Hamiltonian Dynamics
Lecture 1

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Hamiltonian dynamics introduction

- In Hamiltonian mechanics, the equations of motion follow from the Hamiltonian, H, which represents the total energy of a conservative system (the sum of the kinetic energy T and potential energy V).

The Hamiltonian (conservative system)

$$H = T + V$$

Hamilton's equations for H(q,p)

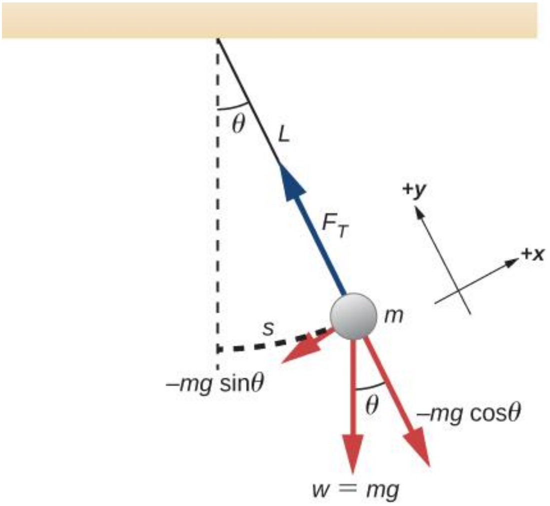
$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

where (q,p) are canonical coordinates.

- Phase space
- Liouville's Theorem
- Action-angle coordinates
- Hamiltonian flow
- Canonical coordinates and transformations
- Symplecticity
- Integrability
- Poisson Brackets
- Lie Algebra

