})}\ln\frac{n_{iq}n_{jr}}{n_{ir}n_{jq}}$$

$$K^{(c)}_{ij;qr} = \frac{4}{n^2_{qr}}\ \left(\frac{n_{iq}n_{jr} + n_{ir}n_{jq}}{n_{iq}n_{jr} - n_{ir}n_{jq}}\ln\frac{n_{iq}n_{jr}}{n_{ir}n_{jq}} - 2\right)$$

Global versus Non-Global: Numerical impact



Numerical solution of RGEs

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! RGEs not yet in a suitable form for implementation in a MC framework

- Change variables from $\mu \to t = \frac{\alpha_s}{4\pi} \log \frac{\mu_h}{\mu_s} + \mathcal{O}(\alpha_s^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathcal{H}(t) | = \langle \mathcal{H}(t) | \, \hat{\Gamma}(t) \qquad \to \text{formal solution} \qquad \langle \mathcal{H}(t) | = \langle \mathcal{H}(0) | \, \mathbb{P} \exp \left[\int_0^t \mathrm{d}t' \, \hat{\Gamma}(t') \right]$$

- Expand anomalous dimension perturbatively $\hat{\Gamma}(t) = \Gamma^{(1)}(t) + \frac{\alpha_s}{4\pi} \Delta\Gamma(t) + \mathcal{O}(\alpha_s^2)$ → Interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathcal{H}^I(t) | = \langle \mathcal{H}^I(t) | \, e^{t\,\Gamma^{(1)}} \left[\frac{\alpha_s}{4\pi} \Delta\Gamma(t) \right] e^{-t\,\Gamma^{(1)}}$$

- Solve RG evolution iteratively including subleading contributions due to $\Delta\Gamma$

$$\sigma \sim \langle \mathcal{H}(t) | \mathcal{S}(t) \rangle = \langle \mathcal{H}(0) | \left[e^{t\,\Gamma^{(1)}} + \int_0^t \mathrm{d}t' \, e^{t'\,\Gamma^{(1)}} \left[\frac{\alpha_s}{4\pi} \Delta\Gamma(t') \right] e^{(t-t')\,\Gamma^{(1)}} \right] | \mathcal{S}(t) \rangle \quad \to \quad \text{suitable for MC}$$

Intermezzo: Toy model I

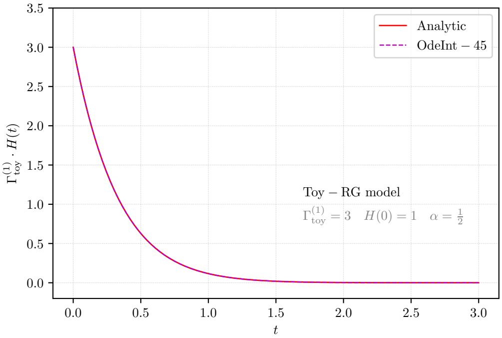
- Consider the first order ODE as toy RGE

$$\frac{d}{dt} H(t) = -\left(\Gamma^{(1)}_{\text{toy}} + \Delta\Gamma^{(2)}_{\text{toy}}(t)\right) H(t) \quad \text{with} \quad \Delta\Gamma^{(2)}_{\text{toy}}(t) = \alpha t$$

1. Separating the variables we obtain an analytical solution

$$H(t) = H(0)\, e^{-t\,\Gamma^{(1)}_{\text{toy}} - \frac{1}{2}\alpha t^2}$$
$$= H(0)\; e^{-t\,\Gamma^{(1)}_{\text{toy}}} \left[1 - \frac{1}{2}\alpha t^2 + \mathcal{O}(\alpha^2)\right]$$

2. Directly use a numerical approach; e.g. adaptive Runge-Kutta methods, etc.



Intermezzo: Toy model I

- Consider the first order ODE as toy RGE

$$\frac{d}{dt}H(t) = -\left(\Gamma^{(1)}_{\rm toy} + \Delta\Gamma^{(2)}_{\rm toy}(t)\right) H(t) \qquad \text{with} \qquad \Delta\Gamma^{(2)}_{\rm toy}(t) = \alpha t$$

