Energy-momentum trace and Hamiltonian decomposition

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XIV International Conference on New Frontiers in Physics
July 17-31, 2025, OAC, Kolumbari, Crete, Greece

ME, PRD **110**, *076020 (2024); Acta Phys. Pol. B* **56**, *3-A2 (2025)*

Outline

QED Lagrangian, dimensional regularization in mass-shell scheme

- Renormalized perturbation theory, dimensional regularization ($d = 4 - 2\epsilon$)

$$\mathcal{L}_0 = \mathcal{L} + \delta\mathcal{L} = -\frac{1}{4}F_0^2 + \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - e_0\bar{\psi}_0 A\!\!\!/_0\psi_0$$

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \mu^\epsilon e\bar{\psi}A\!\!\!/\psi,$$

$$\delta\mathcal{L} = -\frac{1}{4}\delta Z_3 F^2 + \bar{\psi}(i\delta Z_2\partial\!\!\!/ - \delta_m)\psi - \mu^\epsilon e\delta Z_1\bar{\psi}A\!\!\!/\psi,$$

$$\mathcal{L}_0 = \mathcal{L} + \delta\mathcal{L} = -\frac{1}{4}Z_3F^2 + iZ_2\bar{\psi}\partial\!\!\!/\psi - mZ_m\bar{\psi}\psi - eZ_1\mu^\epsilon\bar{\psi}A\!\!\!/\psi$$

- Renormalization constants

$$Z_1 = 1 + \delta Z_1, \quad Z_2 = 1 + \delta Z_2, \quad Z_3 = 1 + \delta Z_3$$

$$mZ_m = m(1 + \delta Z_m) = m + \delta_m, \quad m_0 = mZ_mZ_2^{-1}$$

$$e_0 = \mu^\epsilon Z_3^{-\frac{1}{2}}e, \quad \delta m = m - m_0 = m - mZ_mZ_2^{-1}$$

Energy-momentum tensor

- Symmetric EMT tensor ($D_\alpha = \partial_\alpha + ie_0 A_{0\alpha}$)

$$T_0^{\mu\nu} = \bar{\psi}_0 \gamma^\mu \frac{i}{2} \overleftrightarrow{D}^\nu \psi_0 - F_0^{\mu\alpha} F^\nu_{0\ \alpha} + \frac{1}{4} g^{\mu\nu} F_0^2$$

- **Conserved EMT tensor is not renormalized** $T_0^{\mu\nu} = [T^{\mu\nu}]_R$
- **EMT trace in gauge theories (QED, QCD, etc.) is anomalous**

$$T^\mu_{0\ \mu} = (1 + \gamma_m(e_0))\bar{\psi}_0 m_0 \psi_0 + \frac{\beta(e_0)}{2e_0} F_0^2$$

$$= (1 + \gamma_m(e))[\bar{\psi} m \psi]_R + \frac{\beta(e)}{2e}[F^2]_R$$

- **QED:**
  - One-loop beta-function: $\beta(e)/2e = \alpha/6\pi$
  - One-loop mass anomalous dimension: $\gamma_m(e) = 3\alpha/2\pi$

Mass (energy) - matrix element of Hamiltonian & EMT trace

- $H = \int d^3x T^{00}(x)$ – **Hamiltonian**
- $\Longrightarrow \int d^3x \langle \boldsymbol{p}|T^{00}(x)|\boldsymbol{p}\rangle = E_p\langle \boldsymbol{p}|\boldsymbol{p}\rangle$
- **Translational invariance:** $\langle \boldsymbol{p}|T^{\mu\nu}(x)|\boldsymbol{p}\rangle = \langle \boldsymbol{p}|T^{\mu\nu}(0)|\boldsymbol{p}\rangle$
- $\Longrightarrow \langle \boldsymbol{p}|T^{00}(0)|\boldsymbol{p}\rangle = E_p\frac{\langle \boldsymbol{p}|\boldsymbol{p}\rangle}{V}$
- **Covariant normalization:** $\langle \boldsymbol{p}|\boldsymbol{p}\rangle = 2E_pV$
- $\Longrightarrow \langle \boldsymbol{p}|T^{00}(0)|\boldsymbol{p}\rangle = 2E_p^2$
- **Lorentz invariance:** $\langle \boldsymbol{p}|T^{\mu\nu}(0)|\boldsymbol{p}\rangle = 2p^\mu p^\nu, \quad \langle \boldsymbol{p}|T^{\mu}{}_{\mu}(0)|\boldsymbol{p}\rangle = 2m^2$
- $\int d^3x \langle \boldsymbol{p}|T^{\mu}{}_{\mu}(x)|\boldsymbol{p}\rangle = 2m^2V$
- **Rest frame** $\int d^3x \langle \boldsymbol{0}|T^{\mu}{}_{\mu}(x)|\boldsymbol{0}\rangle = m\langle \boldsymbol{0}|\boldsymbol{0}\rangle$
- **Rest frame & normalization independent expression**

$$E = \frac{\int d^3x \langle \boldsymbol{0}|T^{\mu}{}_{\mu}(x)|\boldsymbol{0}\rangle}{\langle \boldsymbol{0}|\boldsymbol{0}\rangle}, \quad E = \frac{\int d^3x \langle \boldsymbol{0}|T^{00}(x)|\boldsymbol{0}\rangle}{\langle \boldsymbol{0}|\boldsymbol{0}\rangle}$$

