Hamiltonian Dynamics Problem Sheet

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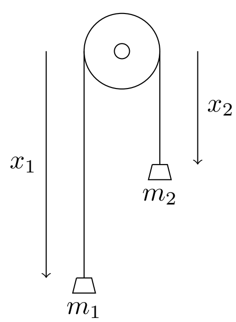
Problem 1. Write down Hamilton's equations for the following Hamiltonians

$$H(q_1, q_2, p_1, p_2; t) = \frac{1}{2}\left(p_1^2\left(p_2^2 + q_2^2\right) + q_1^2\right)$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{\mu m}{r}$$

Problem 2.



Two masses are hanging via a massless string from a frictionless pulley, The kinetic energy of the masses is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2, \tag{1}$$

while the potential energy is

$$V = -m_1 g x_1 - m_2 g x_2. \tag{2}$$

We selected $V = 0$ at the centre of the pulley. The system is subjected to the constraint $x_1 + x_2 = l =$ constant. Write down the Lagrangian using the constraint to reduce the number of variables. Calculate the conjugate momentum and convert to the Hamiltonian. Finally, write down Hamilton's equations.

Problem 3. Show that the following transformation from (q,p) to (Q,P) is canonical

$$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p,\ Q = \ln(1 + \sqrt{q}\cos p)$$

by checking if the Poisson bracket $[Q, P]_{q,p} = 1$. Verify that the following type 3 generating function $F_3(p, Q)$ corresponds to this transformation.

$$F_3(p, Q) = -\left(e^Q - 1\right)^2 \tan p$$

Problem 4. The Hamiltonian of a normal octupole magnet can be written

$$H_4 = \frac{q}{p}\frac{b_4}{4}\left(x^4 - 6x^2z^2 + z^4\right) \tag{3}$$

where q is the particle charge, p is the momentum, (x, z) are the horizontal and vertical coordinates and the octupole strength b_4 is constant along the magnet length L. The coordinates can be written in terms of action-angle coordinates (J_x, ϕ_x), (J_y, ϕ_y) via

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos\phi_x \tag{4}$$

$$y(s) = \sqrt{2J_y\beta_y(s)}\cos\phi_y \tag{5}$$

where $\beta_{x,y}$ are betatron functions. Find the averaged Hamiltonian $\langle H_4 \rangle$ by evaluating

$$\langle H_4 \rangle = \frac{1}{2\pi}\oint H_4(J, \phi)d\phi \tag{6}$$

Note, integrals of the cosine function such as $\oint \cos^4\phi \, d\phi = \frac{3\pi}{4}$ will be useful. Then using the relation for the tune shift in each transverse plane $\Delta Q_{x,y}$

$$\Delta Q_{x,y} = \frac{1}{2\pi} \oint \frac{\partial}{\partial J_{x,y}} \langle H_4 \rangle \, ds \tag{7}$$

show that

$$\Delta Q_x = \frac{q}{p} \frac{3B_4}{8\pi} \left(\beta_x^2 J_x - 2\beta_x \beta_y J_y \right) \tag{8}$$

$$\Delta Q_y = \frac{q}{p} \frac{3B_4}{8\pi} \left(\beta_y^2 J_y - 2\beta_x \beta_y J_x \right) \tag{9}$$

where B_4 is the integrated octupole strength ($B_4 = b_4 L$).

Problem 5. Optional bonus problem! An idealised kick rotator may be represented by the following discrete map

$$\theta_{n+1/2} = \theta_n + 0.5 * K * p_n \tag{10}$$

$$p_{n+1} = p_n - K \sin \theta_{n+1/2} \tag{11}$$

$$\theta_{n+1} = \theta_{n+1/2} + 0.5 * K * p_{n+1} \tag{12}$$

Write a code (e.g. in Python) to iterate this map a few hundred times starting with a set of starting coordinates (p_0, θ_0) that form a regular grid covering the range ($-\pi$, π) in both phase space coordinates. Ensure that tracked coordinates (both p and θ) are always in the range ($-\pi$, π). Here the following python function is useful for wrapping a variable x between bounds (a,b):

$$\text{wrap} = \text{lambda}\, x, a, b : ((x - a) \,\%\, (b - a) + a) \tag{13}$$

Plot the coordinates after each iteration on a single phase space figure (don't join the points for best results). Repeat for various values of K ($K << 1$, $K \sim 1$ and $K > 1$). You should observe bounded motion within a separatrix for $K << 1$ but increasing levels of chaos as K increases above 1.

Note, this map relates to longitudinal dynamics in a stationary bucket where the kick corresponds to RF cavities that impart a change in momentum at discrete points in the ring.