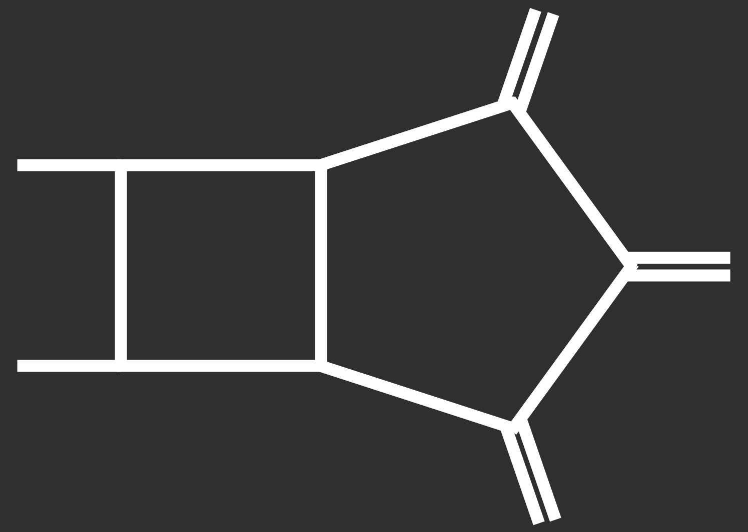
Numerical integration of NNLO virtual corrections to triboson production: the Nf-part

Based on arXiv:2407.18051 in collaboration with Matilde Vicini

Dario Kermanschah
CERN QCD Seminar, 30 September 2024

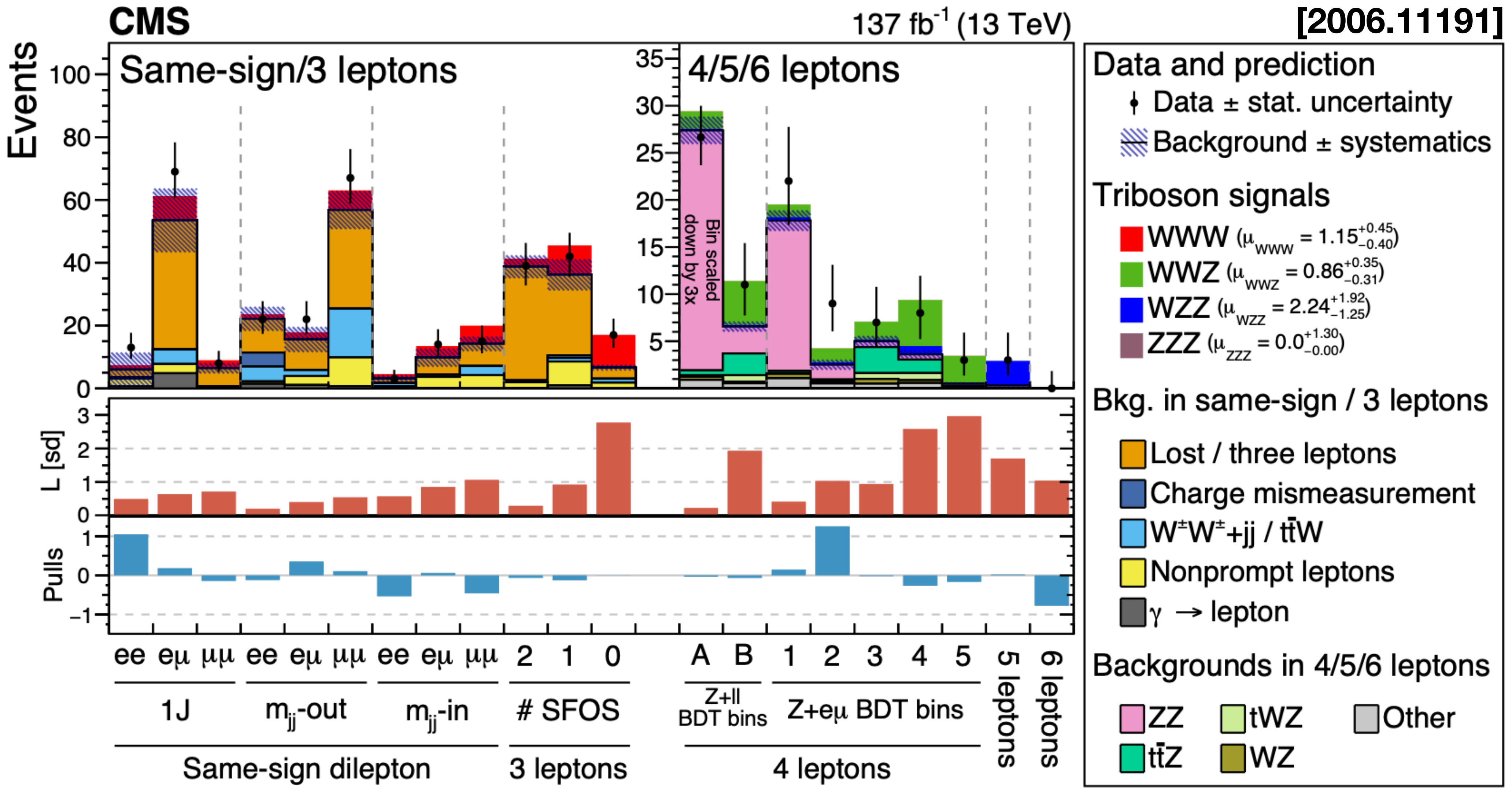
How to conquer multi-scale multi-loop calculations?

- Full two-loop amplitudes beyond 2 → 3 massless particles unavailable
- Overwhelming complexity of IBP reduction & unknown Master Integrals
- NNLO calculations become analytically intractable… resort to numerical methods!



Why vector boson production?

- Uncharted territory: 3 massive bosons at two loops
- Fewer IR singularities: only ISR (no FSR)
- ATLAS and CMS become sensitive to triple Z / W production, test quartic gauge-boson couplings & light-quark Yukawa couplings, BSM…



[2006.11191]

Our approach for the two-loop virtual contribution:
Local subtraction & direct numerical integration

Anastasiou, Haindl, Karlen,
Sterman, Venkata, Yang, Zeng
[2403.13712,2212.12162,
2008.12293,1812.03753]

finite remainder: $R^{(2)} = M^{(2)} \underbrace{- \frac{2\beta_0}{\epsilon} M^{(1)}}_{\text{UV renorm.}} \underbrace{- \mathbf{Z}^{(1)} M^{(1)} - \mathbf{Z}^{(2)} M^{(0)}}_{\text{Catani IR poles}}$

$C_{\mathrm{IR\&UV}}$ from local factorisation

hard scattering amplitude

$$M^{(2)}_{\mathrm{hard}} = M^{(2)} \boxed{- C_{\mathrm{IR\&UV}}}$$

- finite in $D = 4$ dimensions, no dim reg. ($\gamma^5 \ldots$)
- integrate numerically with Monte Carlo
- directly in momentum space
- no IBPs, no Master integrals, no sector decomposition

renormalisation & factorisation scheme change

$$\boxed{+ C_{\mathrm{IR\&UV}}} - \frac{2\beta_0}{\epsilon} M^{(1)} - \mathbf{Z}^{(1)} M^{(1)} - \mathbf{Z}^{(2)} M^{(0)}$$

calculate analytically in $D = 4 - 2\epsilon$ dimensions

$$= c_1 \, M^{(1)}_{\mathrm{hard}} + c_0 \, M^{(0)}$$

interfere with tree & integrate over phase space

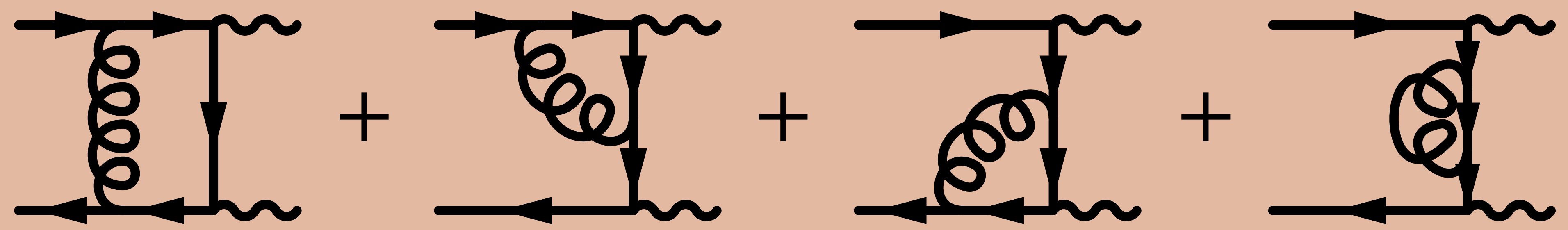
to get the virtual cross section: $\int \mathrm{d}\Pi \, \sum_{\mathrm{hel.}} |M|^2$

hard scattering amplitude

$$M_{\text{hard}} = M - C_{\text{UV\&IR}}$$

one loop

$\mathscr{M}^{(1)}(k) =$



$+\ \text{perm}\,.$

scheme change

$$C_{\text{IR\&UV}} - \text{UV renorm.} - \text{IR fact}\,.$$

Anastasiou, Haindl, Sterman, Yang, Zeng [2008.12293, 2212.12162], DK, Vicini [2407.18051]

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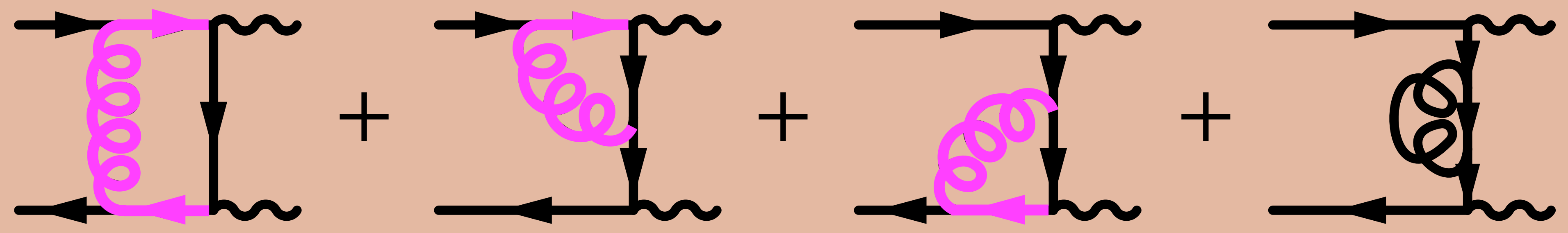
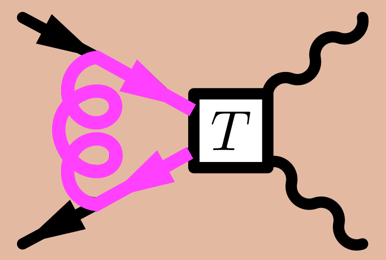


$+ \ \text{perm}.$

$\mathscr{C}^{(1)}_{\text{IR}}(k) =$



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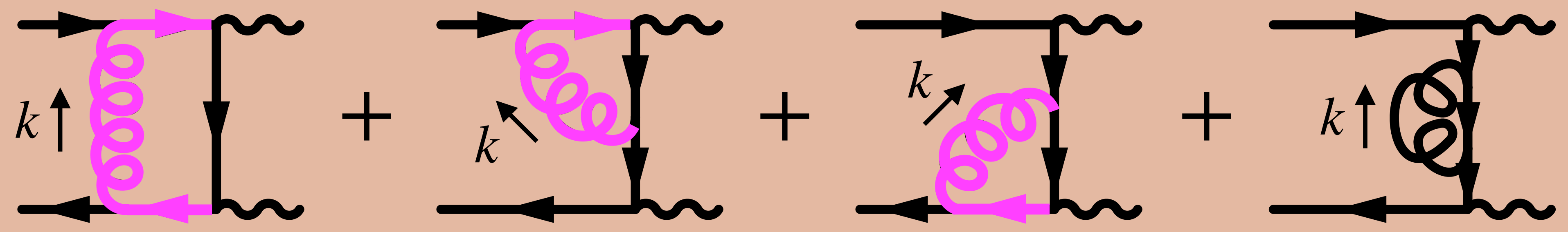

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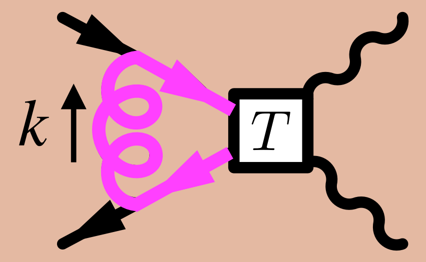

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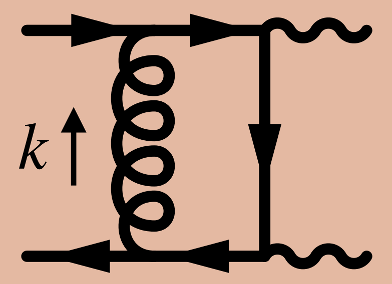

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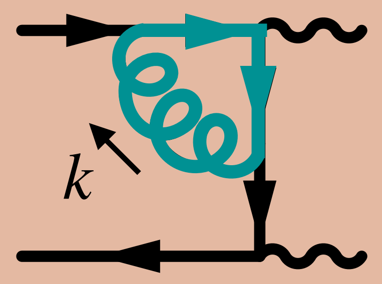

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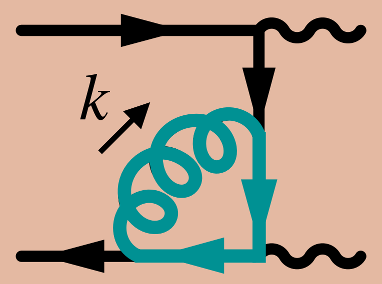

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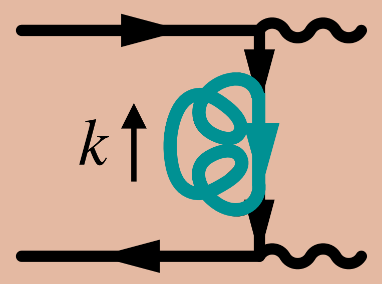

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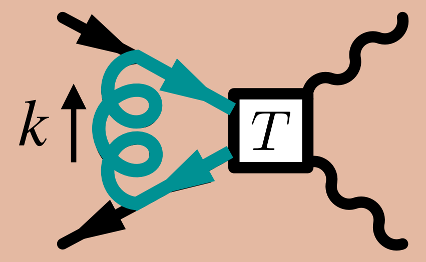
$\mathscr{C}^{(1)}_{\text{IR}}(k) =$



$$\mathscr{C}^{(1)}_{\text{UV}}(k) = \frac{a_3}{(k^2 - M^2)^3} + \frac{a_2}{(k^2 - M^2)^2} \quad \text{(tadpoles)}$$

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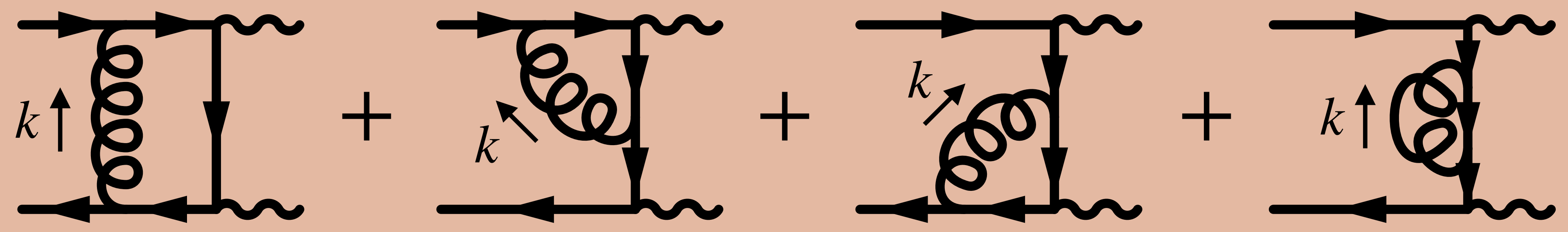

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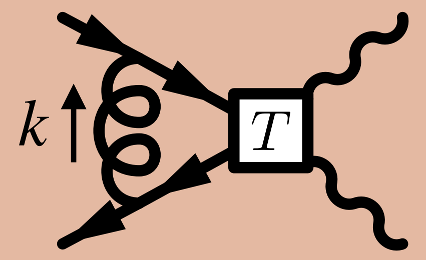

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$$\mathscr{M}^{(1)}_{\text{hard}}(k) = \mathscr{M}^{(1)}(k) - \mathscr{C}^{(1)}_{\text{IR}}(k) - \mathscr{C}^{(1)}_{\text{UV}}(k)$$

one loop

$$C^{(1)}_{\text{IR}} + C^{(1)}_{\text{UV}} - \mathbf{Z}^{(1)} M^{(1)}$$

$$= f_0^{(1)}\left[\ln\left(\frac{\mu^2}{s}\right), \ln\left(\frac{\mu^2}{M^2}\right)\right] \widetilde{M}^{(0)}$$

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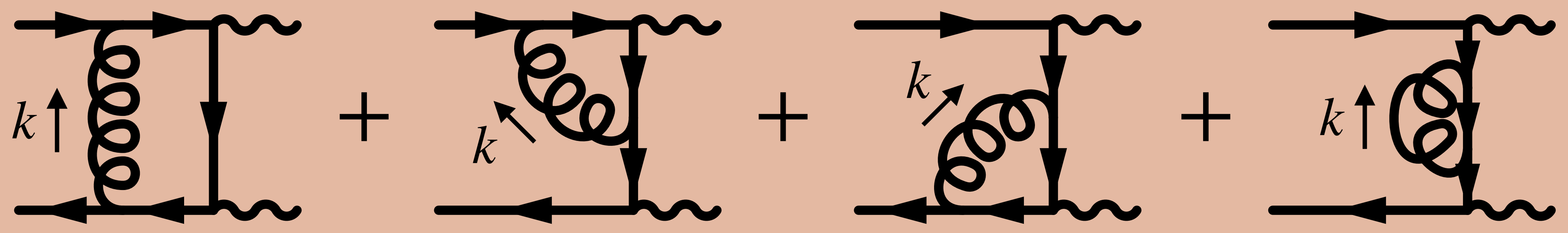

$+$ perm.

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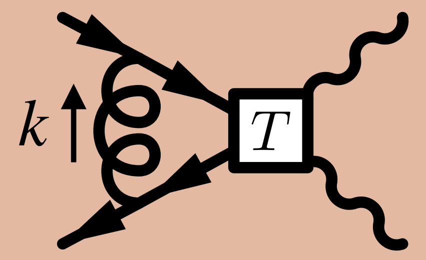

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individually finite in 4 dimensions

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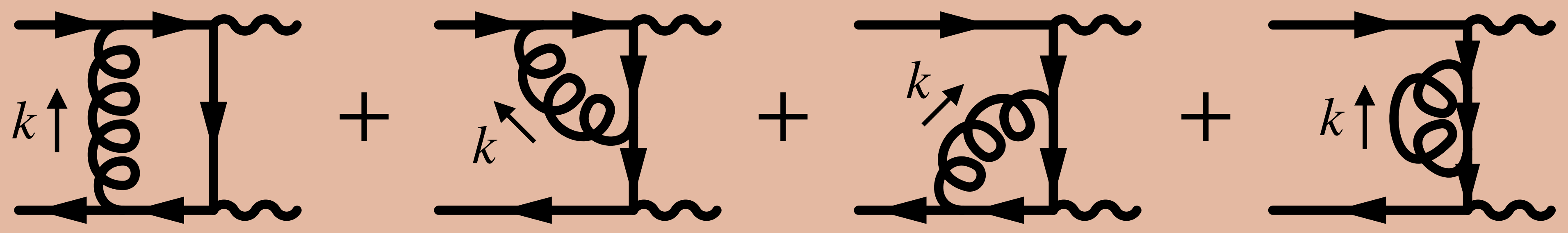

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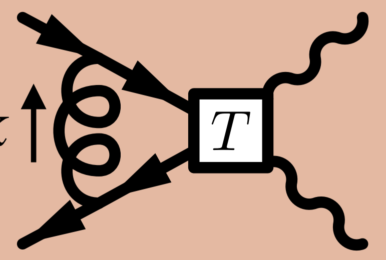

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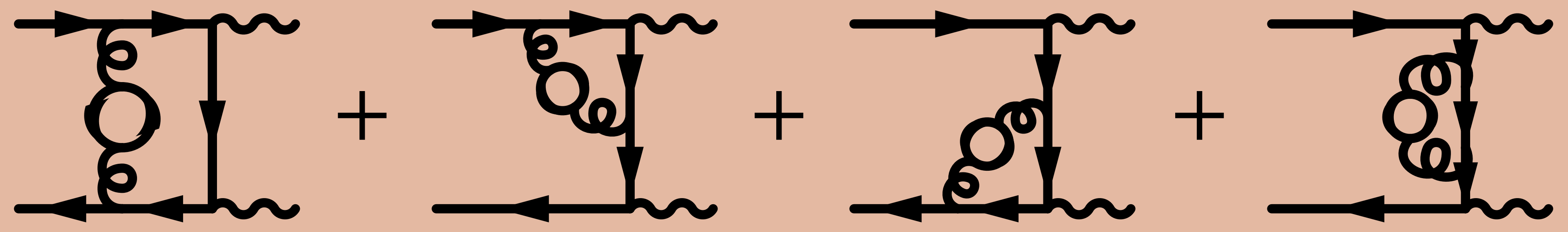
two loop Nf

$\mathscr{M}^{(2,N_f)}(k,l) =$



$+$ perm.

$$\sim \frac{1}{l^2(l+k)^2}\mathscr{M}^{(1)}(k)$$

$$\boxed{\mathscr{M}^{(2,N_f)}_{\text{hard}}(k,l)} \sim \left(\frac{1}{l^2(l+k)^2} - \frac{1}{(l^2 - M^2)^2}\right)\mathscr{M}^{(1)}_{\text{hard}}(k)$$

scheme change

$$C_{\text{IR\&UV}} - \text{UV renorm.} - \text{IR fact.}$$

one loop

$$C^{(1)}_{\text{IR}} + C^{(1)}_{\text{UV}} - \mathbf{Z}^{(1)}M^{(1)}$$

$$= \boxed{f^{(1)}_0\left[\ln\left(\frac{\mu^2}{s}\right), \ln\left(\frac{\mu^2}{M^2}\right)\right]} \, \boxed{\widetilde{M}^{(0)}}$$

individually finite in 4 dimensions

two loop Nf

$$C^{(2,N_f)}_{\text{IR\&UV}} - \frac{2\beta_0}{\epsilon}M^{(1)} - \mathbf{Z}^{(2,N_f)}M^{(0)}$$

$$= \boxed{f^{(2,N_f)}_1\left[\ln\left(\frac{\mu^2}{M^2}\right)\right]} \, \boxed{M^{(1)}_{\text{hard}}}$$

$$+ \boxed{f^{(2,N_f)}_0\left[\ln\left(\frac{\mu^2}{s}\right), \ln\left(\frac{\mu^2}{M^2}\right)\right]} \, \boxed{\widetilde{M}^{(0)}}$$

