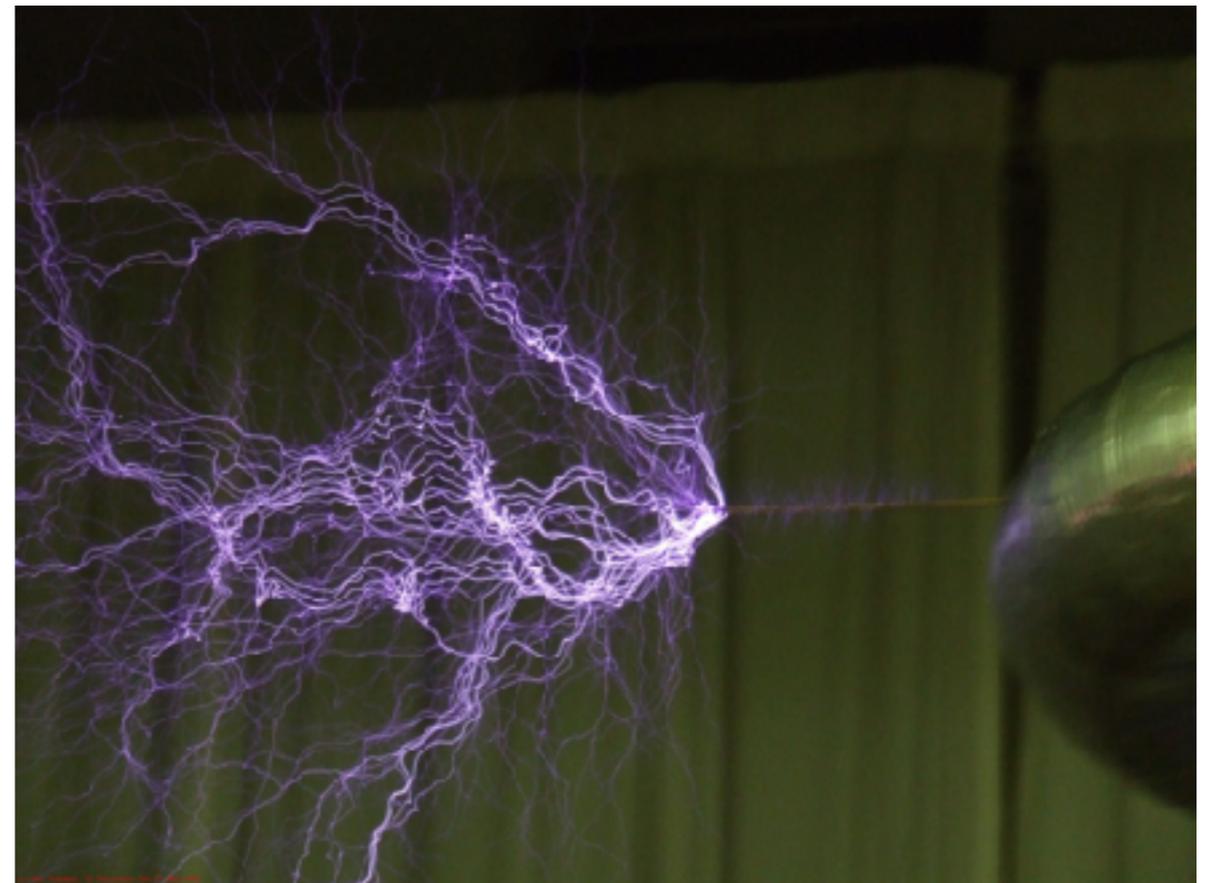# Basic Plasma Physics for Plasma based accelerators

Stuart Mangles

# Why use plasmas in particle accelerators?

- Why are particle accelerators so big (and expensive)?

    - maximum accelerating field in RF accelerators ~100 MV/m

    - higher electric fields produce breakdown

- Use a plasma instead as breakdown already happened

    - plasmas support wave modes suitable for accelerating particles

# What is in this lecture?

- This lecture introduces some key basic plasma physics concepts needed to understand plasma based particle accelerators

  - plasma frequency

  - EM waves in a plasma

    - critical density

    - ponderomotive force

  - driving plasma waves

# What is a plasma?

- The "4th state" of matter

  - NB no phase change... no latent heat of plasma formation...

- *"a quasi-neutral ionized gas that exhibits collective behaviour"*

  - interactions between plasma particles are dominated by long-range (small angle) EM forces

  - neutrality can be broken on short length and time scales... leading to large electric fields

  - Collective effects: currents, magnetic fields, waves and instabilities

## e.g. aluminium

| T[K]$< 958$ | $958 <$ T[K]$< 2765$ | T[K]$> 23000$ |
|:---:|:---:|:---:|
| solid | gas | plasma |
| *melts* | *vaporises* | *ionizes* |

$$-en_e + Zen_i \approx 0$$

# The plasma frequency

- What happens if we displace all the electrons from a region of plasma?

