Accelerator Physics: Longitudinal Beam Dynamics

JAI Michaelmas Term 2024, Lecture 9

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04/11/2024

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Overview

1. Acceleration of Charged Particles
2. Synchrotrons
3. Phase Stability
4. Synchrotron Longitudinal Beam Dynamics
5. Hamiltonian Formalism
6. Synchrotron Oscillations
7. RF Buckets and Phase Space
8. Summary

What is Longitudinal Beam Dynamics?

- Longitudinal Beam Dynamics (LBD) describes the motion of particles in the"longitudinal plane" i.e. in the direction of travel.
- The real-space coordinate varies depending on the context (e.g. z, t, s, ϕ etc.)
- To visualise the LBD, we use the Longitudinal Phase Space (LPS)
- This comprises one real space coordinate (as above), and a coordinate related to energy (e.g. E, p, δ etc.)

LBD concerns itself with particle energies $\rightarrow$ linked to acceleration process.
So what are the relevant interactions in LBD?

Acceleration

In accelerator physics, only interaction of relevance is the electromagnetic interaction:

$$\dot{\vec{p}} = \dot{\gamma} m \vec{v} + \gamma m \dot{\vec{v}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

In order to gain energy:

- $\Rightarrow$ Require $\dot{\gamma} \neq 0$
- $\Rightarrow$ Component of $\dot{\vec{p}} \parallel \vec{v}$
- $\Rightarrow$ 2nd-term on RHS always $\perp \vec{v}$
- $\Rightarrow$ Only $\vec{E}$ can result in energy gain
- $\Rightarrow$ $\dot{p} = qE_z$

Relativistic Kinematics

Units

- Will absorb factors of c into mass and momenta and express them in $[E] =$eV.
- In this convention, $[q] = e$.

Relativistic energy-mass-momentum equivalence:

$$E^2 = m^2 + p^2 \tag{1}$$

Take differential of Equation 1 to obtain key relation:

$$\begin{aligned} 2EdE &= 2pdp \\ \rightarrow dE &= \frac{p}{E}dp \\ &= \frac{\gamma\beta m}{\gamma m}dp \\ &= \beta dp \end{aligned} \tag{2}$$

Relativistic Kinetics

Rate of work done, W, by longitudinal electric field, E_z along particle trajectory:

$$\frac{dW}{dz} = \beta \frac{dp}{dz} = \frac{\dot{p}}{c} = qE_z$$

Total energy gain:

$$W = q \int E_z dz = qV$$

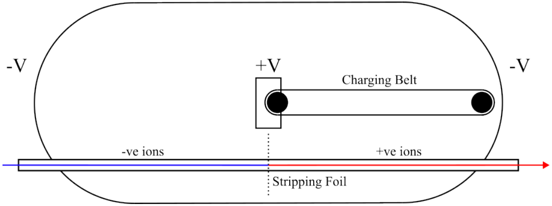
Radius of orbit in constant, homogenous magnetic field ($\vec{B}$):

$$\begin{aligned} \dot{p} = |\dot{\vec{p}}| = q\beta c^2 B = \frac{p\beta c}{\rho} \\ \rightarrow B\rho = \frac{p}{qc} \end{aligned} \tag{3}$$

$B\rho$ is known as the beam rigidity. N.B. ρ is the bending radius of the magnetic field

Methods of Acceleration

Simplest case: **electrostatic field**



- Limited by breakdown of static field
- Not space-efficient
- Fundamentally not applicable for circular accelerators:

$$W = q \oint \vec{E} \cdot d\vec{l} = 0 \tag{4}$$

Methods of Acceleration

An old-fashioned method: **induction accelerator**
While we cannot directly accelerate with B-field, we can use them indirectly.
Faraday's/Lenz's law (& Stokes' theorem):

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = \int_S \dot{\vec{B}} \cdot d\vec{A}$$

First accelerator with constant closed orbit through acceleration.

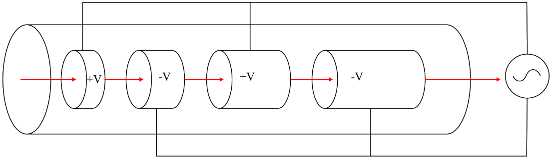
Image sourced from [1].