Key concepts related to Hamiltonian dynamics

Pendulum example



Newtonian approach

$$F = ma = m\ddot{\theta} = -mgsin\theta$$

Hamiltonian approach

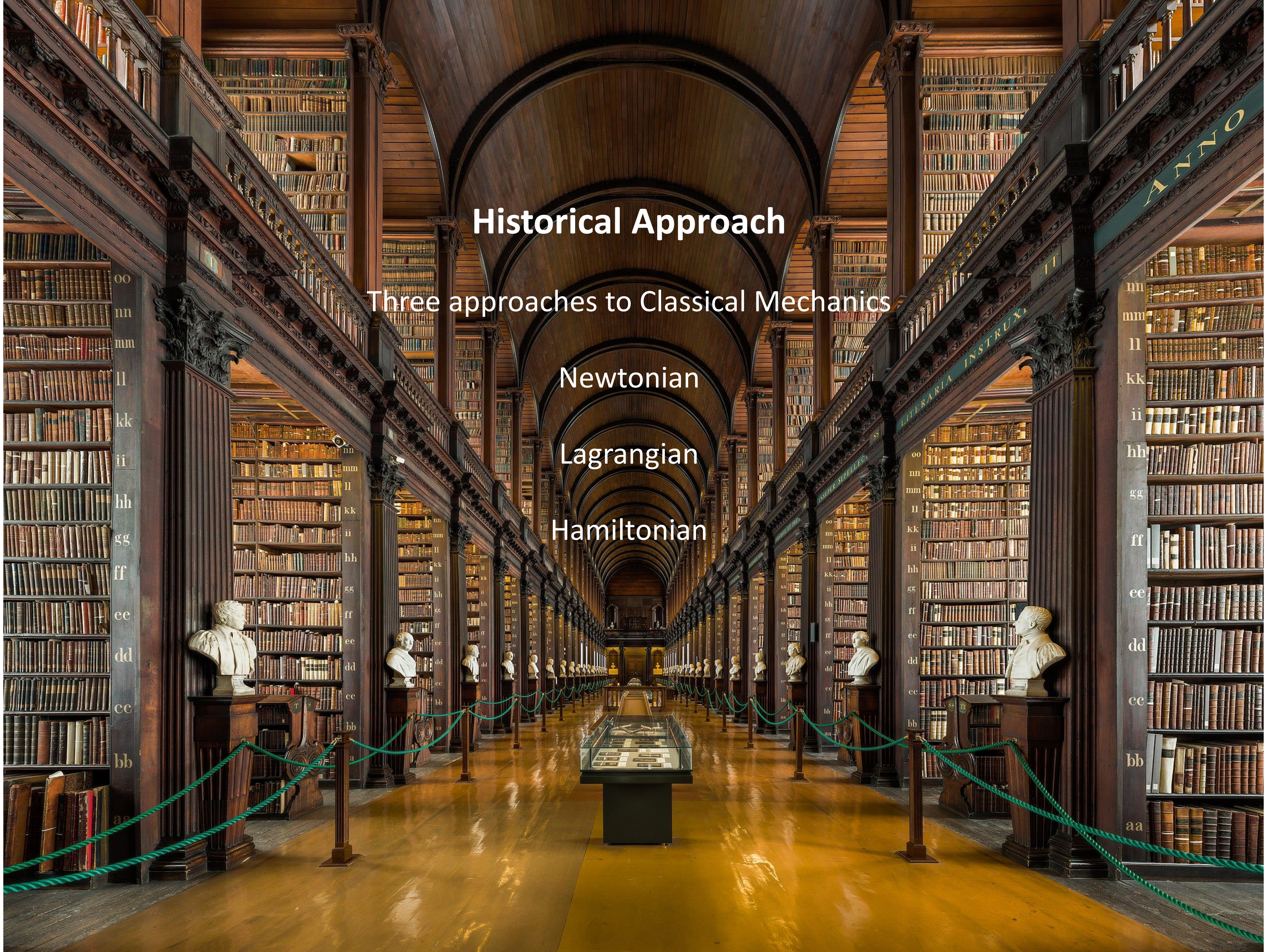
Write the Hamiltonian for the system

$$H(\theta, p_\theta) = T + V = \frac{p_\theta^2}{2mL^2} + mgL(1 - \cos\theta)$$

Hamilton's equations follow

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mL^2}$$

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = mgL\sin\theta$$



Historical Approach

Three approaches to Classical Mechanics

Newtonian

Lagrangian

Hamiltonian

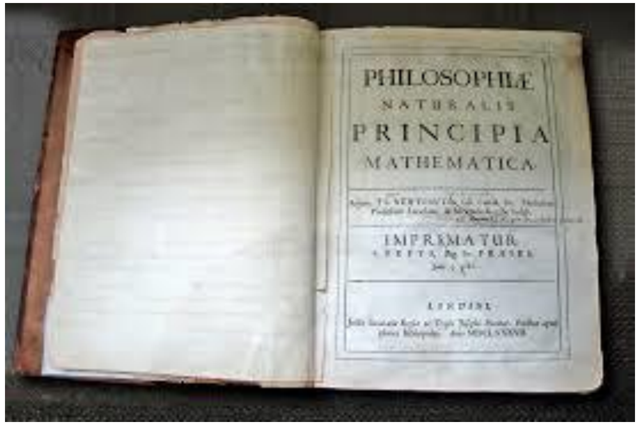
Newtonian Mechanics

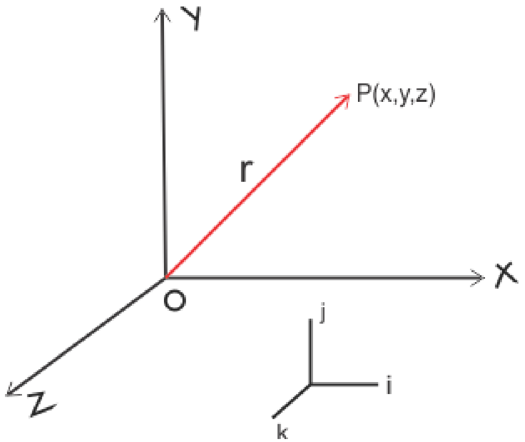
- The key function is the force **F**$(r, \dot{r}, t)$ where r is the position, $\dot{r}$ is the velocity and t is time.
- The equation of motion, in an inertial frame, is

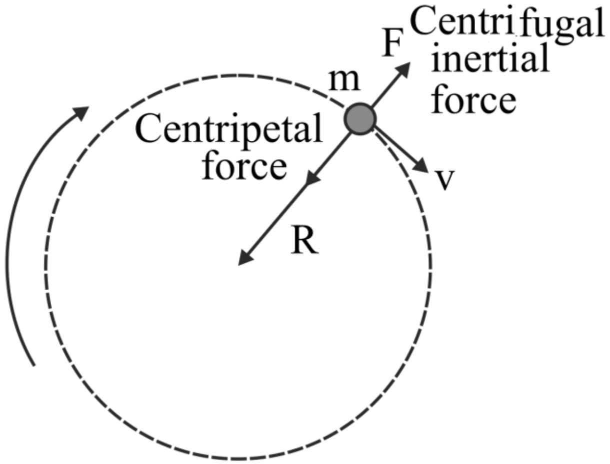
$$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$$

(2^{nd} order differential equation)