  - An electric field is set up that acts to return plasma to quasi-neutrality

  - electrons overshoot original position and we get *electron plasma oscillations*
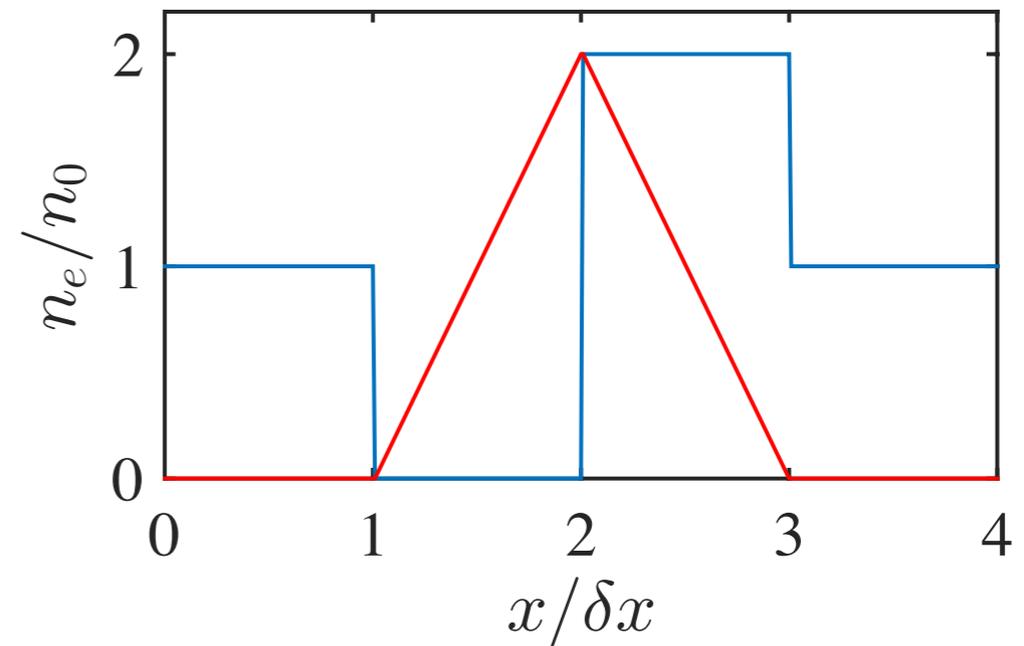
    - oscillations at the plasma frequency $\omega_p$

equilibrium electron density $n_0$

ions    electrons

restoring force
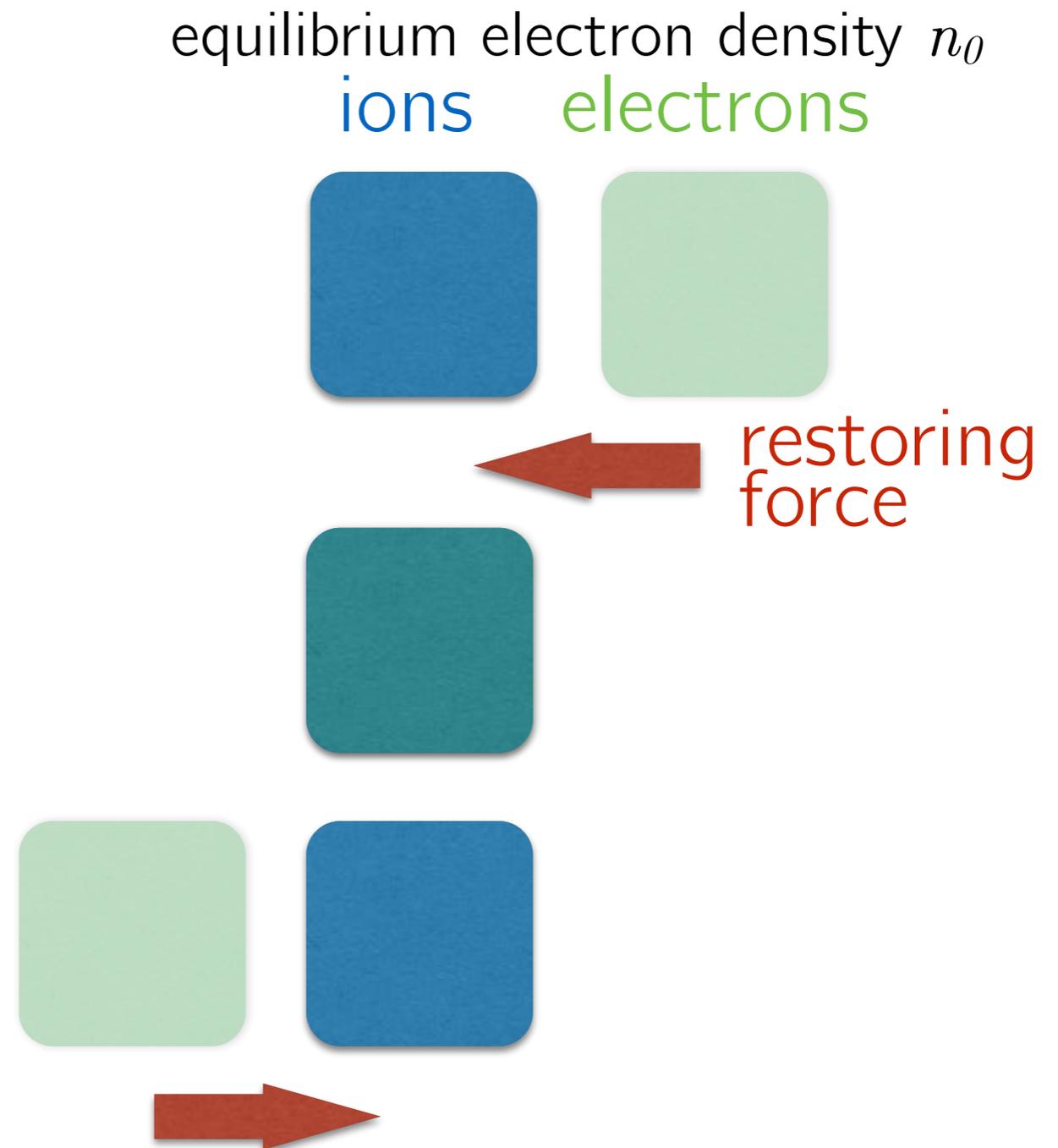
# The plasma frequency

- Displace all electron in small volume by distance $\delta x$

- Find electric field from Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\frac{\mathrm{d}E_x}{\mathrm{d}x} = \frac{Zen_i - en_e}{\epsilon_0}$$