[1] URL: https://en.wikipedia.org/wiki/Betatron

Methods of Acceleration

The modern(ish) way: **radio-frequency (RF) electric fields**

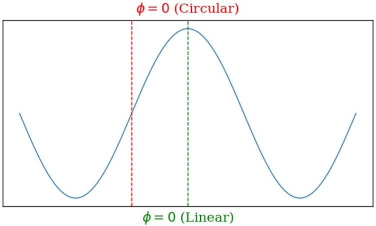
- AC fields allow for larger gradients without breakdown of res. vacuum
- AC fields allow us to have multiple voltage gaps in a much smaller space.
- AC fields allow us to get around Eq. 4
- Typical orbital frequency of beam in accelerator is in RF-band



Comment on Phases

RF phase ($\phi = \omega_{RF} t$) is defined differently depending upon context:

- For linear accelerators, RF voltage is considered cosine-like, with origin at the positive crest of waveform
- For circular accelerators, RF is considered cosine-like, with origin at the positive gradient zero-crossing of the waveform

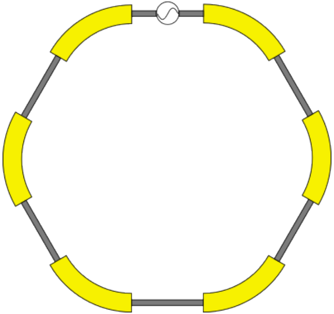


Henceforth, we shall use the circular accelerator convention.

Introduction to Synchrotrons

What is a synchrotron?

- Ring-like structure in which we **accelerate** particle (c.f. some storage rings, accumulator rings), typically over a large energy range
- Constant orbit throughout acceleration cycle
- Bunched beam
- Normally strong focusing



Synchrotrons use RF electric fields to accelerate like the DTL described earlier. Unlike the DTL, RF systems in a synchrotron typically comprise a small number of gaps in a smaller resonant cavity.

Introduction to Synchrotrons

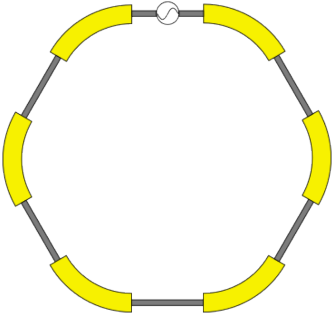
How do we do this?

- The RF frequency is swept to keep synchronicity as the beam accelerates:

$$f_{RF} = hf_r = \frac{h\beta c}{2\pi R_0} \tag{5}$$

- The main magnetic field is increased in-kind to maintain the constant orbit

  $\Rightarrow \rho = \frac{p}{qBc} = const.$ (Reminder: $[p] = \text{eV}$; $[q] = e$)



Notation:

$R_0 = \langle R(\phi) \rangle$, mean radius at center of beam pipe

Energy Ramp

When designing synchrotron, what will the energy ramp look like?
Typically determined by:

- Use case of accelerator
- Type of magnets used (normal- vs super-conducting)
- Required rep-rate

From beam rigidity (Eq. 3) and Equation 2:

$$
\begin{aligned}
\dot{p} &= qc\dot{B}\rho \\
\Delta p_{turn} &= f_r \dot{p} \\
\Delta E_{turn} &= \beta \Delta p_{turn} \\
&= 2\pi R_0 q \dot{B}\rho
\end{aligned}
\tag{6}
$$

Synchronous Particle

In order to accelerate consistently turn-to-turn, the RF frequency must be an integer multiple (harmonic number, h) of the nominal revolution frequency.
Energy gain of a particle accelerated by a sinusoidal AC field:

$$\Delta E = qV \sin\phi$$

Define the **synchronous phase**,

$$\phi_s = \arcsin\left(\frac{2\pi R_0 \dot{B}\rho}{V}\right). \tag{7}$$

A particle crossing the RF gap at ϕ_s sees the same phase on it's return, provided it's energy is the nominal energy. We refer to this particle as the synchronous particle. The synchronous particle gains energy every turn according to Equation 6.
The synchronous particle lies at a **stable fixed-point (SFP) of the LPS**. More on this later!