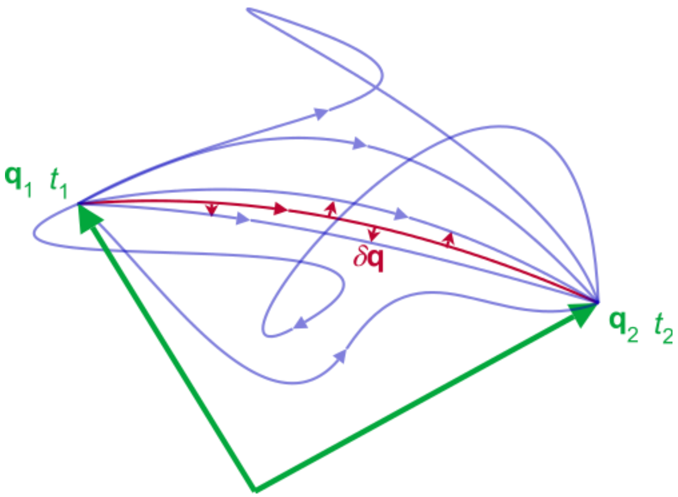
- In a non-inertial frame fictitious forces may need to be considered. In a non-Cartesian coordinate system, the analysis can get more complicated.







Lagrangian Mechanics



- Mechanics can be reformulated in way that avoids specifying a force directly.
- Let us define the action *S*.

$$S = \int_{t1}^{t2} L(q, \dot{q}, t) dt$$

- $L(q, \dot{q}, t)$ is the Lagrangian, a function of generalized coordinates, velocities and time.
- Hamilton's principle (often misleadingly called the "principle of least action") holds that the system evolves such that *S* is stationary,

$$\delta S = 0$$

- The equation of motion (the *Euler-Lagrange* equation) follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

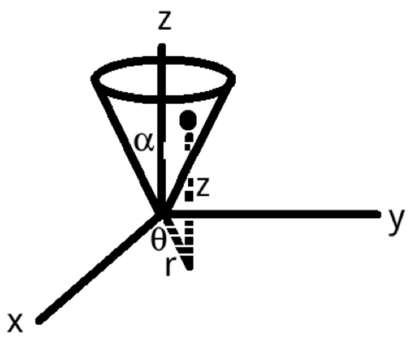
(2nd order differential equation)

- In the case of a conservative force (depends on q only)

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

- Applies in any coordinate system including non-inertial ones.
- Constraints can be incorporated naturally.

Lagrangian example – particle on a cone



- Consider a particle rolling due to gravity in a frictionless cone. The cone's opening angle α places a constraint on the coordinates tanα = r/z. We may write the Lagrangian in cylindrical coordinates

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz$$

- Reduce the number of coordinates by eliminating z via $z = \frac{r}{\tan\alpha}, \ \dot{z} = \frac{\dot{r}}{\tan\alpha}$

$$L = \frac{m}{2}\left((1+\cot^2\alpha)\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cot\alpha$$

- The Euler-Lagrange equation for each coordinate…

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

…can be be solved to obtain the equations of motion

$$\left(1+\cot^2\alpha\right)\ddot{r} - r\dot{\theta}^2 + g\cot\alpha = 0$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

From the Lagrangian to the Hamiltonian

- Perform a *Legendre transformation* to get from the $\mathrm{L}(\mathrm{q_i}, \dot{q}_i, t)$ to $H(q_i, p_i, \mathrm{t})$.
- Defining the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

- The definition of the Hamiltonian follows

$$\boxed{H = \sum_{i=1}^{n} p_i \dot{q}_i - L}$$

Can also write $L = \sum p_i \dot{q}_i - H$

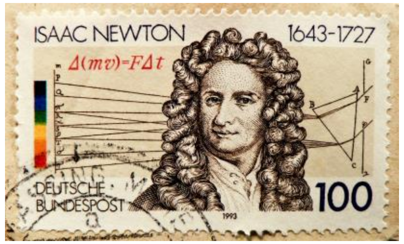
- By comparing the differential of the Hamiltonian and Lagrangian, Hamilton's equations of motion can be found

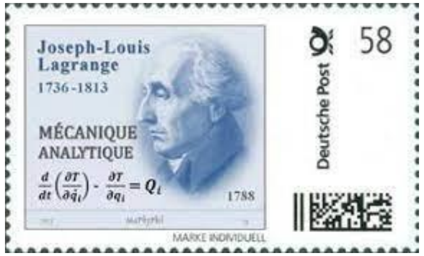
$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

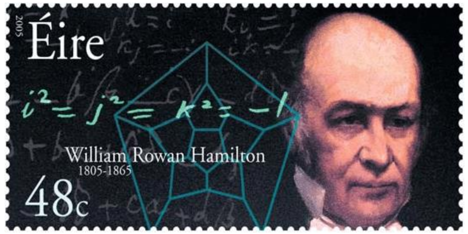
Note – in this case we have a pair of first order differential equations for the phase space coordinates.

