Accelerator Physics: Momentum Effects

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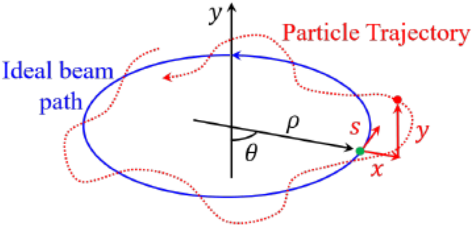
Overview

Curvilinear Coordinates



- (x, y, s), often called the standard co-ordinate system in accelerator physics
- The origin is de
  ned by the vector $\vec{S}(s)$ following the ideal reference path
- $x = r - \rho \qquad s = \rho\theta$
- $X = r\sin\theta = (\rho + x)\sin\theta,\ Y = y,\ Z = r\cos\theta = (\rho + x)\cos\theta$

Transverse Equation of Motion - 1

- Start with the basics

$$
\begin{aligned}
F_x &= m\frac{d^2 r}{dt^2} - \frac{mv^2}{r} \\
&= m\frac{d^2(x+\rho)}{dt^2} - \frac{mv^2}{x+\rho} = -eB_y v
\end{aligned} \tag{1}
$$

- Factorise the equation

$$
m\frac{d^2 x}{dt^2} - \frac{mv^2}{\rho}\left(1+\frac{x}{\rho}\right)^{-1} = -eB_y v \tag{2}
$$

- Utilise the binomial approximation

$$
m\frac{d^2 x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = -eB_y v \tag{3}
$$

Transverse Equation of Motion - 2

- Replace t with s and rearrange

$$mv^2\frac{d^2x}{ds^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -eB_y v \tag{4}$$

$$= \frac{d^2x}{ds^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{eB_y}{mv} \tag{5}$$

- Consider small displacements in x

$$\frac{d^2x}{ds^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{e}{mv}\left(B_0 + x\frac{\partial B_y}{\partial x}\right) \tag{6}$$

Transverse Equation of Motion - 3

- Set field gradient, $g = \frac{\partial B_y}{\partial x}$

$$\frac{d^2x}{ds^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{e}{mv}\left(B_0 + xg\right) \tag{7}$$

  This is a modified Hill's equation

- Consider small momentum offsets $\Delta p = p - p_0 \ll p_0$

$$\begin{aligned} \frac{1}{p_0 + \Delta p} &= \frac{1}{p_0}\left(1 - \frac{\Delta p}{p_0}\right)^{-1} \\ &=\approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2} \end{aligned} \tag{8}$$

Transverse Equation of Motion - 4

- Insert Equation 8 into the modified Hill's equation (Eq. 7)

$$\frac{d^2x}{ds^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{e}{p}\left(B_0 + xg\right)$$

$$\frac{d^2x}{ds^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{e}{p_0}\left(B_0\left(1 + \frac{\Delta p}{p_0}\right) + xg\left(1 + \frac{\Delta p}{p_0}\right)\right) \tag{9}$$

- Remember magnetic rigidity? $B\rho = {}^{p}/_{e}$

$$\frac{d^2x}{ds^2} + \frac{x}{\rho^2} = \frac{1}{\rho}\frac{\Delta p}{p_0} + kx \tag{10}$$

where $k = {}^{eg}/_{p_0}$ and the last term is the product of two small terms (≈ 0)

Transverse Equation of Motion - 5

- Finally a new modified Hill's equation

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho^2} - k\right)x = \frac{1}{\rho}\frac{\Delta p}{p_0} \tag{11}$$

- Compare to the original Hill's equation from transverse lectures

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho^2} - k\right)x = 0 \tag{12}$$

- Particles with different momenta/energy have different orbits

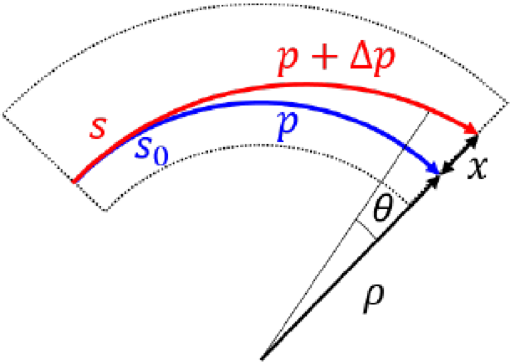
Dispersion

General solution will be of the form $x(s) = x_h(s) + x_i(s)$

- From previous lecture, dispersion is defined as

$$D(s) = \frac{x_i(s)}{\Delta p/p_0} \tag{13}$$

- It is just another orbit and is subject to the focusing properties of the lattice
- The orbit of any particle is the sum of the well-known x_h and dispersion