- **We will use nonrelativistic normalization in calculations**

Irreducible Lorentz Components of EMT

- **Traceless symmetric tensor transforms as irreducible representation of the Lorentz group $\Longrightarrow$ split** $T^{\mu\nu}$ *(Ji, 1995)*

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T^{\alpha}{}_{\alpha}, \qquad \hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} T^{\alpha}{}_{\alpha}, \qquad T^{\mu\nu} = \bar{T}^{\mu\nu} + \hat{T}^{\mu\nu}$$

$$\bar{T}^{\mu}{}_{\mu} = 0, \qquad T^{\mu}{}_{\mu} = \hat{T}^{\mu}{}_{\mu}$$

- **Lorentz invariance:** $\langle \boldsymbol{p}| T^{\mu\nu}(0)|\boldsymbol{p}\rangle = 2p^{\mu}p^{\nu}$, $\langle \boldsymbol{p}| T^{\mu}{}_{\mu}(0)|\boldsymbol{p}\rangle = 2m^2$

$$\Longrightarrow \langle \boldsymbol{p}| \bar{T}^{\mu\nu}(0)|\boldsymbol{p}\rangle = \frac{4p^{\mu}p^{\nu} - m^2 g^{\mu\nu}}{2}, \qquad \langle \boldsymbol{p}| \hat{T}^{\mu\nu}(0)|\boldsymbol{p}\rangle = \frac{m^2}{2} g^{\mu\nu}$$

- $\Longrightarrow \langle \mathbf{0}| \bar{T}^{00}(0)|\mathbf{0}\rangle = \frac{3}{2}m^2, \qquad \langle \mathbf{0}| \hat{T}^{00}(0)|\mathbf{0}\rangle = \frac{1}{2}m^2$

- $\langle \mathbf{0}|H|\mathbf{0}\rangle = m\langle \mathbf{0}|\mathbf{0}\rangle$

$$\Leftrightarrow \int d^3x \langle \mathbf{0}| T^{00}(x)|\mathbf{0}\rangle = \int d^3x \langle \mathbf{0}| \bar{T}^{00}(x) + \hat{T}^{00}(x)|\mathbf{0}\rangle$$

$$\int d^3x \langle \mathbf{0}| \bar{T}^{00}(x)|\mathbf{0}\rangle = \frac{3}{4} m \langle \mathbf{0}|\mathbf{0}\rangle, \qquad \int d^3x \langle \mathbf{0}| \hat{T}^{00}(x)|\mathbf{0}\rangle = \frac{1}{4} m \langle \mathbf{0}|\mathbf{0}\rangle$$

$$\Longrightarrow \int d^3x \langle \mathbf{0}| \hat{T}^{00}(x)|\mathbf{0}\rangle = \frac{1}{4} m \langle \mathbf{0}|\mathbf{0}\rangle$$

- $1/4$ **of particle mass is due to EMT trace!**

Hamiltonian decomposition with account for trace anomaly

- **EMT trace:** $T^{\mu}{}_{\mu} = (1+\gamma_m(e))[\bar{\psi} m \psi]_R + \frac{\beta(e)}{2e}[F^2]_R$
- **Quantum anomalous energy** H_a *(Ji, 1995, 2021)*

$$H = H_c + H_a = \int d^3x \, \bar{T}^{00}(x) + \int d^3x \, \hat{T}^{00}(x)$$
$$= \int d^3x \left[-i\bar{\psi}\boldsymbol{\gamma}\cdot\boldsymbol{D}\psi + m\bar{\psi}\psi + \frac{1}{2}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)\right]_R$$
$$+\frac{1}{4}\int d^3x \left[\gamma_m[\bar{\psi}m\psi(x)]_R + \frac{\beta(g)}{2g}[F^2(x)]_R\right]$$

- $H_\psi = \frac{1}{4}\int d^3x \gamma_m[\bar{\psi}m\psi(x)]_R$, $H_F = \frac{1}{4}\frac{\beta(g)}{2g}[F^2(x)]_R$, $H_a = H_\psi + H_F$
- **All above is valid both in QCD and QED**
- **Does Hamiltonian decomposition with H_a as a separate term makes sense?**
- **QED test for hydrogen energy levels in external field approximation**

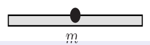
Quantum Anomalous Energy

- **Expectation:** $H_a = H_\psi + H_F$ **contributes to the Lamb shift**

$$\Delta E_A = \frac{\langle \mathbf{0}|H_a|\mathbf{0}\rangle}{\langle \mathbf{0}|\mathbf{0}\rangle} = \frac{\langle \mathbf{0}|H_\psi|\mathbf{0}\rangle}{\langle \mathbf{0}|\mathbf{0}\rangle} + \frac{\langle \mathbf{0}|H_F|\mathbf{0}\rangle}{\langle \mathbf{0}|\mathbf{0}\rangle} = \Delta E_\psi + \Delta E_F$$