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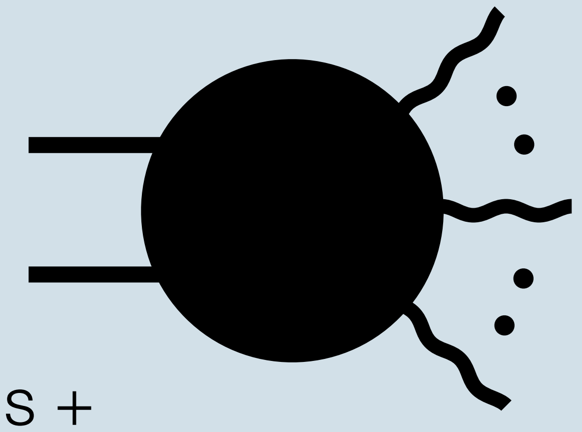
Local singularities of finite loop integrals

$$M_{\text{hard}} = \sum$$

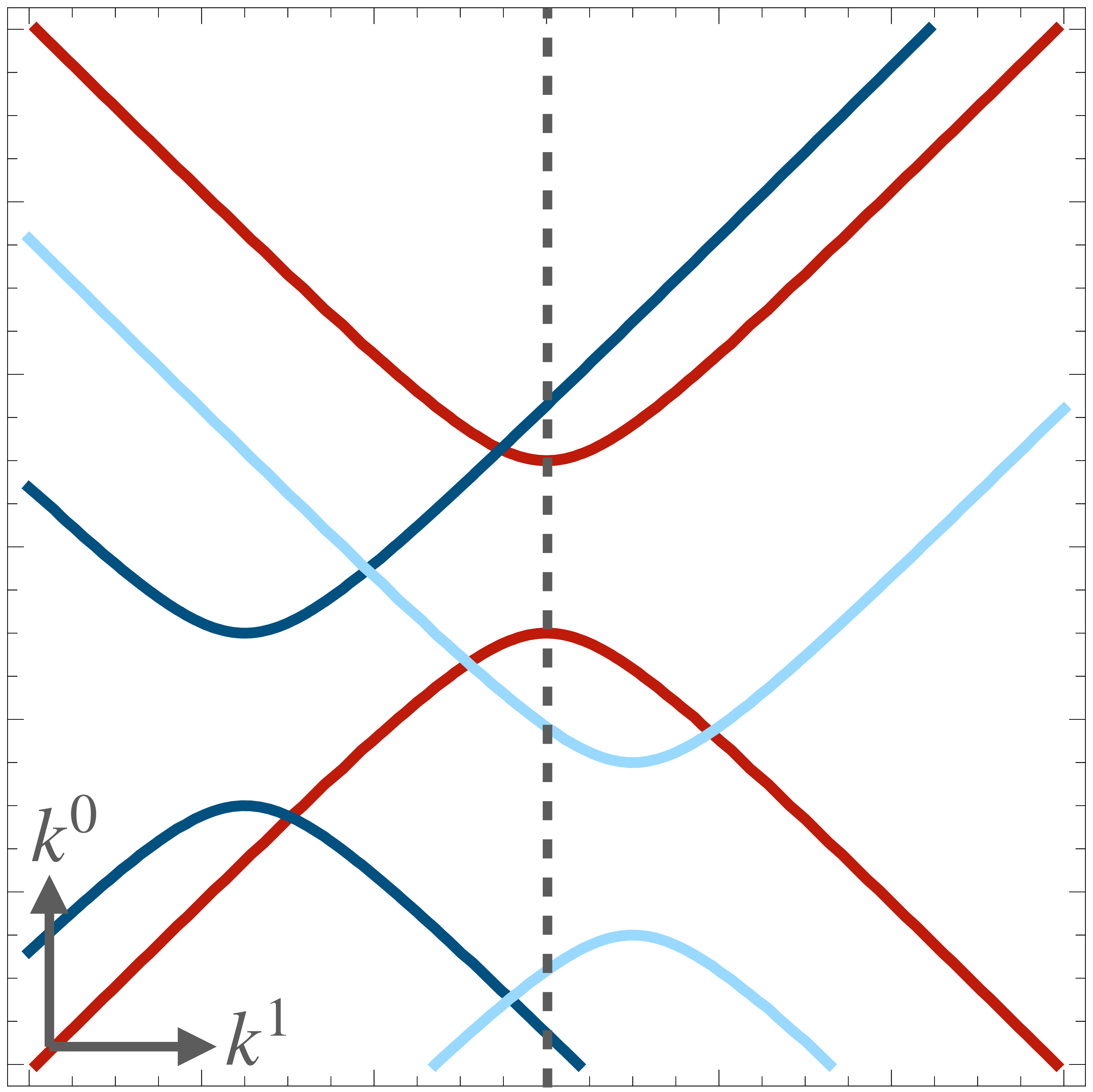


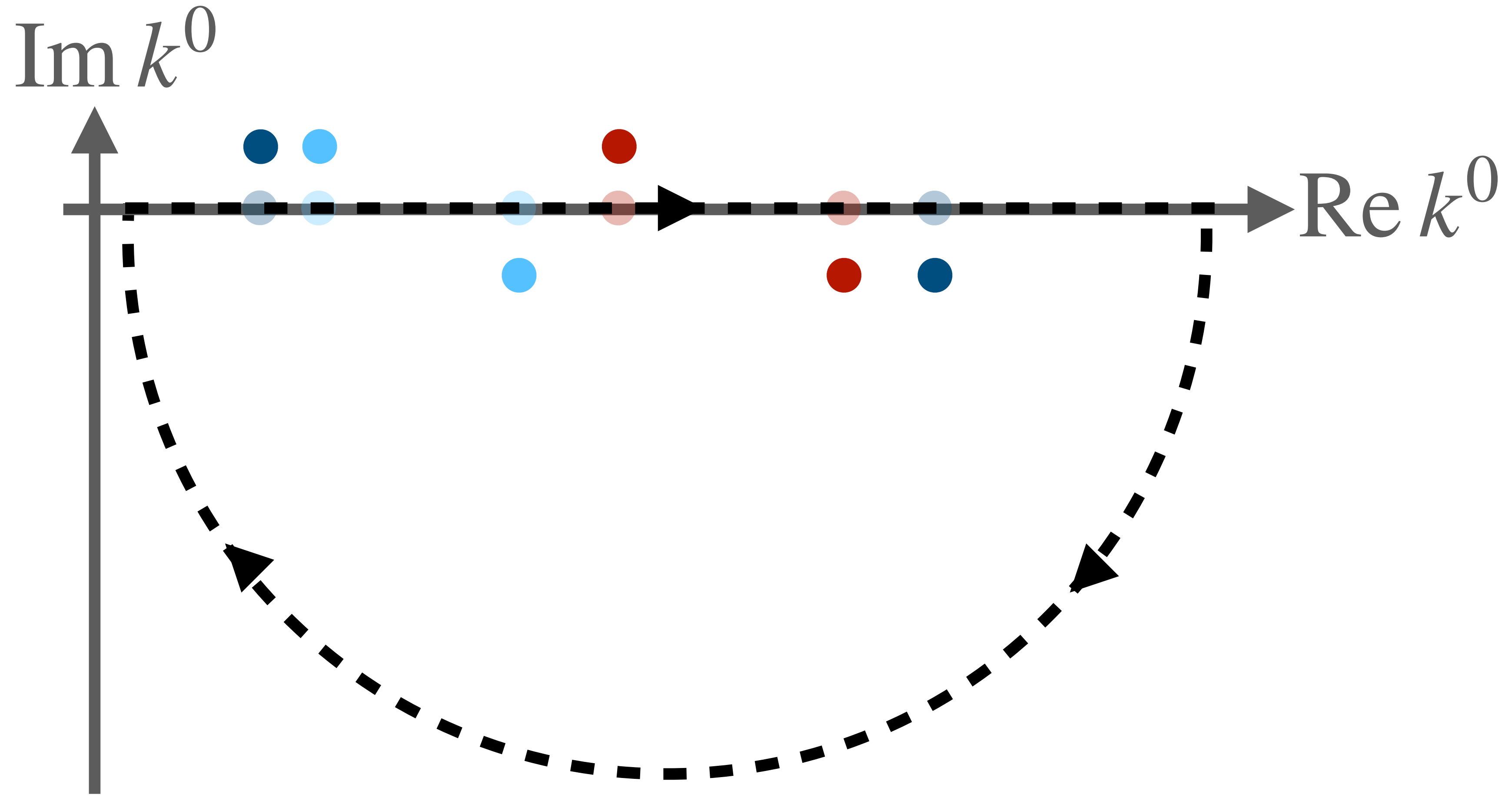
Feyn. diagrams +
local IR & UV CTs

$$= \lim_{\epsilon\to 0} \int [\mathrm{d}^4 k] \sum \cdots \frac{\cdots}{q_i^2 - m_i^2 + \mathrm{i}\epsilon} \cdots$$

- ↯ poles in the integration domain
- ✔ causal prescription
- ⚠ implement causal prescription for numerical integration

  → analytic integration over k^0





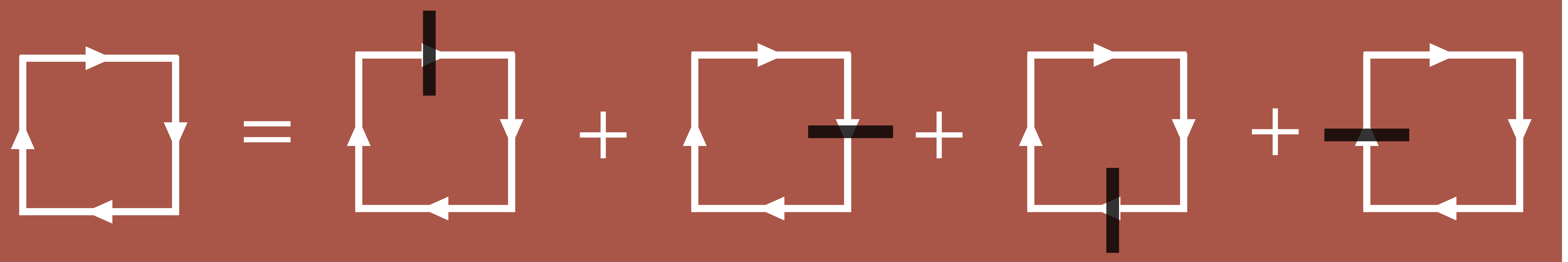
Loop integrals are rational functions in the energy component of the loop momentum

→ integrate using the residue theorem: from D to $D-1$ integration dimensions per loop

Catani, Rodrigo et al. [0804.3170], ETHZ [1906.06138], Mainz [1902.02135], Valencia [2001.03564, 2010.12971]

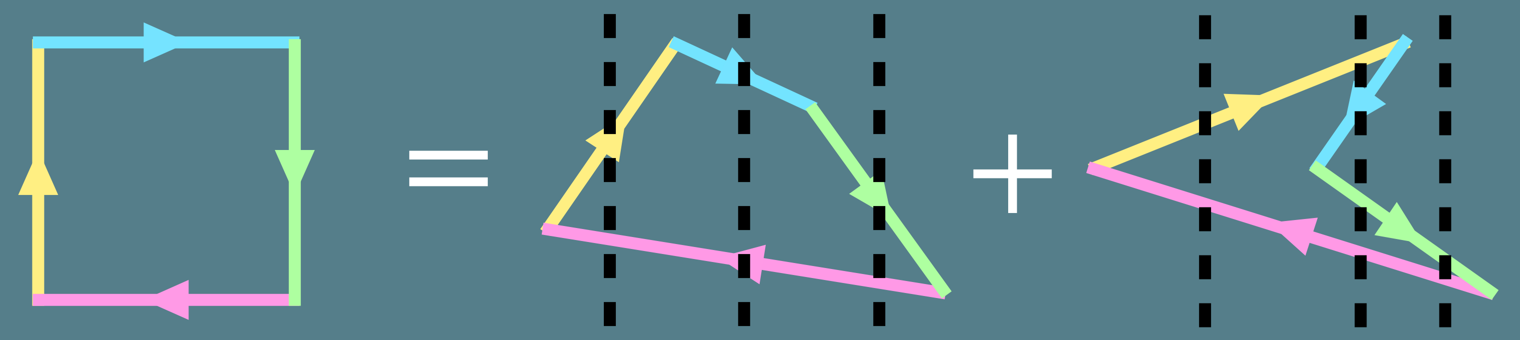
Loop-Tree Duality

compact expression but problematic spurious singularities and derivatives for raised propagators



Time-Ordered Perturbation Theory

still some spurious singularities and more terms

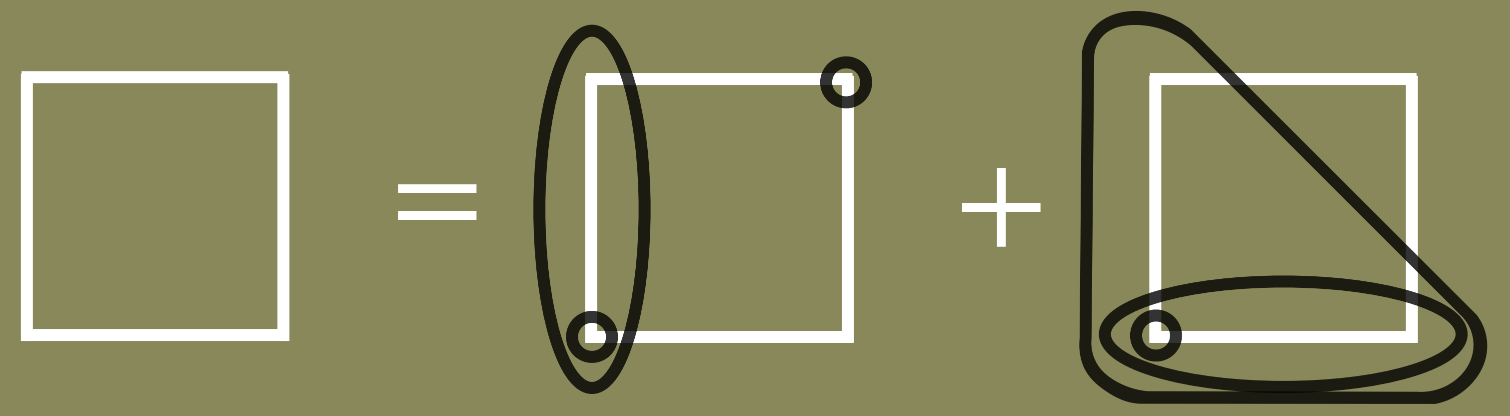


+ [22 others]

Capatti [2211.09653]

Cross-Free Family representation

no spurious singularities



+ [18 others]

+ many others …

ETHZ [2009.05509], Mainz [2208.01060], Stony Brook [2309.13023], Valencia [2006.11217, 2112.09028, 2103.09237, 2102.05062]

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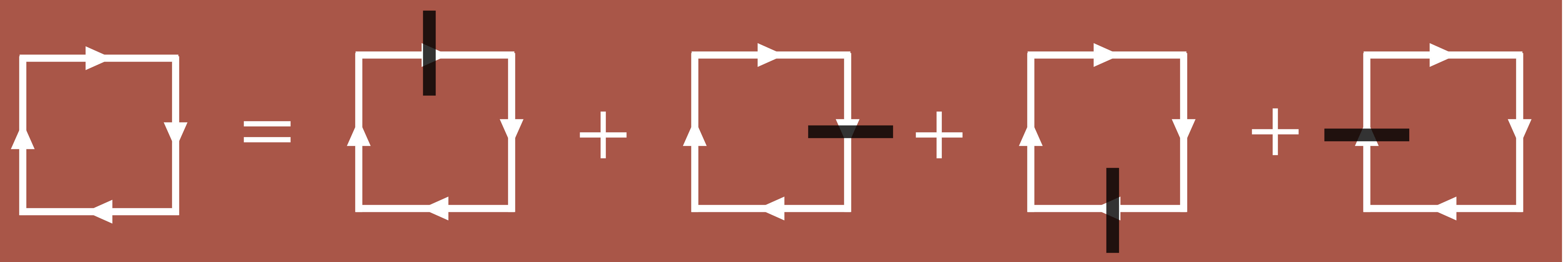

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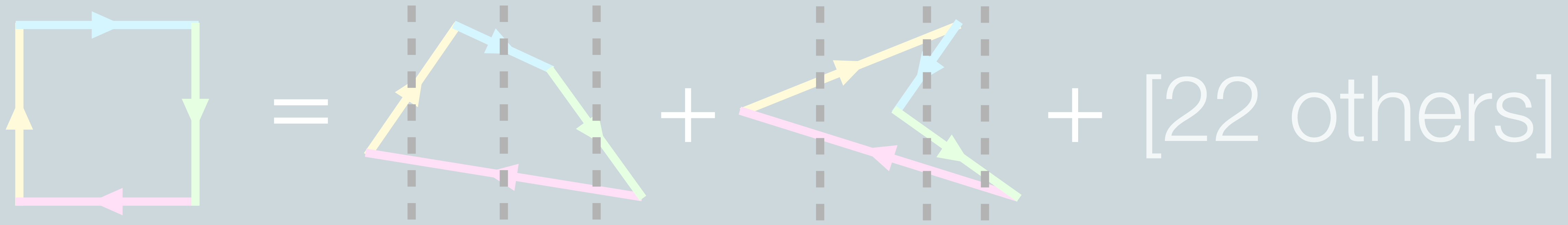

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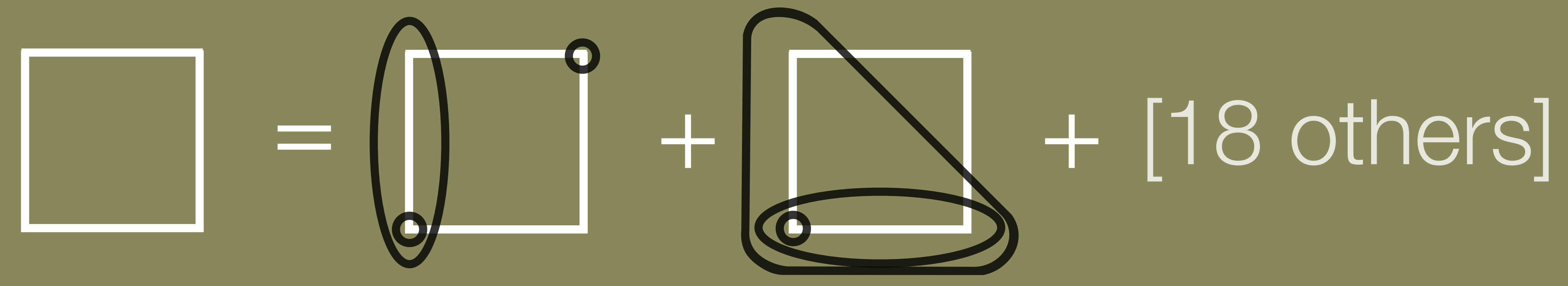

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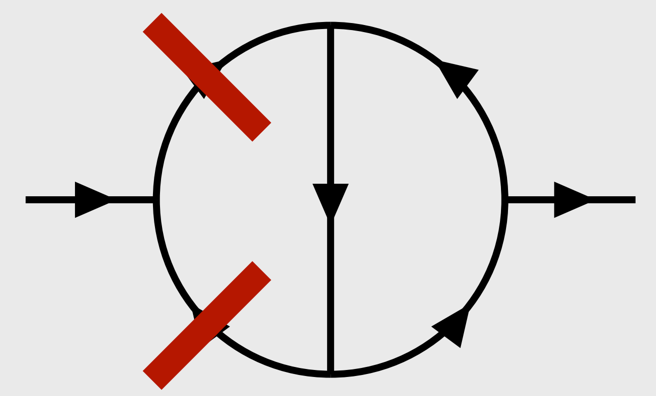
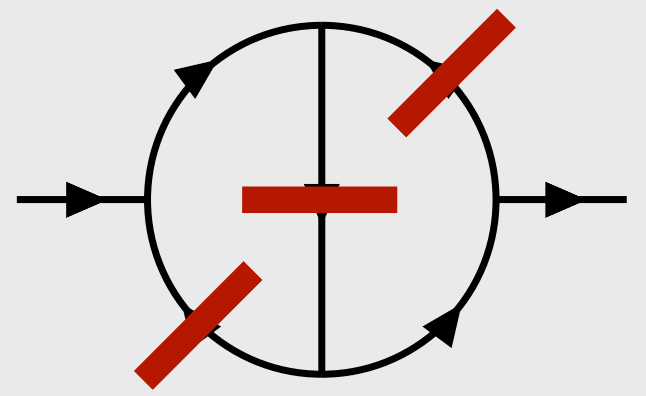
Loop-Tree Duality beyond one loop

