Transverse Dynamics Lectures

JAI lectures - Michaelmas Term 2024

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Outline

Goals of this course

- Introduction to one of the core topics in accelerator physics.
- Explain the basics of the formalism.
- Give an idea of the related phenomenology.
- Full derivations are not included in main lectures.
- Most important thing: learn something and enjoy!

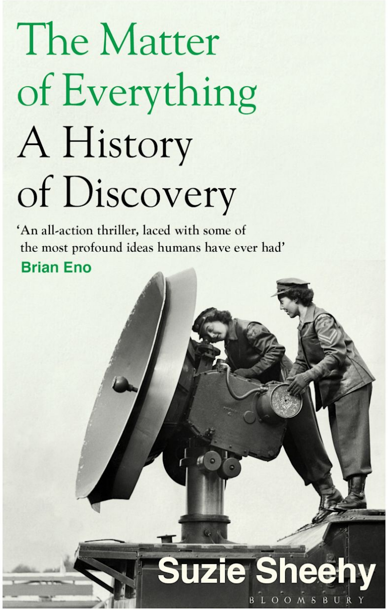
Some references

Books

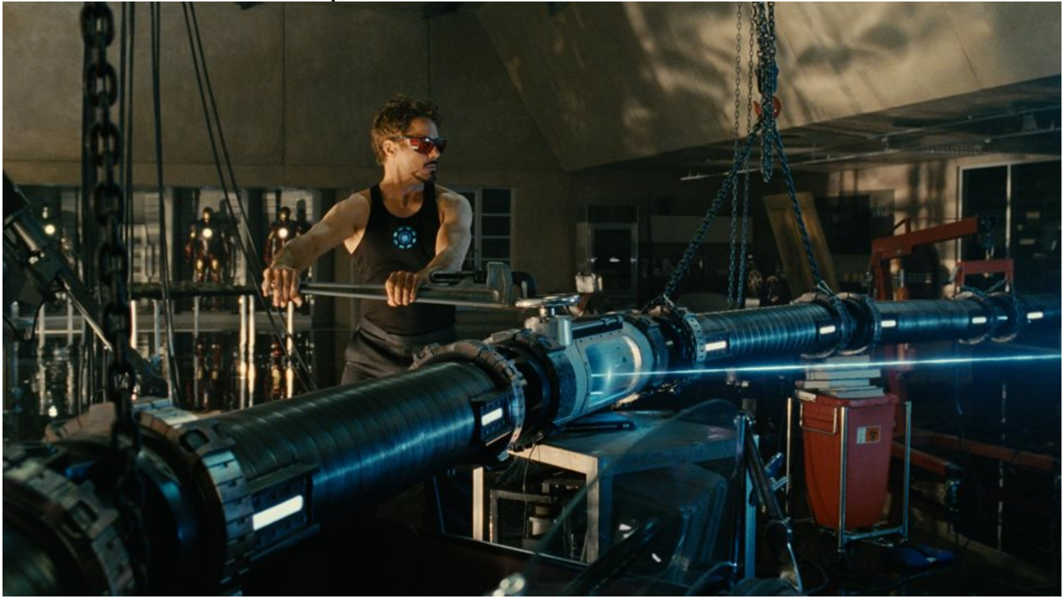
- Wilson, Introduction to Particle Accelerators.
- Lee, Accelerator Physics.
- Wiedemann, Particle Accelerator Physics.
- A, Wolski, Beam Dynamics in High Energy Particle Accelerators.
- E. Forest, Beam Dynamics: A new attitude framework.
- A. Chao, Handbook of Accelerator Physics and Engineering.

Lectures

- A. Latina, JUAS Lectures on Transverse Dynamics (2020).
- H. Garcia, JUAS Lectures on Transverse Dynamics (2021).
- CAS lectures.
- USPAS lectures.


The Matter
of Everything
A History
of Discovery
'An all-action thriller, laced with some of
the most profound ideas humans have ever had'
Brian Eno
Suzie Sheehy
BLOOMSBURY

I did not know how complex an accelerator was...



Why these lectures?

What do we want to study?

High energy particles traveling through intense magnetic fields (usually periodic).

Why transverse dynamics?

- It covers 2/3 of the phase space (4 out of 6 dimensions).
- Magnets act primarily on the transverse plane.
- Main accelerator parameters are determined (at first order) by transverse properties:
  - Luminosity, emittance, brilliance, beam losses, instabilities, tune...

Special relativity recap.

We need to study the motion of charged particles at (very) high energy.

$$E = \sqrt{p^2c^2 + (mc^2)^2} \tag{1}$$

where m is the mass of the particle and p the particle momentum.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2}$$

Ultra-relativistic approximation $\gamma \gg 1$:

$$E = pc \tag{3}$$

What is faster?

1. An electron/positron at LEP ($E = 100$ GeV).
2. A proton in the LHC ($E = 7000$ GeV).

Lorentz Force

The force experienced by a charge q and speed $\mathbf{v}$ under the influence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is given by the Lorentz equation:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4}$$

- Electric field $\mathbf{E}$ for increasing (decreasing) particle speed.
- Magnetic field $\mathbf{B}$ for bending particle trajectory.

Question: Why do we use magnets for bending the trajectory of the beam?