RF Synchronisation

- As the synchronous particle gains energy, its revolution frequency increases.
- RF frequency has to follow:

$$f_{RF} = hf_r = \frac{h\beta(t)c}{2\pi R_0} = \frac{hc}{2\pi R_0}\frac{p(t)}{E(t)} = \frac{qhc^2\rho}{2\pi R_0}\frac{B(t)}{E(t)}$$

- Using Equation 1:

$$f_{RF} = \frac{hc}{2\pi R_0}\sqrt{\frac{B(t)^2}{B(t)^2 + \left(\frac{m}{qc\rho}\right)^2}} \tag{8}$$

$\Rightarrow$ **Intrinsic synchronisation between magnetic and electric fields!**

- Note asymptotic limit of Equation 8: as B increases (i.e. as $\beta \to 1$) $f_{RF} \to \frac{hc}{2\pi R_0}$

$\Rightarrow$ RF frequency sweep is more important at lower energies

Frequency Sweep Examples: LHC Accelerator Chain

The RF requirements of the LHC complex vary massively between the individual accelerators.

Synchrotron	**Initial/Final Energy**	**Min/Max** f_r	$\frac{\Delta f_r}{f_{r,min}}$
PSB	0.16/2.0 GeV	0.99/1.81 MHz	82.3%
PS	2/26 GeV	451/476 kHz	5.5%
SPS	26/450 GeV	43.422/43.488 kHz	0.1%
LHC	450/7000 GeV	11.10340/11.10342 kHz	2.2×10^{-4}%

Revolution frequency shifts through the CERN Acclerator Chain

Phase Stability

We have shown that a particle at (ϕ_s, E_s) sees the same RF phase throughout acceleration.

But a beam is made of many particles, distributed over a range of phases and energies.

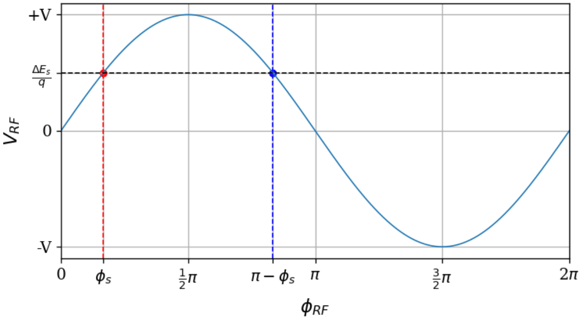
How do we maintain a stable, bunched beam?

Let's start with a simple example: phase stability in a proton drift-tube linac.

We will assume:

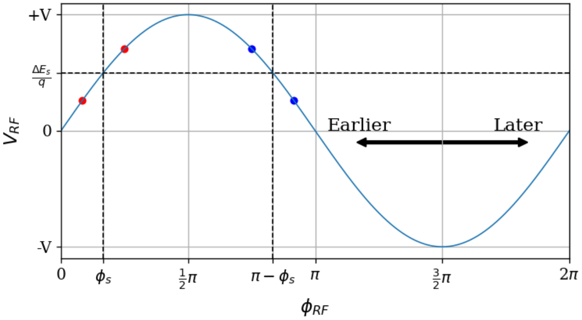
- The linac is a standing-wave structure operating in 2π-mode: the phase of $\vec{E}(t)$ is the same in all gaps at any given time
- The voltage is the same across each gap
- The distance across each gap is negligible

Phase Stability in a Linac



There are two phases (fixed points) in a given RF period at which a particle can remain perfectly synchronous with the RF field. To figure out which is ϕ_s, we must consider small offset about ϕ_s.