Summary of approaches

	Newtonian	Lagrangian	Hamiltonian
Key functional	$\mathbf{F}(r,\dot{r},t)$	$L(q_i, \dot{q}_i, t)$	$H(q_i, p_i, t)$
Equation of motion	$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$	$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$	$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i}$
Strengths	• Can include dissipative forces	• Ease of incorporating constraints • Flexibility of coordinate system	• First order differential equations • Connection to powerful geometric theory that flows from the conservation of energy.

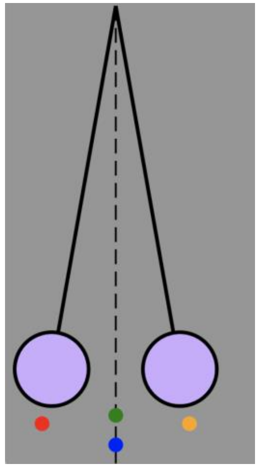


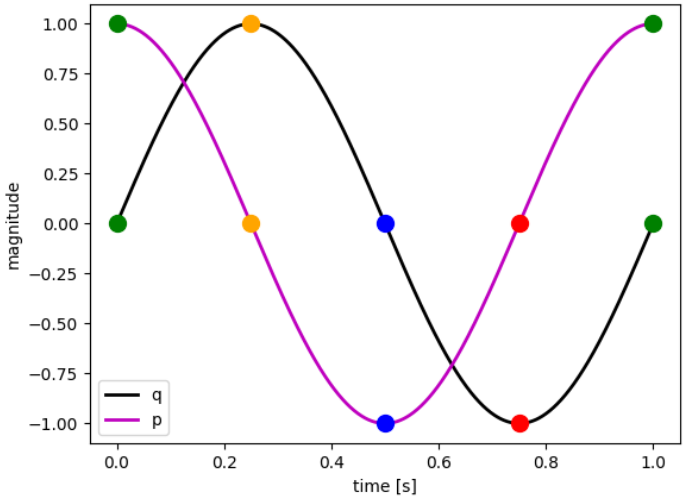




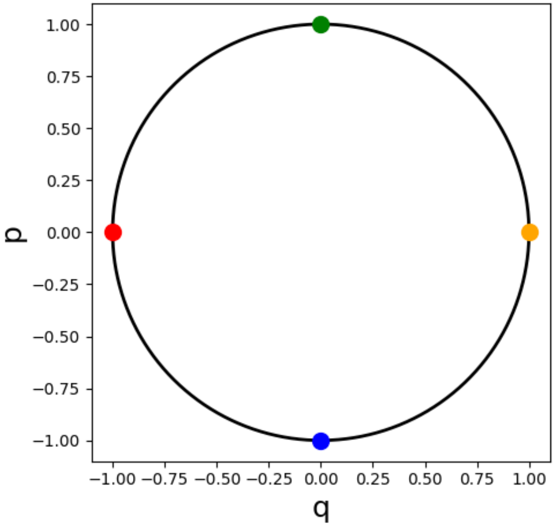
Phase space (1)

- In Hamiltonian mechanics, the canonical momenta p_i are promoted to coordinates on equal footing with the generalized coordinates q_i .
- The coordinates (q, p) are canonical variables, and the space of canonical variables is known as phase space.