Beam rigidity

Lorentz force:

$$F_L = qvB \tag{5}$$

Centripetal force:

$$F_c = m\frac{v^2}{\rho} \tag{6}$$

Null force condition ($\sum F = 0$)

$$F_L = F_c \Rightarrow \frac{p}{q} = B\rho \tag{7}$$

Beam rigidity:

$$B\rho \approx 3.33p[\mathrm{GeV/c}] \tag{8}$$

Applications

- Given size and magnet technology determines physics reach.
- Given magnet technology and physics goals determines required size.
- Given size and physics goal determines technology needed.

Take home exercise

Given current technology ($B_{\mathbf{max}} \sim 10$ T)

- What is the maximum energy of a particle accelerator around the Earth equator?
- and of an accelerator around the Solar System?

Harmonic oscillator is back

Restoring force:

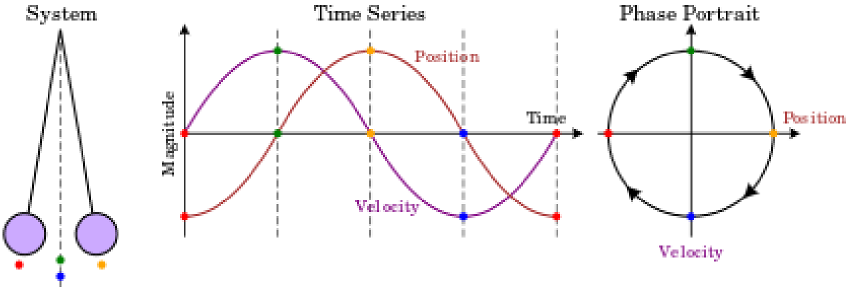
$$F = -ku \qquad (9)$$

Equation of motion:

$$u'' = -\frac{k}{m}u \qquad (10)$$

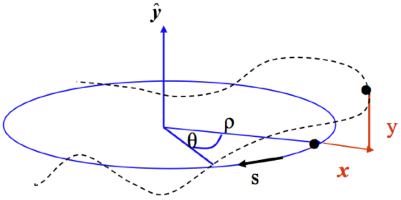
Solution:

$$u = a\cos(\omega t + \phi) \qquad (11)$$



Frenet-Serret reference system

6D phase space: $(x, x,', y, y', z, \delta)$



The coordinates are relative to the reference particle/trajectory.

Coordinate definition:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0} \qquad (12)$$

$$y' = \frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0} \qquad (13)$$

$$\delta = \frac{\Delta P}{P_0} \qquad (14)$$

Pay attention! This is not the set of canonical variables used in Hamilton's equations.

Multipolar expansion

Any magnetic field can be decomposed in:

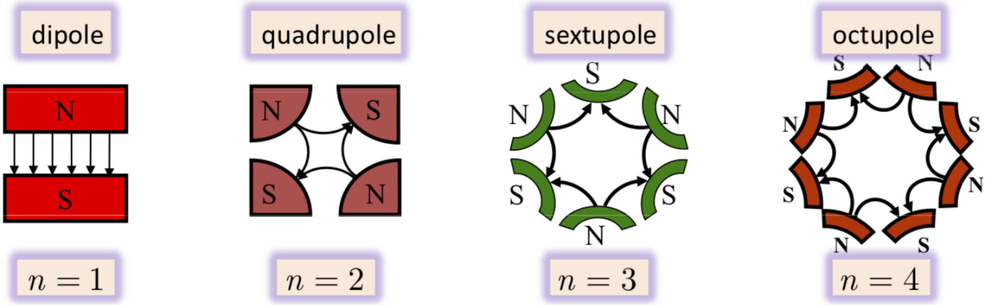
$$B_y + iB_x = \sum_{n=1}^{\infty} c_n (x + iy)^{n-1} \tag{15}$$

where

$$c_n = b_n + ia_n \tag{16}$$

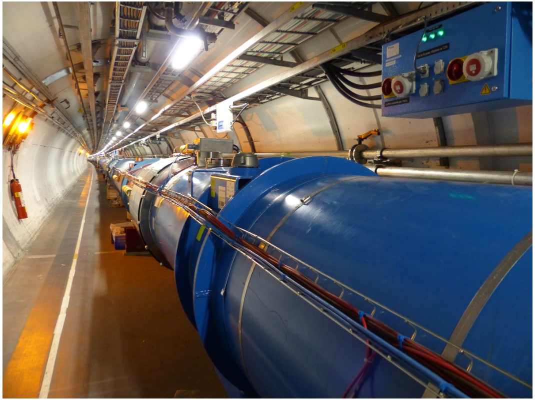
- b_n are the normal coefficients.
- a_n are the skew coefficients.

Magnet types



Magnet types: Dipoles

- Two magnetic poles.
- Bend particle trajectory.
- Provide weak focusing.
- Not required in linear colliders.



Take home exercise: LHC dipoles

The LHC contains 1232 dipole magnets.
Each is 15 m long.

- **What is the length of the full circumference?**

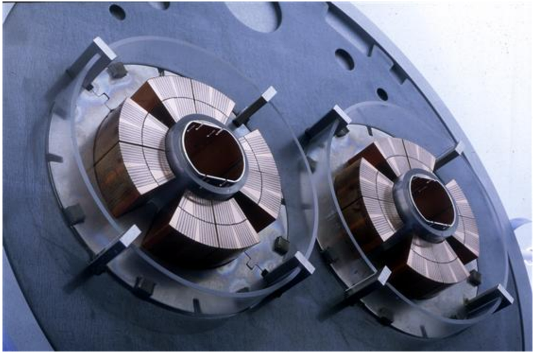
Magnet types: Quadrupoles

- Four poles.
- Focus the beam (horizontally or vertically).