Phase Stability in a Linac



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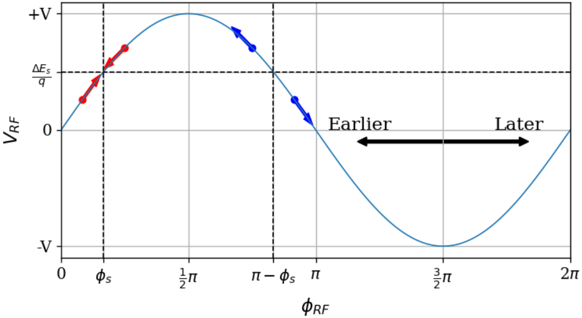
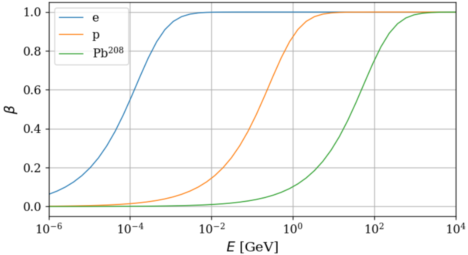
Phase Stability in a Linac



ϕ_s lies at the stable fixed point. Later particles gain more energy, arriving earlier at the next gap (& vice versa). The point $\pi - \phi_s$ is the **unstable fixed point**.

Consequences of Phase Stability in a Linac

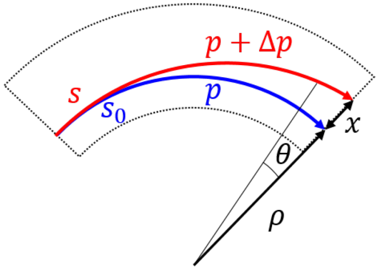
- Phase stability is mediated by velocity differences in particles
- At high γ, the longitudinal position of particles is essentially frozen
  - For electron machines, this happens at relatively low ($\sim$10 MeV) energies
  - For protons, much higher energies are required ($\sim$10 GeV)
  - For lead ions, need $\sim$10 TeV ($\sim$50 GeV u^{-1})



Phase Stability in a Synchrotron

To understand phase stability in a synchrotron, we must dip a toe into transverse dynamics.
Consider a particle with a small relative momentum offset $\delta_p = \frac{p-p_0}{p_0}$ from the nominal momentum of the magnetic field, p_0:

- Particle enters dipole in the ring with transverse coordinate $x = x' = 0$
- Particle takes a wider orbit than the synchronous particle due to **dispersion** (Eq. 3)



Dispersion

- Bending radius increased by

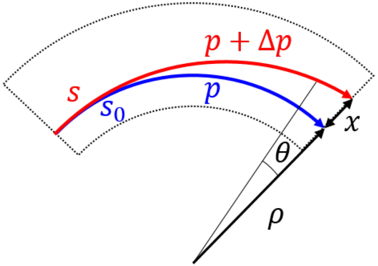
$$\Delta x = D_x \delta_p,$$

where $D_x = D_x(s)$ is the dispersion function

- Path length difference through the magnet:

$$\Delta l = \Delta s - \Delta s_0 = (\rho + \Delta x)\Delta\theta - \rho\Delta\theta = D_x \Delta\theta \delta_p$$

$$\lim_{\Delta \to 0} \frac{\Delta l}{\Delta s_0} = \frac{dl}{ds_0} = \frac{D_x(s_0)}{\rho(s_0)}\delta_p \qquad (9)$$



Momentum Compaction

Integrate Equation (9) to get the total path length difference over full orbit:

$$\Delta C = \oint dl = \delta_p \oint \frac{Dx(s_0)}{\rho(s_0)} ds_0$$

Momentum compaction factor:

$$\alpha_c = \frac{{}^{dL}/_{L}}{{}^{dp}/_{p}} = \frac{1}{\delta_p}\frac{\Delta C}{C_0} = \frac{1}{C_0}\oint \frac{D_x(s_0)}{\rho(s_0)} ds_0 \approx \frac{1}{C_0}\sum_i^N \langle D_x \rangle_i \theta_i,$$

where $\langle D_x \rangle_i$ and θ_i are the average dispersion and bend angle in the i^{th} dipole.