- According to our discussion the LL solution

$$\widehat{H}_{\rm LL}(t) \equiv \Gamma^{(1)}_{\rm toy} \cdot H_{\rm LL}(t) = H(0)\,\Gamma^{(1)}_{\rm toy}\, e^{-t\,\Gamma^{(1)}_{\rm toy}}$$

- Interpret $\int_0^t dt'\,\Gamma^{(1)}_{\rm toy}\, e^{-t'\,\Gamma^{(1)}_{\rm toy}} = e^{-t\,\Gamma^{(1)}_{\rm toy}}$ as a probability

$$e^{-t\,\Gamma^{(1)}_{\rm toy}} = z \quad \Leftrightarrow \quad t = -\frac{1}{\Gamma^{(1)}_{\rm toy}} \ln(z)$$

Pseudo Code

```
# start evolution from t0
t_tot = t0
# generate random time step
delta_t = time(rand(1))
# update time
t_tot += delta_t
# insert weight into a histogram
w = 1.0
hist.insrt(t_tot,w)
```

Intermezzo: Toy model I

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- Consider the first order ODE as toy RGE

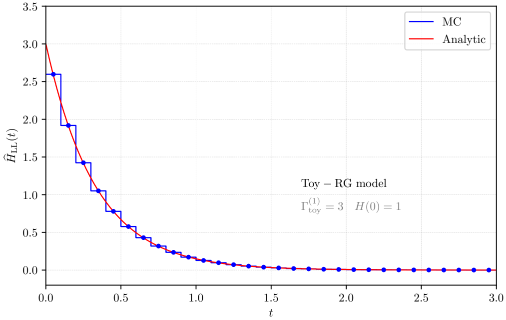
$$\frac{d}{dt} H(t) = -\left(\Gamma^{(1)}_{\text{toy}} + \Delta\Gamma^{(2)}_{\text{toy}}(t)\right) H(t) \quad \text{with} \quad \Delta\Gamma^{(2)}_{\text{toy}}(t) = \alpha\, t$$

- According to our discussion the LL solution

$$\widehat{H}_{\text{LL}}(t) \equiv \Gamma^{(1)}_{\text{toy}} \cdot H_{\text{LL}}(t) = H(0)\, \Gamma^{(1)}_{\text{toy}}\, e^{-t\,\Gamma^{(1)}_{\text{toy}}}$$

- Interpret $\int_0^t dt'\, \Gamma^{(1)}_{\text{toy}}\, e^{-t'\,\Gamma^{(1)}_{\text{toy}}} = e^{-t\,\Gamma^{(1)}_{\text{toy}}}$ as a probability

$$e^{-t\,\Gamma^{(1)}_{\text{toy}}} = z \quad \Leftrightarrow \quad t = -\frac{1}{\Gamma^{(1)}_{\text{toy}}} \ln(z)$$



Intermezzo: Toy model II

◦ Consider the first order ODE as toy RGE

$$\frac{d}{dt}H(t) = -\left(\Gamma^{(1)}_{\text{toy}} + \Delta\Gamma^{(2)}_{\text{toy}}(t)\right) H(t) \qquad \text{with} \qquad \Delta\Gamma^{(2)}_{\text{toy}}(t) = \alpha t$$

- Focus on the solution at NLL accuracy

$$\widehat{H}_{\text{NLL}}(t) \equiv \Gamma^{(1)}_{\text{toy}} \cdot H_{\text{NLL}}(t) =$$

$$H(0)\int_0^t dt' \left[\Gamma^{(1)}_{\text{toy}}\, e^{-t'\,\Gamma^{(1)}_{\text{toy}}}\right] \cdot \frac{\Delta\Gamma^{(2)}_{\text{toy}}(t')}{\Gamma^{(1)}_{\text{toy}}} \cdot \left[\Gamma^{(1)}_{\text{toy}}\, e^{-(t-t')\,\Gamma^{(1)}_{\text{toy}}}\right]$$