- ΔE_A – **quantum anomalous energy**

- **Tree contribution (***Fock, 1930***)**

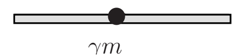


$$T = \langle nj|T^\mu{}_\mu|nj\rangle_{tree} = \int d^3x \langle nj|m\bar{\psi}(x)\psi(x)|nj\rangle$$
$$= m\int d^3x \psi^\dagger_{nj}(x)\gamma^0\psi_{nj}(x) = E_{nj}$$

- E_{nj} – **Dirac energy levels**

Quantum Anomalous Energy

- **Fermion anomalous dimension term**



- **Anomalous mass dimension contribution**

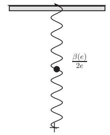
$$\Delta E_\psi = \frac{\gamma_m}{4} E_{nj} = \frac{\gamma_m m}{4}\left[1+\left(\frac{Z\alpha}{n-(j+\frac{1}{2})+\sqrt{(j+\frac{1}{2})^2-(Z\alpha)^2}}\right)^2\right]^{-\frac{1}{2}}$$

$$\approx \frac{3\alpha}{8\pi}\left[m-\frac{m(Z\alpha)^2}{2n^2}+\frac{m(Z\alpha)^4}{n^4}\left(\frac{3}{8}-\frac{n}{2j+1}\right)+\dots\right]$$

- **Notice:**
  1. **Terms of order αm, $\alpha(Z\alpha)^2 m$, etc.**
  2. **Even powers of n**
  3. **Depends on total angular momentum j**

Quantum Anomalous Energy

- Anomalous dimension β-function term



- **Two-prong vertex insertion in the external photon line**
- β**-function contribution**

$$(\beta/(2e))F^{\alpha\beta}F_{\alpha\beta} \to 4(\beta/(2e))(-\boldsymbol{q}^2)$$

$$\Delta E_F = \frac{\alpha(Z\alpha)^2 m}{6\pi}\frac{n-\left(j+\frac{1}{2}\right)+\frac{\left(j+\frac{1}{2}\right)^2}{\sqrt{\left(j+\frac{1}{2}\right)^2-(Z\alpha)^2}}}{\left(\left[\sqrt{\left(j+\frac{1}{2}\right)^2-(Z\alpha)^2}+n-\left(j+\frac{1}{2}\right)\right]^2+(Z\alpha)^2\right)^{3/2}}$$

$$\approx \frac{\alpha(Z\alpha)^2 m}{6\pi n^2} - \frac{\alpha(Z\alpha)^4 m}{4\pi n^4}\left[1-\frac{4n}{3\left(j+\frac{1}{2}\right)}\right]+\dots.$$

Quantum Anomalous Energy

- **Notice:**
  1. **Term of order $\alpha(Z\alpha)^2 m$**
  2. **Even powers of n**
  3. **Depends on total angular momentum j**
- **Lamb shift:**
  1. **Leading contribution of order $\alpha(Z\alpha)^4 m$**
  2. **Depends on n^3**
  3. **Depends on orbital momentum ℓ, not on total angular momentum j**
- **ΔE_F and ΔE_ψ – wrong dependence on n, depends on total angular momentum j and not on ℓ**
- **$\Longrightarrow$ QAE contribution completely cancels with contributions of other terms in the Hamiltonian**
- **Conclusion: Quantum Anomalous Energy H_a should not be included in the Hamiltonian decomposition as a separate term!**

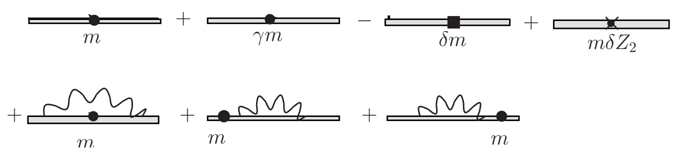
EMT trace in one-loop approximation

- **One-loop matrix element**

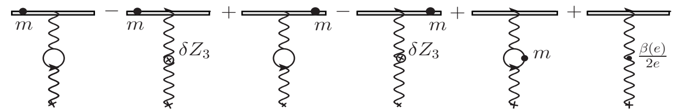
$$T \equiv \int d^3r \langle n\ell | T^{\mu}{}_{\mu}(r) | n\ell \rangle$$

$$\approx \int d^3r \langle n\ell | [m - \delta m + m\gamma_m(e) + m\delta Z_2] \bar{\psi}(r)\psi(r) + \frac{\beta(e)}{2e} F^2(r) | n\ell \rangle$$

- **One-loop perturbation theory diagrams**
  - **Tree and self-energy type diagrams:**