⚠ no loops ⚠ ⚠ no forest ⚠

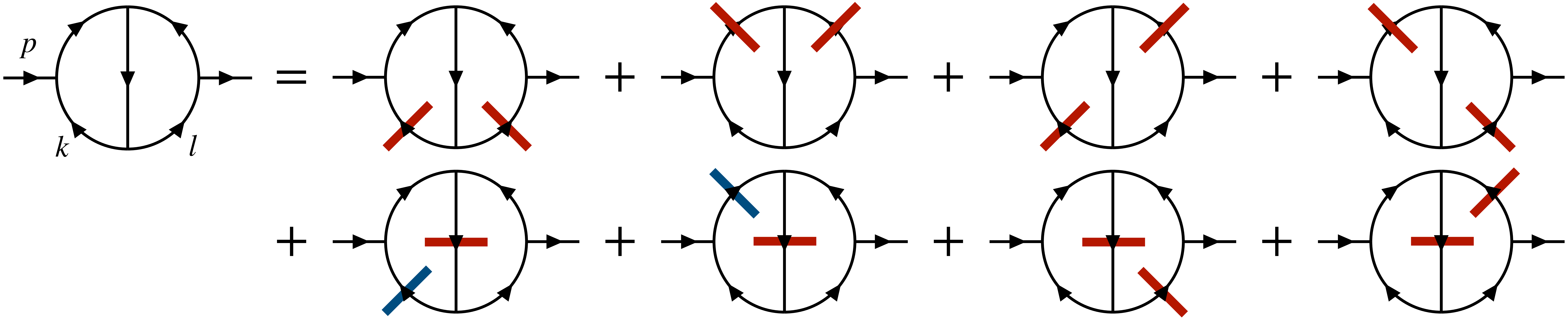




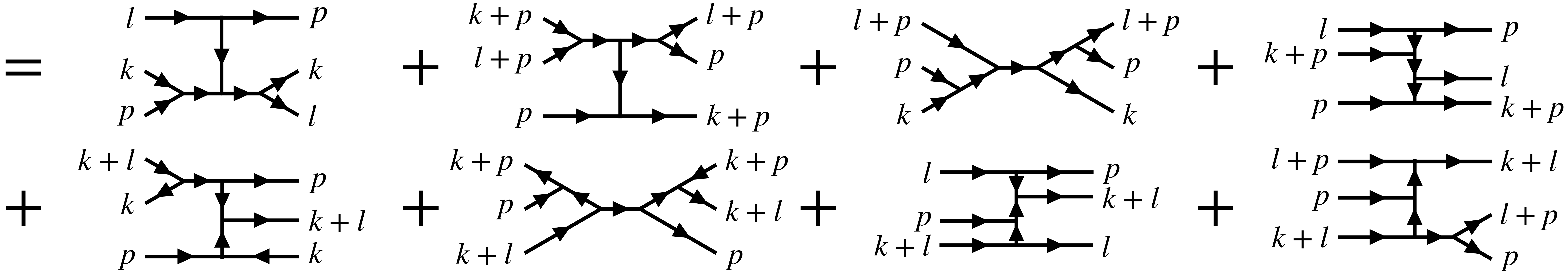
⊕ ▬ positive OS energy

⊖ ▬ negative OS energy

integrated over spatial loop momenta

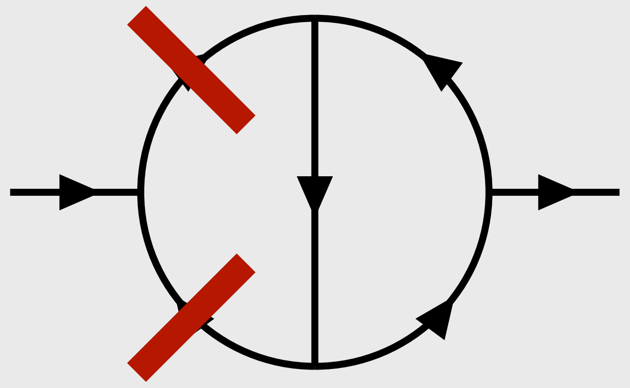


integrated over phase space of additional external on-shell particles and summed over helicities

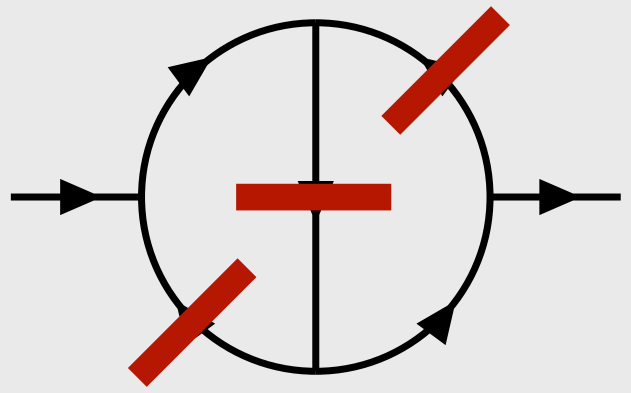


Loop-Tree Duality beyond one loop

⚠ no loops ⚠



⚠ no forest ⚠



⊕ positive OS energy

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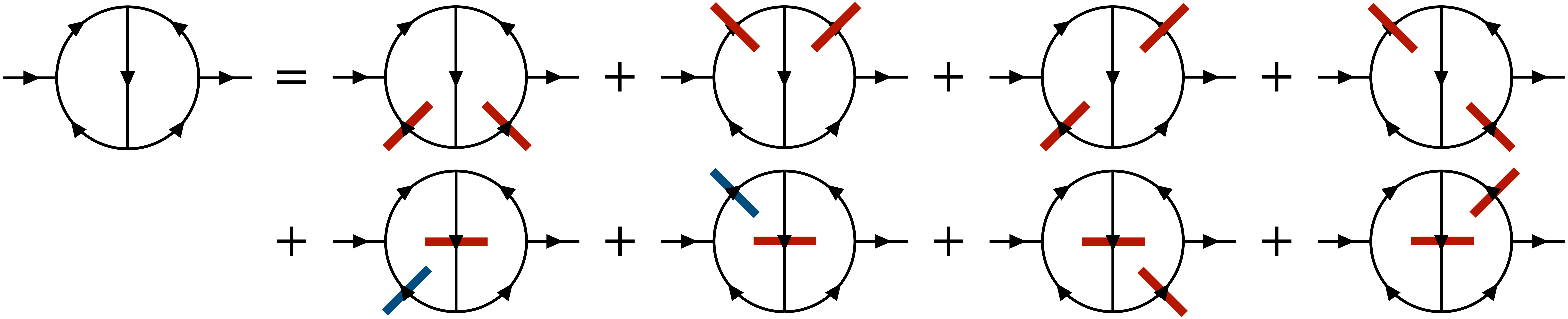
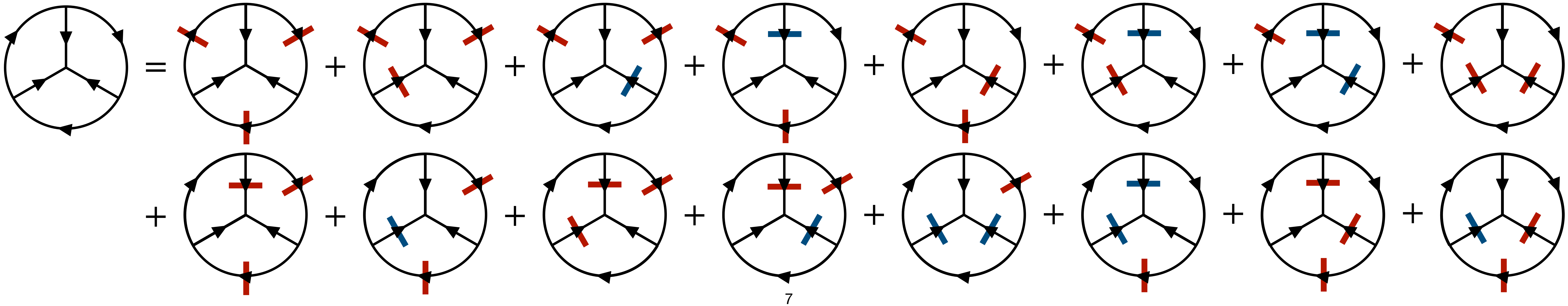
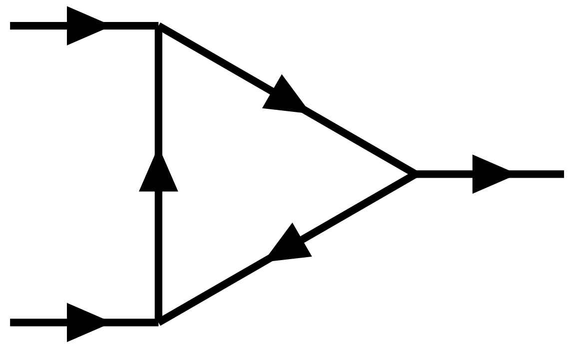




Singular surfaces in Loop-Tree Duality

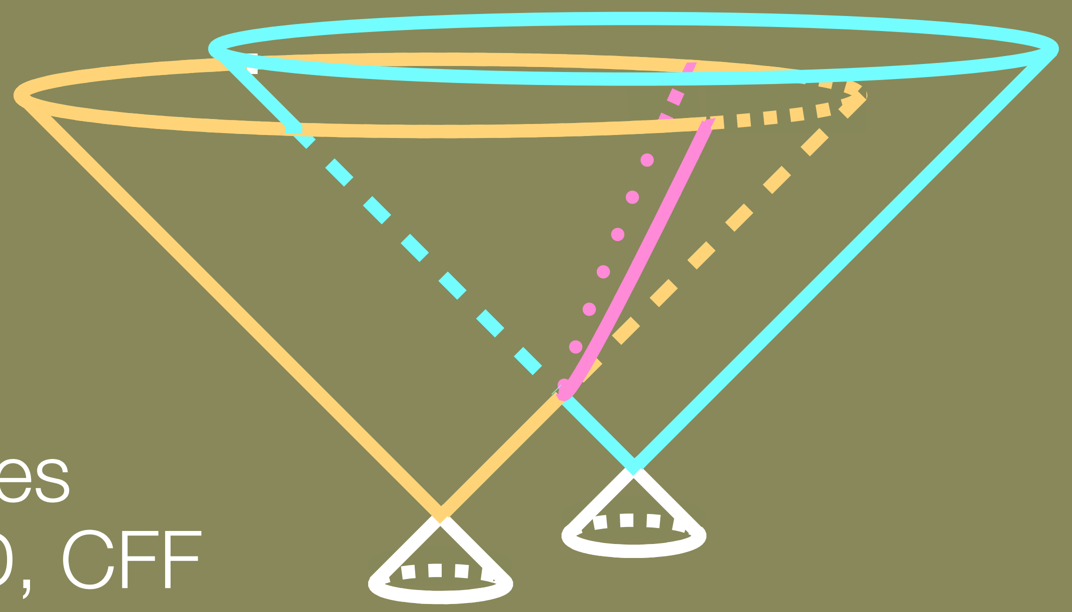


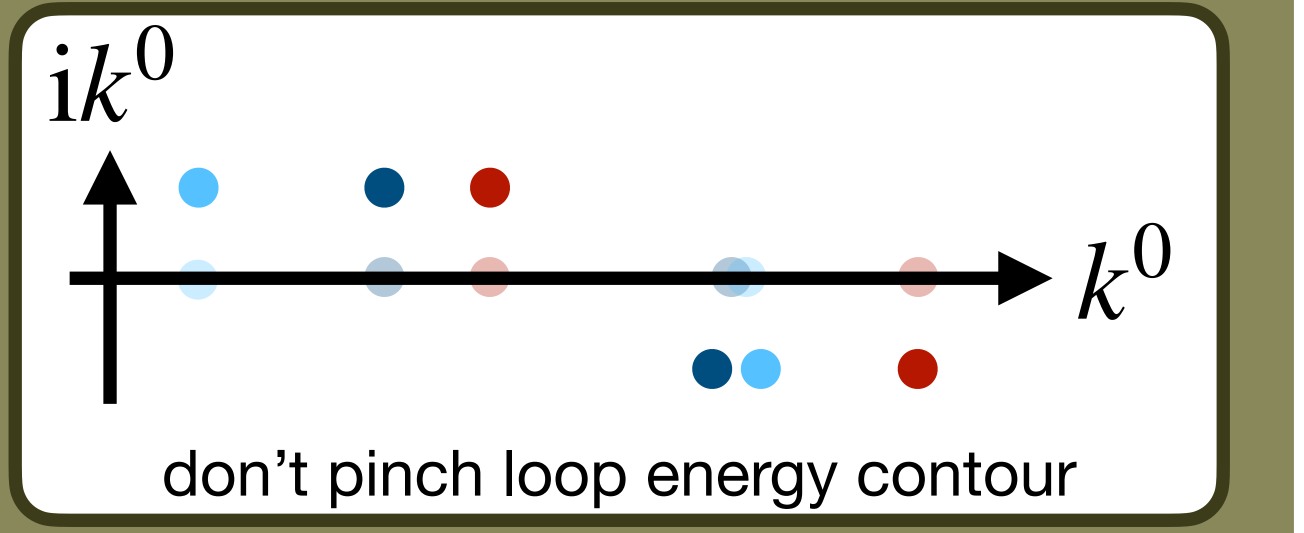
$$\sim \lim_{\epsilon\to 0}\int \mathrm{d}^3\vec{k} \left\{ \begin{aligned} & \frac{1}{2E_3}\,\frac{1}{E_3 - E_1 + p_1^0}\,\frac{1}{E_3 + E_1 + p_1^0}\,\frac{1}{E_3 - E_2 - p_2^0}\,\frac{1}{E_3 + E_2 - p_2^0} \\ + & \frac{1}{E_1 - E_3 - p_1}\,\frac{1}{E_1 + E_3 - p_1}\,\frac{1}{2E_1}\,\frac{1}{E_1 - E_2 - p_1 - p_2^0}\,\frac{1}{E_1 + E_2 - p_1 - p_2^0} \\ + & \frac{1}{E_2 - E_3 + p_2^0}\,\frac{1}{E_2 + E_3 + p_2^0}\,\frac{1}{E_2 - E_1 + p_2^0 + p_1^0}\,\frac{1}{E_2 + E_1 + p_2^0 + p_1^0}\,\frac{1}{2E_2} \end{aligned} \right\}$$

spurious singularities

Hyperboloid

may cause numerical instabilities
not present in (P)TOPT, causal LTD, CFF



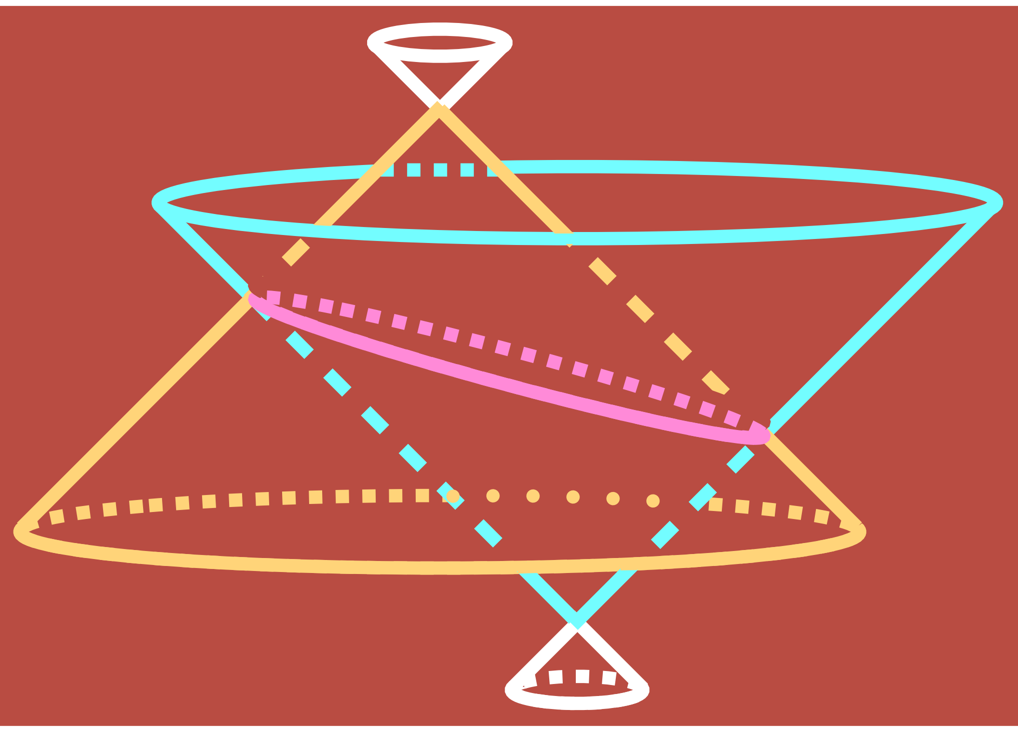


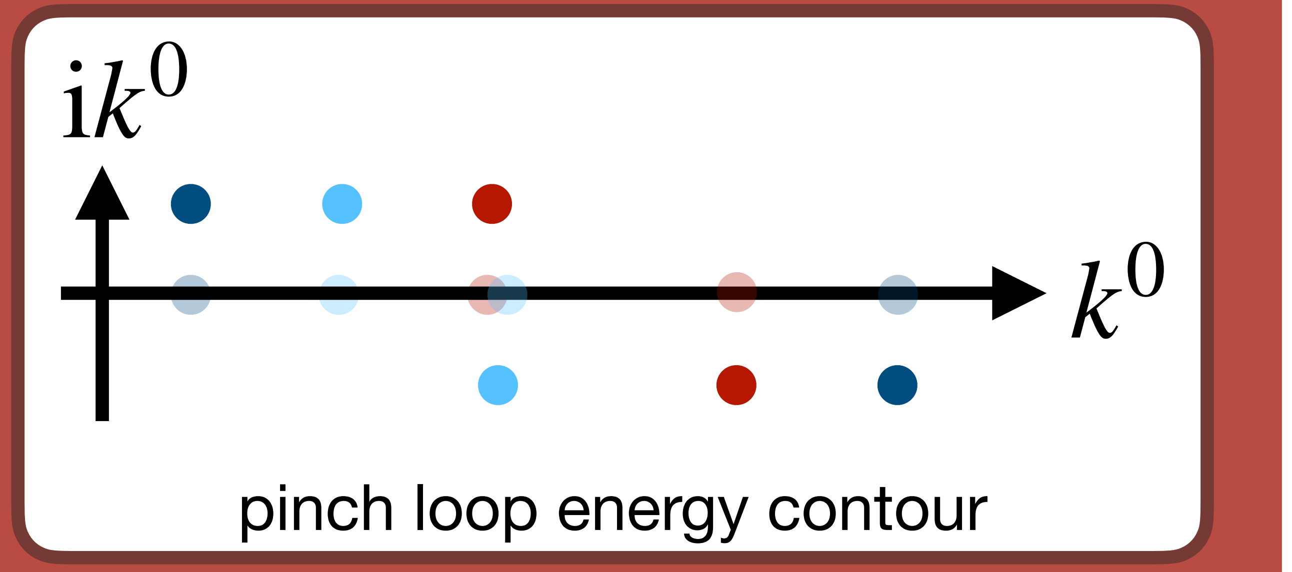
don't pinch loop energy contour

threshold singularities

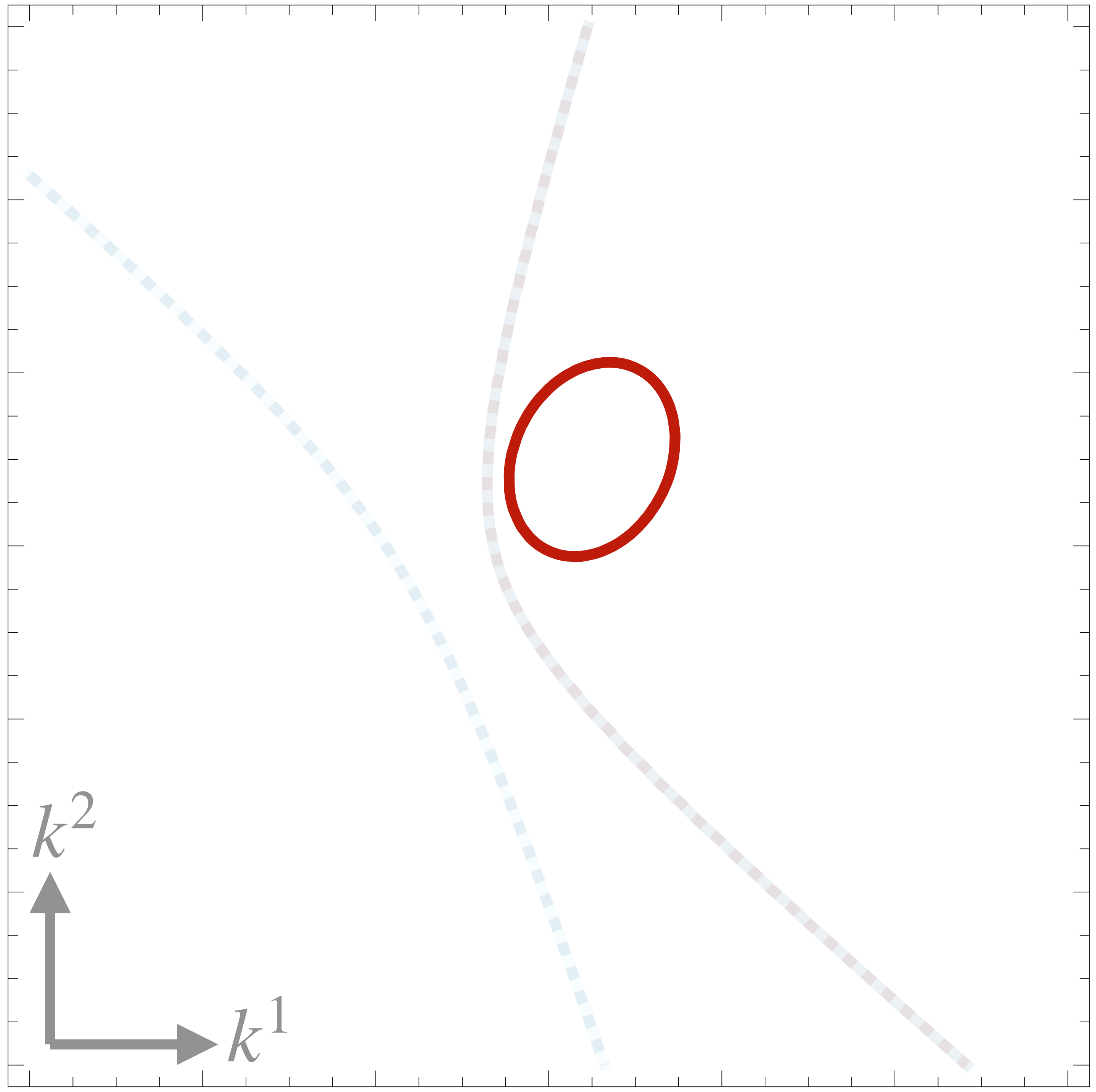
Ellipsoid

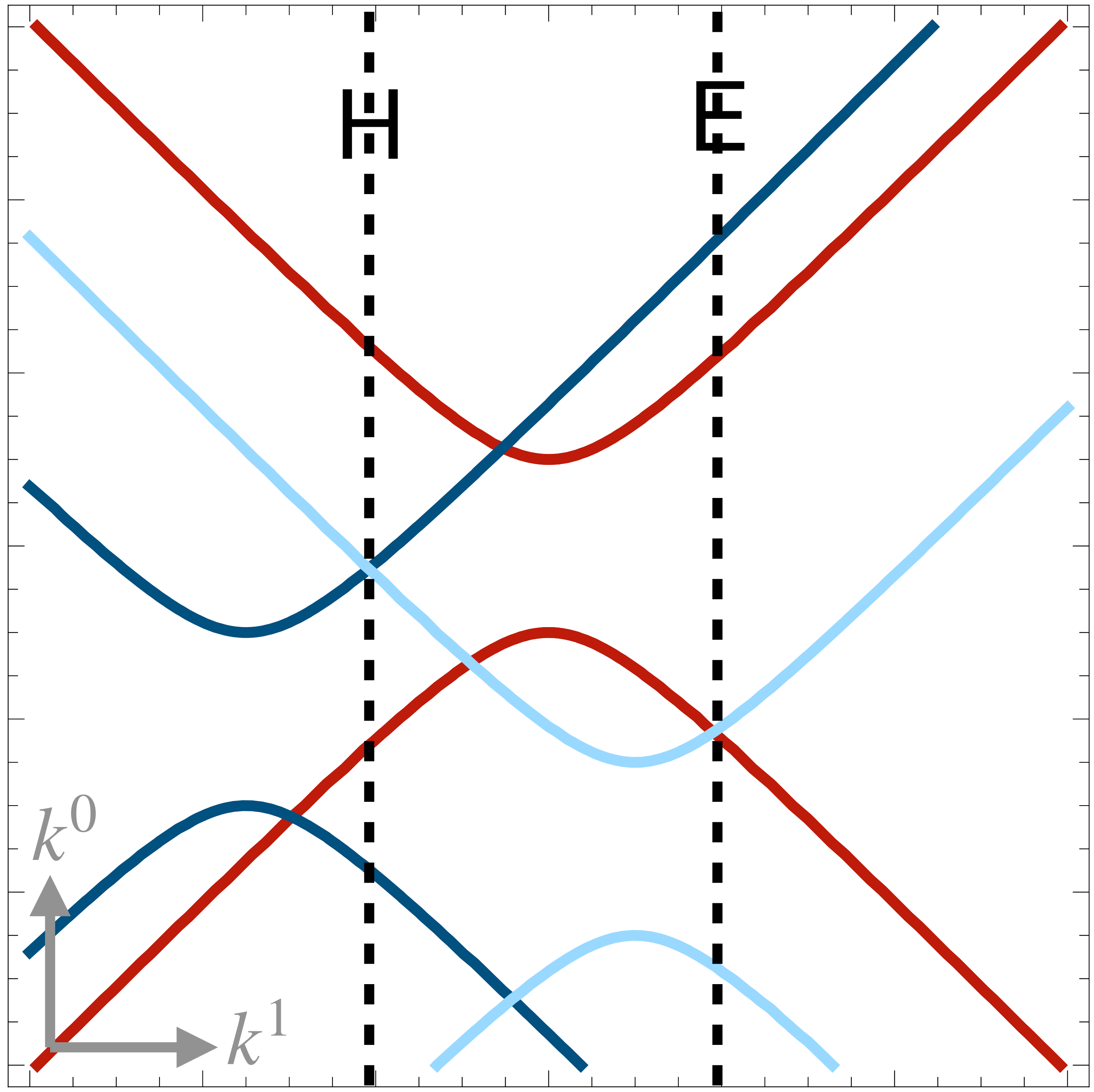
dictated by unitarity
regularised by causality