Normalized focusing strength:

$$k = \frac{G}{P/q}[\mathrm{m}^{-2}] \tag{17}$$

$$k[\mathrm{m}^{-2}] \approx 0.3\frac{G[\mathrm{T/m}]}{P[\mathrm{GeV/c}]/q[e]} \tag{18}$$



Magnet types: Quadrupoles

The focal length of a quadrupole is:

$$f = \frac{1}{k \cdot L}[\mathrm{m}] \qquad (19)$$

where L is the length of the quadrupole.

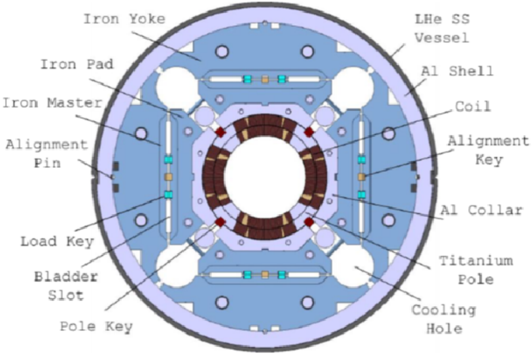
Example: Q1 LHC

$$L = 6.37\mathrm{m}$$

$$kL = -5.54 \times 10^{-2}\mathrm{m}^{-1}$$

Magnet types: Quadrupoles

- The LHC upgrade will require stronger focusing at IP1 and IP5.
- New quadrupole magnets with stronger gradients are required.
- Successful tests on short models.

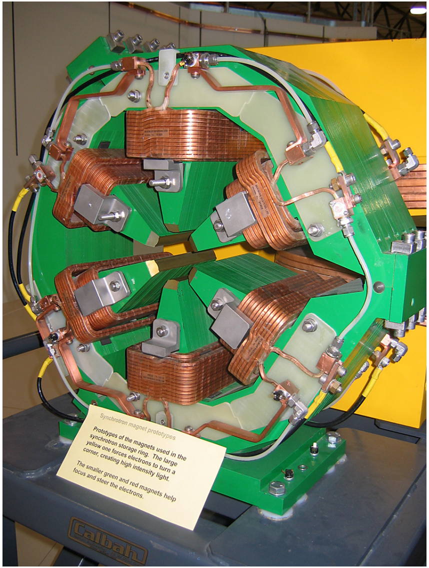


Magnet types: Sextupoles

- Six poles.
- Correct chromatic aberrations.
- Usually distributed along the arcs.
- Essential for accelerator performance.

Other multipoles

- Octupoles.
- Decapoles.
- Dodecapoles.



Hamiltonian approach

Hamiltonian of a particle with mass m, charge q and momentum p in presence of an electromagnetic field $(\phi, \mathbf{A})$:

$$H = c\sqrt{(\mathbf{p} - q\mathbf{A}) + m^2c^2} + q\phi \tag{20}$$

Hamilton equation:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{21}$$

Equation (20) will be explained in future lectures including the derivation of the dynamics.

Hill's equation

- We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position s along the ring.
- The linear motion (dipoles and quadrupoles) can be described by:

$$u'' + K(s)u = 0 \tag{22}$$

where $K(s) = \left(\frac{1}{\rho^2} + k\right)$ is composed by linear fields only (dipole and quadrupole).

Hill's equation

$$u'' + K(s)u = 0 \tag{23}$$

Some remarks

- $K(s)$ is a non-constant (s-dependent) restoring force.
- $K(s)$ is a periodic function with period $L \Rightarrow K(s+L) = K(s)$
- Usually in the vertical plane $1/\rho = 0$, therefore $K_y = k_y$.
- In a quadrupole $1/\rho = 0$ and $K_x = -K_y$ i.e. a horizontal focusing quadrupole defocuses in the vertical plane (and vice versa).
- In a bending magnet $k = 0$ so $K = 1/\rho^2$.

Hill's equation: general solution

For $K(s) = K(s+L)$:

$$u = \sqrt{2J_u\beta_u(s)}\sin(\phi_u(s) - \phi_{u0}) \tag{24}$$

$$u' = -\frac{\sqrt{2J_u}}{\beta_u(s)}[\cos(\phi_u(s) - \phi_{u0} + \sin(\phi_u(s) - \phi_{u0})] \tag{25}$$

where $u = x, y$.

Integration constants

- Action: J is a constant (related to emittance).
- Phase constant: ϕ_0.

- Beta-function: $\beta(s)$, periodic function:

$$\beta(s+L) = \beta(s) \tag{26}$$

- Phase advance: $\phi(s_0|s) = \int_{s_0}^{s} \frac{ds'}{\beta(s')}$

Weak focusing and cyclotrons

In cyclotrons, only dipole magnets are used.
But still there is some focusing effect.

$$u'' + \left(\frac{1}{\rho^2} + k\right) u = 0 \xrightarrow[k=0]{} u'' + \frac{1}{\rho^2} u = 0 \tag{27}$$

- Small and low energy accelerators.
- Example: mass spectrometer.

