Transverse Beam Dynamics - Tutorial

JAI lectures 2024 - Michaelmas Term

1 Preliminary exercices

1. Watch this Iron Man clip and discuss the main accelerator physics concepts involved either if they are properly represented or not in the movie.
2. Go through the short questions posted during lectures and try to answer them.

2 To think about

1. How can we measure β^*(β-function at the IP) in the LHC?

We cannot measure it directly because we do not have BPMs a the IP. However using K-modulation technique, the strength of the last quadrupole before the IP is modulated. This modulation produces a measurable tune shift. The tune shift is linerly related to the β-function at the quadrupole location.

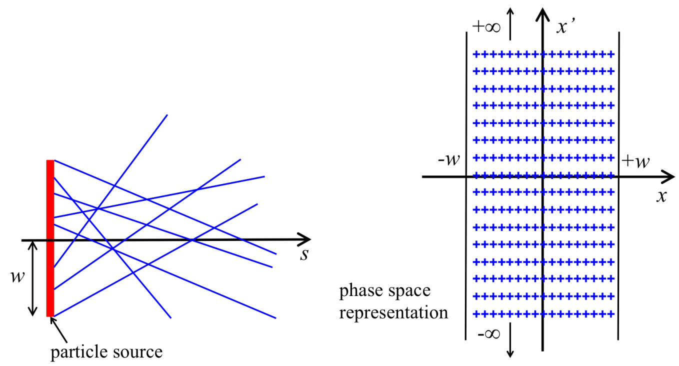
$$\Delta Q = \frac{\beta_q \Delta K}{4\pi}$$

By transporting the measured β-function at the quadrupole to the IP we can have an estimation of the β-function at that location.

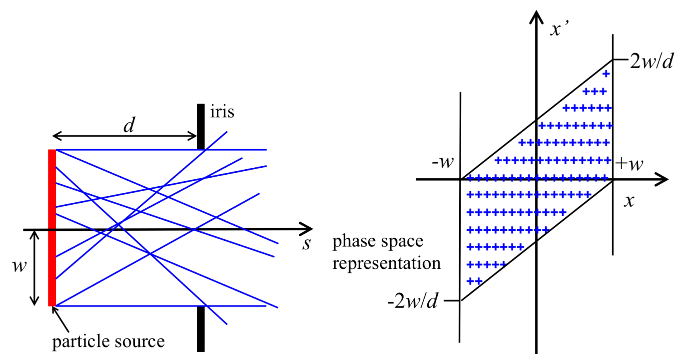
1. What are the possible effects of ground motion in the beam?
   (a) Orbit distorsion.
   (b) Emittance growth.
2. What can we do if there is a small object partially blocking the beam aperture?
   (a) Orbit bump.
   (b) Evaporate it.
   (c) Open the machine and remove it.

3 Exercise: Understanding the phase space concept

1. Phase Space Representation of a Particle Source:
   - Consider a source at position s_0 with radius w emitting particles. Make a drawing of this setup in the configuration space and in the phase space. Which part of the phase space can be occupied by the emitted particles?
     Answer. Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.

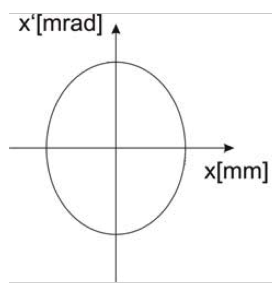


- Any real beam emerging from a source like the one above will be collimated. This can be modelled by assuming that a distance d away from the source there is an iris with opening radius $R = w$. Draw this setup in the configuration space and in the phase space. Which part of the phase space is occupied by the beam, right after the collimator?
  Answer. Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.

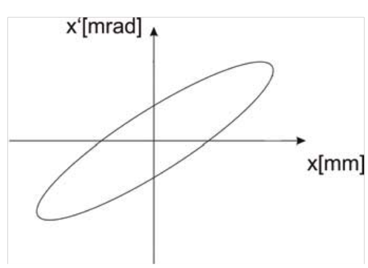


2. Sketch the emittance ellipse of a particle beam in:

(I) horizontal x-x' phase space at the position of a transverse waist,
Answer. Beam at the position of a transverse (x) waist

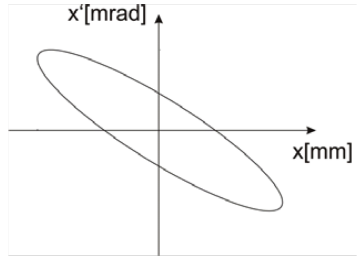


(II) when the beam is divergent, and
Answer. Divergent beam (positive slope):



(III) when the beam is convergent.