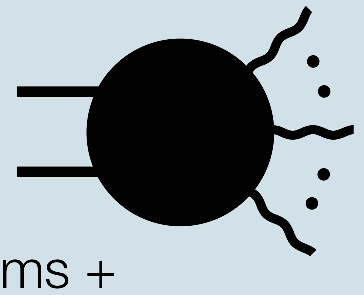
pinch loop energy contour





Threshold singularities

$$M_{\text{hard}} = \sum$$



$$= \lim_{\epsilon\to0}\int [\mathrm{d}^4k] \sum \cdots \frac{\cdots}{q_i^2 - m_i^2 + \mathrm{i}\epsilon} \cdots$$

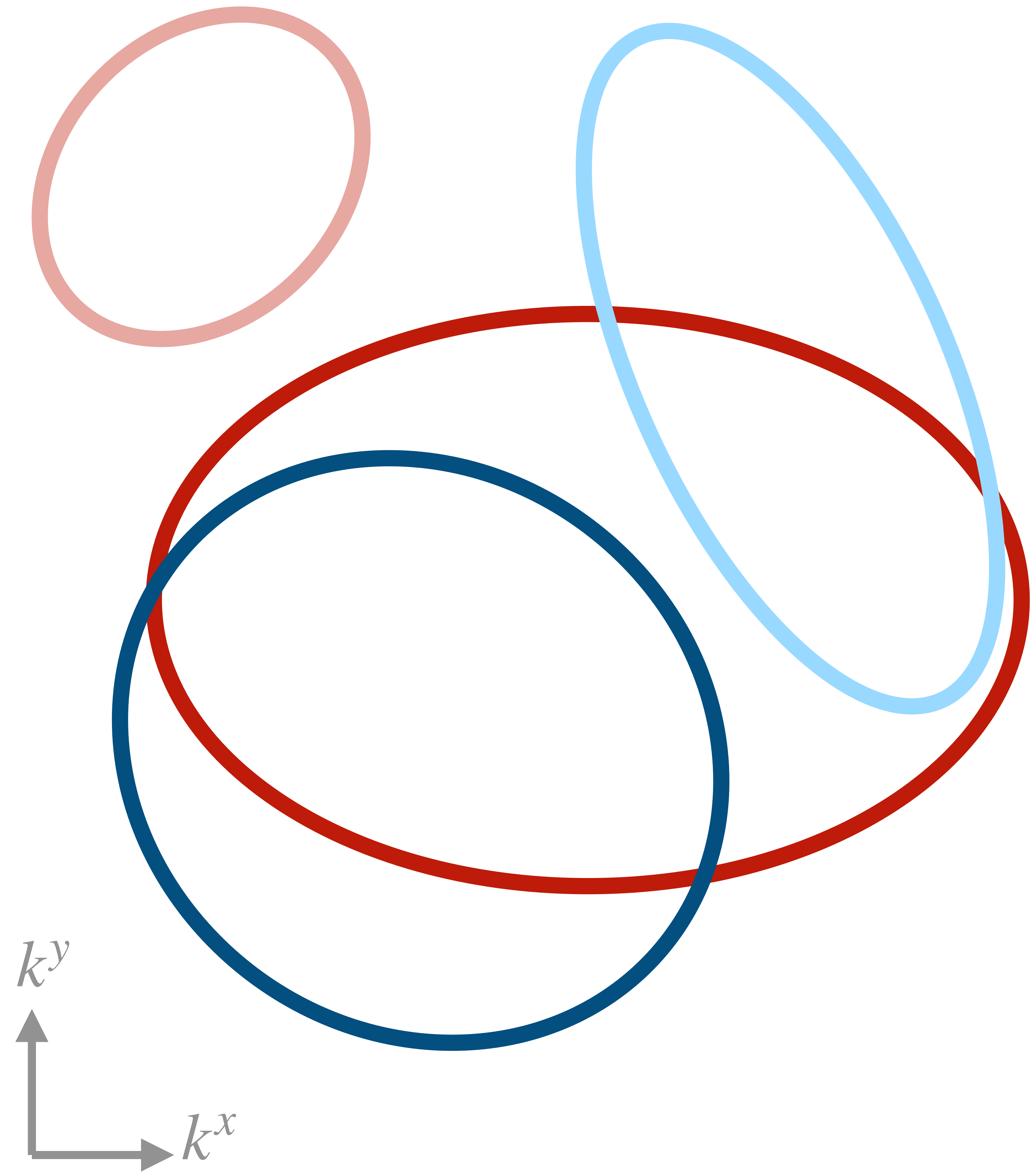
Feyn. diagrams +
local IR & UV CTs

$$= \lim_{\epsilon\to0}\int [\mathrm{d}^3\vec{k}] \sum \cdots \frac{\cdots}{E_1 + E_2 - \mathrm{i}\epsilon} \cdots$$

$$E_i = \sqrt{\vec{q}_i^{\,2} + m_i^2}$$

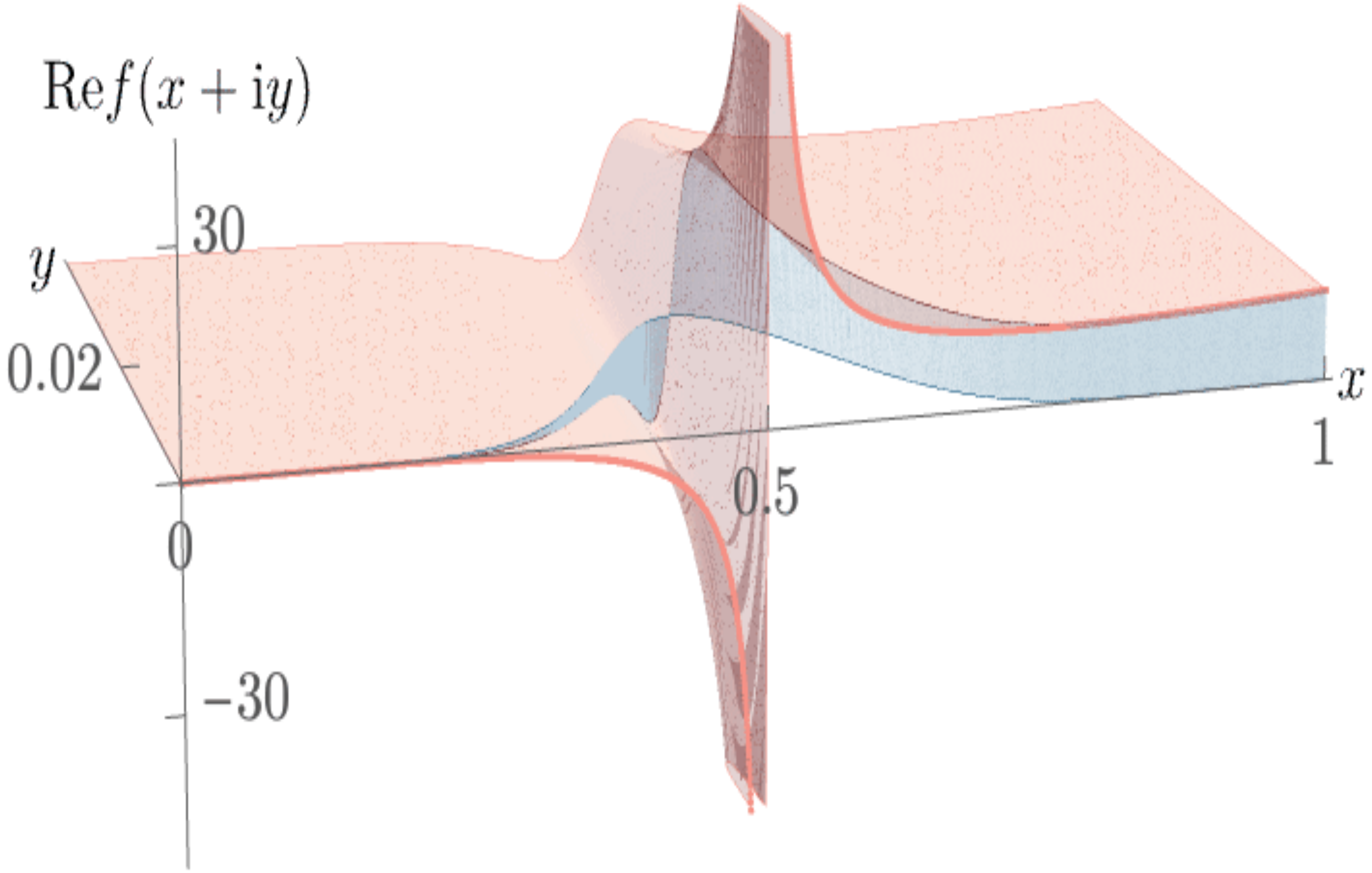
- ↯ poles in the integration domain
- ✔ causal prescription
- ⚠ implement causal prescription for numerical integration

→ Same problems? Yes but fewer integration dimensions & fewer integrand singularities in compact region!



contour deformation

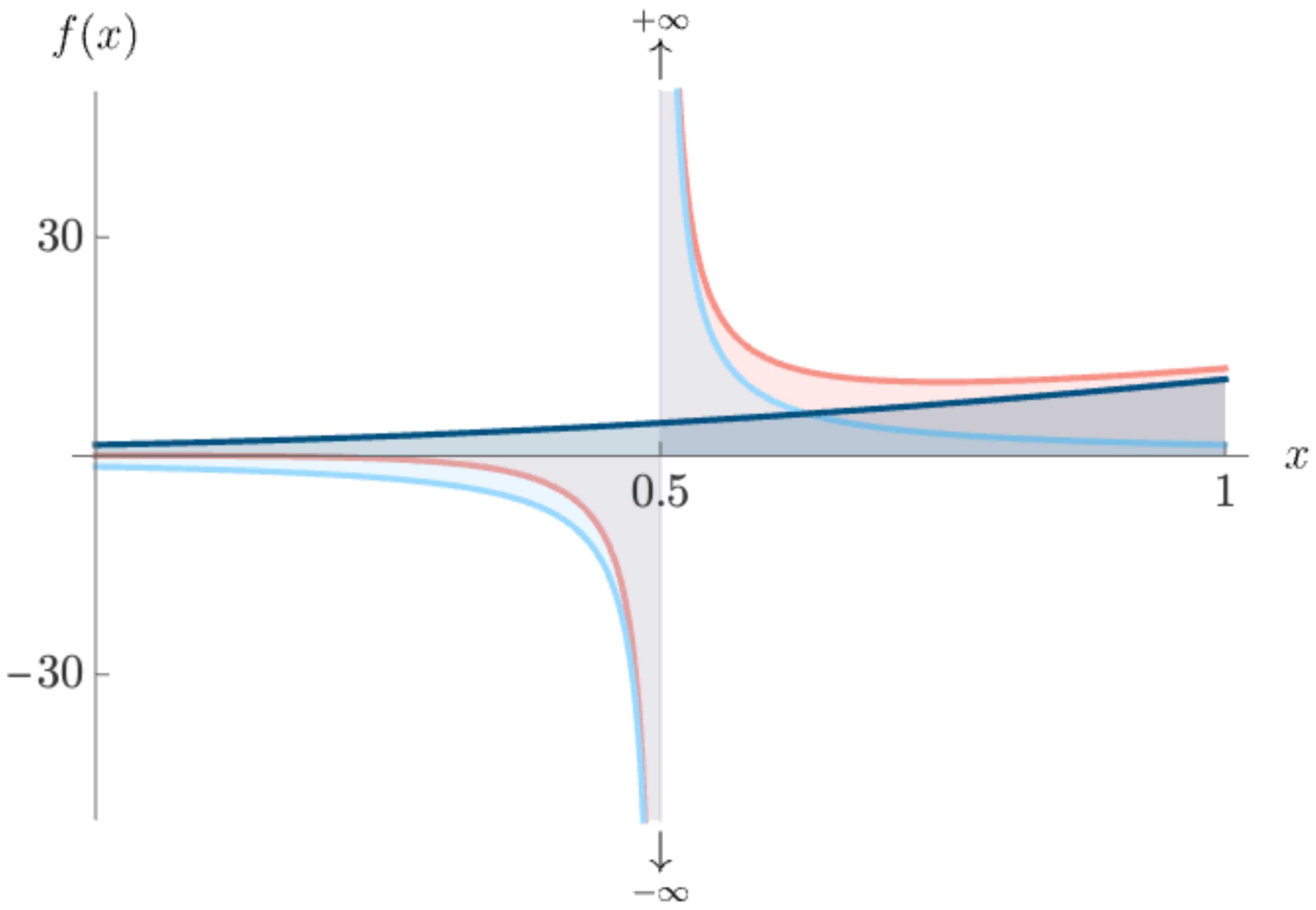
$$\mathbb{R} \to \mathbb{C}$$



$$\frac{1}{1000000}\sum_{i=1}^{1000000} \mathrm{Re} f(z(x_i)) j_z(x_i) = 4.9948$$

subtraction

$$\frac{1}{x - x_0 + \mathrm{i}\epsilon} = \mathrm{PV}\frac{1}{x - x_0} - \mathrm{i}\pi\delta(x_0)$$



$$\frac{1}{1000000}\sum_{i=1}^{1000000} (f(x_i) - f_{\mathrm{ct}}(x_i)) = 5.0008$$

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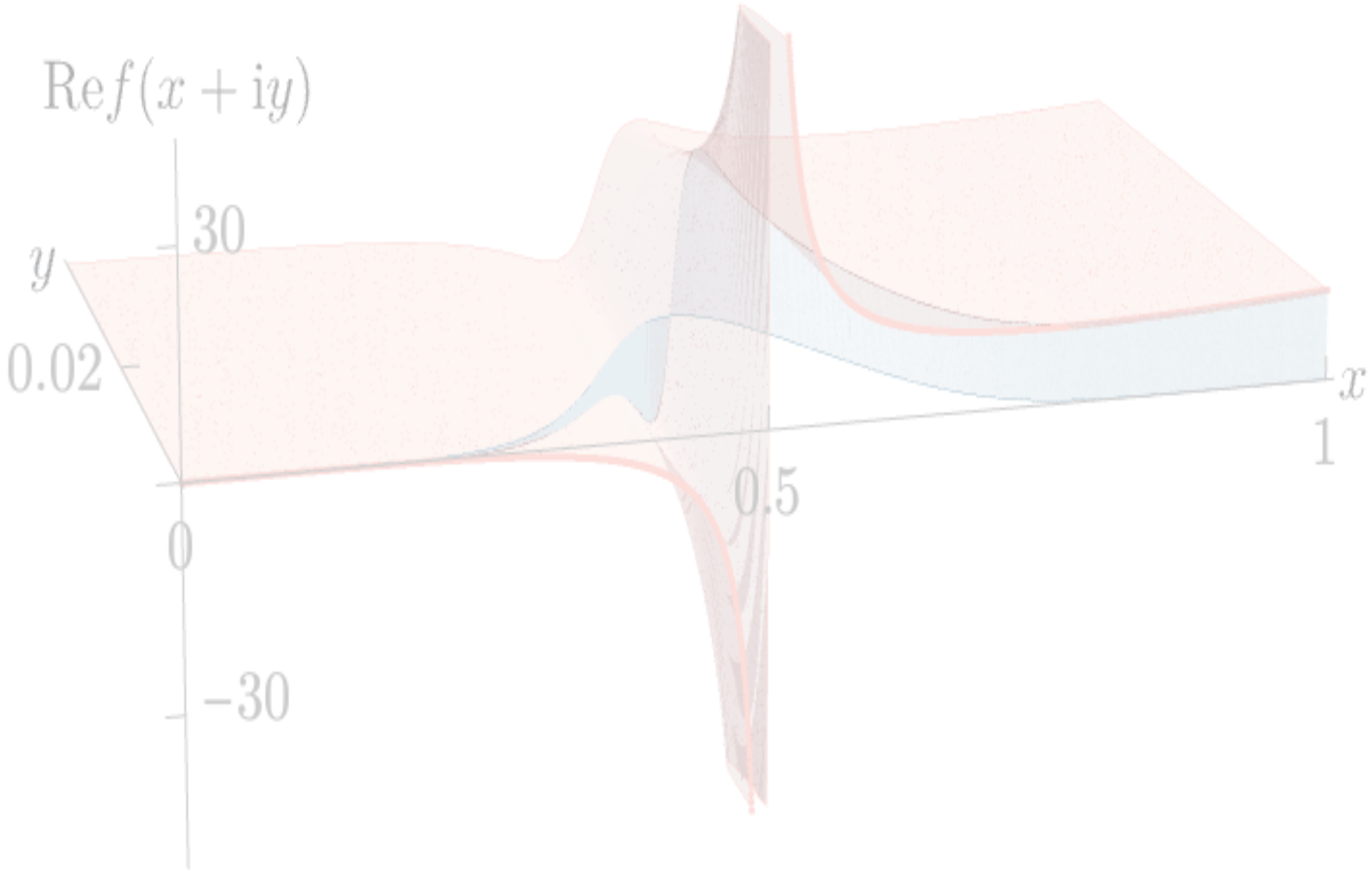

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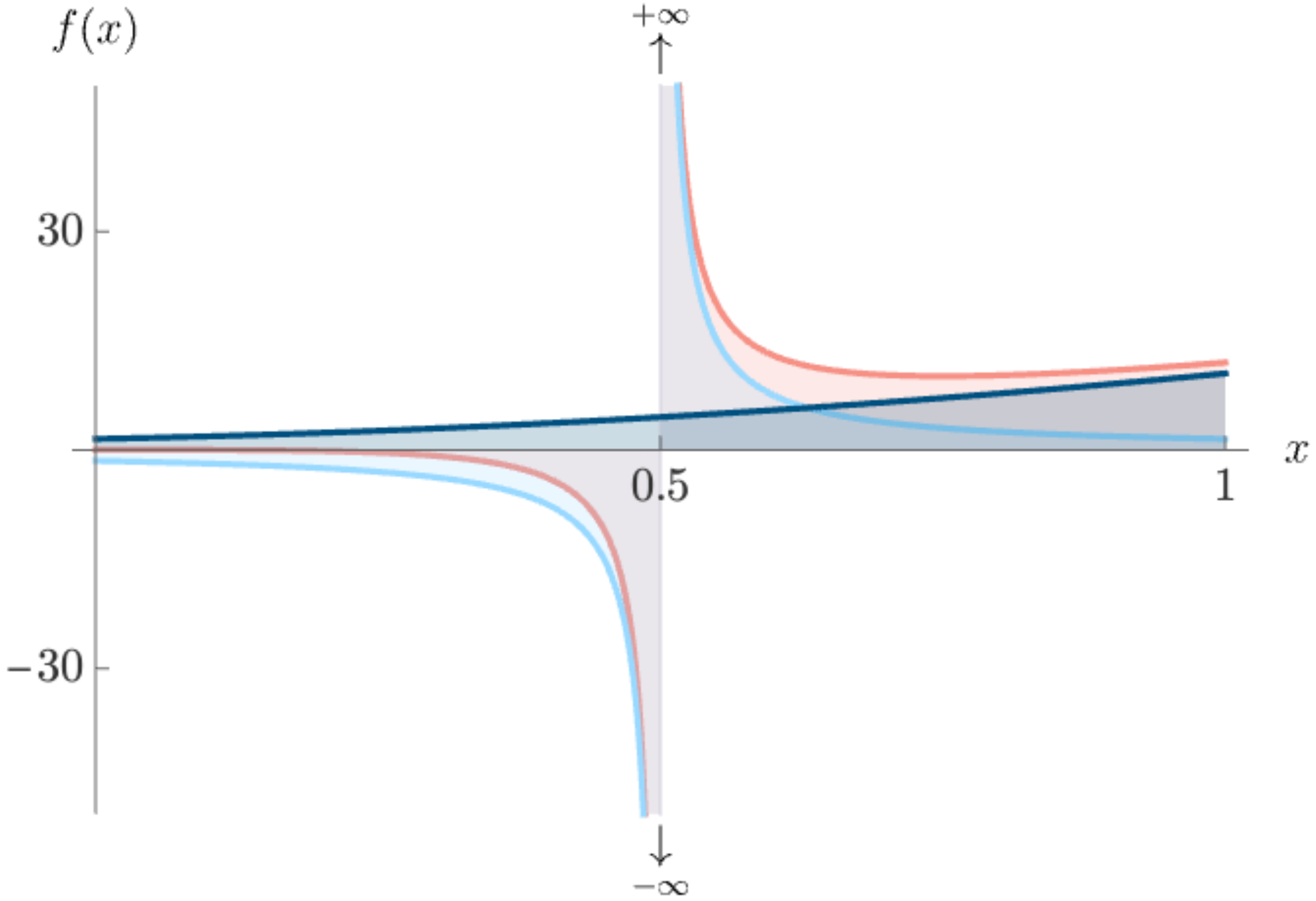