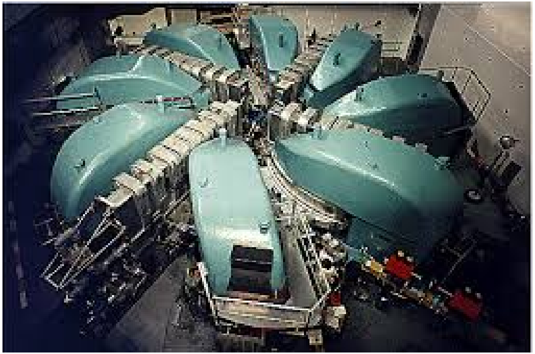


Figure: PSI cyclotron (250 MeV protons)

Strong focusing ($K > 0$)

Initial conditions: $x = x_0, x' = x'_0$ Solution:

$$x(s) = x_0 \cos(\sqrt{K}s) + \frac{x'_0}{\sqrt{K}} \sin(\sqrt{K}s) \tag{28}$$

$$x'(s) = -x_0\sqrt{K} \sin(\sqrt{K}s) + x'_0 \cos(\sqrt{K}s) \tag{29}$$

Matrix formalism for a focusing quadrupole of length L:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) \\ -\sqrt{K} \sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \tag{30}$$

Strong focusing ($K < 0$)

Initial conditions: $x = x_0, x' = x'_0$ Solution:

$$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x'_0}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \tag{31}$$

$$x'(s) = -x_0\sqrt{|K|} \sinh(\sqrt{|K|}s) + x'_0 \cosh(\sqrt{|K|}s) \tag{32}$$

Matrix formalism for a defocusing quadrupole of length L:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}L) \\ -\sqrt{|K|} \sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \tag{33}$$

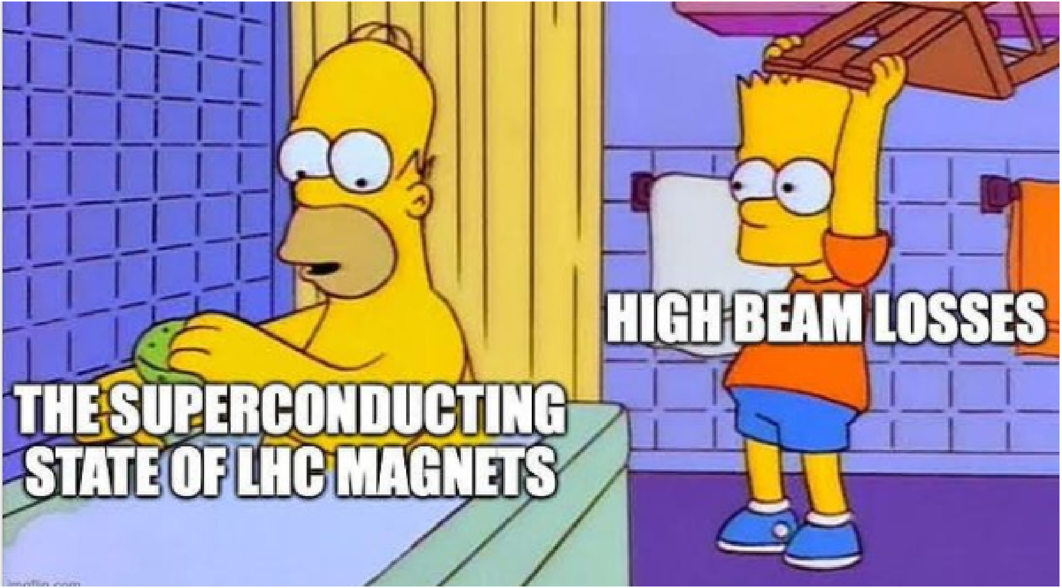
Recap.

- Special relativity and magnetic properties.
- Reference system and Hill's equation (without deviation).
- Solution of linear homogeneous Hill's equations.
- Weak and strong focusing.
- Matrix formulation for dipoles and quadrupoles.

Next episode

- Generalization of matrix formalism.
- Twiss parameters in detail.
- Phase space.
- Example: FODO.
- Dispersion and chromaticity.

End of the section meme



Part II

General matrix formalism

The transformation between $x(s_0)$ and $x(s)$ can be expressed in a general way:

$$x(s) = M(s|s_0)x(s_0) \tag{34}$$

where the application $M(s|s_0)$ can be expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \tag{35}$$

where C and S are the cosine-like and sine-like functions and their derivatives C' and S' with respect to s.

Element concatenation

The transfer matrices for different elements of the lattice can be concatenated to find the full transfer matrix between two locations s_0 and s,

$$x(s_n) = M_n(s_n|s_{n-1}) \ldots M_2(s_2|s_1)M_1(s_1|s_0)x_0 \tag{36}$$

Remember to multiply matrices in reverse order!

Lattice design lectures

We will se more about how lattices are designed in practice in MADX.