Matrix Formalism

- Recall transfer matricies from transverse lectures and add dispersion

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p_0} \begin{pmatrix} D \\ D' \end{pmatrix} \tag{14}$$

where $C = \cos\sqrt{|k|}s$, $S = \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}s$, $C' = \frac{dC}{ds}$, $S' = \frac{dS}{ds}$, and $D' = \frac{x_i'(s)}{\Delta p/p_0}$

- Solving Equation 11, one can show that

$$D(s) = S(s)\int_{s_0}^{s_1} \frac{1}{\rho} C(s) ds - C(s)\int_{s_0}^{s_1} \frac{1}{\rho} S(s) ds \tag{15}$$

Examples of Dispersion - 1

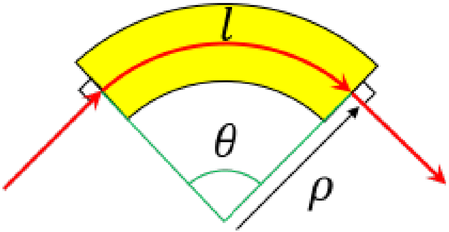
- Start with something simple, a drift!

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \qquad C(s) = 1,\ S(s) = l \tag{16}$$

- Importantly $\rho = \infty$ so immediately $D_{drift} = 0$
- Okay, so how about a pure sector dipole?

$$M_{dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

$$C(s) = \cos\frac{l}{\rho},\ S(s) = \rho\sin\frac{l}{\rho} \tag{17}$$



Examples of Dispersion - 2

- Putting this in the equation for dispersion

$$\begin{aligned} D_{dipole}(s) &= \sin\frac{l}{\rho}\int_0^l \cos\frac{s}{\rho}ds - \cos\frac{l}{\rho}\int_0^l \sin\frac{s}{\rho}ds \\ &= \sin\frac{l}{\rho}\left[\rho\sin\frac{s}{\rho}\right]_0^l + \cos\frac{l}{\rho}\left[\rho\cos\frac{s}{\rho}\right]_0^l \\ &= \rho\sin^2\frac{l}{\rho} + \rho\cos\frac{l}{\rho}\left(\cos\frac{l}{\rho} - 1\right) \\ &= \rho\left(1 - \cos\frac{l}{\rho}\right) \end{aligned} \tag{18}$$

- And $D'_{dipole}(s) = \sin\frac{l}{\rho}$

Examples of Dispersion - 3

- Assuming $\theta = {}^{l}/_{\rho}$ is small we can expand this

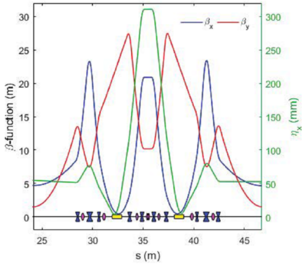
$$
\begin{aligned}
D_{dipole}(s) &= \rho\left(1 - \cos\frac{l}{\rho}\right) \\
&\approx \rho\left(1 - \left[1 - \frac{1}{2}\left(\frac{l}{\rho}\right)^2\right]\right) \\
&\approx \frac{\rho}{2}\left(\frac{l}{\rho}\right)^2 = \frac{\rho\theta^2}{2}
\end{aligned}
\tag{19}
$$

Matrix Formalism Continued

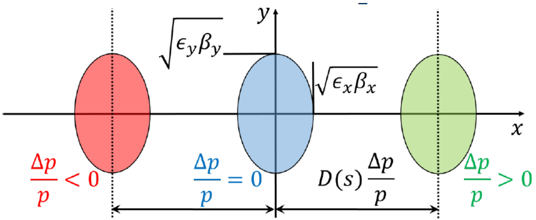
Can now expand the transfer matrix to include dispersion

$$\begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_1 = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_0 \tag{20}$$

- Dispersion can be calculated by an optics code for a real machine
- $D(s)$ is created by the dipoles...
- ...and focused by the quadrupoles
- Diamond DBA Example ⇒



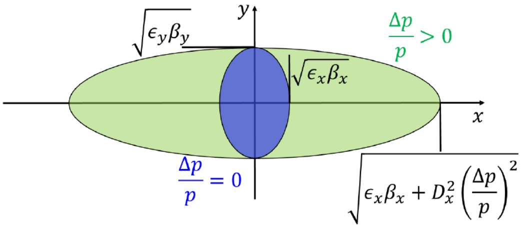
Dispersed Beam Orbits



- These are 2D ellipses defining the beam
- The central and extreme momenta are shown (there is a distribution in between)
- The vacuum chamber must accommodate the full spread
- With dispersion the dispersed closed orbit for a given particle is (assuming $D_y = 0$)