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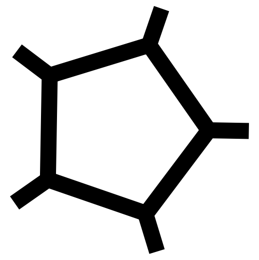
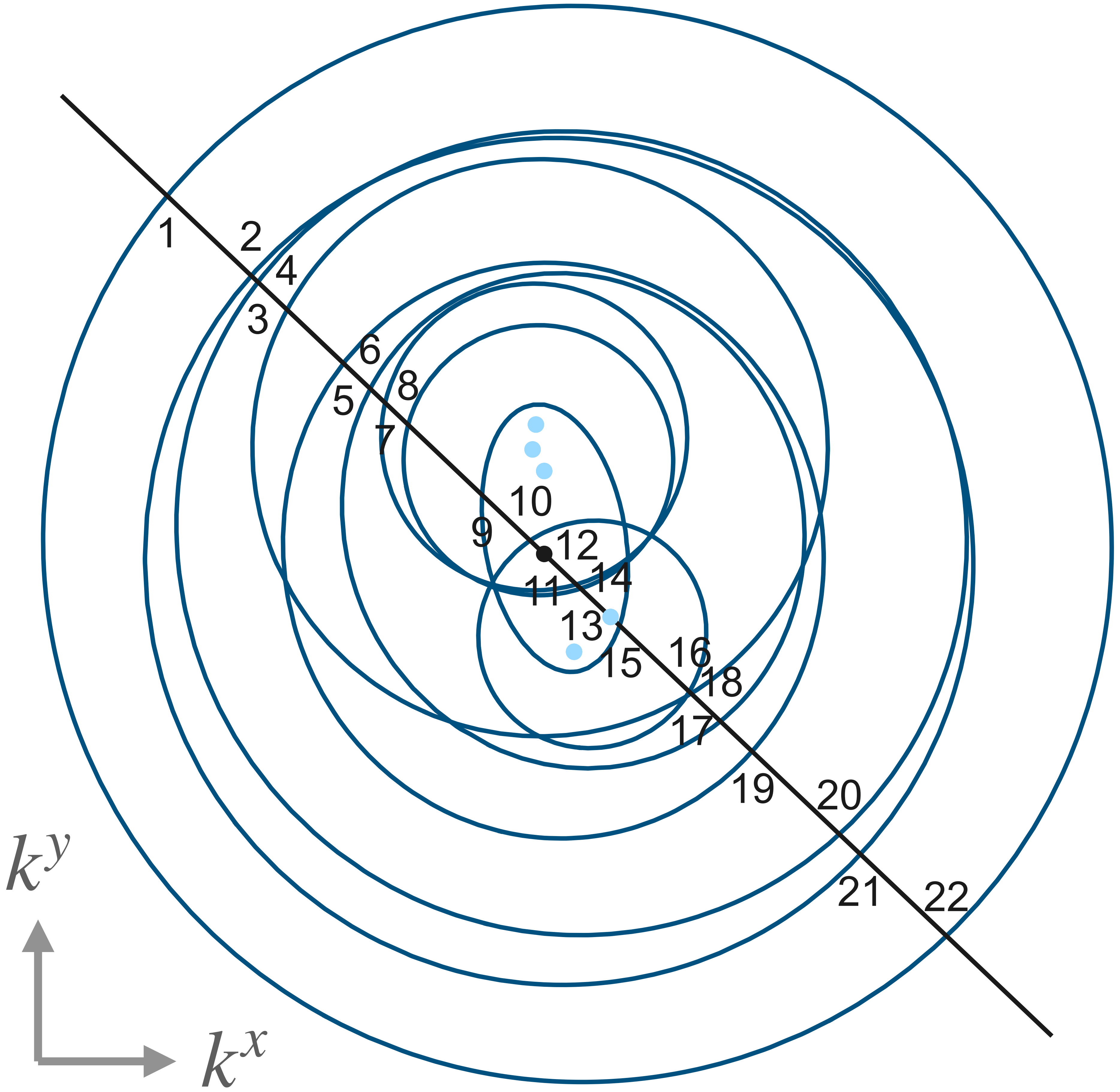
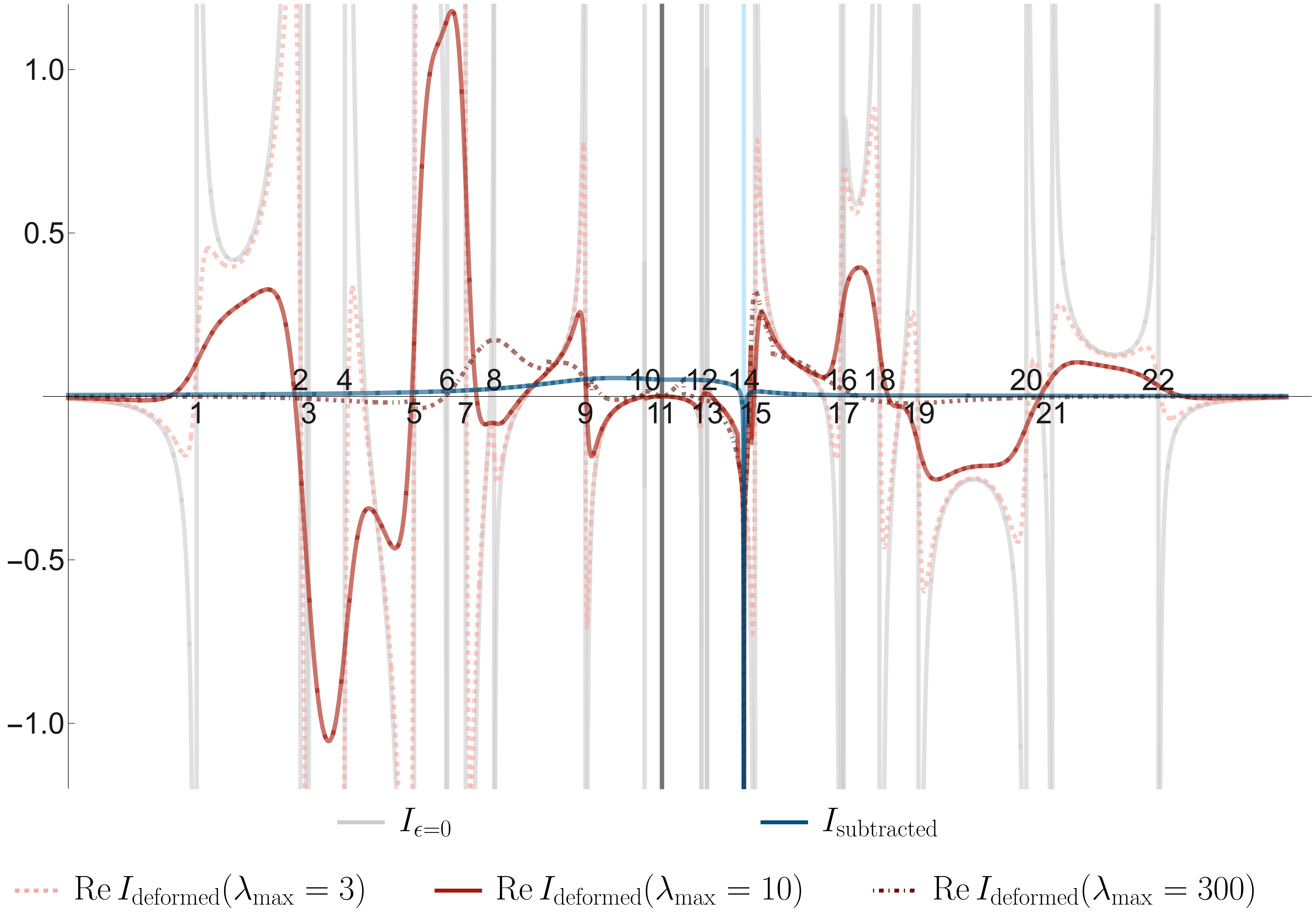
Contour deformation vs. subtraction of

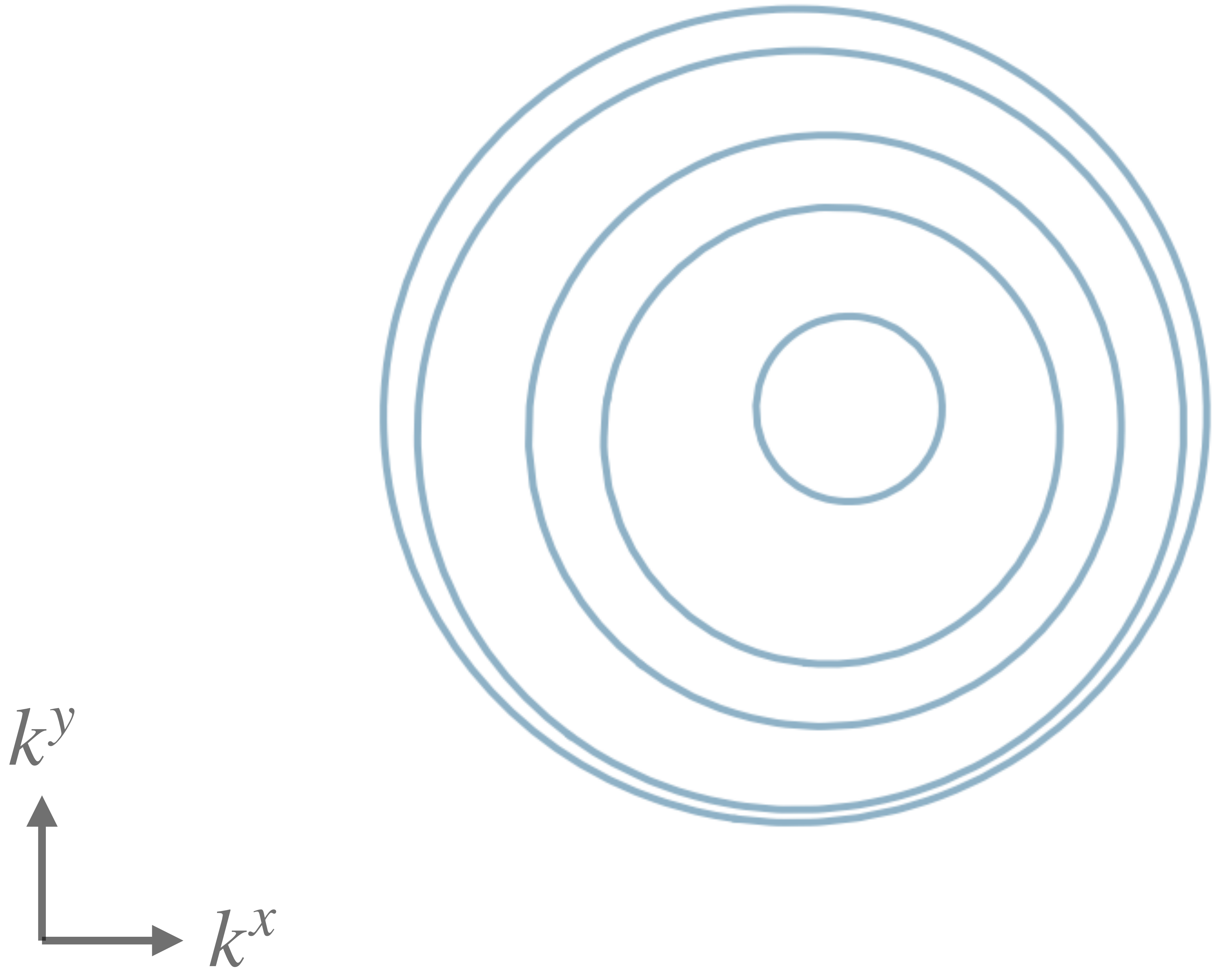


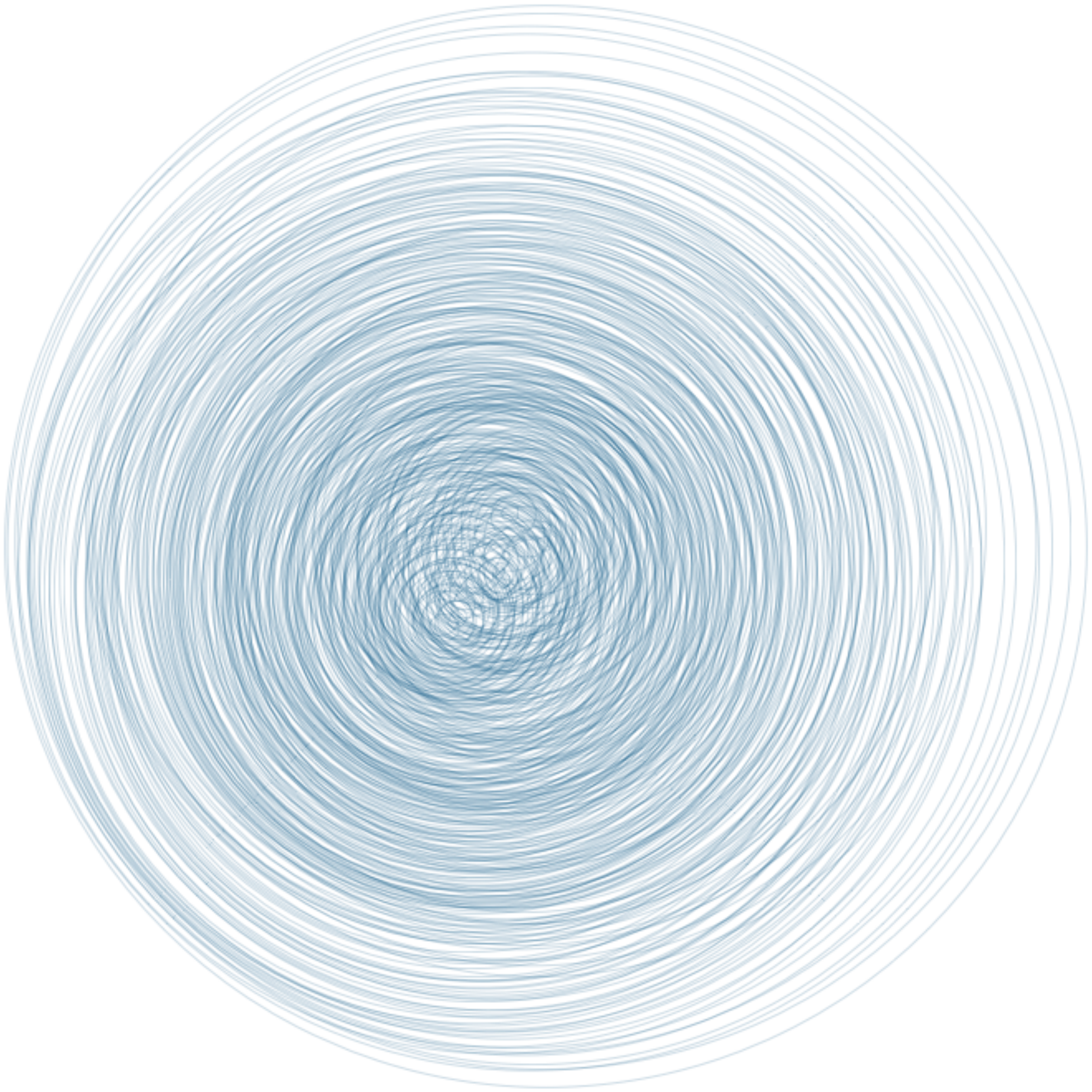




Threshold subtraction is stable for high multiplicities of external legs

Topology	Kin.	N_E	N_G	N_G^{max}	N_P	Phase	Exp.	Reference	Numerical	Δ $[\sigma]$	Δ [%]	Δ [%]|·|
 Triacontagon	1L30P.I	5	1	1	10^9	Re	-02	-1.007398	-1.007449 +/- 0.001467	0.035	0.005	0.002
					10^9	Im		3.175180	3.175183 +/- 0.000085	0.030	8e-05	
	1L30P.II	6	1	1	10^9	Re	-12	-4.166377	-4.165527 +/- 0.006697	0.127	0.020	0.016
					10^9	Im		3.413930	3.413917 +/- 0.000075	0.182	4e-04	
	1L30P.III	408	15	354	10^9	Re	-09	-2.991654	-2.984733 +/- 0.026977	0.257	0.231	0.231
					10^9	Im		-0.000000	-0.000001 +/- 0.003831	3e-04		
	1L30P.IV	408	15	354	10^9	Re	-07	-1.757748	-1.757913 +/- 0.002169	0.076	0.009	0.009
					10^9	Im		-0.000000	0.000001 +/- 0.000199	0.007		





DK: [2110.06869]

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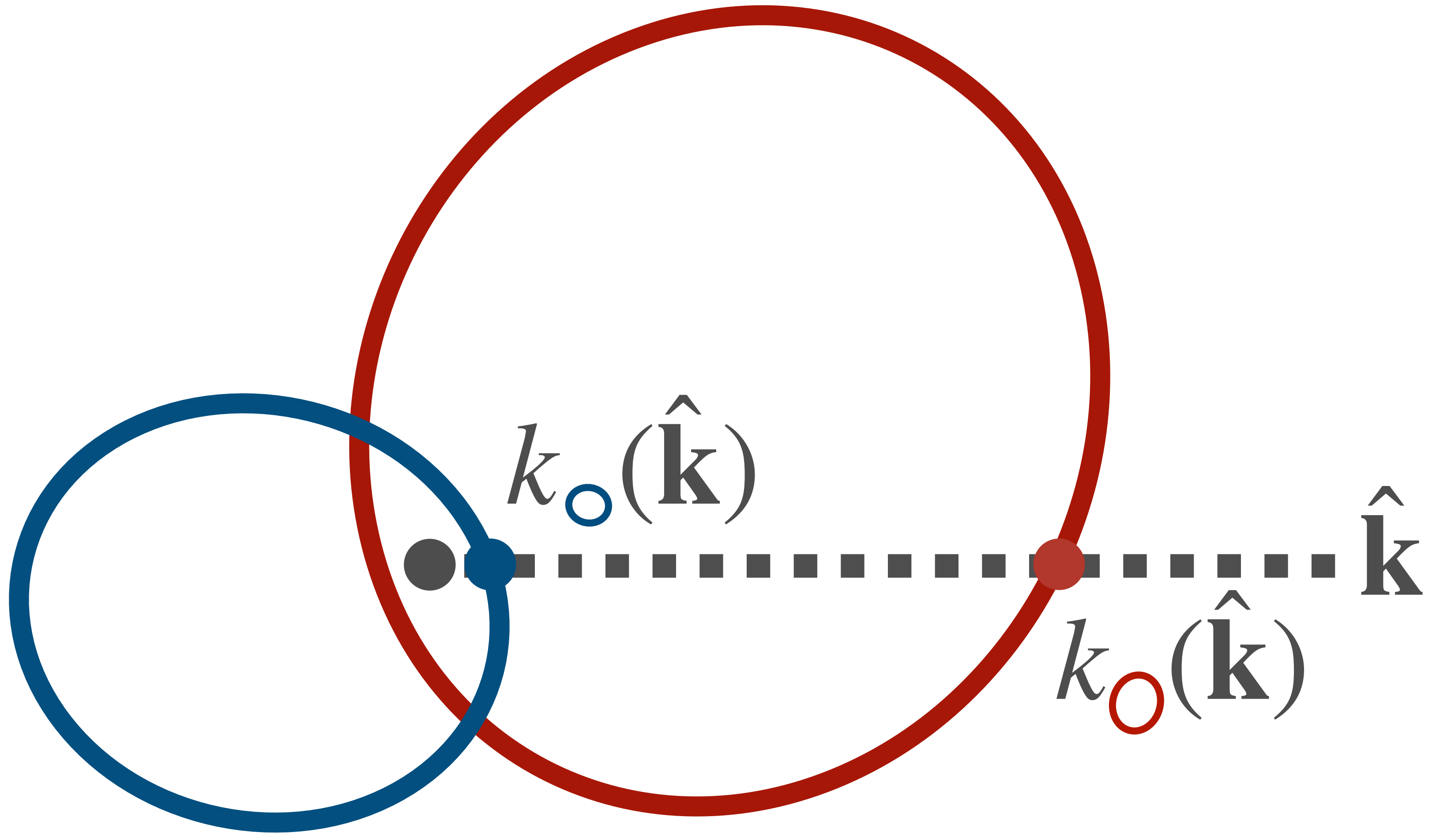
DK, Matilde Vicini [2407.21511]