Thin lens approximation

When the focal length f of a quadrupole is much larger than the magnet itself L_q the transfer matrices can be rewritten as,

$$M_{\text{foc}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \tag{37}$$

$$M_{\text{def}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \tag{38}$$

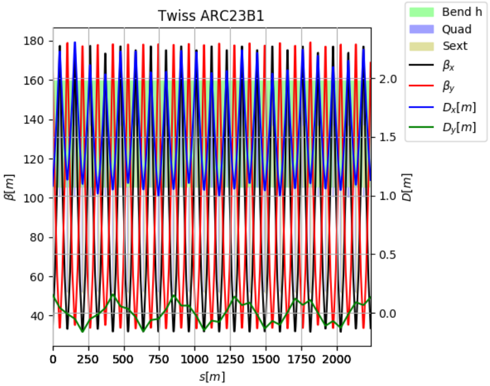
Take home exercise

Derive the limits for the thin lens approximation and find the new matrices for quadrupoles in thin lens approximation.

Twiss parameters

$$u(s) = \sqrt{2J_u\beta_u(s)}\sin(\phi_u(s) - \phi_{u0}) \tag{39}$$

$\beta_u(s)$ is a periodic function given by the periodic properties of the lattice.



$$\phi(s|s_0) = \int_{s_0}^{s} \frac{ds}{\beta(s')} \tag{40}$$

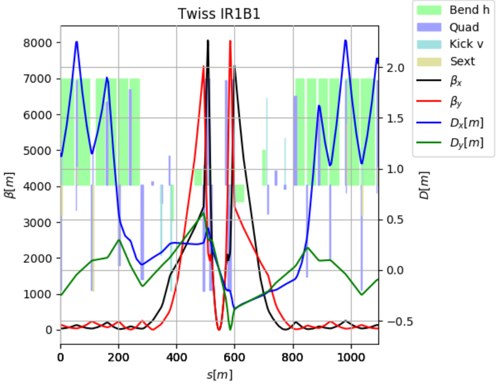
$$\alpha_u(s) = -\frac{1}{2}\frac{d\beta_u}{ds} \tag{41}$$

$$\gamma_u(s) = \frac{1+\alpha_u^2(s)}{\beta_u(s)} \tag{42}$$

Twiss parameters

$$u(s) = \sqrt{2J_u\beta_u(s)}\sin(\phi_u(s) - \phi_{u0}) \tag{43}$$

$\beta_u(s)$ is a periodic function given by the periodic properties of the lattice.



$$\phi(s|s_0) = \int_{s_0}^{s} \frac{ds}{\beta(s')} \tag{44}$$

$$\alpha_u(s) = -\frac{1}{2}\frac{d\beta_u}{ds} \tag{45}$$

$$\gamma_u(s) = \frac{1 + \alpha_u^2(s)}{\beta_u(s)} \tag{46}$$

Transfer matrix in terms of Twiss parameters

Aim: express M in terms of the initial and final Twiss parameters (instead of magnetic properties).

Taking $s(0) = s_0$ and $\phi(0) = \phi_0$ we can obtain,

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\phi_s + \alpha_0 \sin\phi_0) & \sqrt{\beta_s\beta_0}\sin\phi_s \\ \frac{(\alpha_0-\alpha_s)\cos\phi_s-(1+\alpha_s\alpha_0)\sin\phi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\phi_0 - \alpha_s\sin\phi_s) \end{pmatrix} \tag{47}$$

This expression is very useful when Twiss parameters are known at two different

locations.

How do we measure β and ϕ

Phase ϕ

- Harmonic analysis of oscillations.

Betatron tune Q

- FFT of transverse beam position over many turns.

Beta function β

- β from phase.
- β from amplitude.
- K-modulation.

One matrix to rule them all

If we take matrix M and consider the case for one full turn (i.e. $\beta_s = \beta_0$ and $\alpha_s = \alpha_0$) the matrix simplifies,

$$\mathcal{M} = \begin{pmatrix} \cos\phi_L + \alpha_0 \sin\phi_L & \beta_0 \sin\phi_L \\ \gamma_0 \sin\phi_L & \cos\phi_0 - \alpha_0 \sin\phi_L \end{pmatrix} \tag{48}$$

The tune Q is the phase advance of the full ring in 2π units.

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)} = \frac{\phi_L}{2\pi} \tag{49}$$

then, the one turn matrix $\mathcal{M}$ can be rewritten,

$$\mathcal{M} = \begin{pmatrix} \cos(2\pi Q) + \alpha_0 \sin(2\pi Q) & \beta_0 \sin(2\pi Q) \\ \gamma_0 \sin(2\pi Q) & \cos(2\pi Q) - \alpha_0 \sin(2\pi Q) \end{pmatrix} \tag{50}$$

Properties of transfer matrices

1. Phase space area preservation.

$$\det(M) = 1 \tag{51}$$

2. Motion is stable over $N \to \infty$

$$|\text{trace}(M)| < 2 \tag{52}$$

Stability condition (derivation)

Let's consider the transfer matrix M for a periodic system:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{53}$$

we want the motion to be stable over $N \to \infty$ turns.

$$x_N = M^N x_0 \tag{54}$$

How can we compute M^N?

Stability condition (derivation)

$$x_N = M^N x_0 \tag{55}$$

- $\det(M) = ad - bc = 1$
- $\mathrm{tr}(M) = a + d$

If we diagonalise M, we can rewrite it as,

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot U^T \tag{56}$$

where U is some unitary matrix and λ_1 and λ_2 its eigenvalues.