  - **Vacuum polarization type diagrams:**

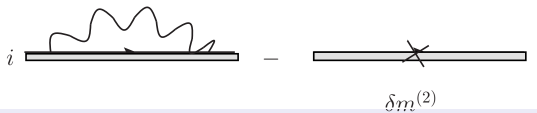


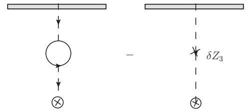
EMT trace in one-loop approximation

- **Diagrams for trace are different from the textbook Lamb shift diagrams**
- **Two questions:**
  1. **How these diagrams reproduce Lamb shift?**
  2. **Why two different sets of diagrams produce the same result?**

Classical Lamb shift calculation

- **Furry picture: QED in external Coulomb field**
- Two sets of one-loop diagrams: self-energy and polarization





One-loop self-energy diagrams. Furry Picture

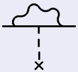
- **General formula**

$$\Delta E_{SE} = \int d^3r d^3r' \langle n\ell|\bar{\psi}(\boldsymbol{r}) \left[\Sigma_{reg}(\boldsymbol{r}, \boldsymbol{r}', E_n, A_0) - \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')\delta m\right] \psi(\boldsymbol{r}')|n\ell\rangle$$

- **Leading in $Z\alpha$ contribution**

$$\Delta E_{SE}(n, \ell = 0) = \frac{4}{3}\frac{\alpha(Z\alpha)^4 m}{\pi n^3}\left[\ln\frac{1}{(Z\alpha)^2} + \frac{5}{6} - \ln k_0(n, 0)\right]$$

- **How this result arises?**
- **External field expansion, leading effective potential**



$$V_{eff,ff} = \frac{4\alpha(Z\alpha)}{3m^2}\ln\left(\frac{1}{(Z\alpha)^2}\right)\delta^{(3)}(\boldsymbol{r})$$

- Dimension of effective potential in mass units: $[V_{eff,ff}] = 1$

One-loop self-energy. Furry picture

- **Leading contribution**

$$\Delta E_{ff}(n,\ell) = \langle n\ell | V_{eff,ff} | n\ell\rangle = \frac{4\alpha(Z\alpha)}{3m^2}\ln\left(\frac{1}{(Z\alpha)^2}\right)|\psi_{n\ell}(0)|^2$$

$$= \frac{4\alpha(Z\alpha)^4 m}{3\pi n^3}\ln\left(\frac{1}{(Z\alpha)^2}\right)\delta_{\ell 0}$$

- **Full effective potential ($[V_{eff}]=1$)**

$$V_{eff} = \frac{4\alpha(Z\alpha)}{3m^2}\left[\ln\frac{1}{(Z\alpha)^2} + \frac{5}{6} - \ln k_0(n,0)\right]\delta^{(3)}(\boldsymbol{r})$$

- Self-energy contribution

$$\Delta E_{SE}(n,0) = \langle n0|V_{eff}|n0\rangle = \frac{4\alpha(Z\alpha)}{3m^2}\left[\ln\frac{1}{(Z\alpha)^2} + \frac{5}{6} - \ln k_0(n,0)\right]|\psi_{nl}(0)|^2$$

$$= \frac{4}{3}\frac{\alpha(Z\alpha)^4 m}{\pi n^3}\left[\ln\frac{1}{(Z\alpha)^2} + \frac{5}{6} - \ln k_0(n,0)\right]$$

One-loop polarization. How the result above arises?

- **Leading order in $Z\alpha$ contribution**

$$\Delta E_{VP} = \langle n\ell|H_{int}|n\ell\rangle = e\int d^3r\langle n\ell|\bar{\psi}\gamma_0\psi A^0_{ext}(\boldsymbol{r})|n\ell\rangle_{one\ \ loop}$$

- **External field ($\Pi_R(-\boldsymbol{q}^2) = \Pi(-\boldsymbol{q}^2) - \Pi(0)$, $\delta Z_3 = \Pi(0)$)**

$$A^0_{ext,one\ \ loop}(\boldsymbol{r}) = -Ze\int \frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{\Pi_R(-\boldsymbol{q}^2)}{\boldsymbol{q}^2}$$

- **Polarization operator**

$$\Pi_R^{(2)}(-\boldsymbol{q}^2) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\ln\frac{m^2}{x(1-x)\boldsymbol{q}^2+m^2}_{|\boldsymbol{q}^2/m^2\to 0}$$

$$\to \frac{2\alpha}{\pi}\int_0^1 dx x^2(1-x)^2\frac{\boldsymbol{q}^2}{m^2} = \frac{\alpha}{15\pi}\frac{\boldsymbol{q}^2}{m^2}$$

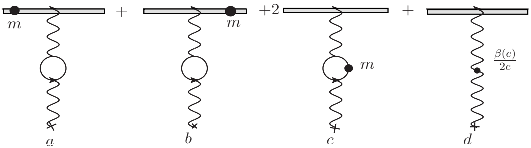
- **Effective potential ($[V_{pol,eff}] = 1$)**

$$V_{pol,eff} = eA^0_{ext,one\ \ loop}(\boldsymbol{r}) = -\frac{4\alpha(Z\alpha)}{15m^2}\delta^{(3)}(\boldsymbol{r})$$