Subtraction of threshold singularities

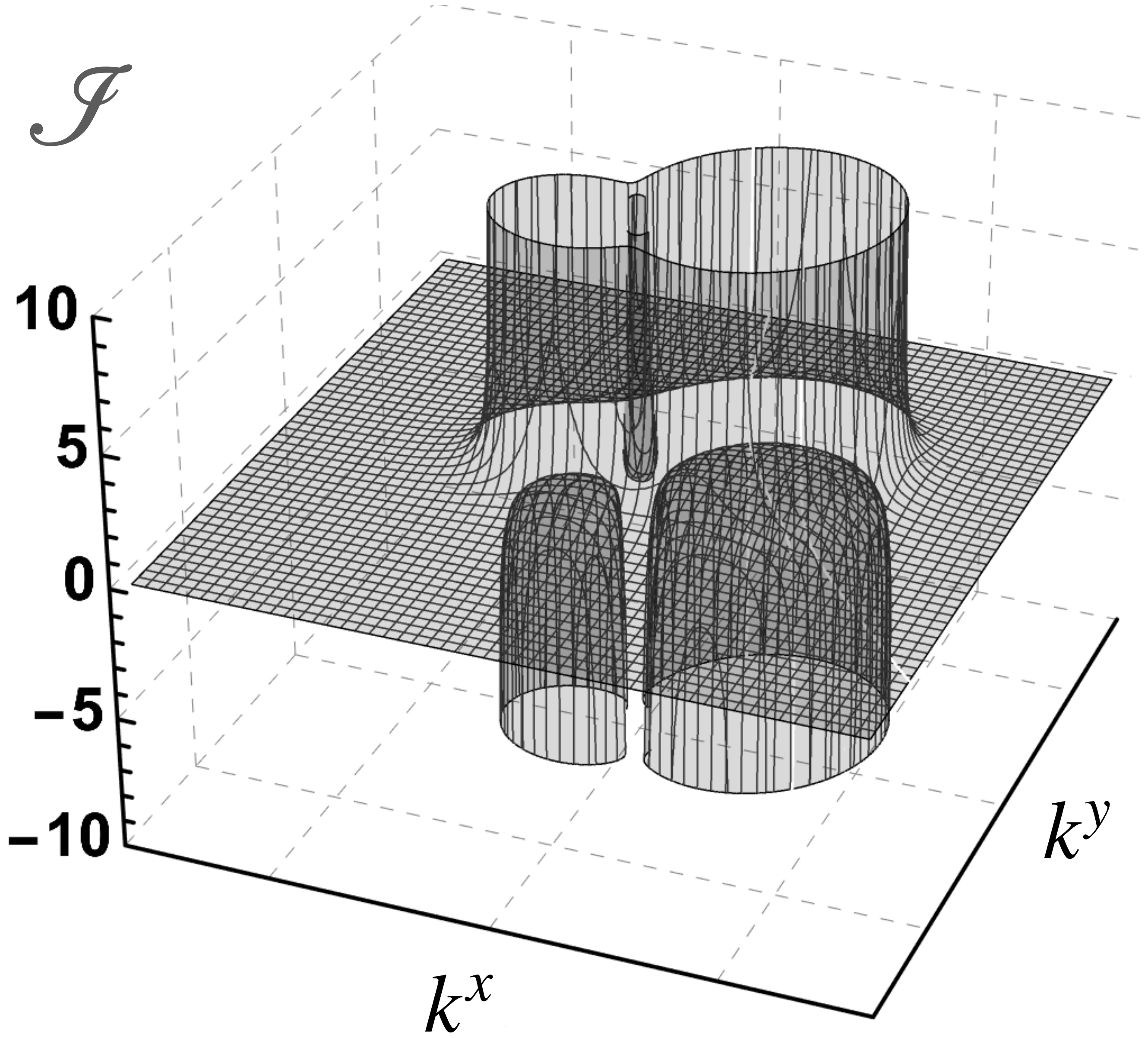
around a threshold the integrand behaves as

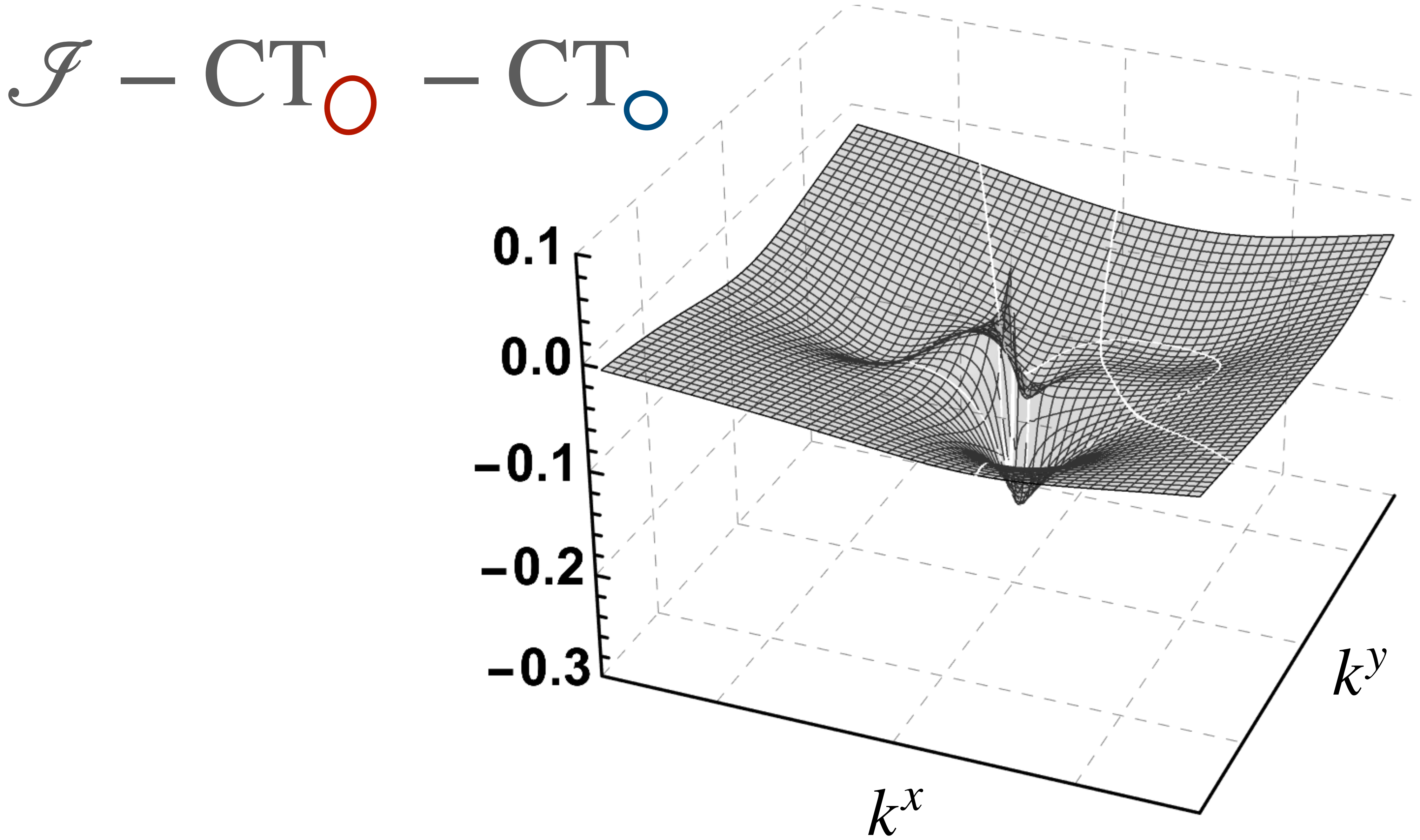
$$\mathcal{J} \sim \frac{\mathrm{Res}_i\, \mathcal{J}}{|\,\mathbf{k}\,| - k_i(\hat{\mathbf{k}}) - \mathrm{i}\epsilon} \to \mathrm{CT}_i$$ threshold counterterm

$$\mathrm{Re}\, I = \int \mathrm{d}^{3n}\mathbf{k} \left(\mathcal{J} - \sum_i \mathrm{CT}_i \right)$$ dispersive part



well suited for numerical integration!





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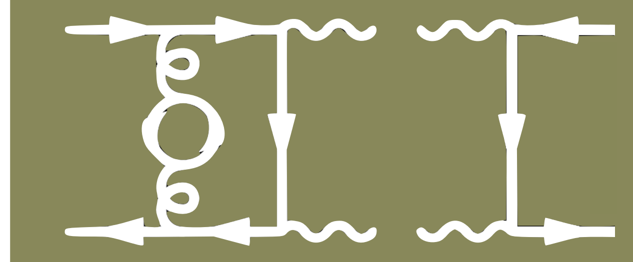
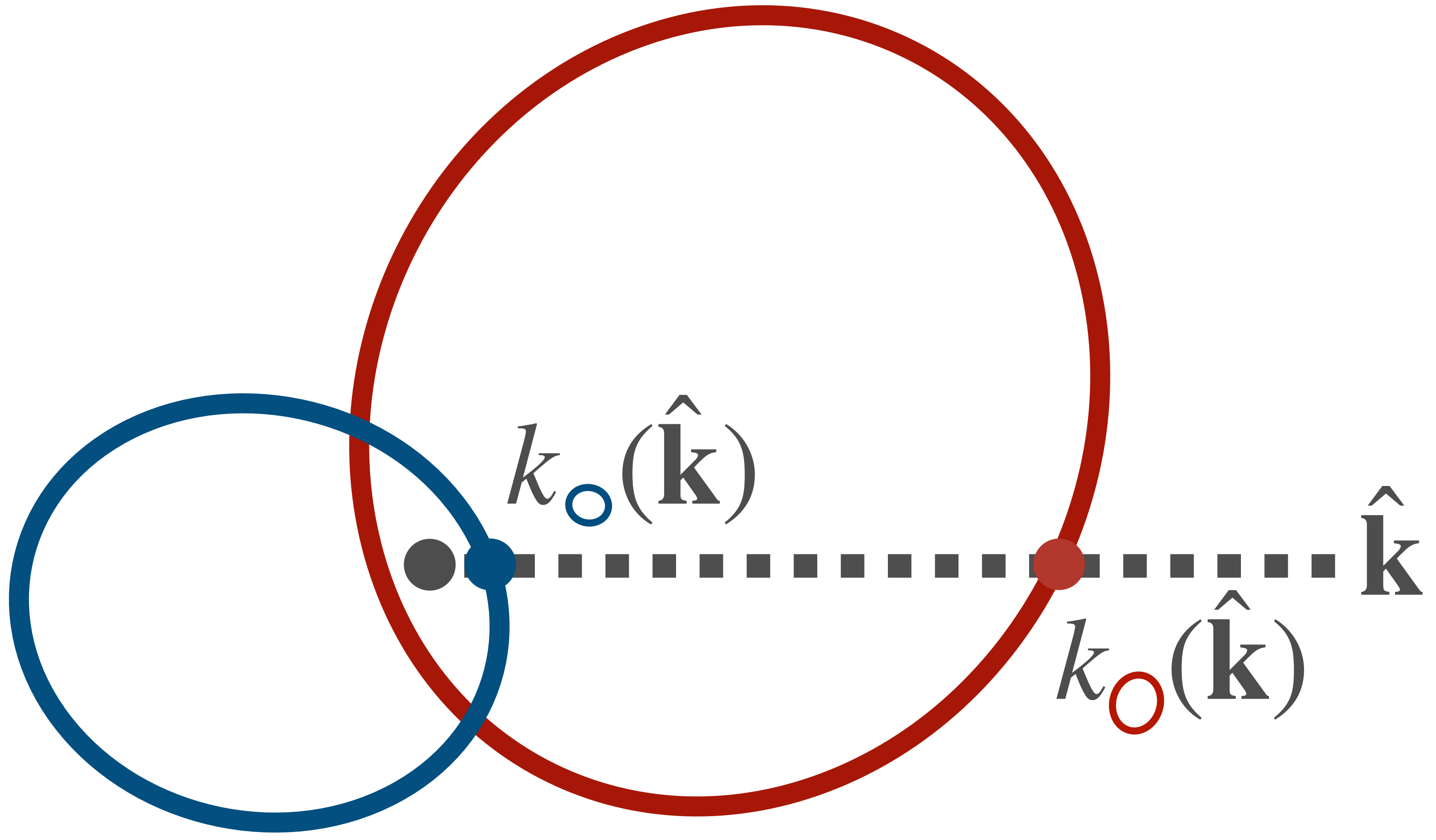
for NLO and NNLO Nf-part
we will only need the dispersive part!

$$\int \text{d}\Pi\, \text{d}^3\vec{k}\, \text{d}^3\vec{l} \sum_{\text{hel.}} 2\, \text{Re} \Big[$$

$$+ \cdots \Big]$$



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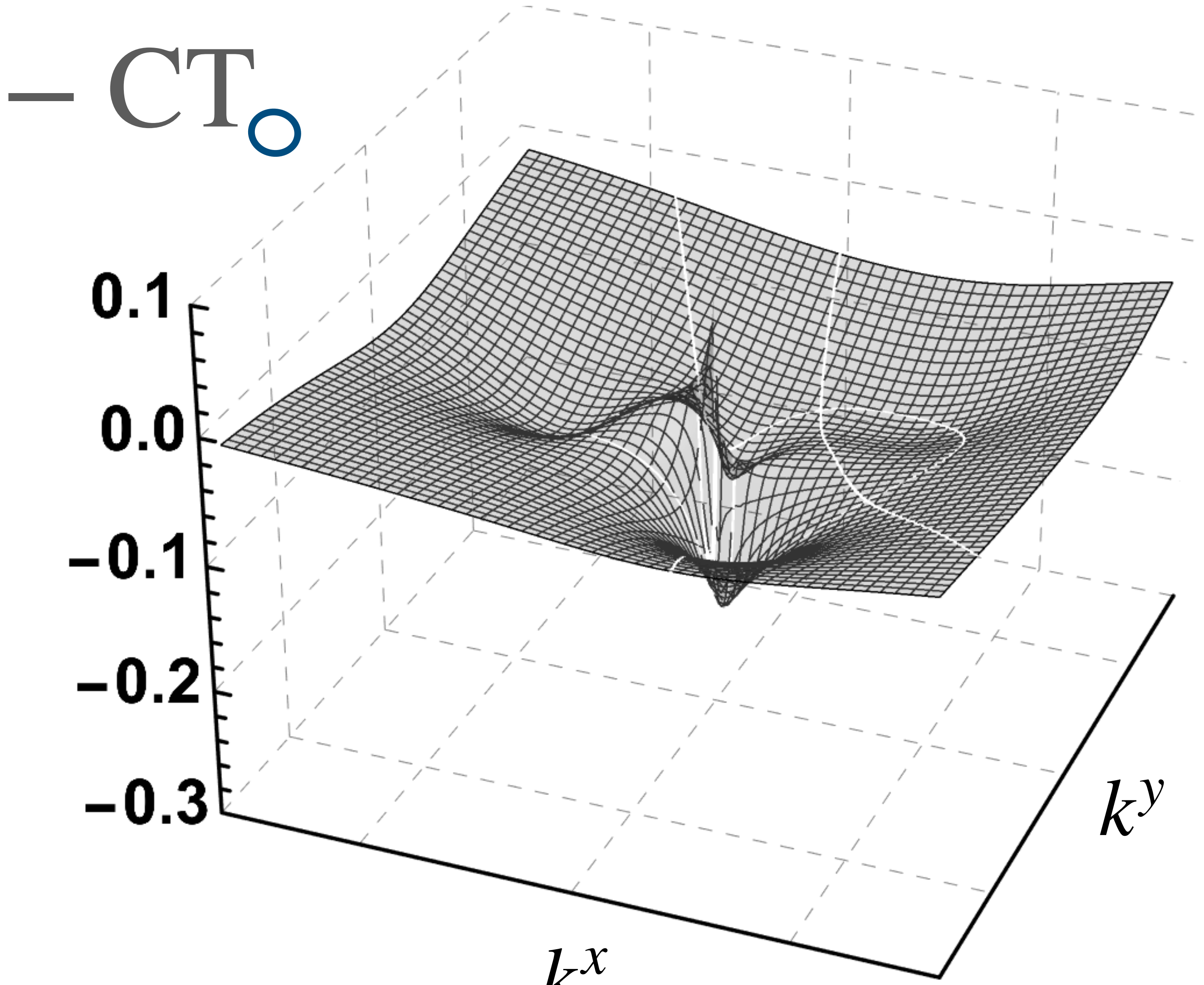
$\mathcal{I} - \text{CT}_{\circ} - \text{CT}_{\circ}$



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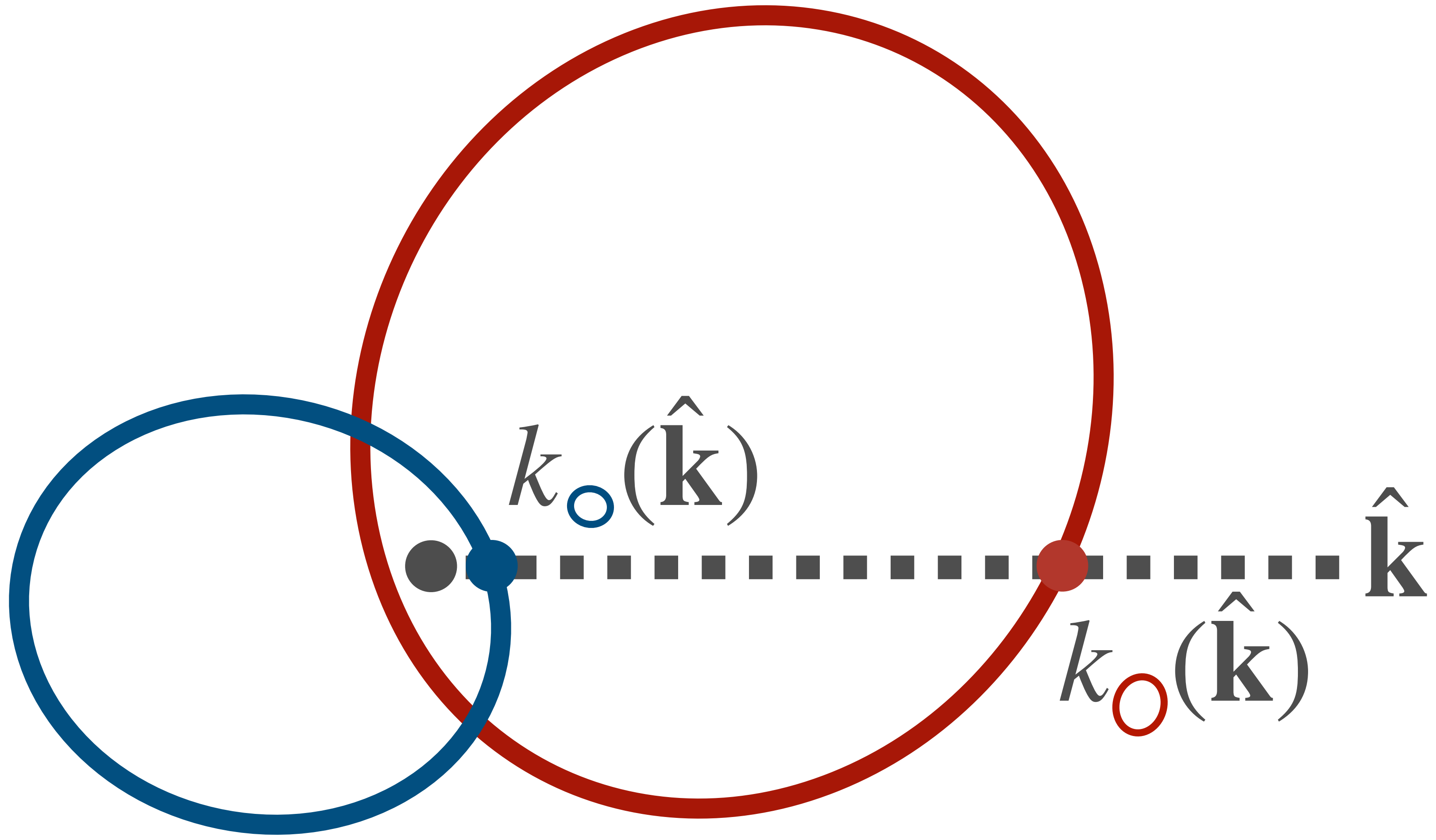

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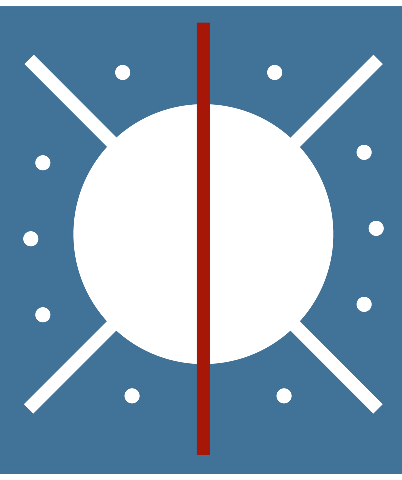
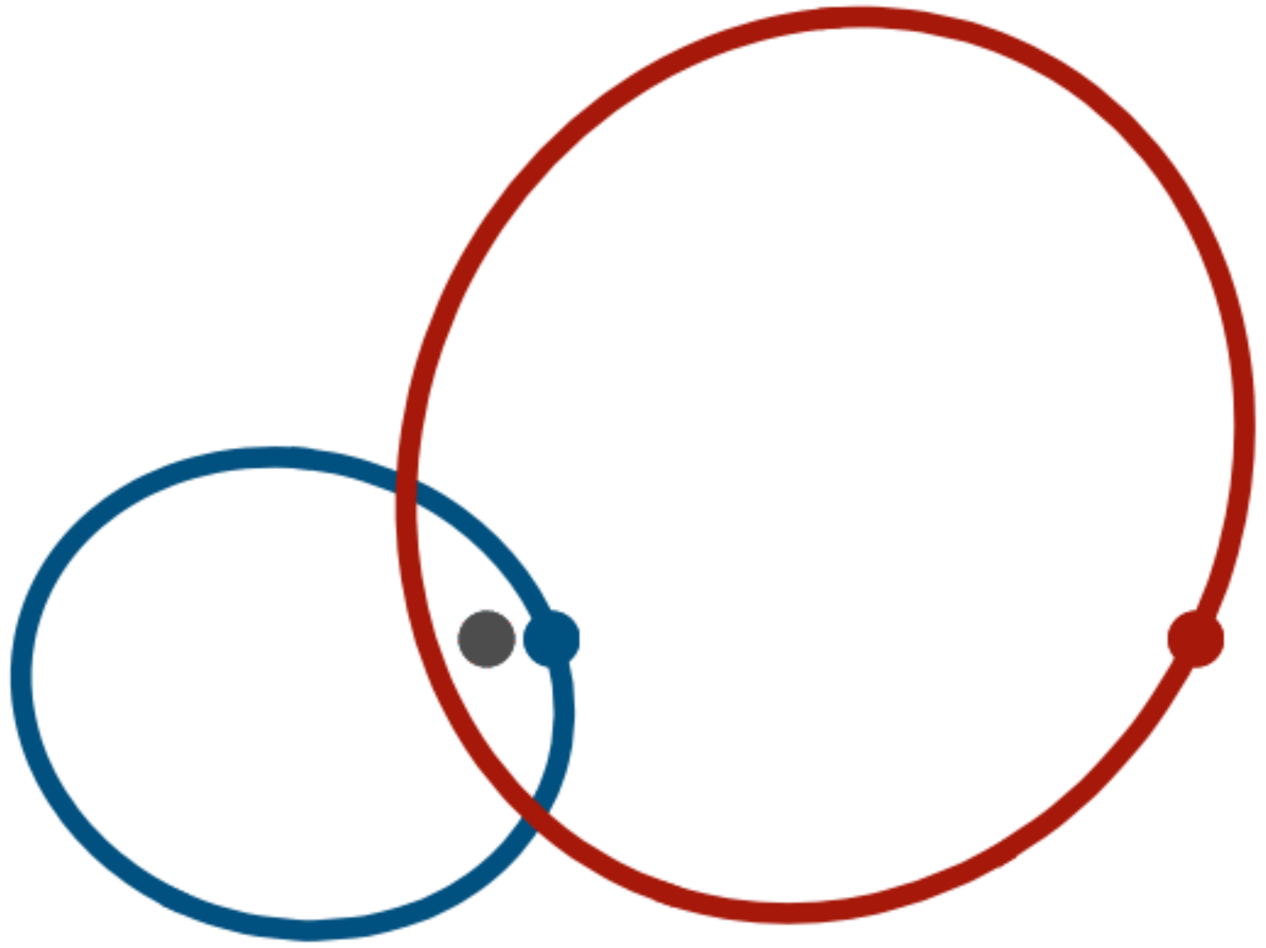