Answer. Convergent beam (negative slope):



4 Moon Collider

In the science-fiction novel \textit{Firstborn} written by Arthur C. Clarke, the Alpehtron is described as a particle accelerator wrapping around the lunar equator. Let's consider our magnet technology at that time reaches 20 T at 20 m long dipoles. The goal is to produce collisions at 1 PeV (10^{15}eV) in the center of mass. ($R_{\text{moon}} = 1737$ km)

1. What is the minimum filling factor (fraction of the accelerator filled with dipoles) required in order to reach the desired energy with the technology available?

$$B\rho = 3.33p[\text{GeV}/c]$$

Let's consider that the full circunference is filled with dipole magnets. That means that the bending radius of the dipoles is the radius of the Moon. For this bending radius, we can compute the required magnetic field:

$$B \approx 2\text{T}$$

this is a factor 10 smaller than the maximum field provided by our technology. Thus, the minimum filling factor is about 10%.

1. Enumerate two advantages and two advantages of building a particle accelerator on the surface of the Moon.

   (a) Advantages: Vacuum and cryogenics comes for free. There is no need to build a tunnel.

   (b) Disadvantages: Reasonable cost of bringing materials to the Moon. 2 seconds delay operation. Synchrotron radiation?

5 Stability condition

Consider a lattice composed by a single 2 meters long quadrupole, with $f = 1$ m

- Prove that if the quadrupole is defocusing, then a lattice is not stable
- Prove that if the quadrupole is focusing, then the lattice is stable

Solution:

This quadrupole is clearly thick. Therefore one should use the thick quadrupole matrices. However, I post the thin lens calculation for comparison.

- In the case of a defocusing quadrupole:

$$M_{\text{QD}} = \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix}$$

  which can be computed to be

$$M_{\text{QD}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

  has trace $\text{Tr}\left(M_{\text{QD}}\right) = 4$, which does not fulfill the stability requirement:

$$\left|\text{Tr}\left(M_{\text{QD}}\right)\right| \leq 2$$

- In the case of a focusing quadrupole:

$$M_{\mathrm{QF}} = \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which clearly satisfies the stability criterion.

6 Twiss functions evolution

Which of the optics parameters can be constant

1. In a drift.
2. In a quadrupole with constant strength K.

Justify the response.

Hint: The differential equation representing the evolution of the β-function reads,

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 K = 1$$

Solution

Let's consider the two cases separately:

1. In a drift $K = 0$,

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 = 1$$

therefore, β cannot be constant ($\beta' = \beta'' = 0$).
Taking into account that $-2\alpha = \beta'$, if $\alpha = \text{const.}$, $\beta'' = 0$ and β must evolve linearly with s.

2. In a quadrupole, $K \neq 0$
therefore β is constant if

$$\beta = \frac{1}{\sqrt{K}}$$

In addition, ifα is constant,

$$-\alpha^2 + \beta^2 K = 1$$

$$K = \frac{1 + \alpha^2}{\beta^2}$$

7 Bump and Orbit Control

Given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at L_{cell} distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x_i,\, x_i') = (0,\, 0)$, an arbitrary offset at the end of the cell while preserving its angle, $\left(x_f,\, x_f'\right) = (x_{\mathrm{arbitrary}},\, 0)$.