- **Polarization contribution**

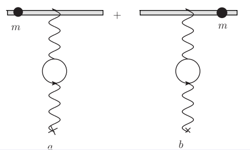
$$\Delta E_{VP}(n,\ell) = \langle n\ell|V_{eff}|n\ell\rangle = -\frac{\alpha(Z\alpha)}{15m^2}|\psi_{n\ell}(0)|^2 = -\frac{4\alpha(Z\alpha)^4m}{15\pi n^3}\delta_{\ell 0}$$

Trace polarization type diagrams



- *a*) and *b*) – two diagrams with sidewise insertions of external field with renormalized polarization loop
- *c*) – 2 diagrams with the scalar vertex m insertion in the external field polarization loop
- *d*) – diagram with anomalous term $(\beta/2e)F^2$ insertion in the external field

Sidewise insertions of polarization loop



- **Contributions of diagrams** $a)$ **and** $b)$ **are equal**

$$\Delta E_a(nj) = \Delta E_b(nj)$$

$$= \int d^3r d^3r' \psi_{njl}(\boldsymbol{r}) V_{pol,eff}(\boldsymbol{r})[-iG_r(\boldsymbol{r},\boldsymbol{r}',E_n)] m\gamma_0 \psi_{nj\ell}(\boldsymbol{r}')$$

- **Reduced Dirac-Coulomb Green function is defined as**

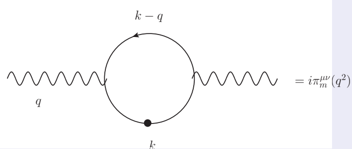
$$G_r(\boldsymbol{r},\boldsymbol{r}',E) = \langle \boldsymbol{r}| \left(\frac{i}{E-H}\right)' \gamma_0 |\boldsymbol{r}'\rangle = \langle \boldsymbol{r}| \sum_{k\neq n} \frac{i}{E-E_k} \gamma_0 |\boldsymbol{r}'\rangle$$

$H = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m + V, \qquad V = -\frac{Z\alpha}{r}$ **– Coulomb potential**

- **Use** *Shabaev (1991)* **methods to calculate** ΔE_a **and obtain**

$$\Delta E_a(n\ell) = \Delta E_b(n\ell) = -\frac{3}{2}\frac{4\alpha(Z\alpha)^4 m}{15\pi n^3}\delta_{\ell 0} = \frac{3}{2}\Delta E_{VP}(n\ell)$$

Scalar vertex insertion in polarization loop



- $\pi_m^{\mu\nu}(q^2)$ **is transverse**

$$\pi_m^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\pi_m(q^2)$$

- **where**

$$\pi_m(q^2) = i\frac{4\alpha m^2}{\pi}\int_0^1 dx \frac{x(1-x)^2}{-x(1-x)q^2+m^2}$$

- **Contribution to the energy level (an extra factor 2 below is due to insertion of the scalar vertex in each fermion line)**

$$\Delta E_c(nj) = i4\pi Z\alpha\int d^3r\psi^\dagger_{njm}(\boldsymbol{r})\psi_{njm}(\boldsymbol{r})\int\frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{2\pi_m(-\boldsymbol{q}^2)}{\boldsymbol{q}^2}$$

Scalar vertex insertion in polarization loop

- **Low momentum expansion**

$$\pi_m(-\boldsymbol{q}^2)_{|\frac{\boldsymbol{q}^2}{m^2}\to 0} \to \frac{2\alpha i}{\pi}\left(\frac{1}{6} - \frac{\boldsymbol{q}^2}{30m^2} + \frac{\boldsymbol{q}^4}{140m^4} + \dots\right)$$

- **Finally**

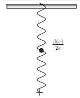
$$\Delta E_c(nj\ell) \approx -\frac{2\alpha(Z\alpha)^2}{3\pi n^2} + \frac{\alpha(Z\alpha)^4 m}{\pi n^4}\left[1 - \frac{4n}{3\left(j+\frac{1}{2}\right)}\right] + \frac{8\alpha(Z\alpha)^4 m}{15\pi n^3}\delta_{\ell 0}$$

$$= -\frac{2\alpha(Z\alpha)^2}{3\pi n^2} + \frac{\alpha(Z\alpha)^4 m}{\pi n^4}\left[1 - \frac{4n}{3\left(j+\frac{1}{2}\right)}\right] - 2\Delta E_{VP}(n\ell)$$

$$= \Delta E_{c1}(nj) - 2\Delta E_{VP}(n\ell)$$

- **Last term is $-2\Delta E_{VP}(n\ell)$!**

Anomalous term $(\beta/2e)F^2$ insertion in external field



- **Contribution to the energy level**

$$\Delta E_d(nj) = 4\pi Z\alpha \frac{4\beta}{2e} \int d^3r \psi^\dagger_{njm}(\boldsymbol{r})\psi_{njm}(\boldsymbol{r}) \int \frac{d^3q}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \frac{1}{\boldsymbol{q}^2}$$

$$= \frac{2\alpha(Z\alpha)}{3\pi} \int d^3r \frac{\psi^\dagger_{njm}(\boldsymbol{r})\psi_{njm}(\boldsymbol{r})}{r} = \frac{2\alpha(Z\alpha)^2 m}{3\pi n^2} - \frac{\alpha(Z\alpha)^4 m}{\pi n^4}\left[1 - \frac{4n}{3\left(j+\frac{1}{2}\right)}\right]$$