- Use probabilistic interpretation for $\Gamma^{(1)}_{\text{toy}}\, e^{-t'\,\Gamma^{(1)}_{\text{toy}}}$

Pseudo Code

```
# start evolution from t0
t_tot = t0
w = 1.0
# LL step; thereby generating Δt₁
ll(t_tot,w)
# insertion weight at t_ins = t_tot + Δt₁
weight_nll = ΔΓ(t_ins) / Γ
# second LL step; insert nll weight
ll(t_ins,weight_nll)
```

Intermezzo: Toy model II

- Consider the first order ODE as toy RGE

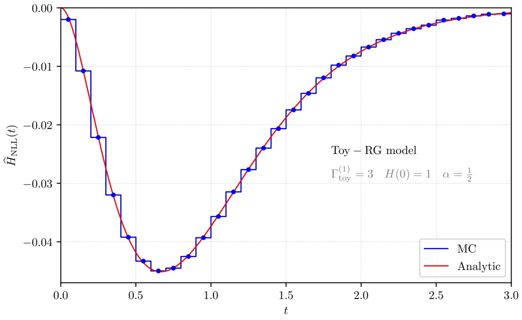
$$\frac{d}{dt}H(t) = -\left(\Gamma^{(1)}_{\text{toy}} + \Delta\Gamma^{(2)}_{\text{toy}}(t)\right) H(t) \qquad \text{with} \qquad \Delta\Gamma^{(2)}_{\text{toy}}(t) = \alpha t$$

- Focus on the solution at NLL accuracy

$$\widehat{H}_{\text{NLL}}(t) \equiv \Gamma^{(1)}_{\text{toy}} \cdot H_{\text{NLL}}(t) =$$

$$H(0)\int_0^t dt' \left[\Gamma^{(1)}_{\text{toy}}\, e^{-t'\Gamma^{(1)}_{\text{toy}}}\right] \cdot \frac{\Delta\Gamma^{(2)}_{\text{toy}}(t')}{\Gamma^{(1)}_{\text{toy}}} \cdot \left[\Gamma^{(1)}_{\text{toy}}\, e^{-(t-t')\Gamma^{(1)}_{\text{toy}}}\right]$$

- Use probabilistic interpretation for $\Gamma^{(1)}_{\text{toy}}\, e^{-t'\Gamma^{(1)}_{\text{toy}}}$



Perturbative setup $q\,\overline{q} \to Z$

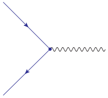
- The differential cross section for Z – Boson production is written as a convolution between the partonic cross section $\mathrm{d}\hat{\sigma}_{ij}$ and the PDFs f_l

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{4\pi^2\alpha}{N_c\, s} \sum_{i,j} \int \mathrm{d}x_1 \mathrm{d}x_2\, f_i(x_1)\, f_j(x_2)\, \frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}Q^2}$$

$$\frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}Q^2} = \int \mathrm{d}\Pi_f\, |\mathcal{M}_{ij}|^2\, \delta\Big(z - \frac{Q^2}{\hat{s}}\Big) = \hat{\Sigma}^{(0)}_{ij} + \Big(\frac{\alpha_S}{4\pi}\Big)\hat{\Sigma}^{(1)}_{ij} + \mathcal{O}(\alpha_s^2)$$

- The LO partonic cross section is expressed in terms of a delta-distribution in $z = \frac{Q^2}{\hat{s}}$

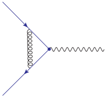
$$\frac{\hat{\Sigma}^{(0)}_{q\overline{q}}}{e_q^2} \sim \int \mathrm{d}\Pi_f \left| \quad \right|^2 \sim \delta(1-z)$$

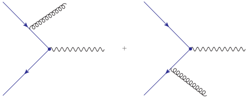


NLO corrections to $q\,\overline{q} \to Z$

- At NLO we obtain virtual and real contributions $\Rightarrow$ dimensional regularisation in $d = 4 - 2\epsilon$ to make both UV and IR divergences explicit, e.g. the $q\,\overline{q}$ - channel yields

$$\frac{\hat{\Sigma}^{(1)}_{q\overline{q}}}{e_q^2} \sim \int \mathrm{d}\Pi_f \left| \quad \right|^2 + \int \mathrm{d}\Pi_f \left| \quad + \quad \right|^2$$