Stability condition (derivation)

After N turns,

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T \tag{57}$$

Given that $\det(M) = 1$,

$$\lambda_1 \lambda_2 = 1 \rightarrow \lambda_{1,2} = e^{\pm ix} \tag{58}$$

To have stable motion, $x \in \mathbb{R}$.
To find the eigenvalues, use characteristic equation,

$$\det(M - \lambda \mathbb{I}) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \tag{59}$$

Solve it,

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

$$\lambda^2 + \text{trace}(M)\lambda + 1 = 0$$

$$\text{trace}(M) = \lambda + \frac{1}{\lambda} = e^{ix} + e^{-ix} = 2\cos x$$

Since $x \in \mathbb{R}$,

$$|\text{trace}(M)| \leq 2$$

Twiss transport matrix and Twiss parameters evolution

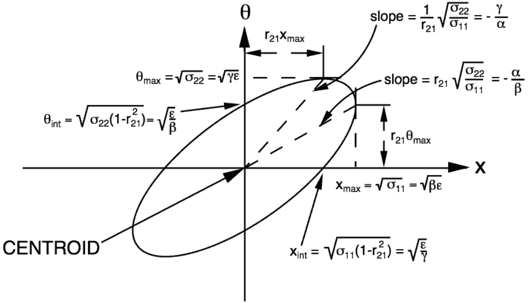
Instead of transporting the coordinates x and x' we can transport the Twiss parameters (β, α, γ),

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0 \tag{60}$$

- Given the Twiss parameters at any point in the lattice we can transform them and compute their values at any other point in the ring.
- The transfer matrix is given by the focusing properties of the lattice elements, the same matrix elements to compute single particle trajectories.

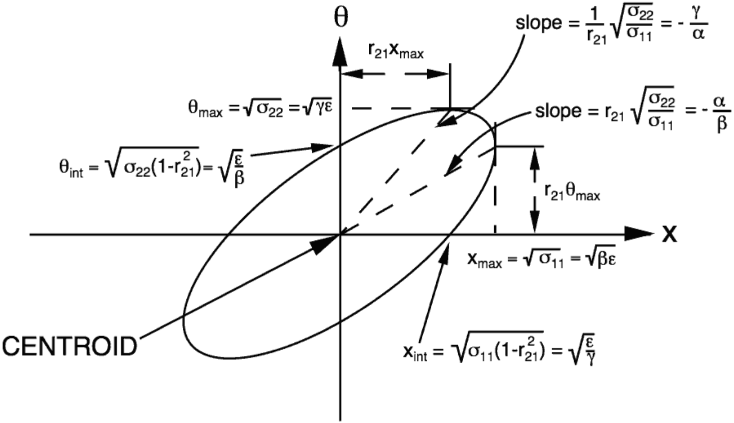
Phase space properties

- Area is preserved.
- Beam size: $\sigma_u = \sqrt{J_u \beta_u}$.
- When σ_u is large $\sigma_{u'}$ is small.
- In a β minimum/maximum $\alpha = 0$ and the ellipse is not tilted.



$$J = \gamma x^2 + 2\alpha x x' + \beta x'^2 \qquad (61)$$

Phase space properties

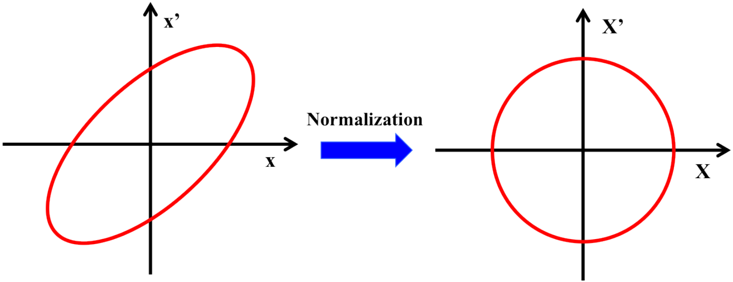


$$J = \gamma x^2 + 2\alpha x x' + \beta x'^2 \tag{62}$$

Normalized phase space

Can we use another reference frame so it is simpler to describe the system?

$$\mathcal{M} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{63}$$



For linear systems is fine but it gets much more complex when non-linearities are included (we will see more details in the tutorial).

Beam emittance: single particle definition

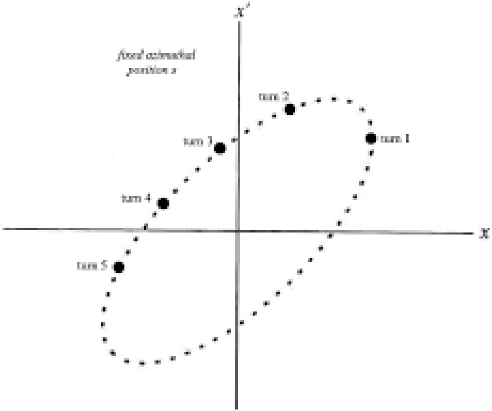
The geometric emittance is a constant of motion only if the beam energy is preserved (conservative system). This quantity is related to the action J that appeared in the solution of the Hill's equation.