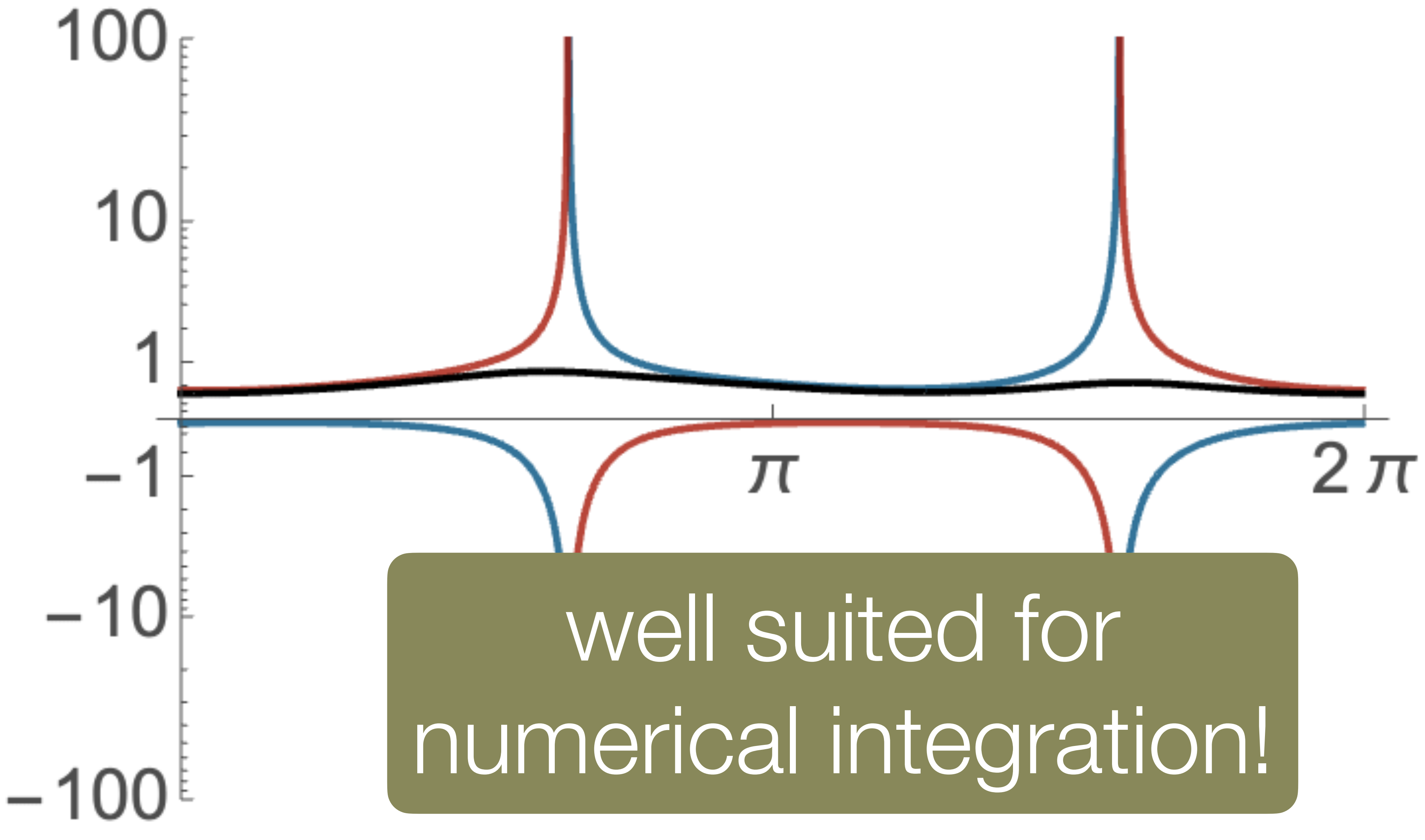
phase space integral

$$\mathrm{Im}\,I = \pi \int \mathrm{d}^{3n-1}\hat{\mathbf{k}} \sum_i \mathrm{Res}_i\,\mathscr{I}$$ absorptive part



parameterisation aligns singularities

$$\mathrm{Res}_{\circ}\,\mathscr{I} + \mathrm{Res}_{\circ}\,\mathscr{I}$$



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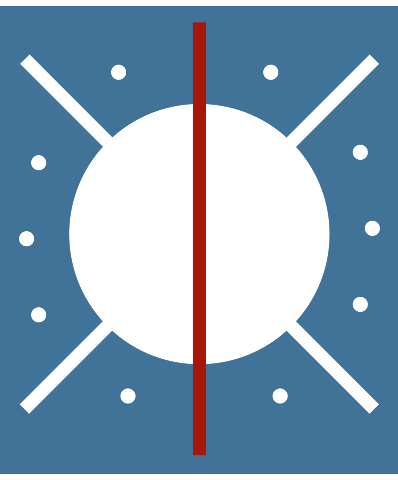

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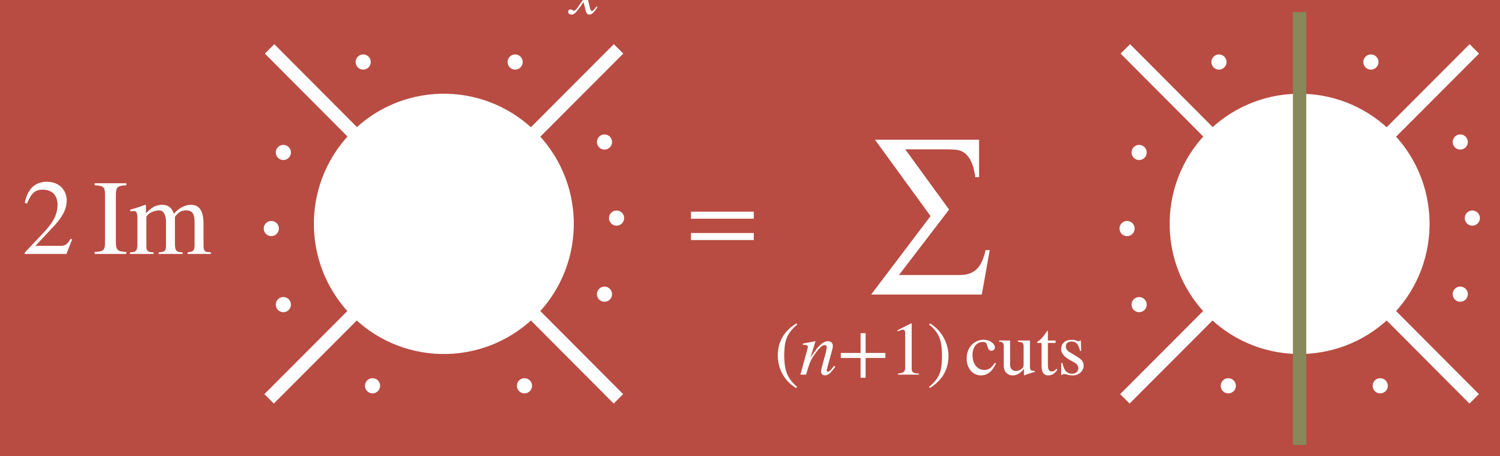

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aligns singularities incl. FSR IR between real and virtual

similar to: [Soper: hep-ph/9804454, hep-ph/9910292],
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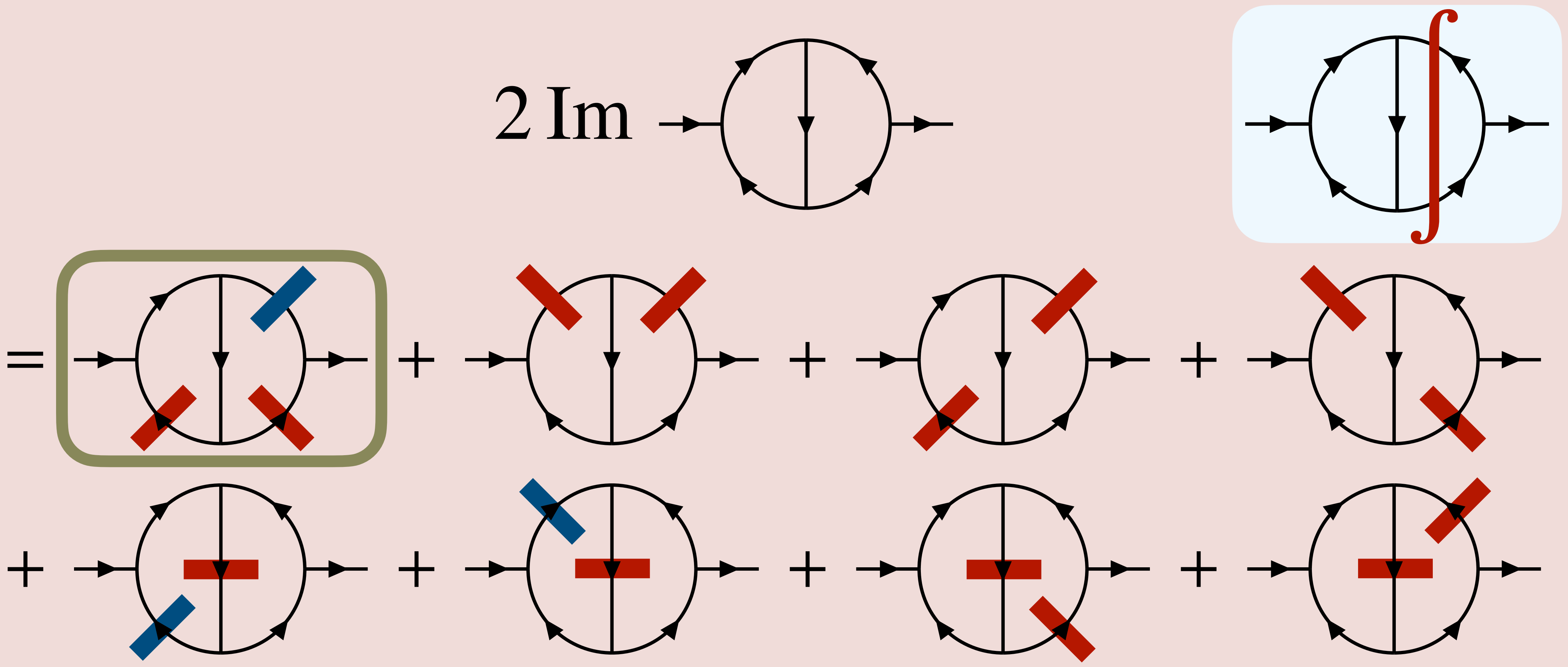

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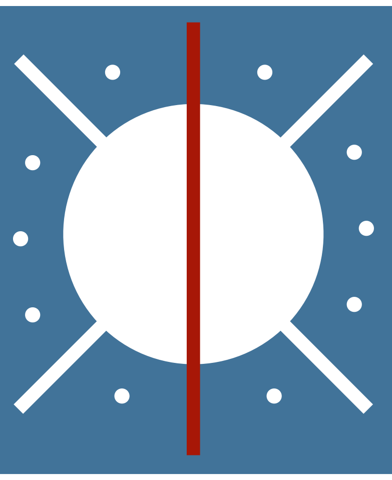

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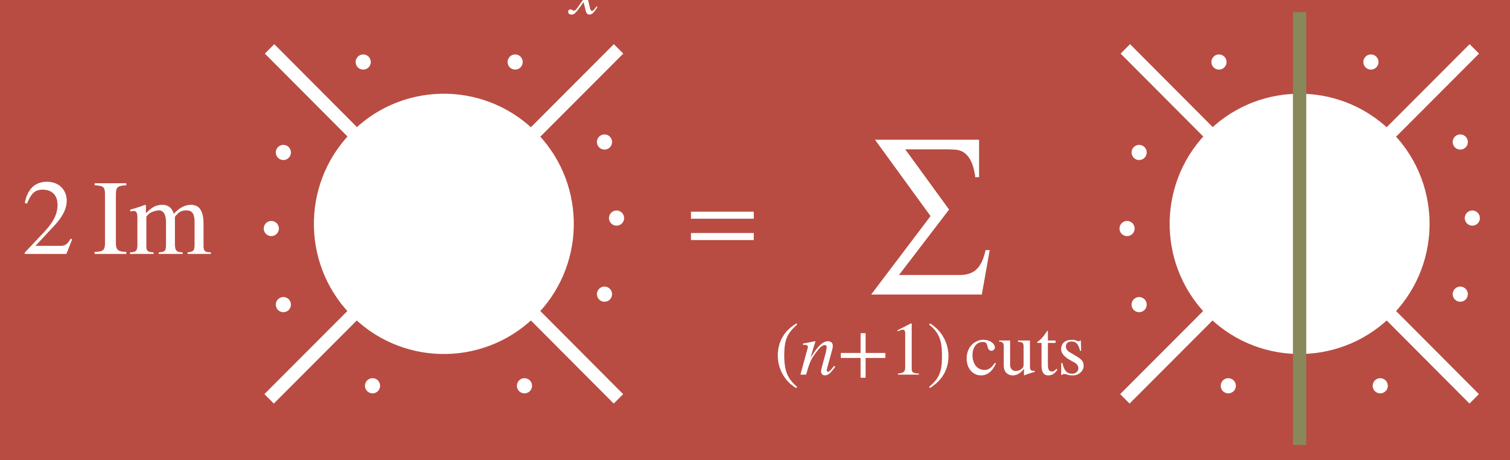

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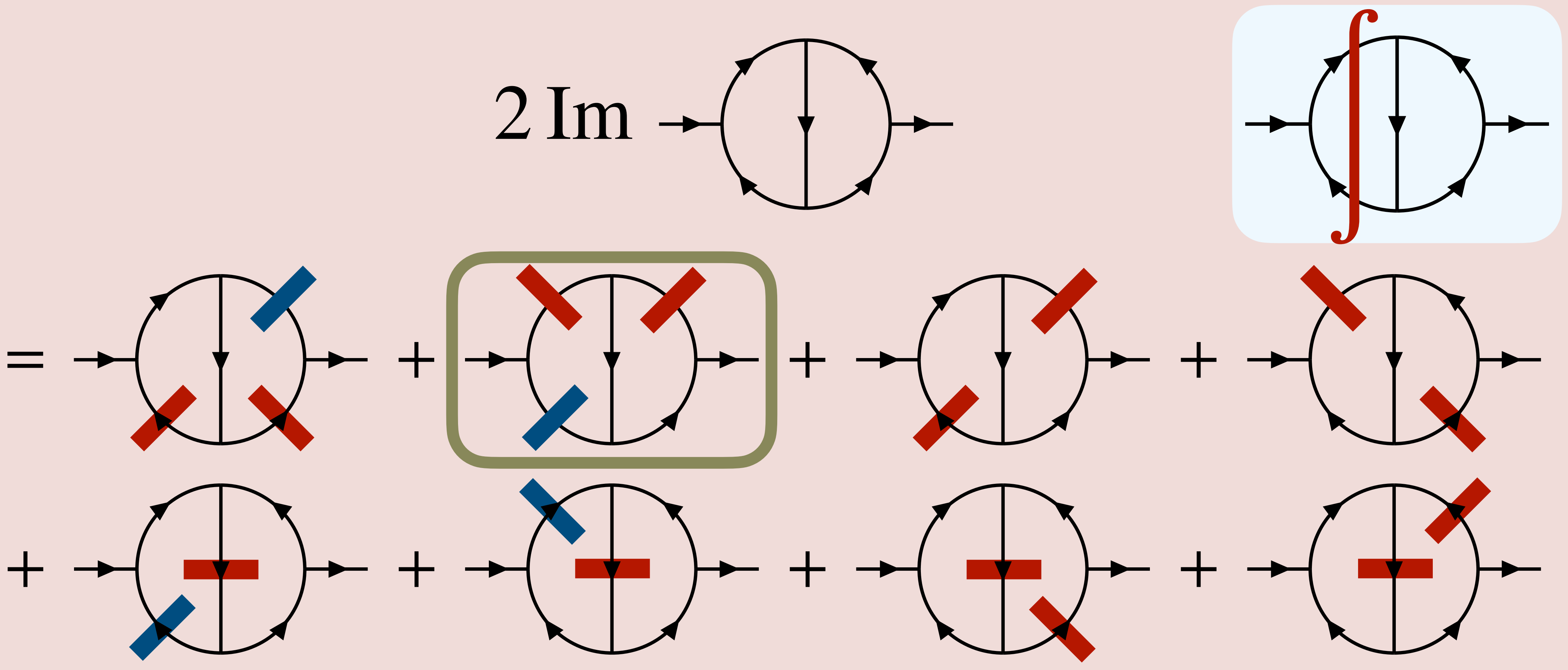

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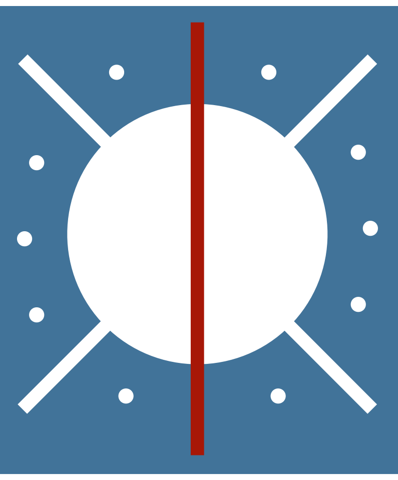

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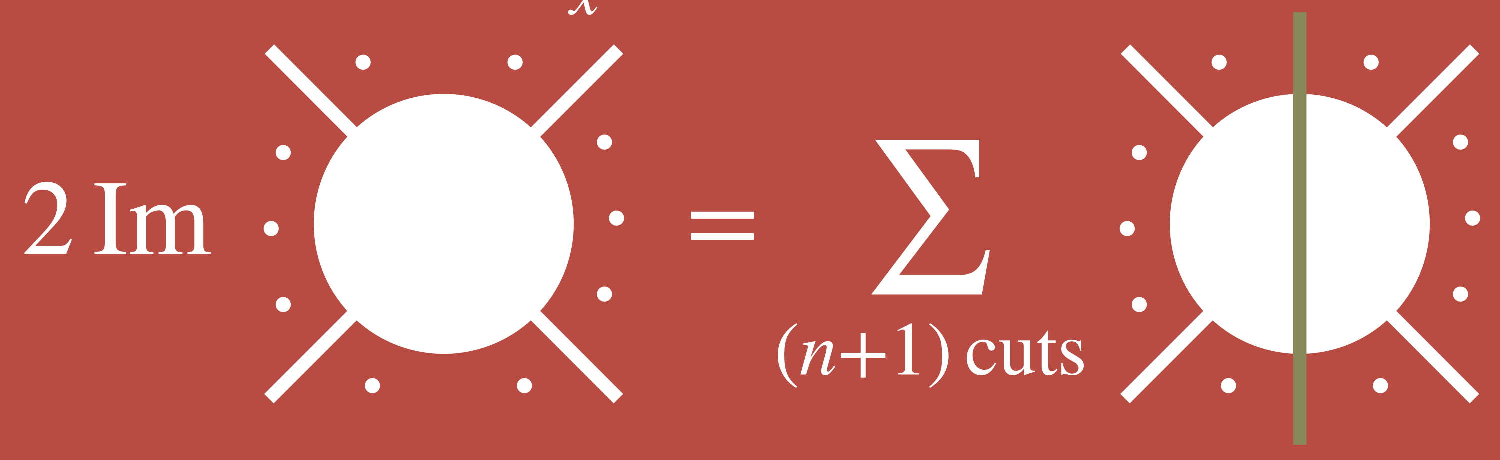