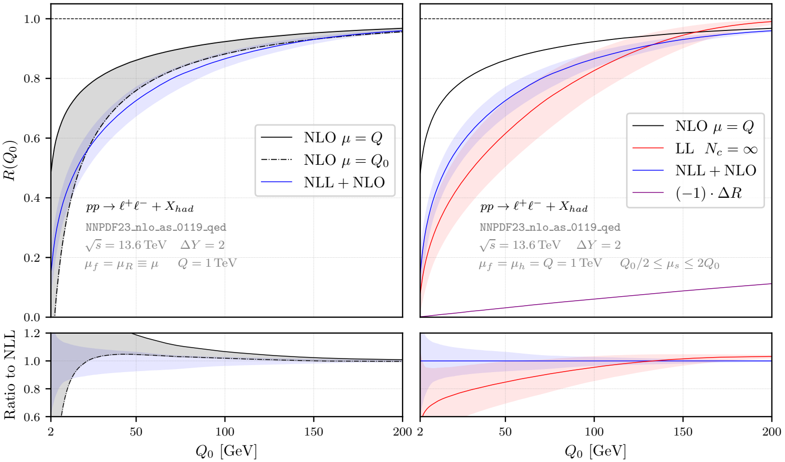
$$\sim C_F \Big\{ \big(\delta(y) + \delta(1-y)\big) \Big[\delta(1-z)\big(2\zeta_2 - 4\big) + 4 \Big[\tfrac{\log(1-z)}{1-z}\Big]_+ - 2\,(1+z)\log(1-z) - \tfrac{(1+z^2)}{1-z}\log(1-z) + 1 - z \Big]$$

$$+ \Big[(1+z^2) \Big[\tfrac{1}{1-z}\Big]_+ \Big(\Big[\tfrac{1}{y}\Big]_+ + \Big[\tfrac{1}{1-y}\Big]_+ \Big) - 2\,(1-z) \Big] + \mathcal{O}(\epsilon) \Big\}$$

❗ New channel opens up: $\hat{\Sigma}^{(1)}_{qg}$ also needs to be taken into account

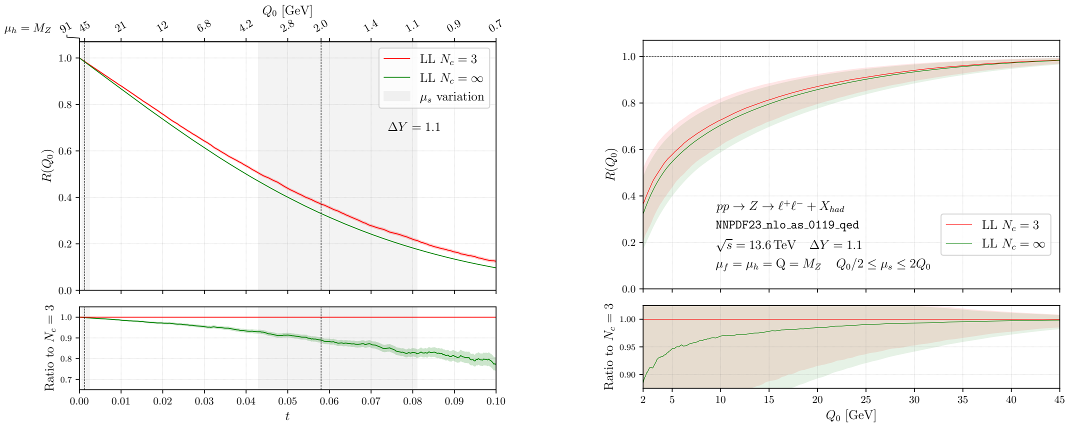
Fixed order versus resummation

$\mu = Q$ or $\mu = Q_0$?



Finite-N_c corrections

- Obtained result for $N_c = 3$ from [Hatta and Ueda; 1304.6930 + improved numerics]



Validation: $n_F = 0$