- **Cancels first term in ΔE_c**

$$\Delta E_{c1}(nj) + \Delta E_d(nj) = 0$$

$$\Longrightarrow \Delta E_c(nj\ell) + \Delta E_d(nj) = -2\Delta E_{VP}(n\ell)$$

- **The sum is 2 times larger and has opposite sign to $\Delta E_{VP}(n\ell)$!**

Sum of all polarization type diagrams for EMT trace

$$\Delta E_{VP,trace}(n\ell) = \Delta E_a(n\ell) + \Delta E_b(n\ell) + \Delta E_c + \Delta E_d$$

$$-\frac{3}{2}\frac{4\alpha(Z\alpha)^4 m}{15\pi n^3}\delta_{\ell 0} - \frac{3}{2}\frac{4\alpha(Z\alpha)^4 m}{15\pi n^3}\delta_{\ell 0} + \frac{8\alpha(Z\alpha)^4 m}{15\pi n^3}\delta_{\ell 0}$$

$$= 3\Delta E_{VP}(n\ell) - 2\Delta E_{VP}(n\ell) = \Delta E_{VP}(n\ell)$$

- **Sum of all polarization type diagrams for trace is equal standard polarization contribution**
- **Can we understand diagrammatically why and how this happened?**

Heuristic considerations

- ΔE_{VP} **is linear in mass** $\Longrightarrow \Delta E_{VP} = m\frac{d\Delta E_{VP}}{dm}$

$$\Delta E_{VP}(n\ell) = -\frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}|\psi_{n\ell}(0)|^2$$

$$\Delta E_{vp}(n,\ell) = m\frac{\Delta E_{vp}(n,\ell)}{dm} = -\frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}|\psi_{n\ell}(0)|^2$$

$$= -\left[m\frac{d\left(\frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}\right)}{dm}|\psi_{n\ell}(0)|^2 + \frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}m\frac{d|\psi_{n\ell}(0)|^2}{dm}\right]$$

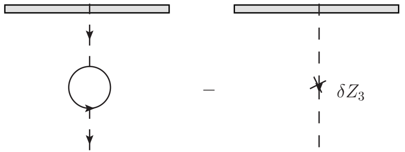
Heuristic considerations

- **For dimensional reasons**

$$= -\left[-2\frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}|\psi_{nl}(0)|^2 + 3\frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}|\psi_{nl}(0)|^2\right]$$
$$= 2\frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}|\psi_{nl}(0)|^2 - 3\frac{4}{15}\frac{\alpha(Z\alpha)}{m^2}|\psi_{nl}(0)|^2$$

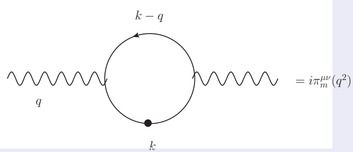
- **Hypothesis:**

1. **The term with derivative of the polarization operator corresponds to the diagrams with m insertion in the polarization loop**
2. **The term with derivative of the state vector corresponds to the diagrams with sidewise m insertions**
3. **The term with derivative of δZ_3 corresponds to the diagram with anomaly $\beta/(2e)F^2$ insertion**



Logarithmic derivative of polarization diagram

- Logarithmic mass derivative of polarization loop produces two diagrams with mass insertion in each branch of the fermion loop



- Minor subtlety: to obtain loop with the mass insertion we should differentiate regularized, not renormalized loop

$$\Pi^{(2)}_{reg}(-\boldsymbol{q}^2) = \Pi^{(2)}_{reg}(0) + \Pi^{(2)}_{R}(-\boldsymbol{q}^2)\ \ (\Pi^{(2)}_{reg}(0) = \delta Z_3)$$

- After an easy calculation

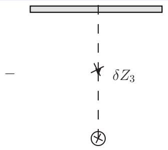
$$m\frac{\partial}{\partial m}\Pi^{(2)}_{reg}(-\boldsymbol{q}^2)_{(\boldsymbol{q}^2/m^2)\to 0} \to \frac{2\alpha}{3\pi} - \frac{2\alpha}{15\pi}\frac{\boldsymbol{q}^2}{m^2}$$