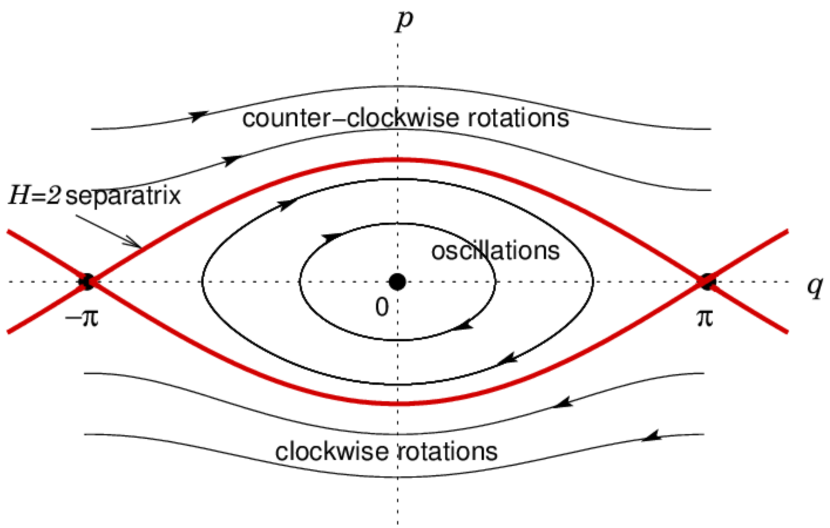
Time series

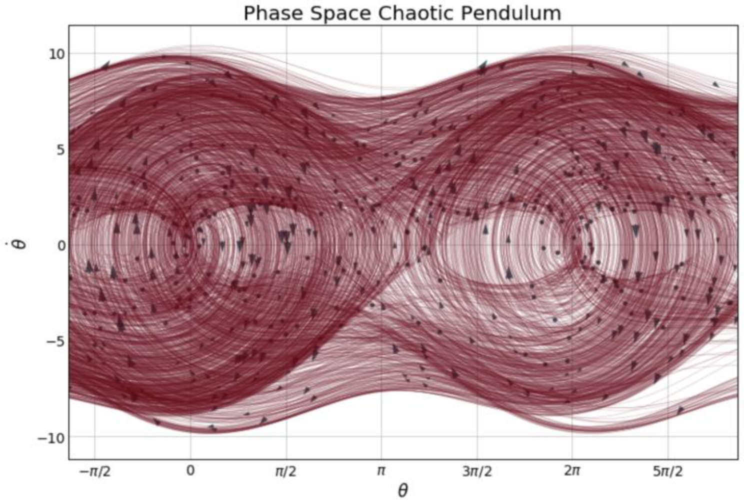


phase space

Phase space (2)

- The phase space may exhibit features such as bounded/unbounded motion, regular or chaotic motion, stable and unstable fixed points, resonances etc.



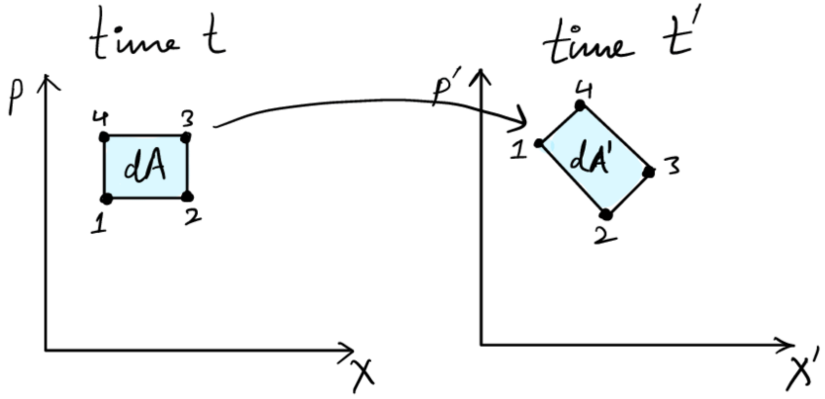


Liouville’s theorem

- Consider the particle distribution function, $f(p_i,q_i;t)$.
- Liouville’s theorem states that, for a system subject only to non-dissipative forces ($dH/dt = 0$) the phase space distribution function is constant along the trajectory of the motion, i.e.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) = 0$$

- The phase space acts like an incompressible fluid. The phase space density cannot be increased unless a non-conservative (dissipative) force is added (e.g. charge exchange injection).



Symplecticity

- A map M is used to track particles from one part of a ring to another or turn-by-turn. Quantities such as betatron tune and other optics parameters can be obtained from the map itself.

$$\begin{bmatrix} x_f \\ p_f \end{bmatrix} = M \begin{bmatrix} x_i \\ p_i \end{bmatrix}$$

- How do we ensure the map is consistent with the Hamiltonian? Let's write Hamilton's equations in matrix form

$$\begin{bmatrix} \dot{q}_i \\ \dot{p}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{bmatrix}$$

- Define a vector ζ =(q_i, p_i) and write Hamilton's equations in vector form

$$\dot{\zeta} = \Omega \nabla H(\zeta) \qquad \text{where } \Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

- It can be shown that the corresponding map M given by $\zeta(t) = M\zeta(t_0)$

(Ω is a skew-symmetric matrix)

has the symplectic property

$$M^T \Omega M = \Omega$$

Canonical transformations

- It often proves useful to transform from one set of phase space coordinates (q,p) to another (Q,P). The transformation is said to be canonical if it preserves the form of Hamilton's equations.
- Consider the transformation from H(q, p, t) to K(Q, P, t). From the gauge invariance of the Lagrangian we can write

$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{dF}{dt}$$

(Assume the case λ=1)

- The function F is a generating function that can depend on various combinations of old and new phase space coordinates.
- Consider the case F = F_1(q,Q,t), known as a type 1 generating function. Then by the partial derivative chain rule

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{\partial F}{dq_i}\dot{q}_i + \frac{\partial F}{dQ_i}\dot{Q}_i + \frac{\partial F_i}{dt}$$