$$E_x = \int_0^{\delta x} \frac{en_0}{\epsilon_0} \mathrm{d}x$$

$$\approx \frac{en_0 \delta x}{\epsilon_0}$$

# The plasma frequency

- Use this field in equation of motion for an electron

$$m_e \frac{\mathrm{d}^2 \delta x}{\mathrm{d}t^2} = -eE$$

$$= -\frac{e^2 n_0 \delta x}{\epsilon_0}$$

$$\frac{\mathrm{d}^2 \delta x}{\mathrm{d}t^2} + \left( \frac{e^2 n_0}{m_e \epsilon_0} \right) \delta x = 0$$

$$\omega_p = \left( \frac{e^2 n_0}{m_e \epsilon_0} \right)^{\frac{1}{2}}$$

# The plasma frequency

- Conventional RF accelerator operates at GHz frequencies:

$$f_p(n_e[\text{cm}^{-3}]) = \omega_p/(2\pi) \approx 9000\sqrt{n_e} \text{ Hz}$$

$$f_p(n_e = 1 \times 10^{18}\text{cm}^{-3}) = 9 \times 10^{13} \text{ Hz}$$



- Wavelength of conventional RF structure is

$$\lambda_{\text{RF}} = \frac{c}{f}$$

$$\approx 0.25 \text{ m}$$

- Wavelength of plasma accelerator structure will be:

$$\lambda_p = \frac{c}{f_p}$$

$$\lambda_p(n_e = 10^{18} \text{ cm}^{-3}) \approx 33 \ \mu\text{m}$$

# Plasma's support very strong electric fields

- Consider Electric field strength associated with a plasma density perturbation with wavenumber $k_p = \omega_p/c$

- For large amplitude plasma waves we get very large electric fields

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$k_p E_x = -e \frac{\delta n_e}{\epsilon_0}$$

$$E_x = \frac{\delta n_e}{\epsilon_0} \frac{c}{\omega_p} \times \frac{n_e}{n_e}$$

$$= \frac{\delta n_e}{n_e} \frac{m_e c \omega_p}{e}$$

$$E_x \left( n_e = 10^{18} \text{ cm}^{-3} \right) \approx 10^{11} \text{ Vm}^{-1} = 100 \text{ GVm}^{-1}$$

# Electro magnetic waves in a plasma

- One way to excite a plasma wave with a phase velocity close to $c$ is to use an intense laser pulse — **laser wakefield acceleration**

- Need to know how lasers propagate inside plasmas: the plasma dispersion relation from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{E} = E_x e^{i(kz-\omega t)}\hat{\mathbf{x}}$$

# Electro magnetic waves in a plasma

- Take time derivative of Ampere's law

- Insert Faraday's law and use vector triple product

- Look for transverse plane waves of form:

$$\mathbf{E} = E_x e^{i(kz-\omega t)}\hat{\mathbf{x}}$$

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$-\nabla \times (\nabla \times \mathbf{E}) = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$-k^2 E - \underbrace{\nabla(\nabla \cdot \mathbf{E})}_{0} = \mu_0 \boxed{\frac{\partial j}{\partial t}} - \frac{\omega^2}{c^2}E$$

the plasma response

# Electromagnetic waves in a plasma

- current density is:

$$j_x = -n_e e v_x$$

$$\frac{\partial j_x}{\partial t} = -n_e e \frac{\partial v_x}{\partial t}$$

- Acceleration of electron in x is due to wave's electric field

$$m_e \frac{\partial v_x}{\partial t} = -e E_x$$

$$\frac{\partial j_x}{\partial t} = \frac{n_e e^2}{m_e} E_x$$
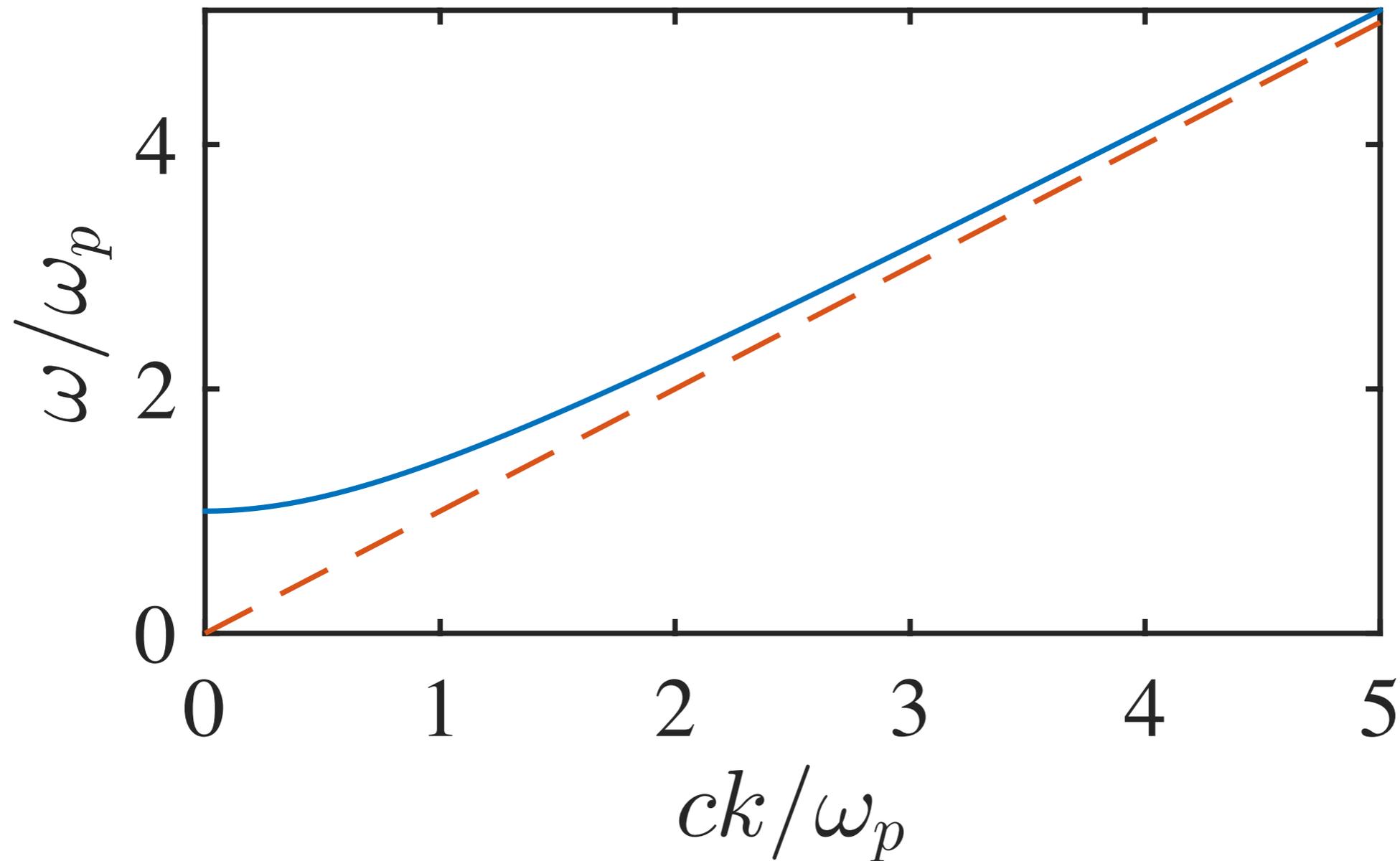
- So our wave equation becomes:

$$-k^2 E_x = \mu_0 \frac{n_e e^2}{m_e} E_x - \frac{\omega^2}{c^2} E_x$$
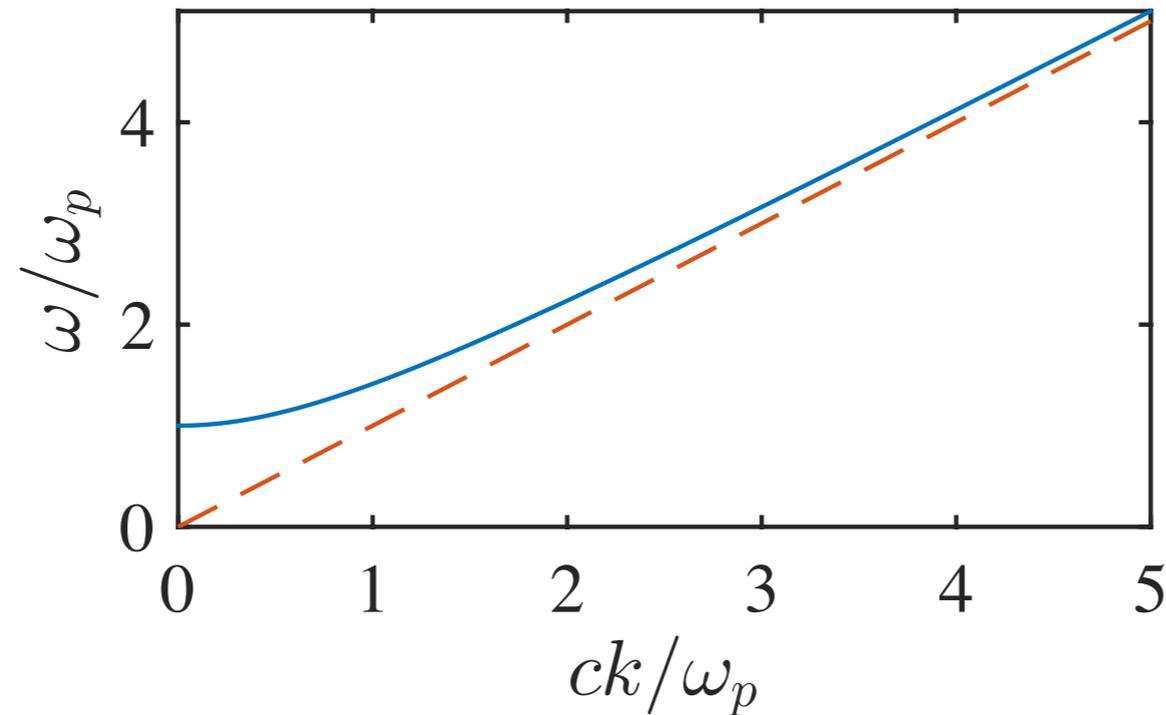
- Dispersion relation for EM waves in plasma:

$$\omega^2 = \omega_p^2 + c^2 k^2$$

# Electromagnetic waves in a plasma

$$\omega^2 = \omega_p^2 + c^2 k^2$$

# Group and Phase velocity of EM waves in plasma



$$\omega^2 = \omega_p^2 + c^2 k^2$$

$$\omega \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{\frac{1}{2}} = ck$$

$$v_p = \frac{\omega}{k} = c \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{-\frac{1}{2}}$$