Normalized emittance takes into account beam energy. It is a constant of motion even if energy is not constant:

$$\epsilon_n \equiv \beta_{\text{rel}} \gamma_{\text{rel}} \epsilon \tag{64}$$

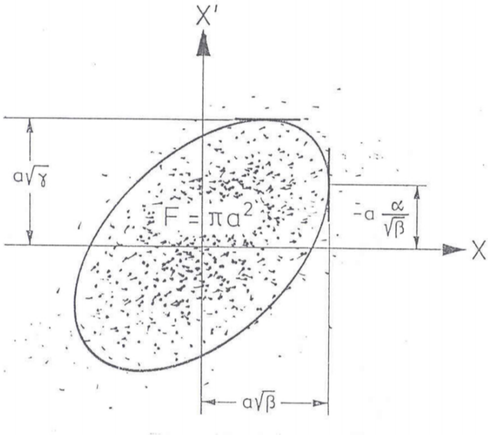
The beam size at any location of the lattice is given by,

$$\sigma = \sqrt{\epsilon \beta} \tag{65}$$



Beam emittance: statistical definition

The beam is composed of particles distributed in phase space.



Statistical emittance is defined by,

$$\epsilon_{\text{rms}} = \sqrt{\sigma_u^2 \sigma_{u'}^2 + \sigma_{uu'}^2} \tag{66}$$

The rms emittance of a ring in phase space, i.e. particles uniformly distributed in phase ϕ at a fixed action J, is,

$$\epsilon_{\text{rms}} = \langle J \rangle. \tag{67}$$

If the accelerator is composed of linear elements, and no dissipative forces act ϵ_{rms} is invariant.

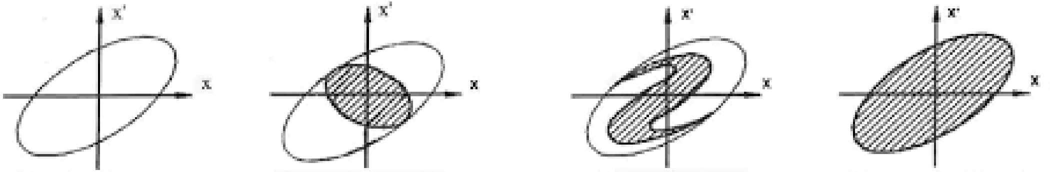
Beam emittance: phenomenology

What determines beam emittance

- Amount of particles.
- Injector manipulation.
- Beam transfer efficiency.

Sources of emittance growth

- Intrabeam scattering.
- Beam-beam interaction.
- Residual gas scattering.
- Optics missmatch.
- Nonlinearities and resonances.
- Ground motion and PS ripple.



(a) machine phase space (b) unmatched beam injected (c) filamenting beam (d) fully filamented beam

Liouville's theorem and symplectic condition

Liouville's equation describes the time evolution of the phase space distribution function $\rho(q, p; t)$,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0 \tag{68}$$

where (q_i, p_i) are the canonical coordinates of the Hamiltonian system.

Symplectic condition

Liouville's theorem ⇒ invariant volume in phase space. The symplectic condition reads,

$$M^T J M = J \tag{69}$$

where J is the 6D sympelctic matrix

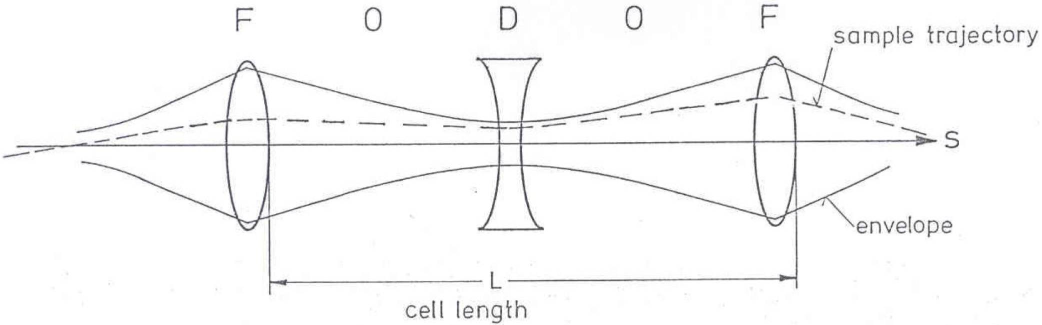
$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{70}$$

Take home exercise

Prove that Eq. (69) holds for the matrices described above.

FODO lattice (The "Hello World" example)

The FODO lattice is a sequence of a Focusing magnet (F), a Drift space (O), a Defocusing magnet (D) and a second drift space.



$$M_{\text{FODO}} = M_0 M_{\text{def}} M_0 M_{\text{foc}} = \begin{pmatrix} 1+\frac{L}{2f} & L+\frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{pmatrix} \tag{71}$$

FODO lattice (The "Hello World" example)

Take-home exercise

Prove that the stability condition for a FODO lattice is given by:

$$f > \frac{L}{4} \tag{72}$$

What if

We take the FODO lattice and replace drifts by bending magnets?
We will see this in next lectures...

The end of the ideal world

So far, we have considered ideal linear systems.
While, in the real world...

- Dispersion.
- Chromaticity.
- Misalignment.
- Magnetic errors.
- ...

Some of these topics will be covered in next lectures.