Rearranging terms

$$\left(p_i - \frac{\partial F}{dq_i}\right)\dot{q}_i - \left(P_i + \frac{\partial F}{dQ_i}\right)\dot{Q}_i + K - (H + \frac{\partial F_i}{dt}) = 0$$

To allow separately independent coordinates the coefficients must be zero

$$p_i = \frac{\partial F_1}{\partial q_i},\ P_i = -\frac{\partial F_1}{\partial Q_i},\ K = H + \frac{\partial F_1}{\partial t}$$

Canonical transformation – generating functions

Generating function	Transformation equations	
$F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$	$P_i = -\frac{\partial F_1}{\partial Q_i}$
$F_2(q, P, t)$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$
$F_3(p, Q, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$P_i = -\frac{\partial F_3}{\partial Q_i}$
$F_4(p, P, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$

Action-angle coordinates (1)

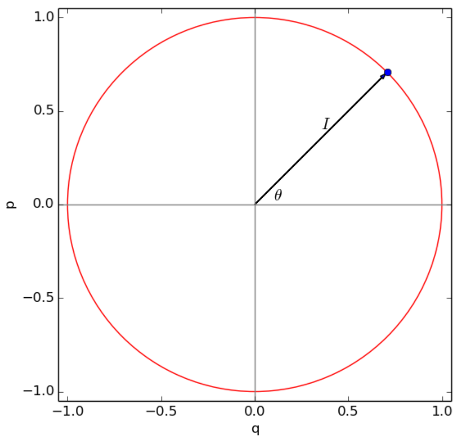
- The canonical transformation to action-angle coordinates helps simplify the dynamics. Define canonical variables (θ,I) such as the Hamiltonian depends only on action, H = H(I). Then

$$\dot{I} = -\frac{\partial H}{\partial \omega} = 0, \ \dot{\theta} = \frac{\partial H}{\partial I} = \omega(I)$$

- Let's apply this transformation for the case of a simple harmonic oscillator with Hamiltonian

$$H = \frac{\omega}{2}(q^2 + p^2)$$

- Try a transformation to action-angle coordinates



$$q = \sqrt{\frac{2}{\omega}} f(P) \sin Q, \ p = \sqrt{\frac{2}{\omega}} f(P) \cos Q$$

=> $p = q \cot Q, \ K = H = f^2(P) \left(\sin^2 Q + \cos^2 Q\right) = f^2(P)$

This is independent of f(P), and has the form of the F1(q,Q,t) type of generating function

$$p = \frac{\partial F_1}{\partial q}$$

Action-angle coordinates (2)

- The corresponding generating function is given by

$$F_1(q,Q) = \frac{1}{2}q^2 \cot Q$$

=> $P = -\frac{\partial F_1(q,Q)}{\partial Q} = \frac{1}{2}\frac{q^2}{\sin^2 Q}$

- Rearrange for q

$$q = \sqrt{2P}\sin Q$$

- By comparing with equation for q on previous slide, we obtain f(P) and K.