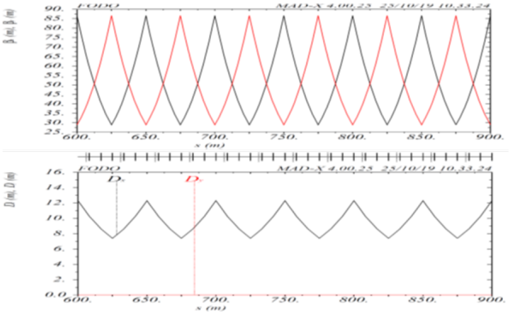
$$y(s) = y_{\beta_y}(s), \qquad x = x_{\beta_x}(s) + D(s)\frac{\Delta p}{p_0} \tag{21}$$

Dispersed Beam Size



- Dispersion also contributes to the beam size
- Therefore we can measure the dispersion by measuring beam sizes at different locations with different amounts of dispersion and different βs

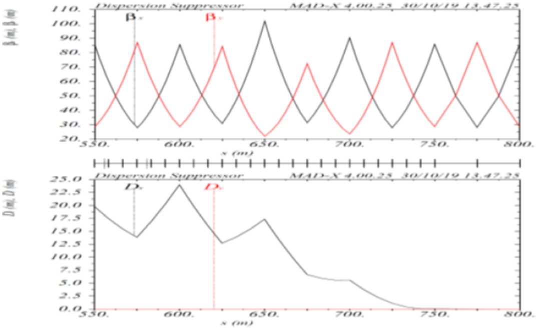
Dispersion Suppression



- Given a periodic lattice what can we do about dispersion?
- We can't get rid of it completely as it's produced by the dipoles
- Answer: suppress the dispersion elsewhere

Dispersion Suppression: Easy Option

- Use extra quadrupoles to match $D(s)$ and $D'(s)$
- Given an optical solution in the arc, suppressing dispersion can be achieved with 2 additional quadrupoles
- But that's not enough! Need to match the Twiss, optical parameters too
- An extra 4 quadrupoles are needed to match α and β



Dispersion Suppression: Easy Option

Advantages:

- Straight forward
- Works for any phase advance per cell
- Ring geometry is unchanged
- Flexible! Can match between different lattice structures

Disadvantages:

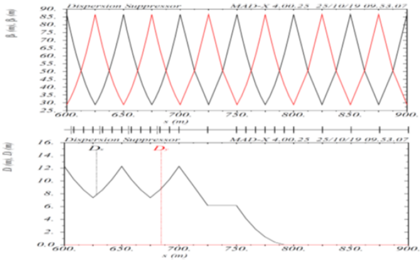
- Additional quadrupole magnets and power supplies required
- The extra quadrupoles are, in general, stronger
- The β function increases so the aperture increases

Dispersion Suppression: Missing Bend

- Start with $D = D' = 0$ and create dispersion such that the conditions are matched in the first regular quadrupoles
- Utilise n cells without dipole magnets at the end of an arc, followed by m arc cells
- Hence "missing bend" dispersion suppression

Condition:

$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2} \qquad (22)$$



where Φ_C is the hase advance per cell,
$\sin\frac{m\Phi_C}{2} = \frac{1}{2}$, $k = \mathrm{even}$
or $\sin\frac{m\Phi_C}{2} = -\frac{1}{2}$, $k = \mathrm{odd}$

Dispersion Suppression: Missing Bend

Advantages:

- No additional quadrupoles or new power supplies
- Aperture requirements are the same as those in the arc as β is unchanged

Disadvantages:

- Only works for certain phase advances restricting optics options in the arc
- The geometry of the ring is changed

Dispersion Suppression: Half Bend

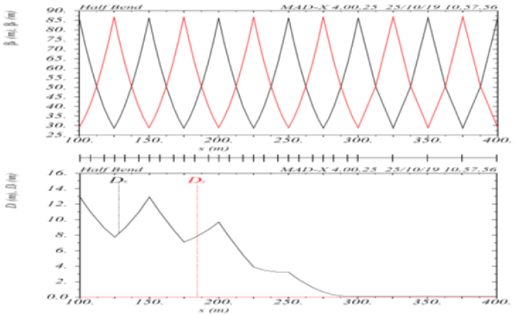
- How about inserting different strength dipoles? Does it help?
- Assume you have a FODO arc cell, a lattice insertion and then a dispersion free section without dipoles
- Condition for vanishing dispersion can be calculated for n cells with dipole strength δ_{sup}

$$2\delta_{sup}\sin^2\left(\frac{n\Phi_C}{2}\right) = \delta_{arc} \tag{23}$$

- So if we require $\delta_{sup} = \frac{1}{2}\delta_{arc}$ we get

$$\sin^2\left(\frac{n\Phi_C}{2}\right) = 1 \quad \Rightarrow \quad \sin\left(n\Phi_c\right) = 0$$
$$\Rightarrow n\Phi_C = k\pi, \quad k = \mathrm{odd} \tag{24}$$