Transition Energy

So we now have two factors affecting the revolution frequency of off-momentum particles in a synchrotron:

- A **decrease** in revolution time due to an **increase** in velocity
- An **increase** in revolution time due to an increase in the orbit length with **increasing** momentum

Combining these:

$$\frac{df_r}{f_r} = \frac{C_0}{\beta_s}\left(d\beta - \frac{dC}{C_0^2}\right) = \frac{d\beta}{\beta_s} - \alpha_c \delta_p \tag{10}$$

Recognizing that β is itself also a function of p, and using Equation 2:

$$\begin{aligned}\frac{d\beta}{\beta_s} &= \frac{dp}{p_s} - \frac{d\gamma}{\gamma_s} \\ &= \frac{dp}{p_s} - \frac{dE}{E_s} \\ &= \left(1 - \beta^2\right)\delta_p \end{aligned} \tag{11}$$

Transition Energy

Combining Equations 10 and 11 together:

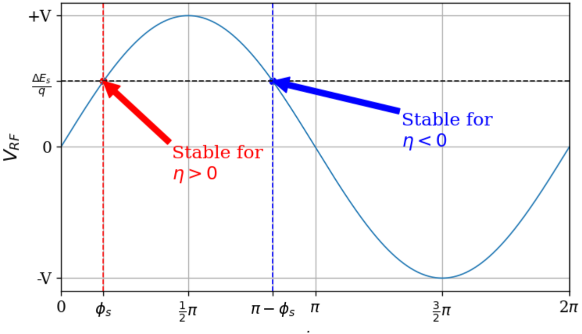
$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\delta_p = \eta\delta_p, \tag{12}$$

where η is the **phase slip factor**.
N.B some references define η with a minus sign.
Inspection of 12 shows a clear transition regime. We define the transition energy with $\gamma_t = {}^1/\sqrt{\alpha_c}$.

- Below transition, the phase slippage is velocity dominated
- Above transition, phase slippage is dispersion dominated
- Navigating transition is a challenge for high-energy synchrotrons

Phase Stability in a Synchrotron

$$\frac{df_r}{f_r} = \eta\delta_p = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}\right)\delta_p$$



Consequences of Phase Stability in a Synchrotron

- Phase stability is determined by both velocity and orbit length differences, which are in opposition to one another
- During acceleration, synchrotron may pass through **transition**, where stable region of LPS changes rapidly
- Potentially problematic for overall accelrator performance
- Generally more relevant for hadron and ion machines than electron machines

Variables in Synchrotron LBD

LBD, also known as **synchrotron motion** in the context of circular machines, describes the motion of particles in the longitudinal plane through the coupled variables of energy and a longitudinal spatial coordinate. For circular machines, the RF phase is a convenient choice for this coordinate.
We will first begin by introducing some reduced variables, centered on the energy and phase of the synchronous particle:

- $\Delta E = E - E_s$, particle energy
- $\Delta p = p - p_s$, particle momentum
- $\Delta\theta = \theta - \theta_s$, particle azimuthal angle
- $\Delta\phi = \phi - \phi_s$, particle RF phase
- $\Delta f_r = f_r - f_{r,s}$, particle RF phase

First Energy-Phase Relation

RF phase is related to azimuthal position around the ring by:

$$\Delta\theta = -\frac{1}{h}\Delta\phi_{RF}$$

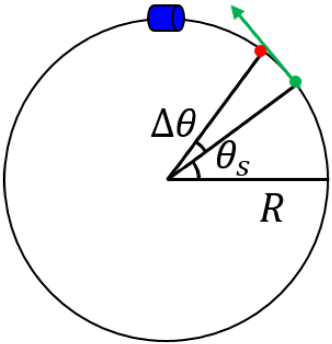
The change in revolution frequency is then related to the rate-of-change of ϕ by:

$$\Delta\omega = \frac{d\Delta\theta}{dt} = -\frac{1}{h}\dot{\phi}_{rf}$$

Combining with the phase slip factor (Eq. 12):

First Energy-Phase Relation

$$\dot{\phi} = h\omega_r\eta\delta_p = \frac{h\omega_r^2\eta}{\beta^2 E}\left(\frac{\Delta E}{\omega_r}\right) \tag{13}$$