- Derivative diagrams contribution

$$\Delta E(nj\ell) = -\frac{2\alpha(Z\alpha)^2}{3\pi n^2} + \frac{\alpha(Z\alpha)^4 m}{\pi n^4}\left[1 - \frac{4n}{3\left(j+\frac{1}{2}\right)}\right] + \frac{8\alpha(Z\alpha)^4 m}{15\pi n^3}\delta_{\ell 0}$$
$$= \Delta E_{c1}(nj) - 2\Delta E_{VP}(n\ell)$$

coincides with the contribution of the respective trace diagram

Logarithmic derivative of the counterterm

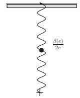


- **Recall**

$$\delta Z_3 = \Pi^{(2)}_{reg}(0) = -\frac{\alpha}{3\pi}\ln\left(\frac{\mu^2}{m^2}\right)$$

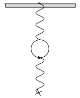
$$\Longrightarrow m\frac{d\delta Z_3}{dm} = -\mu\frac{d\delta Z_3}{d\mu} = \frac{2\beta(e)}{e} \approx \frac{2\alpha}{3\pi}$$

- **Contribution of the diagram with the logarithmic derivative of δZ_3 coincides with the contribution of the trace diagram with the anomaly insertion**



$$\Delta E(n\ell) = \frac{2\alpha(Z\alpha)^2 m}{3\pi n^2} - \frac{\alpha(Z\alpha)^4 m}{\pi n^4}\left[1 - \frac{4n}{3\left(j+\frac{1}{2}\right)}\right]$$

Logarithmic derivative of state vectors



- **Classical polarization contribution to the Lamb shift (renormalized polarization operator) can be written as**

$$\Delta E_{VP} = \langle n|V_{eff}|n\rangle$$

- **Up to this moment we differentiated V_{eff}, we also need to differentiate state vectors**

$$m\frac{d\Delta E_{VP}}{dm}_{(st-vect)} = \left(m\frac{d}{dm}\langle n|\right) V|n\rangle + \langle n|V\left(m\frac{d}{dm}|n\rangle\right)$$

$$= \sum_k \left(m\frac{d}{dm}\langle n|\right) k\rangle\langle k|V|n\rangle + \sum_k \langle n|V|k\rangle\langle k\left(m\frac{d}{dm}|n\rangle\right)$$

- **The term $k = n$ does not contribute**

$$\left(m\frac{d}{dm}\langle n|\right) n\rangle\langle n|V|n\rangle + \langle n|V|n\rangle\langle n\left(m\frac{d}{dm}|n\rangle\right) = \langle n|V|n\rangle m\frac{d\langle n|n\rangle}{dm} = 0$$

Logarithmic derivative of state vectors

Auxiliary calculations: $\langle k \left(m\frac{d|n\rangle}{dm} \right)$

- **Dirac Hamiltonian** $H = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m + V$

$$H|n\rangle = E_n|n\rangle, \qquad \langle k|H|n\rangle_{k\neq n} = 0$$

- **At** $k \neq n$

$$\left(m\frac{d\langle k|}{dm}\right) H|n\rangle + \langle k|\beta m|n\rangle + \langle k|H\left(m\frac{d|n\rangle}{dm}\right)$$

$$= E_n\left(m\frac{d\langle k|}{dm}\right) n\rangle + \langle k|\beta m|n\rangle + E_k\langle k\left(m\frac{d|n\rangle}{dm}\right) = 0$$

- **Also:** $m\frac{d\langle k|n\rangle}{dm} = \left(m\frac{d\langle k|}{dm}\right) n\rangle + \langle k\left(m\frac{d|n\rangle}{dm}\right) = 0$

$$\Longrightarrow \langle k\left(m\frac{d|n\rangle}{dm}\right) = \frac{\langle k|\beta m|n\rangle}{E_n - E_k}$$

$$\Longrightarrow \sum_{k\neq n} |k\rangle\langle k\left(m\frac{d|n\rangle}{dm}\right) = \sum_{k\neq n} |k\rangle\frac{\langle k|\beta m|n\rangle}{E_n - E_k}$$

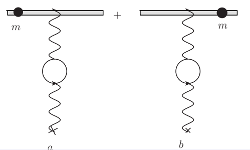
- **Subtracted Dirac-Coulomb Green function**

$$G_r(E_n) = \left(\frac{i}{E_n - H}\right)_{red}\gamma_0 = \sum_{k\neq n}\frac{|k\rangle\langle k|}{E_n - E_k} i\gamma_0$$

Auxiliary calculations: $\langle k\left(m\frac{d|n\rangle}{dm}\right)$

$$\Longrightarrow \langle n|V\left(m\frac{d|n\rangle}{dm}\right)=\sum_{k\neq n}\langle n|V|k\rangle\langle k\left(m\frac{d|n\rangle}{dm}\right)=\langle n|V(-iG_r(E_n))m|n\rangle$$

- Similar formula holds for $\left(m\frac{d\langle n|}{dm}\right)k\rangle$
- Differentiating state vectors we obtain
  $m\frac{d\Delta E_{VP}}{dm}_{(st-vect)}=\langle n|mV(-iG_r(E_n))|n\rangle+\langle n|V(-iG_r(E_n))m|n\rangle$
- These are exactly two sidewise diagrams from the EMT trace