$$\frac{v_p}{c} \approx 1 + \frac{1}{2} \frac{\omega_p^2}{\omega^2}$$

$$\omega^2 = \omega_p^2 + c^2 k^2$$

$$2\omega \frac{\mathrm{d}\omega}{\mathrm{d}k} = 2c^2 k$$

$$v_g = c^2 \frac{k}{\omega} = \frac{c^2}{v_p}$$

$$v_g = c \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{\frac{1}{2}}$$

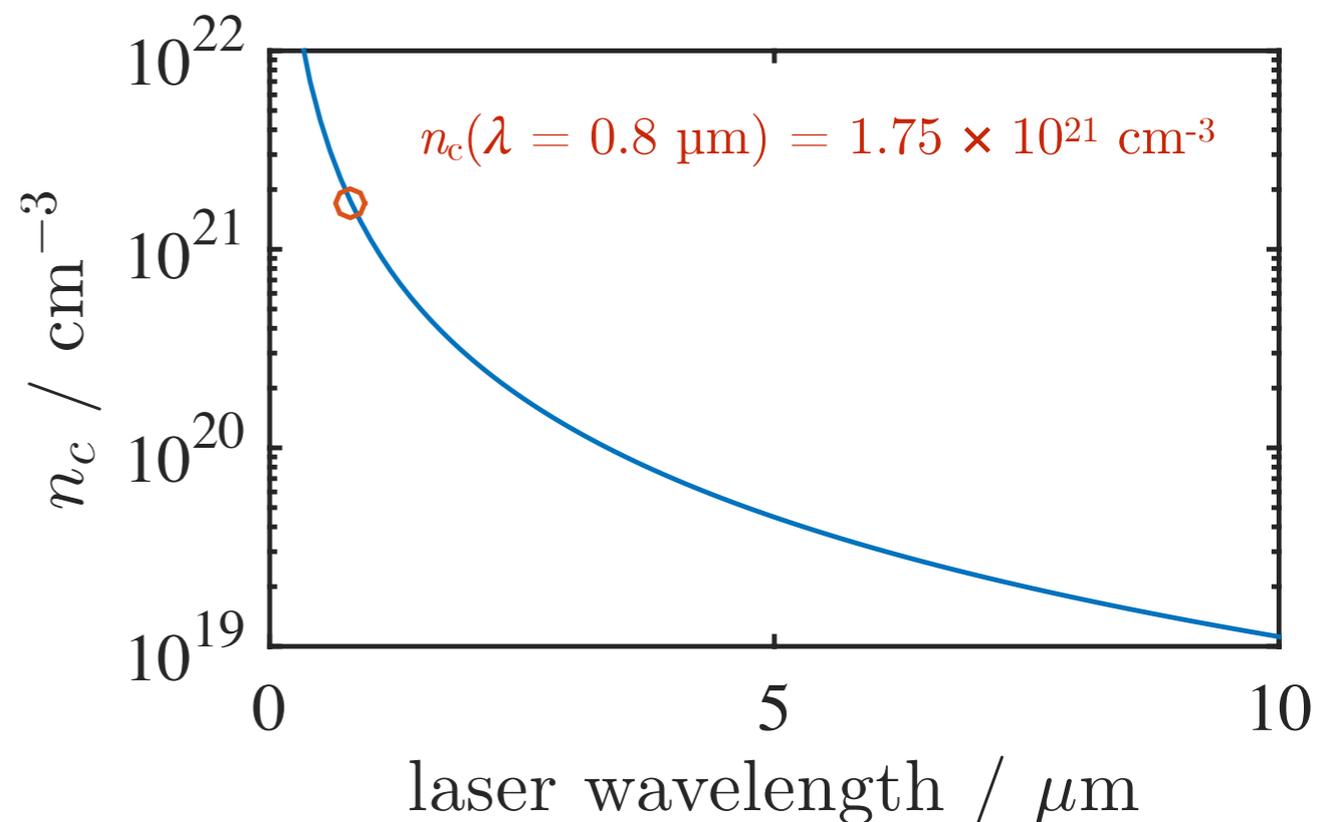$$\frac{v_g}{c} \approx 1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2}$$

- phase velocity $v_p > c$ (!)

- group velocity $v_g < c$

# Critical Density

- Dispersion relation tells us

  - if $\omega = \omega_p$; $v_g = 0$; i.e. EM wave cannot propagate

  - Using definition of $\omega_p$ we can define the critical density, $n_c$

  - Plasmas with density > critical are opaque
    - *laser light reflects: ion acceleration techniques*

  - Plasmas with density < critical are transparent
    - *laser light travels through plasma: wakefield acceleration*

$$\omega^2 = \frac{n_c e^2}{m_e \epsilon_0}$$

$$n_c = \frac{m_e \epsilon_0 \omega^2}{e^2}$$



$n_c(\lambda = 0.8 \text{ µm}) = 1.75 \times 10^{21} \text{ cm}^{-3}$

# Laser strength parameter

$$E_x = E_0 \cos(kz - \omega t)$$

- consider motion of a single electron in a plane EM wave

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$A = \int E \, dt$$

- Ignore the B field (v << c)

$$A = \frac{E_0}{\omega} \sin(kz - \omega t) = A_0 \sin(kz - \omega t)$$

- electron motion is a transverse oscillation (called a "quiver")

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -eE_x$$

$$p_x = -e \int E_x \mathrm{d}t$$

- maximum normalised quiver momentum of electron is given by laser strength parameter, $a_0$