Second Energy-Phase Relation

From Equation 6, we can get the rate-of-change of energy:

$$\dot{E} = f_r q V \sin\phi$$

From this, we can quickly reach the second energy-phase relation:

Second Energy-Phase Relation

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_r}\right) = \frac{qV}{2\pi}\left(\sin\phi - \sin\phi_s\right) \tag{14}$$

Longitudinal Equation of Motion

Combining Equations 13 and 14, we can obtain an equation of motion:

$$\frac{d}{dt}\left(\frac{\beta^2 E}{h\omega_r^2\eta}\dot{\phi}\right) + \frac{qV}{2\pi}(\sin\phi - \sin\phi_s) = 0 \tag{15}$$

A non-linear 2$^{\text{nd}}$-order differential equation.
N.B. the parameters β, E, ω_r, and η are all relatively slowly varying.

Longitudinal Hamiltonian

We can also derive the enrgy-phase relations from a Hamiltonian. Use the canonical variables (ϕ, W), where $W = \frac{\Delta E}{\omega_r}$:

Longitudinal Hamiltonian

$$\mathcal{H} = \frac{h\omega_{r,s}^2\eta}{2\beta^2 E_s} W^2 + \frac{q}{2\pi} U(\phi) \tag{16}$$

$$U(\phi) = \int_{\phi_s}^{\phi} \left(V(\phi') - V(\phi_s)\right) d\phi' \tag{17}$$

Equations 13 and 14 can then be obtained from Equation 16 by

$$\dot{W} = -\frac{\partial \mathcal{H}(\phi, W)}{\partial \phi} \qquad \dot{\phi} = \frac{\partial \mathcal{H}(\phi, W)}{\partial W} \tag{18}$$

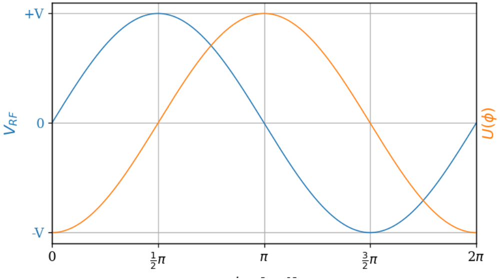
Longitudinal Hamiltonian Dynamics Example

Simple example: **single-harmonic RF**

$$V(\phi) = V_1 \sin\phi$$

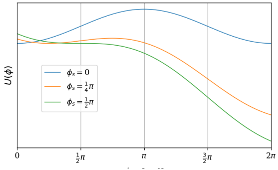
In this example, the potential (Eq. 17) is given by:

$$U(\phi) = V_1\left(\cos\phi_s - \cos\phi - (\phi - \phi_s)\sin\phi_s\right)$$



Longitudinal Hamiltonian Dynamics Example

- We have created a **potential well in the longitudinal plane**
- Shape of the well changes with the synchronous phase

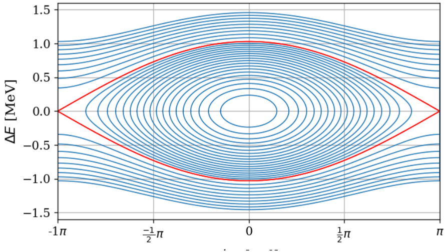


Longitudinal Hamiltonian Dynamics Example

Single Harmonic RF Hamiltonian

$$\mathcal{H}(\phi, W) = \frac{h\omega_{r,s}^2\eta}{2\beta^2 E_s} W^2 + \frac{qV_1}{2\pi}\left(\cos\phi_s - \cos\phi - (\phi - \phi_s)\sin\phi_s\right)$$

- Particles travel along contours of constant $\mathcal{H}$ under conservative forces
- The synchronous particle, is located at the **stble fixed point (SFP)**, $(\phi_s, 0)$
- The stable region is bounded by the **separatrix**
- The separatrix passes through the **unstable fixed point (UFP)** at $W = 0$
- Let's first look at particles close to ϕ_s



Small Amplitude Oscillations

We can re-write Equation 15 as

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0, \tag{19}$$

with the **synchrotron frequency**

$$\Omega_s^2 = \frac{h\eta\omega_r^2 qV\cos\phi_s}{2\pi\beta^2 E_s},$$

where the slowly-varying parameters η, ω_r, β, and E_s have been assumed to be constant. If we take a small deviation in phase away from ϕ_s, such that:

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \approx \Delta\phi\cos\phi_s,$$

we can reduce Equation 19 to simple harmonic motion:

$$\ddot{\phi} + \Omega_s^2\Delta\phi = 0.$$

Small Amplitude Oscillations

Particles close to ϕ_s exhibit simple harmonic motion in the LPS, with characteristic frequency Ω_s.