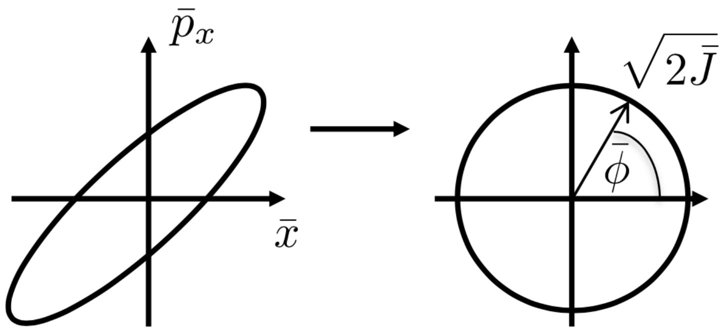
$$f = \sqrt{\omega P},\ K = \omega P$$

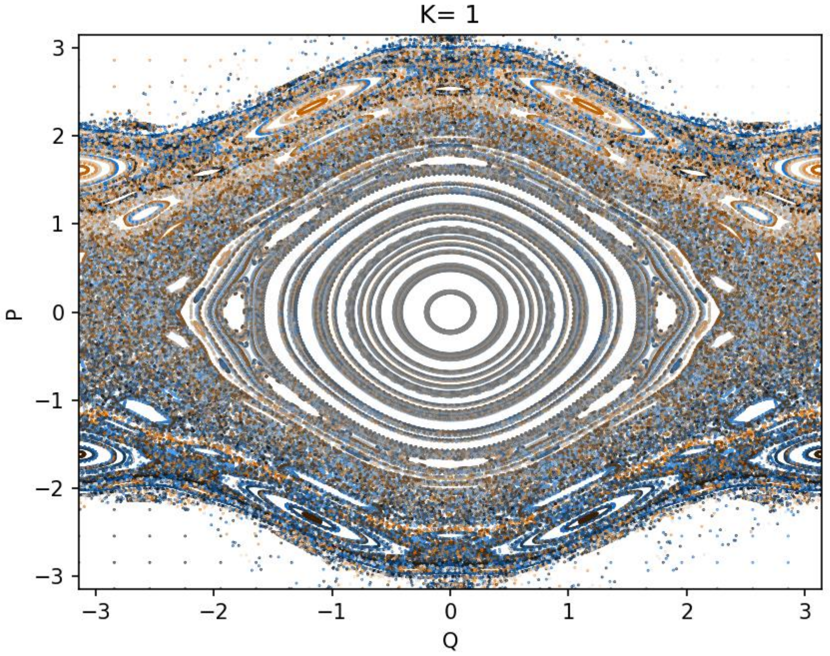
- From the equations of motion for P, Q we see action P is constant and depends on energy, while angle Q increases monotonically in time.

$$P = \frac{K}{\omega} \qquad \dot{Q} = \frac{\partial K}{\partial P} = \omega,\ Q = \omega t + C$$

Integrability

- The Liouville-Arnold theorem states that existence of n invariants of motion is enough to fully characterize a for an n degree-of-freedom system. In that case a canonical transformation exists to action angle coordinates in which the Hamiltonian depends only on the action.
- For an ideal linear lattice, the motion in both horizontal and vertical planes can be separately transformed into action-angle coordinates. The motion remains bounded and regular indefinitely in this case.





Poisson Brackets

- Introduce functions of the canonical variables u(q,p) and v(q,p). The Poisson bracket of u and v is defined as

$$[u, v]_{p,q} = \frac{\partial u}{\partial q}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial q}$$

- For the phase space coordinates we have

$$[q_i, q_j] = [p_i, p_j] = 0$$
$$[q_i, p_j] = -[p_i, q_j] = \delta_{i,j}$$

- Poisson bracket is invariant under canonical transformation.

$$[u, v]_{p,q} = [u, v]_{P,Q}$$

Poisson Brackets – Hamilton's equations

- Start with the total differential of a function u = (q_i, p_i, t)

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t}$$

- Making use of Hamilton's equations

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}$$

- Rewriting in terms of a Poisson bracket

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

- Setting u= q or u = p, and assuming no explicit time dependence, Hamilton's equations follow

$$\dot{q} = [q, H], \quad \dot{p} = [p, H]$$

Lie operator

- The Lie operator for function $f(q_i, p_i)$ is defined

$$:f: = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}$$

- The Lie operator f operating on the function g is equivalent to the Poisson bracket of the two functions.

$$:f:g = [f, g]$$

- Powers of Lie operators

$$: f :^0 g = g \qquad : f :^1 g =: f : g = [f, g]$$

$$: f :^2 g =: f : g = [f, [f, g]]$$

Lie operators of phase space variables

$$:q_i: = \frac{\partial q_i}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial}{\partial q_i} = \frac{\partial}{\partial p_i}$$

$$:p_i: = \frac{\partial p_i}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial p_i}{\partial p_i}\frac{\partial}{\partial q_i} = -\frac{\partial}{\partial q_i}$$

$$:q_i p_i: = \frac{\partial (q_i p_i)}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial (q_i p_i)}{\partial p_i}\frac{\partial}{\partial q_i} = p_i\frac{\partial}{\partial q_i} - q_i\frac{\partial}{\partial p_i}$$

$$:q_i^2: = \frac{\partial q_i^2}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial q_i^2}{\partial p_i}\frac{\partial}{\partial q_i} = 2q_i\frac{\partial}{\partial p_i}$$

$$:p_i^2: = 2p_i\frac{\partial}{\partial q_i}$$

Credit: Todd Satagota (USPAS lectures)

Lie Transform

- The exponential operator is known as a Lie Transformation (allows us to build symplectic transfer maps!)

$$e^{:f:} = \sum_{k=0}^{\infty} \frac{:f:^k}{k!} \qquad \exp(:f:)g = \sum_{k=1}^{\infty} \frac{:f:^k\, g}{k!} = g + [f,g] + [f,[f,g]]/2! + \ldots$$

Symplectic map

- Define a map M (e.g. transfer matrix) that updates the coordinates over some increment

$$(q_{i+1}, p_{i+1}) = M(q_i, p_i)$$

- The map is symplectic if

$$M^T \Omega M = \Omega$$

$$\text{where } \Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

Taylor series map

- The phase space coordinates can be expressed as a Taylor power series of the initial coordinates

$$z(i,1) = \sum_{j=1}^{6} R_{ij} z(j,0) + \sum_{j,k=1, j\leq k}^{6} T_{ijk} z(j,0) z(k,0) + \dots$$

where R, T are the 1st and 2nd order transfer map matrices, (zi,0) and (zi,1) are the phase space coordinates at the entrance and exit of a lattice element, respectively. In general, the map is not symplectic when truncated at some order.