$$= -\frac{eE_0}{\omega} \sin(kz - \omega t)$$

$$\frac{p_x}{m_e c} = -\frac{eE_0}{m_e \omega c} \sin(kz - \omega t)$$
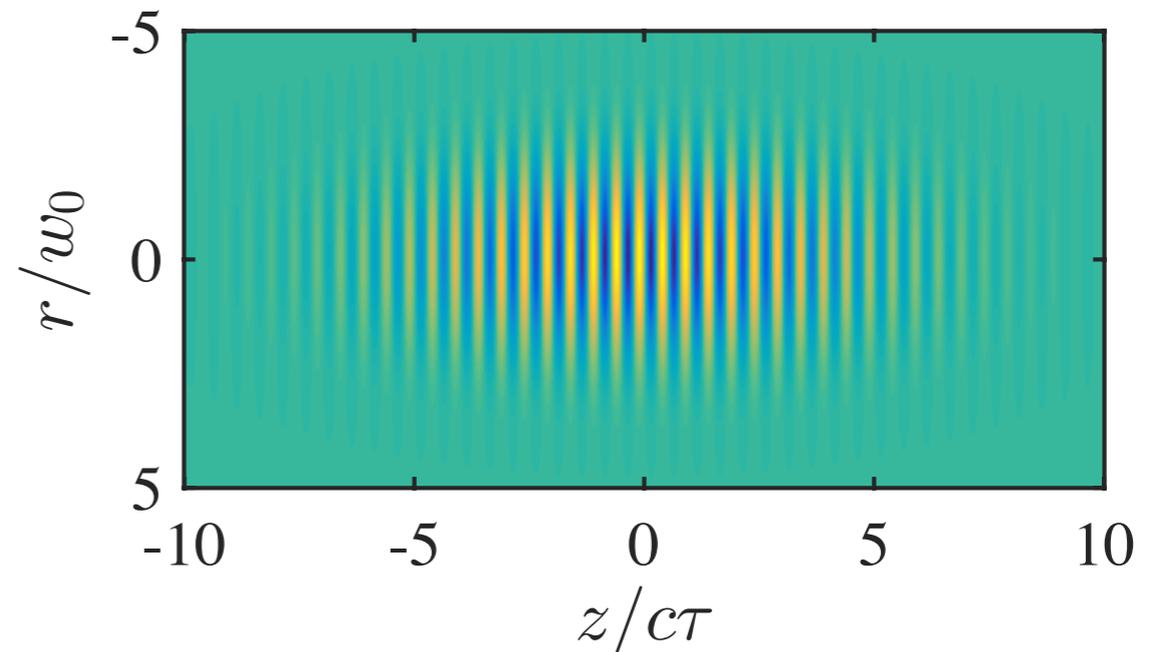
$$= -\frac{eA_0}{m_e c} \sin(kz - \omega t)$$

$$a_0 = \frac{eA_0}{m_e c} = \frac{eE_0}{m_e \omega c} \propto \sqrt{I\lambda^2}$$

$$= a_0 \sin(kz - \omega t)$$

# Ponderomotive force

$$a(r,z) = a_0 0 \exp\left[-\frac{r^2}{w_0^2}\right] \exp\left[-\frac{(z - v_g t)^2}{c^2 \tau^2}\right] \sin(kz - \omega t)$$

$$= a_0(r,z) \ \sin(kz - \omega t)$$

- "real" lasers have a spatially varying intensity profile

- consider average kinetic energy of an electron at r,z

  - We can think of this position dependent energy as a potential energy

  - Gradient in this potential energy is a force — *the ponderomotive force*

  - Ponderomotive force pushes charged particles away from regions of high laser intensity



$$\langle K \rangle = \frac{p_x^2}{2 m_e}$$

$$= \frac{1}{2} m_e c^2 a_0^2$$

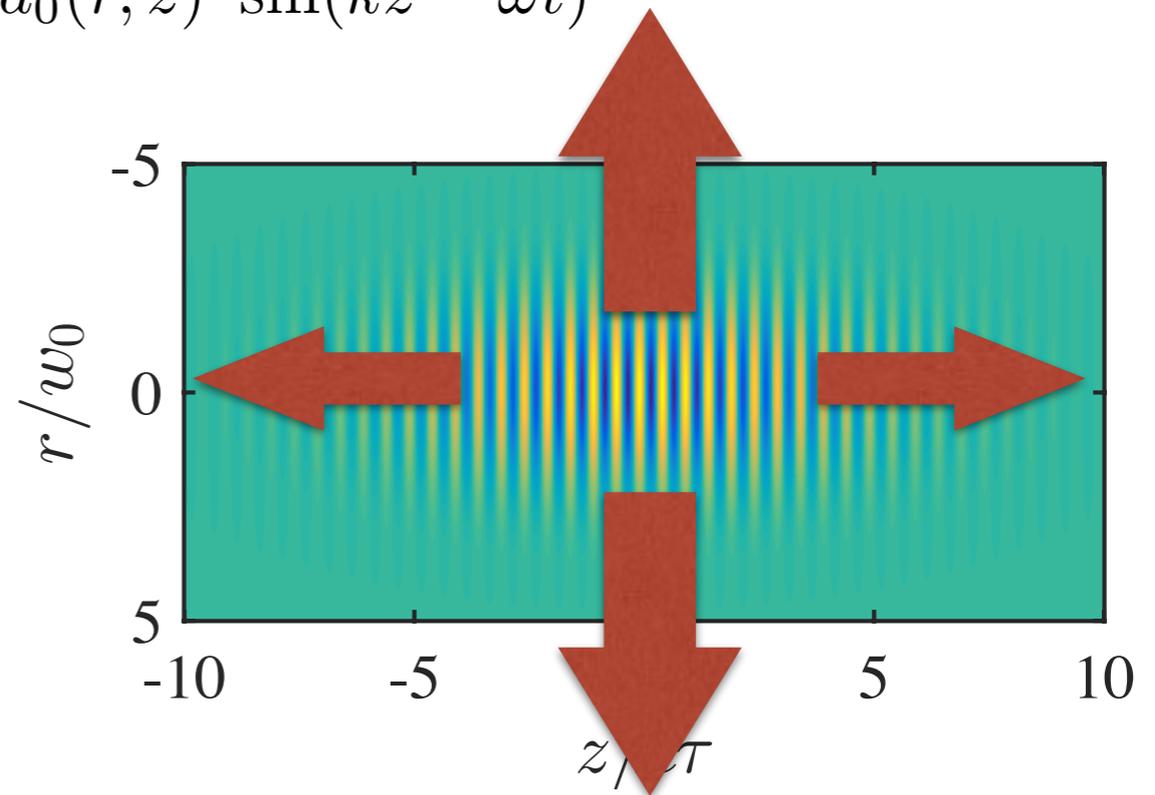$$\mathbf{F}_p = -\frac{1}{2} m_e c^2 \nabla a_0^2$$

**this is a non-relativistic "derivation" but full result is very similar!

# Ponderomotive force

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**this is a non-relativistic "derivation" but full result is very similar!