Dispersion

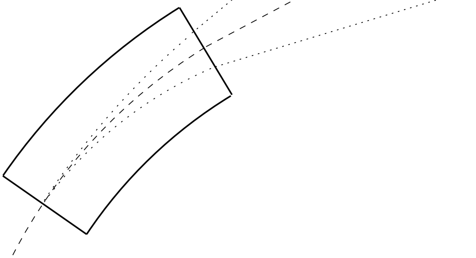
What if particles in a bunch have different momenta?
Remember beam rigidity:

$$B\rho = \frac{P}{q} \tag{73}$$

Orbit:

$$x(s) = D(s)\frac{\Delta P}{P_0} \tag{74}$$

where $D(s)$ is the dispersion function, an intrinsic property of dipole magnets.



Dispersion

Inhomogeneus Hill's equation:

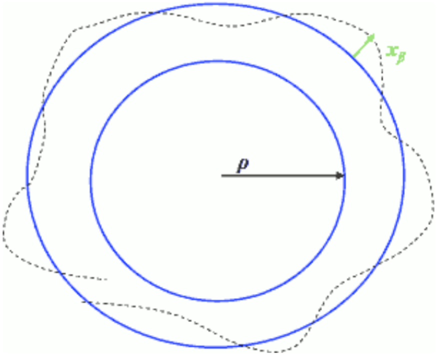
$$u'' + \left(\frac{1}{\rho^2} + k\right) u = \frac{1}{\rho}\frac{\Delta P}{P_0} \qquad (75)$$

Particle trajectory:

$$u(s) = u_\beta(s) + u_D(s) = $$
$$= u_\beta(s) + D(s)\frac{\Delta P}{P} \qquad (76)$$

where $D(s)$ is the solution of:

$$D''(s) + K(s)D(s) = \frac{1}{\rho} \qquad (77)$$



Dispersion

Solution:

$$U(s) = C(s)u_0 + S(s)u_0' + D(s)\frac{\Delta P}{P} \quad (78)$$

this can be added to the transfer matrix representation,

$$M = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \quad (79)$$

Dipole transfer matrix:

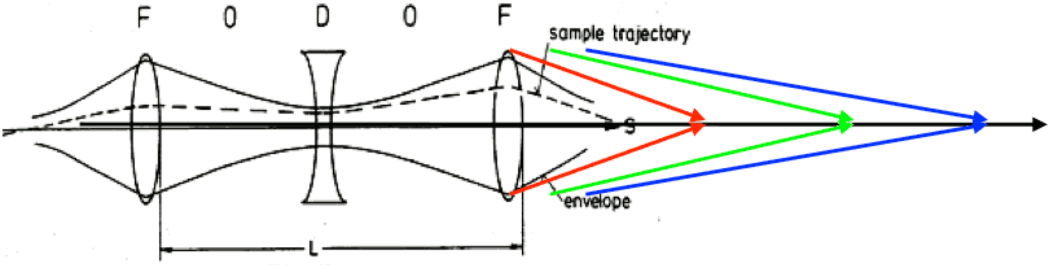
$$\begin{pmatrix} \cos\left(\frac{L}{\rho}\right) & \rho\sin\left(\frac{L}{\rho}\right) & \rho\left(1-\cos\left(\frac{L}{\rho}\right)\right) \\ -\frac{1}{\rho}\sin\left(\frac{L}{\rho}\right) & \cos\left(\frac{L}{\rho}\right) & \sin\left(\frac{L}{\rho}\right) \\ 0 & 0 & 1 \end{pmatrix} \quad (80)$$

Quadrupole transfer matrix (expanded):

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) & 0 \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (81)$$

Chromaticity

All particles do not have the same energy. Therefore, they focalize at different points.

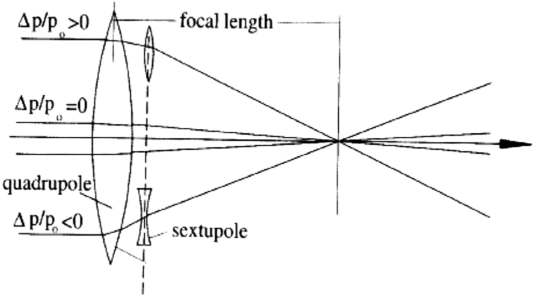


This defines chromaticity,

$$\xi = -\frac{1}{4\pi} \oint \beta(s) k(s) ds \quad (82)$$

How to correct chromaticity

Sextupoles, through a non-linear magnetic field, correct the effect of energy spread and focuses particles at a single location.



- Located in dispersive regions.
- Usually in arcs.
- Sextupole families.

Now is when the party starts

- Sextupoles introduce non-linear fields.
- ...i.e. they induce non-linear motion.
- resonances, tune shifts, chaotic motion.

Chromaticity correction

- Chromatic aberrations must be compensated in both planes.

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x(s)[k(s) - S_F D_x(s) + S_D D_x(S)]ds \tag{83}$$

$$\xi_y = -\frac{1}{4\pi}\oint \beta_y(s)[k(s) + S_F D_x(s) - S_D D_x(S)]ds \tag{84}$$

- To minimise sextupole strength they must be located near quadrupoles where βD is large.
- For optimal independent correction S_F should be located where β_x/β_y is large and S_D where β_y/β_x is large.

Recap.

- Optics functions and parameters.
- Phase space and emittance.
- Example: FODO lattice.
- Dispersion and chromaticity.

What do we do with this?

- We have covered the basic aspects of transverse dynamics.
- I skipped most of the derivations. You can follow references.
- In the next two weeks: lattice design and tutorials for a more complete picture.
- Now you are ready to take the following lectures to become accelerator experts.

Thank you very much!