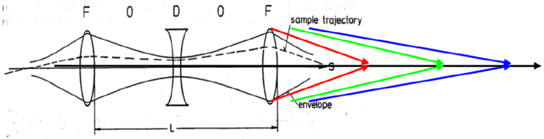
Dispersion Suppression: Half Bend



Advantages and disadvantages are the same as for the missing bend only there is an extra disadvantage:
A special half strength dipole is required which may add extra cost to the design N.B.
This is not an exhaustive list of dispersion suppression techniques, just a taster!

Chromaticity - 1

- What about off-momentum effects through quadrupoles?
- The focusing strength of a quadrupole depends on the momentum of the particle $1/f \propto 1/p$



- Particles with $\Delta p > 0$, $\Delta p < 0$, $\Delta p = 0$
- Off-momentum particles oscillate around a chromatic closed orbit NOT the design orbit

Chromaticity - 2

- Normalised quadrupole strength $k = \frac{g}{p/e}$
- In case of a momentum spread

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{eg}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) = k_0 + \Delta k \tag{25}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0 \tag{26}$$

- This acts like a quadrupole error in the machine and leads to a tune spread

$$\Delta Q = \frac{1}{4\pi} \oint \Delta k(s)\beta(s)ds = -\frac{1}{4\pi}\frac{\Delta p}{p_0} \oint k_0(s)\beta(s)ds \tag{27}$$

Chromaticity - 3

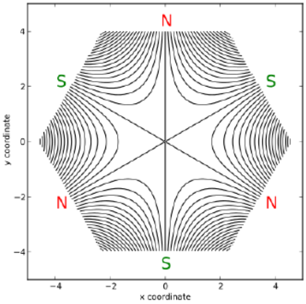
- This spread in tune is expressed via chromaticity, Q' or the normalised chromaticity, ξ

$$Q' = \frac{\Delta Q}{\Delta p / p_0}, \qquad \xi = \frac{\Delta Q / Q}{\Delta p / p_0} \tag{28}$$

- Note that chromaticity is produced by the lattice itself
- It is determined by the focusing strength of all the quadrupoles
- The "natural" chromaticity is negative and can lead to a large tune spread and consequent instabilities
- For example, for a FODO lattice $\xi \approx 1$

Correcting Chromaticity - 1

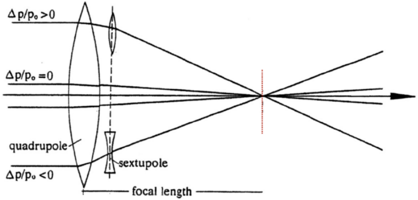
- Want to "sort" the particles by their momentum
- Utilise dispersive trajectory! Apply magnetic field that is zero at small amplitudes and rises quickly outward
- Use sextupoles!



$$B_x = \tilde{g}xy, \qquad B_y = \frac{1}{2}\tilde{g}\left(x^2 - y^2\right) \tag{29}$$

- This results in a linear gradient in x, $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x$
- And a normalised sextupole strength $k_{sext} = \frac{\tilde{g}x}{p/e} = m_{sext}x = m_{sext}D^{\Delta p}/_{p_0}$

Correcting Chromaticity - 2



- This all results in a corrected chromaticity

$$Q' = -\frac{1}{4\pi} \oint [k(s) - mD(s)]\, \beta(s) ds \quad (30)$$

- Chromatic sextupoles: Sextupoles at nonzero dispersion can correct natural chromaticity
- Usually 2 families, one horizontal and one vertical
- Place where $\beta_{x/y} D$ is large to minimise their strength

Summary

- Reminder of co-ordinate system
- Transverse equation of motion: modified Hill's equation with momentum spread
- Dispersion revisited in matrix form
- Effect of dispersion on beam orbit and beam size
- Dispersion suppression
- Chromaticity and chromatic tune spread
- Chromatic sextupoles and chromaticity correction

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