# Driving Plasma Waves with lasers or particle beams

$$m_e \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \, \mathbf{v} \right) = -\frac{1}{2} m_e c^2 \nabla a_{\text{laser}}^2 - e\mathbf{E}_{\text{beam}} - e\mathbf{E}_{\text{plasma}}$$

- Treat plasma as a fluid (use momentum equation and continuity equation)

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}) = 0$$

- Consider two drive terms

  - laser (through ponderomotive force)

  - charged particle beam (through electric field)

$$E = \cancel{E_0}^{\ 0} + E_1$$
$$v = \cancel{v_0}^{\ 0} + v_1$$
$$n = n_0 + n_1$$

- Do in one dimension

- linearise equations (ignore products of small terms)

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial v_1}{\partial x} = 0$$

$$m_e \frac{\partial v_1}{\partial t} = -\frac{1}{2} m_e c^2 \nabla a_{\text{laser}}^2 - eE_{\text{beam}} - eE_1$$

# Driving Plasma Waves with lasers or particle beams

- take d/dt of continuity and d/dx of momentum equation and combine

$$\frac{\partial^2 n_1}{\partial t^2} + n_0 \frac{\partial^2 v_1}{\partial x \partial t} = 0$$

$$m_e \frac{\partial^2 v_1}{\partial t \partial x} = -\frac{1}{2} m_e c^2 \frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - e \frac{\partial E_{\text{beam}}}{\partial x} - e \frac{\partial E_1}{\partial x}$$

$$m_e \frac{\partial^2 n_1}{\partial t^2} - n_0 \frac{1}{2} m_e c^2 \frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - e n_0 \frac{\partial E_{\text{beam}}}{\partial x} - e n_0 \frac{\partial E_1}{\partial x} = 0$$

- use Gauss's law to find E field due to plasma response

$$\frac{\partial E}{\partial x} = \frac{\rho}{\epsilon_0} = \frac{Z e n_i - e n_e}{\epsilon_0}$$
$$= \frac{Z e n_i - e(n_0 + n_1)}{\epsilon_0}$$
$$= -\frac{e n_1}{\epsilon_0}$$

$$\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \frac{n_1}{n_0} = -\frac{c^2}{2} \frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - \omega_p^2 \frac{n_{\text{beam}}}{n_0}$$

# Driving Plasma Waves with lasers or particle beams

$$\left( \frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \frac{n_1}{n_0} = -\frac{c^2}{2} \frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - \omega_p^2 \frac{n_{\text{beam}}}{n_0}$$

- Solution for this linear non-evolving driver moving at speed v is:
  - sinusoidal density pertubation
  - sinusoidal potential
  - sinusoidal electric field
  - valid if $n_1 << n_0$
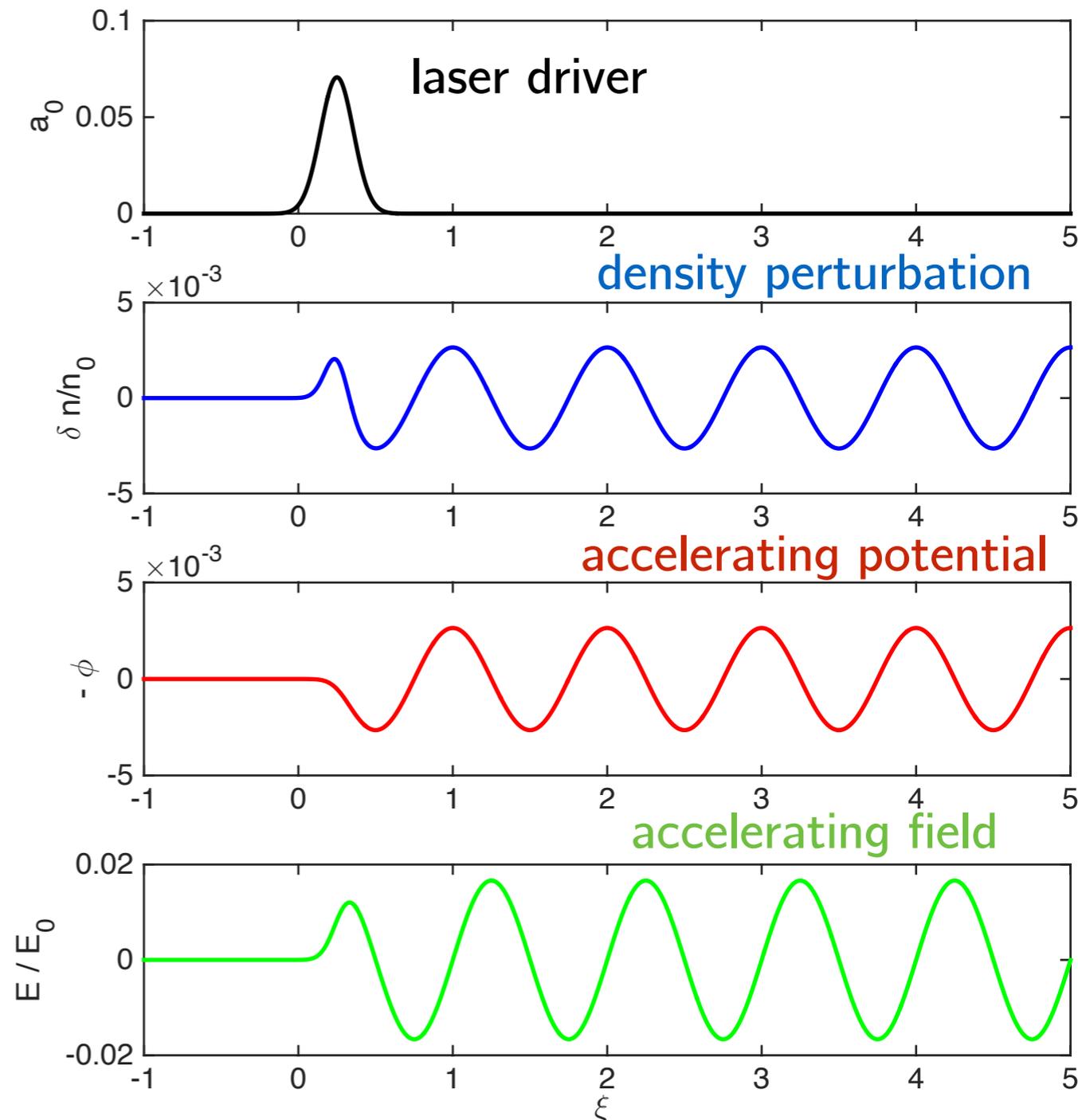
# Driving Plasma Waves with lasers or particle beams