The **synchrotron tune**, $Q_s = {}^{\Omega_s}/_{\omega_r}$, is typically small:

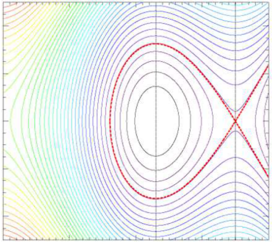
- $Q_s \sim 10^{-3}$ for proton machines
- $Q_s \sim 10^{-1}$ for electron machines

Implicit in the previous analysis is the stability condition we recognised by inspection previously:

- $\Omega_s^2 > 0 \quad \rightarrow \quad \eta \cos \phi_s > 0$
- $\gamma < \gamma_t \quad \rightarrow \quad \eta > 0 \quad \rightarrow \quad 0 < \phi_s < {}^{\pi}/_{2}$
- $\gamma > \gamma_t \quad \rightarrow \quad \eta < 0 \quad \rightarrow \quad {}^{\pi}/_{2} < \phi_s < \pi$

Large Amplitude Oscillations

- For large $\Delta\phi$, Equation 15 is non-linear
- Particles follow hyperbolic trajectories close to UFP.
- Can use $\mathcal{H}$ to calculate separatrix



Separatrix

- First, using Equation 17, identify the UFP, $(\phi^\star, 0)$, as

$$\left.\frac{dU}{d\phi}\right|_{\phi^\star} = 0 \qquad \phi^\star = \pi - \phi_s$$

- Calculate Hamiltonian of separatrix from Equation 16:

$$\mathcal{H}_{sep} = \frac{qV_1}{2\pi}\left(2\cos\phi_s - (\pi - 2\phi_s)\sin\phi_s\right)$$

- Put back into Equation 16 to obtain the separatrix equation

Separatrix Equation

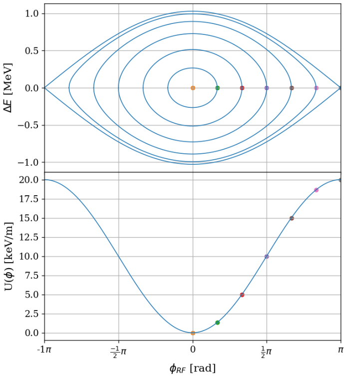
$$\Delta E_{sep} = \sqrt{\frac{qV_1\beta^2 E_s}{\pi h \eta}\left(\cos\phi_s + \cos\phi + (\phi - \pi + \phi_s)\sin\phi_s\right)}$$

Synchrotron Oscillations

Particles within the stable region of the LPS oscillate around ϕ_s. These are called **synchrotron oscillations**.

- Amplitude of oscillation depends upon initial coordinate of a particle
- Higher amplitude oscillations are typically lower frequency
- Synchrotron frequency is typically much lower than betatron frequency

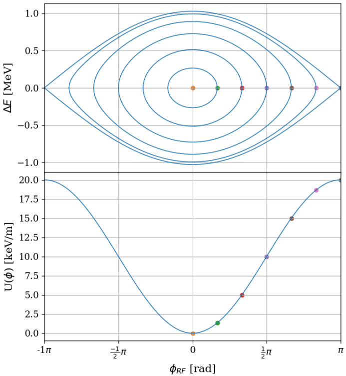
$\Rightarrow$ Force imparted by RF $\ll$ quadrupole magnetic field