Solution

The transfer matrix of a periodic cell is:

$$M = \begin{pmatrix} \cos\varphi + \alpha\sin\psi & \beta\sin\varphi \\ -\gamma\sin\varphi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

Substituting the value for the phase advance we get the matrix to apply to the beam right after the first kick k_1:

$$\begin{pmatrix} x_f \\ x_f' \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1+\alpha & \beta \\ -\gamma & 1-\alpha \end{pmatrix} \begin{pmatrix} 0 \\ k_1 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} \beta k_1 \\ (1-\alpha) k_1 \end{pmatrix}$$

From this we see that to achieve an arbitrary x_f we need:

$$k_1 = \frac{\sqrt{2}x_f}{\beta}$$

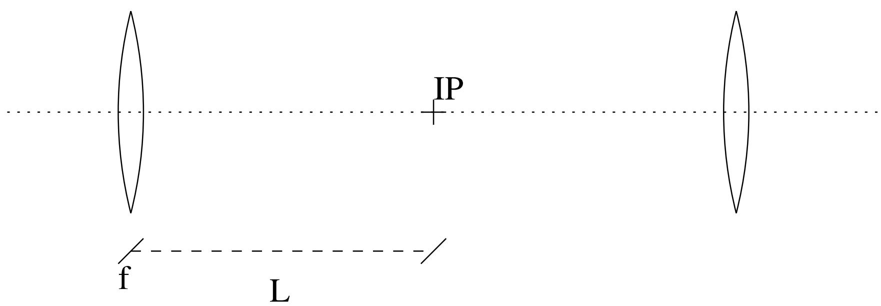
The second kick, k_2, has only to remove the final tilt:

$$k_2 = -x'_f = -\frac{(1-\alpha)}{\sqrt{2}}k_1$$

Notice that one can reduce the strength of the kickers by placing them close to a focusing quadrupoles, where β is maximum.

8 Low-Beta Insertion

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP:



The beam enters the quadrupole with Twiss parameters $\beta_0 = 20$ m and $\alpha_0 = 0$. The drift space has length $L = 10$ m.

(i) Determine the focal length of the quadrupole in order to locate the waist at the IP.

(ii) What is the value of $\beta^\star$?

(iii) What is the phase advance between the quadrupole and the IP?

Solution.

$$M = \begin{pmatrix} 1-\frac{L}{f} & L \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{\text{IP}} = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

$$\begin{pmatrix} \beta_{\text{IP}} & 0 \\ 0 & 1/\beta_{\text{IP}} \end{pmatrix} = M \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \cdot M^T$$

We get a system of equations:

$$\begin{cases} \beta_{\text{IP}} = & \beta_0 \left(1-\frac{L}{f}\right)^2 + \frac{L^2}{\beta_0} \\ \frac{1}{\beta_{\text{IP}}} = & \frac{\beta_0}{f^2} + \frac{1}{\beta_0} \end{cases}$$

multiplying them:

$$1 = \left(\beta_0 \left(1-\frac{L}{f}\right)^2 + \frac{L^2}{\beta_0}\right)\left(\frac{\beta_0}{f^2} + \frac{1}{\beta_0}\right)$$

and solving for f:

$$f = \frac{\beta_0\sqrt{(\beta_0^2-4L^2)} + \beta_0^2}{2L}$$

from which one finds:

$$f = 20 \text{ m}$$

and substituting back into one of the equations in the system:

$$\beta_{\mathrm{IP}} = 10 \text{ m.}$$

The phase advance can be computed remembering that

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

In this case, $\alpha_0 = \alpha_{\mathrm{IP}} = 0$,

$$\mathrm{Trace}\,(M) = \frac{3}{2} = \left(\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}\right)\cos\Delta\mu$$

$$\Delta\mu = \arccos\left(\frac{3}{2}\cdot\frac{1}{\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}}\right) = \arccos\left(\frac{3}{2}\cdot\frac{1}{2.1213}\right) = 45 \text{ degrees}$$

Alternatively, given that the system:

$$M = Q\cdot D\cdot D\cdot Q$$

is indeed periodic, one can say:

$$M = \begin{pmatrix} 1-\frac{2L}{f} & 2L \\ \frac{2L}{f^2}-\frac{2}{f} & 1-\frac{2L}{f} \end{pmatrix}$$

$$\cos\Delta\mu_{\mathrm{twice}} = \frac{1}{2}\mathrm{Trace}\,(M) = \frac{1}{2}\mathrm{Trace}\left(2-\frac{4L}{f}\right) = 0$$

$$\Delta\mu_{\mathrm{twice}} = 90 \text{ degrees} \qquad \Rightarrow \Delta\mu = 45 \text{ degrees}$$