Map from Lie Transformations

- Symplectic maps can be created using Lie transformations

$$z(t) = exp(t : H :)z_0$$

with M=exp(t:H:). A map to a given order in phase space coordinates can be created by composition

$$M = e^{:f1:}e^{:f2:}e^{:f3} \dots e^{:fk} + \mathcal{O}(k)$$

The map can be truncated at order k and it remains symplectic (Dragt-Finn factorisation theorem). Make use of the Backer-Campbell-Hausdoff (BCH) formula

$$e^{:A:}e^{:B:} = e^{:C:}$$

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] \dots$$

Lie Operators for a drift

The map for a drift is simply $M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$ Since there is no potential term, $H_{drift} = \frac{p^2}{2}$

The equivalent Lie operator is $exp(: Lp^2/2 :)$

To show this expand the transformation as follows

$$exp(: Lp^2/2 :)x = x + [Lp^2/2, x] + [Lp^2/2, [[Lp^2/2, x]]/2 + ...$$
$$exp(: Lp^2/2 :)p = p + [Lp^2/2, p] + [Lp^2/2, [[Lp^2/2, p]]/2 + ...$$

Noting $[Lp_x^2/2, p_x] = Lp_x \frac{\partial p_x}{\partial x} = Lp_x,\ [Lp_x^2/2, p_x] = Lp_x \frac{\partial p_x}{\partial x} = 0$ and the higher order terms are zero,

Error here!

Credit: Todd Satagota (USPAS lectures)

Lie Operators for Accelerator elements

Table 1: Lie Operators for Common Accelerator Elements

Element	Map	Lie Operator
Drift space	$x = x_0 + Lp_0$ $p = p_0$	$\exp(:-\frac{1}{2}Lp^2:)$
Thin-lens quadrupole	$x = x_0$ $p = p_0 - \frac{1}{f}x_0$	$\exp(:-\frac{1}{2f}x^2:)$
Thin-lens kick	$x = x_0$ $p = p_0 + \lambda n x_0^{n-1}$	$\exp(:\lambda x^n:)$
Thick focusing quad	$x = x_0 \cos\sqrt{k}L + \frac{p_0}{\sqrt{k}}\sin\sqrt{k}L$ $p = -kx_0 \sin\sqrt{k}L + p_0\cos\sqrt{k}L$	$\exp(:-\frac{1}{2}L(kx^2 + p^2):)$
Thick defocusing quad	$x = x_0 \cosh\sqrt{k}L + \frac{p_0}{\sqrt{k}}\sinh\sqrt{k}L$ $p = -kx_0 \sinh\sqrt{k}L + p_0\cosh\sqrt{k}L$	$\exp(:-\frac{1}{2}L(kx^2 - p^2):)$
Coordinate shift	$x = x_0 - b$ $p = p_0 + a$	$\exp(:ax + bp:)$
Coordinate rotation (Phase advance μ)	$x = x_0\cos\mu + p_0\sin\mu$ $p = -x_0\sin\mu + p_0\cos\mu$	$\exp(:-\frac{\mu}{2}(x^2 + p^2):)$
Full-turn Hamiltonian	(lots of things)	$\exp(C:H_{\text{eff}}:)$ or $\exp(:-\frac{\mu}{2}(\gamma x^2 + 2\alpha xp + \beta p^2):)$

Credit: Todd Satagota (USPAS lectures)

Cheat Sheet

- Hamiltonian formal definition, the Legendre transform $H = \sum_{i=1}^{n} p_i \dot{q}_i - L$
- For a conservative system H = T + V
- Hamilton's equations… $\dot{q} = \frac{\partial H}{\partial p}, \; \dot{p} = -\frac{\partial H}{\partial q}$
- …may be expressed in terms of Poisson brackets

$$\dot{q} = [q, H], \; \dot{p} = [p, H]$$

- The following fundamental concepts follow
  - **Liouville's theorem:** $\frac{\partial f}{\partial t} + [f, H] = 0$
  - **Symplecticity** $M^T \Omega M = \Omega$ where $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$
  - **Integrability** (#invariants = # degrees of freedom)

Symplectic maps can be constructed using Lie Transformations

$$z(t) = exp(t : H :) z_0$$