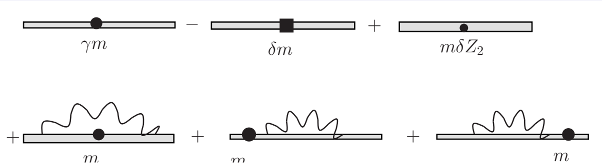
Summary on polarization diagrams

- Sum of polarization type EMT trace diagrams reproduces classical contribution of polarization diagram to Lamb shift
- Polarization type EMT trace diagrams are logarithmic mass derivatives of the classical polarization diagram for the Lamb shift

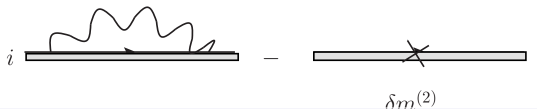
Self-energy type trace diagrams

- **Tree and one-loop diagrams above arise from the matrix element**

$$T \approx \int d^3r \langle H|[m - \delta m + m\gamma_m(e) + m\delta Z_2]\bar{\psi}(\boldsymbol{r})\psi(\boldsymbol{r})|H\rangle$$

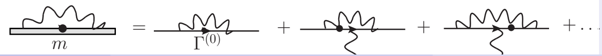


- **Standard one-loop self-energy Lamb shift diagrams**



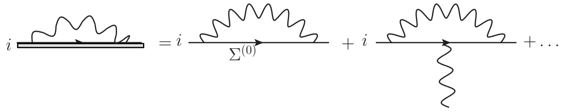
What is the connection between two sets of diagrams?

- **External field expansion**



What is the connection between two sets of diagrams?

- **External field expansion of self-energy**



- **There are two relationships for logarithmic mass derivatives (*ME, 2023*)**

$$m\frac{d\Sigma^{(0)}_{reg}(\not{p}=m)}{dm} = \Gamma^{(0)}(\not{p}=m) + m\delta Z_2$$

$$m\frac{d\delta m^{(2)}}{dm} = \delta m^{(2)} - m\gamma_m$$

- **Logarithmic mass derivative of $(\Sigma^{(0)}_{reg} - \delta m^{(2)})$ generates first four diagrams from the trace**
- **On the other hand $\delta m^{(2)} = \Sigma^{(0)}_{reg}(\not{p}=m)$**
- **$\Longrightarrow$ First three diagrams and the leading term in the external field expansion of the 4th diagram for the trace cancel each other!**

$$-\delta m + m\gamma_m(e) + m\delta Z_2 = -\Gamma^{(0)}_m(m)$$

What is the connection between two sets of diagrams?

- **Everything else goes as in the case of polarization**
  1. **Logarithmic mass derivative of the remaining contribution to Σ generates the remaining contributions to Γ and contribution of the Γ diagram to the Lamb shift is $-2\Delta E_{SE}$ for dimensional reasons (compare contribution of the polarization loop with mass insertion)**
  2. **Logarithmic mass derivative of the state vectors in the matrix element of the renormalized Σ produce two diagrams with sidewise mass insertions and their contribution to the Lamb shift is $3\Delta E_{SE}$**
- **Self-energy type trace diagrams arise from the standard self energy diagrams in extrnal field as logartihmic mass derivatives and total contribution of these diagrams to the Lamb shift is just ΔE_{SE}**

Summary

- **Quantum Anomalous Energy H_a should not be included in the Hamiltonian decomposition as a separate term!**
- **Bound state energy can be calculated as a matrix element of the EMT trace**
- **Sum of self-energy type trace diagrams coincides with the standard diagrams self-energy diagram for the Lamb shift**
- **Sum of vacuum polarization type trace diagrams coincides with the standard diagrams vacuum polarization diagram for the Lamb shift**
- **Equality of contributions of two type of diagrams was expected from the trace anomaly and Lorentz invariance**
- **Technically equality of contributions of two different sets of diagrams arises due to their linearity in mass and because diagrams of one set are logarithmic mass derivatives of the diagrams from another set**

This collection of papers is devoted to Dmitry Diakonov (1949–2012), Victor Petrov (1955–2021), and Maxim Polyakov (1966–2021), who made original, innovative and well recognized by the community contributions to the development of high-energy theory. Many of their results will remain for a long time. Unfortunately, untimely deaths interrupted their productive work at the peak of their creative powers. This collection of papers by some of their friends and colleagues is a small tribute to their achievements and their memory.
[from *the Preface*]

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Acta Physica Polonica B Vol. 56 (2025) Number 3–4
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PL ISSN 0587-4254

JAGIELLONIAN UNIVERSITY
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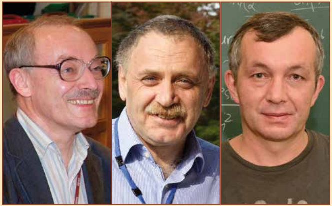

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Vol. 56 — Number 3–4
Cracow
APOBB 56 (3–4) 2025

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