Synchrotron Motion

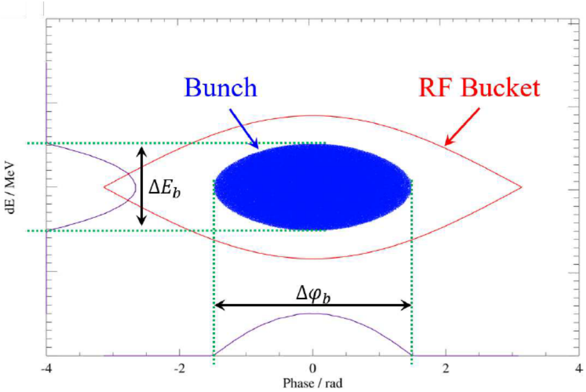
The motion in the LPS is called **synchrtron motion**.

- Particles follow contours of constant $\mathcal{H}$
- Particles orbit the SFP in an anti-clockwise fashion
- Energy is exchanged between phase-dependent potential and kinetic energy analogue $\propto W^2$



RF Bucket

- The region bounded by the separatrix is also known as the RF bucket
- The area of LPS occupied by the bunch is the **longitudinal emittance**, ε_ϕ
- The longitudinal emittance is often much smaller than the bucket area
- $\varepsilon_\phi = 4\pi\sigma_\phi\sigma_{\Delta E}$
- N.B. Other references may use different definitions of the emittance



Energy Acceptance

The height of the bucket is also known as the **energy acceptance**:

Energy Acceptance

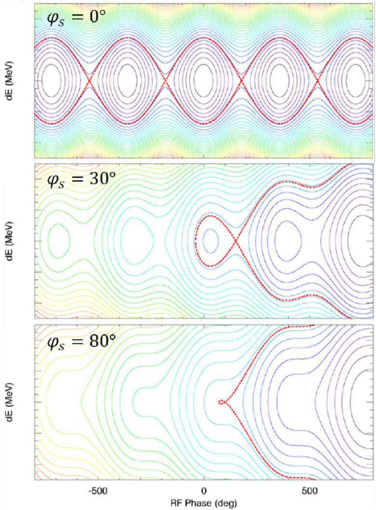
$$\left(\frac{\Delta E}{E_s}\right)_{max} = \sqrt{\frac{qV\beta^2}{\phi\eta E_s}\left(2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right)}$$

- Depends strongly upon ϕ_s
- Bucket height decreases during acceleration due to changes in ϕ_s
- As $V \uparrow$, $\Delta E_{max} \uparrow$
- Higher harmonics result in a smaller energy acceptance

Accelerating Bucket

- Bucket area dramatically reduced with increasing ϕ_s

$\Rightarrow$ Need considerably more gap volts than the minimum to practically accelerate a bunch



Summary

- Methods of acceleration
- Synchrotrons
- Momentum compaction and dispersion (more on this later!)
- Phase stability
- Equations of Motion
- Hamiltonian formalism
- RF Buckets and Emittance
- What next? To be seen in tutorials!

References

- Frank Tecker, Longitudinal Beam Dynamics in Circular Accelerators, CERN Accelerator School 2018, https://cas.web.cern.ch/schools/constanta-2018
- Edward J.N. Wilson, An Introduction to Particle Accelerators, Oxford University Press, 2001
- S.Y. Lee, Accelerator Physics, 2nd edition, World Scientific, 2007
- H. Wiedemann, Particle Accelerator Physics I, 2nd edition, Springer, 2003
- A. Chao, K.H. Mess, M. Tigner, F. Zimmerman, Handbook of Accelerator Physics and Engineering, 2nd edition, World Scientific, 2013
- Mario Conte, William W MacKay, An Introduction to the Physics of Particle Accelerators, 2nd edition, World Scientific, 2008
- Klaus Wille, The Physics of Particle Accelerators: An Introduction, Oxford University Press, 2005
- D.A. Edwards, M.J. Syphers, An Introduction to the Physics of High Energy Accelerators, Wiley-VCH, 2004
- Philip J. Bryant and Kjell Johnsen, The Principles of Circular Accelerators and Storage Rings, Cambridge University Press, 2008
- Stanley Humphries, Jr., Principles of Charged Particle Acceleration, Dover Publications, 2012