- A proper non-linear treatment also possible:

  - density spikes

  - quadratic potential

  - linear electric field

  - lengthening of wavelength (a relativistic shift of $\omega_p$)

  - occurs when $n_1 \sim n_0$
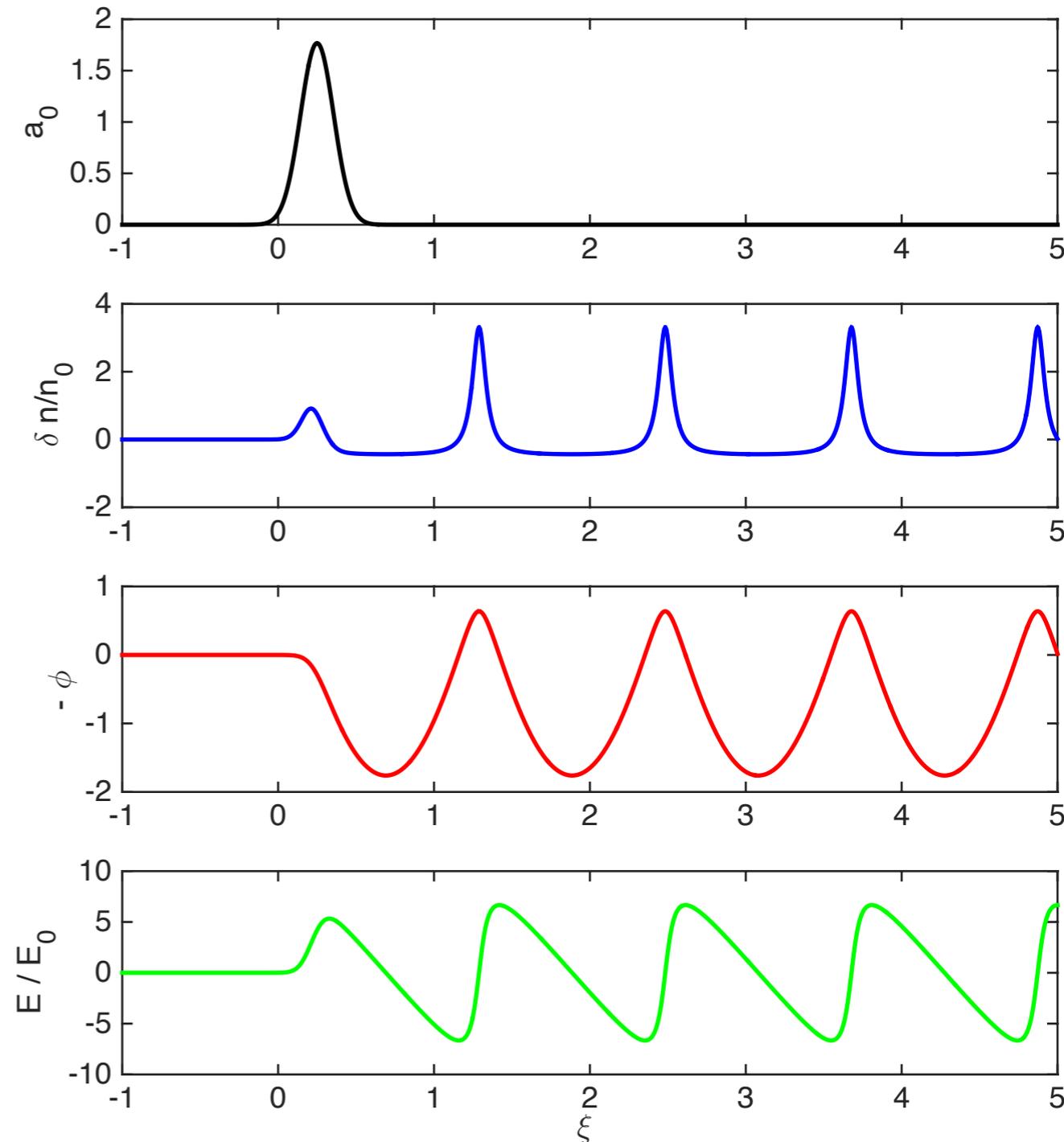
# Summary - and what's next?

Introduced some key concepts you will encounter in plasma based accelerators:

- plasma oscillations (the plasma wave in a plasma accelerator)

- ponderomotive force (what drives the plasma wave)

- driving plasma waves

Next time with me we will look at:

- Key limits to plasma based accelerators

- non-linear 3D plasma waves: the blow out or bubble regime

- Radiation generation in plasma accelerators