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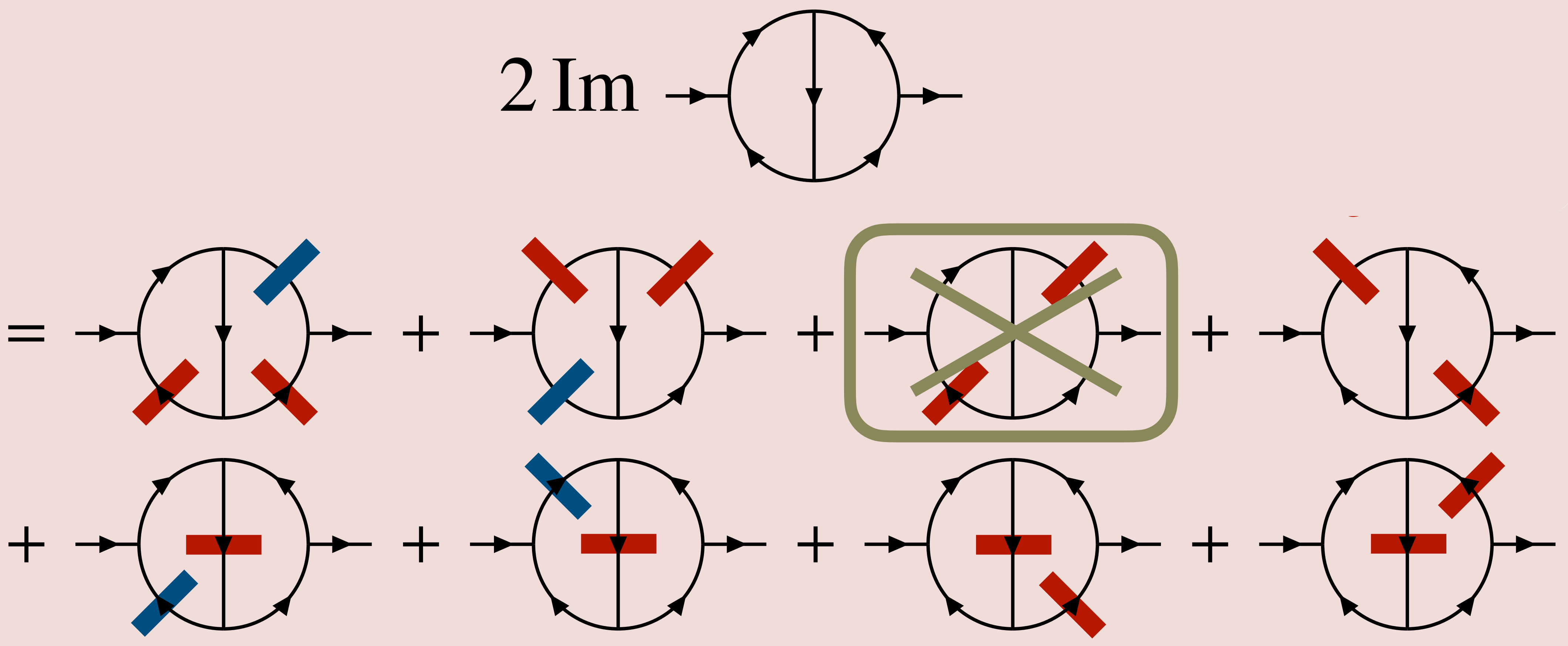

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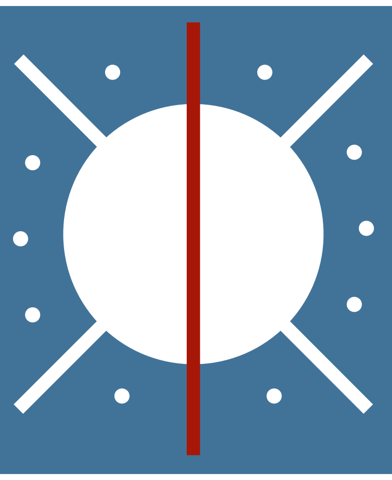

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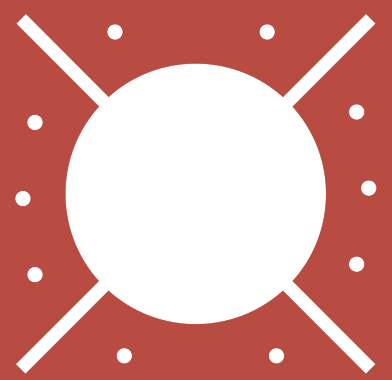
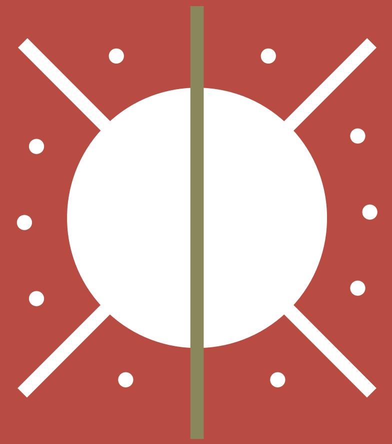
$2\,\mathrm{Im}$



$= \sum_{(n+1)\text{ cuts}}$



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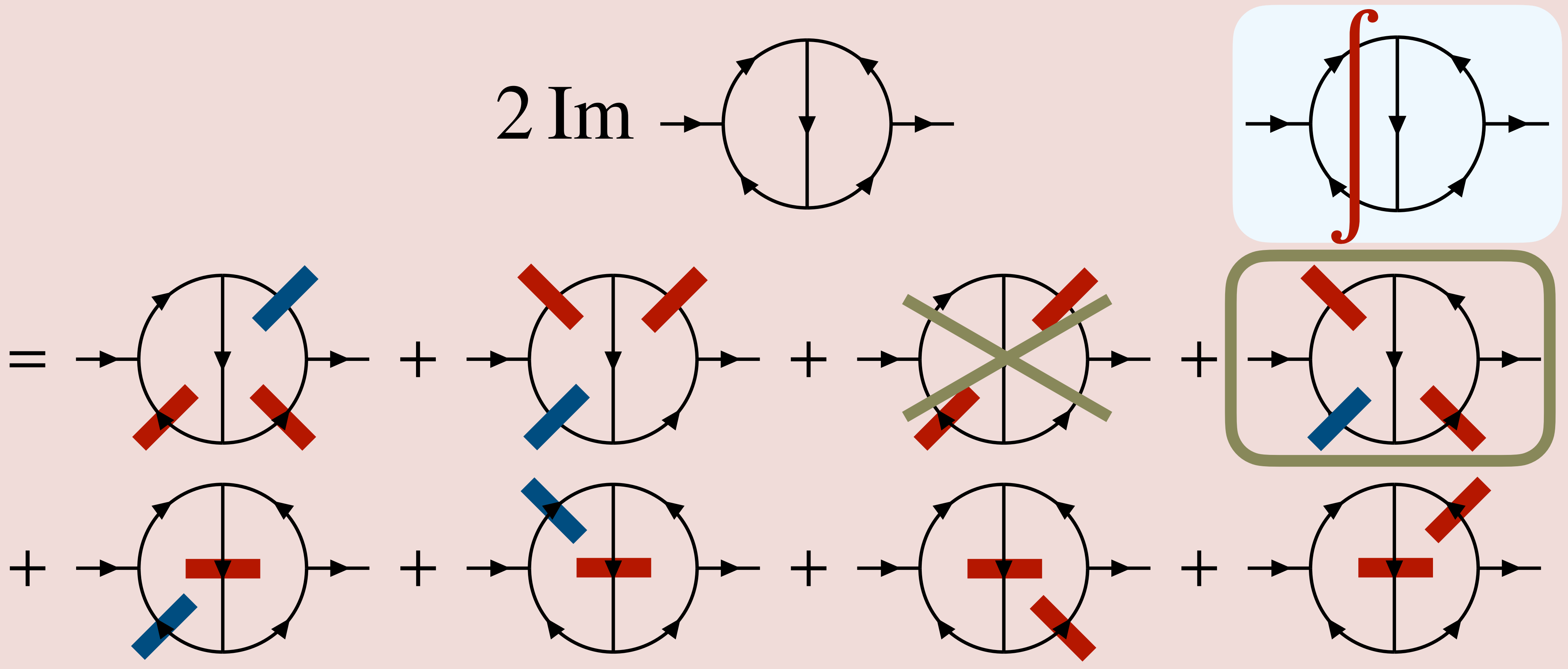

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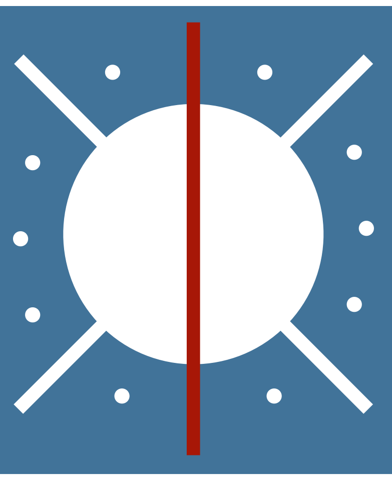

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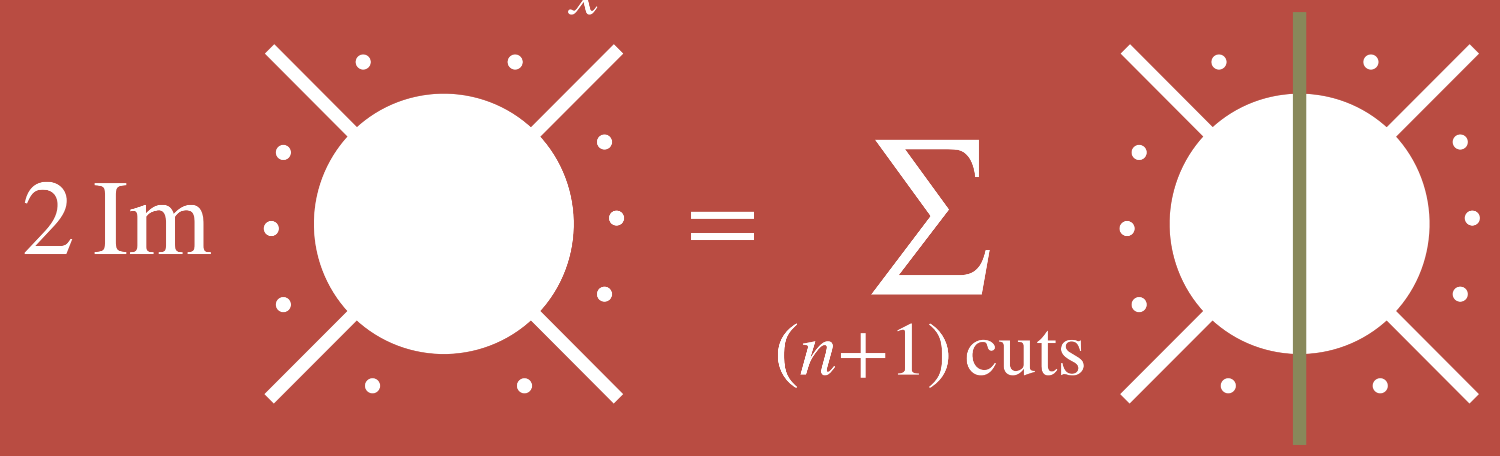

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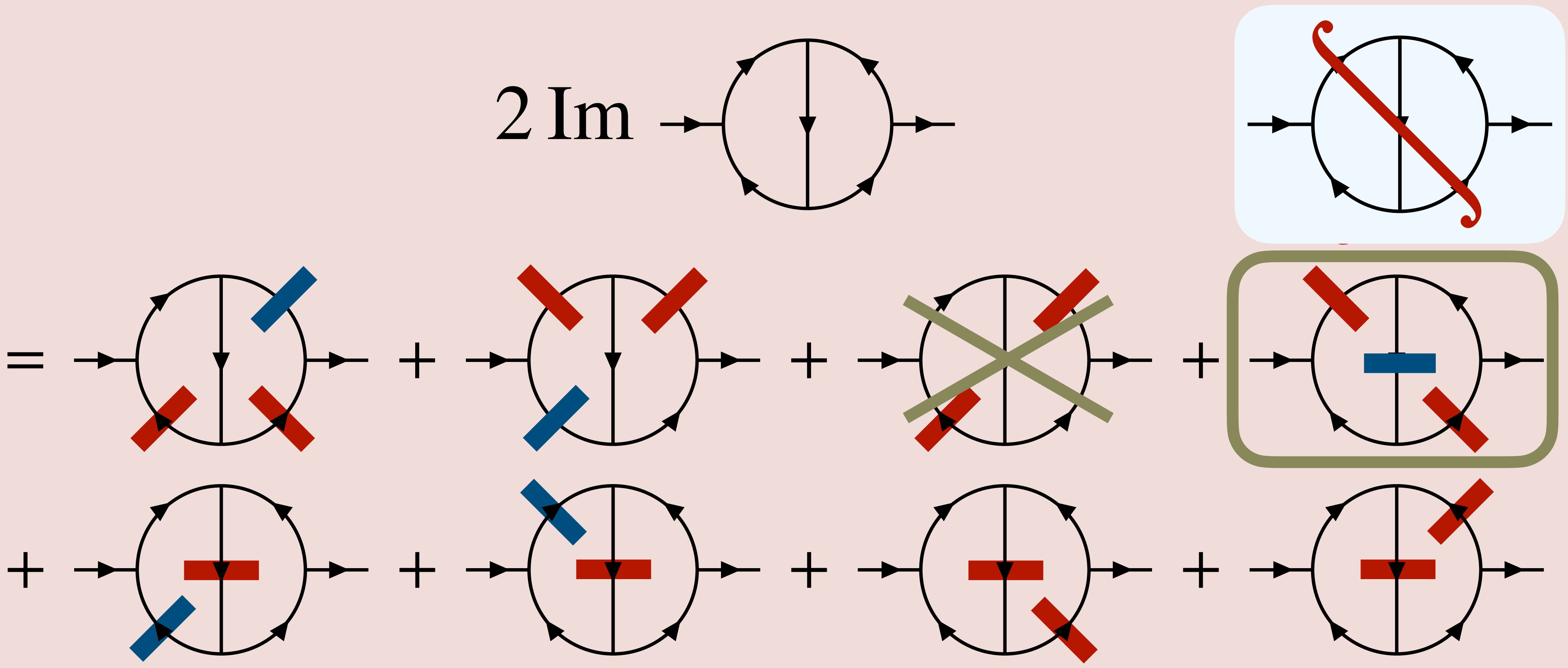

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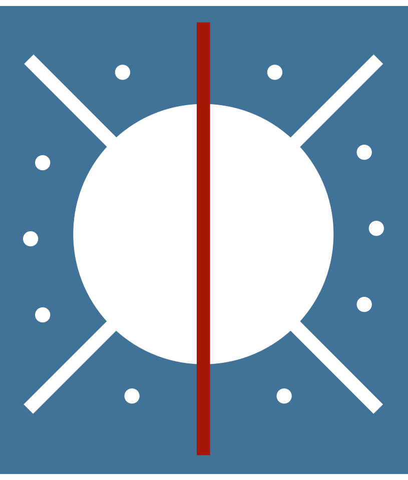

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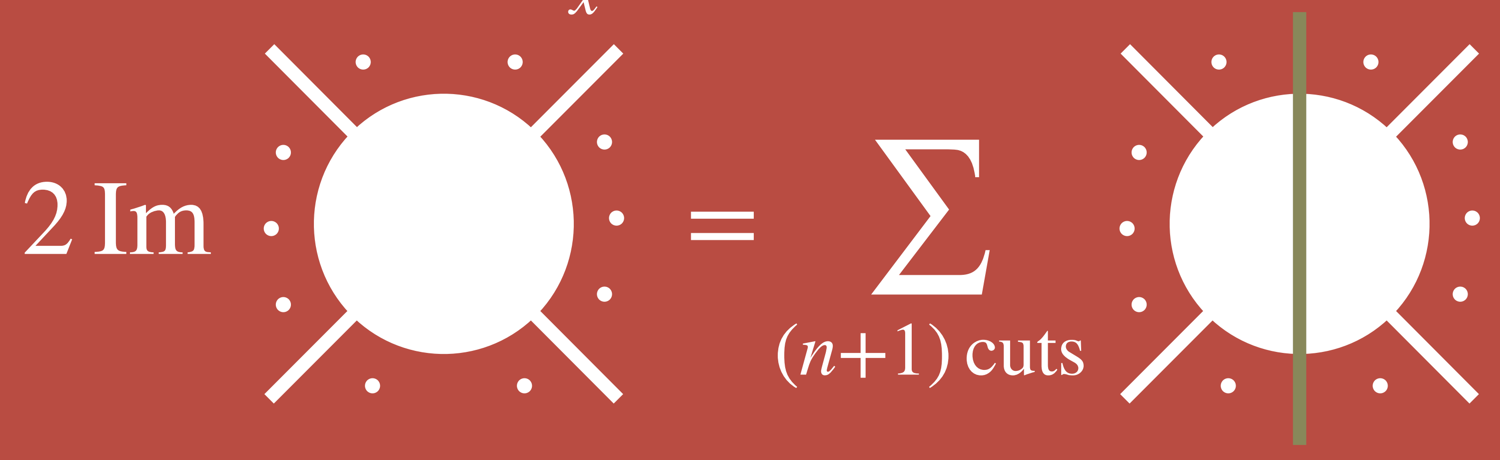

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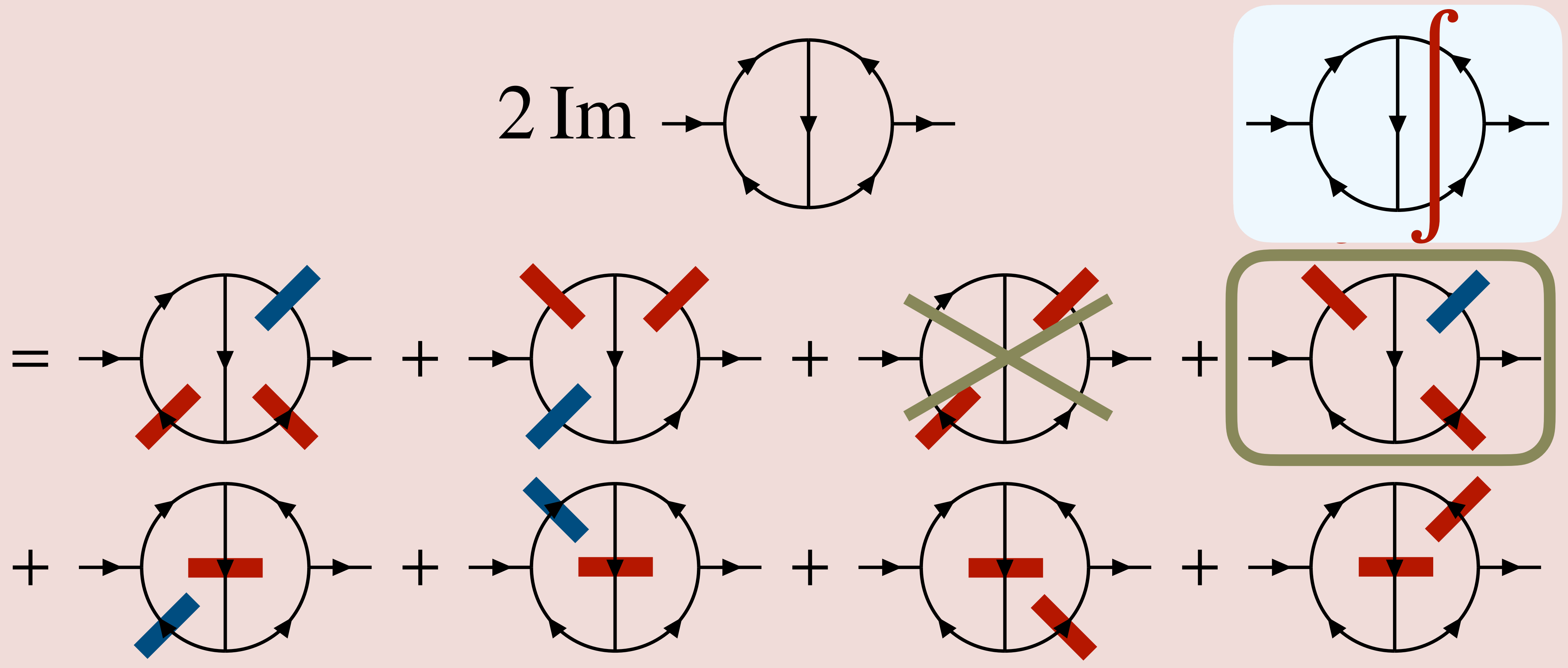

DK: [2110.06869]
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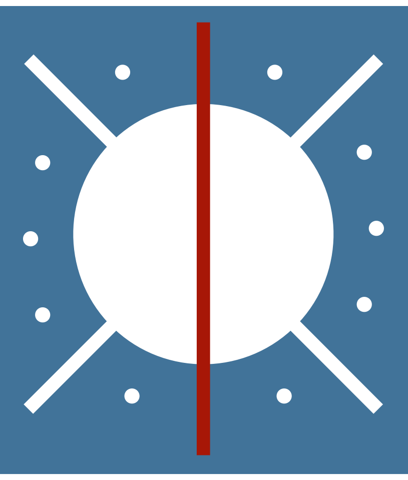

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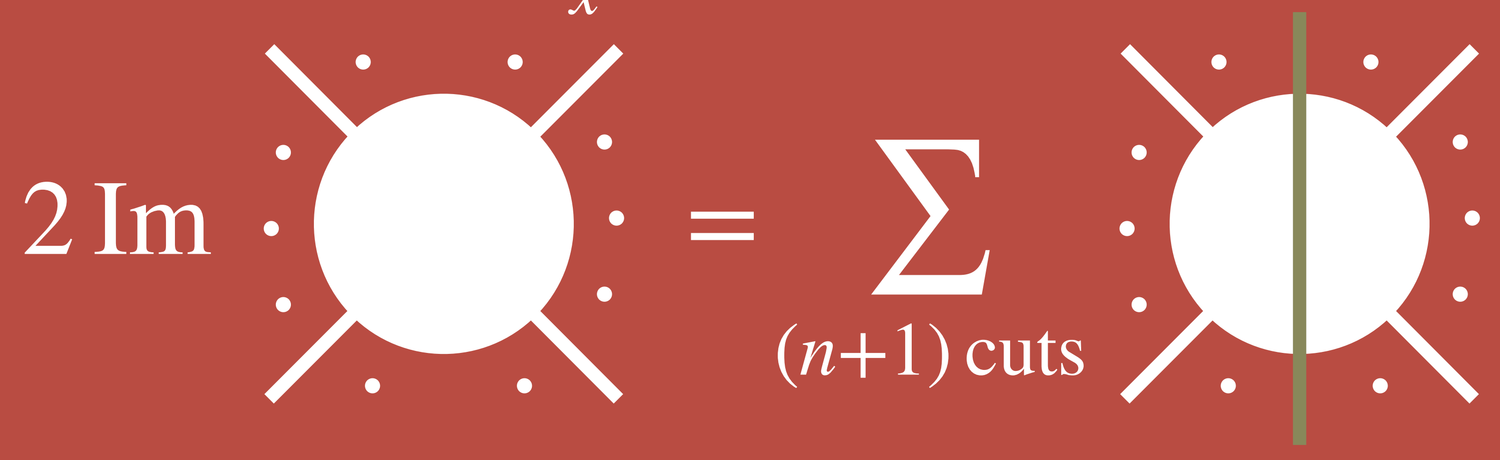

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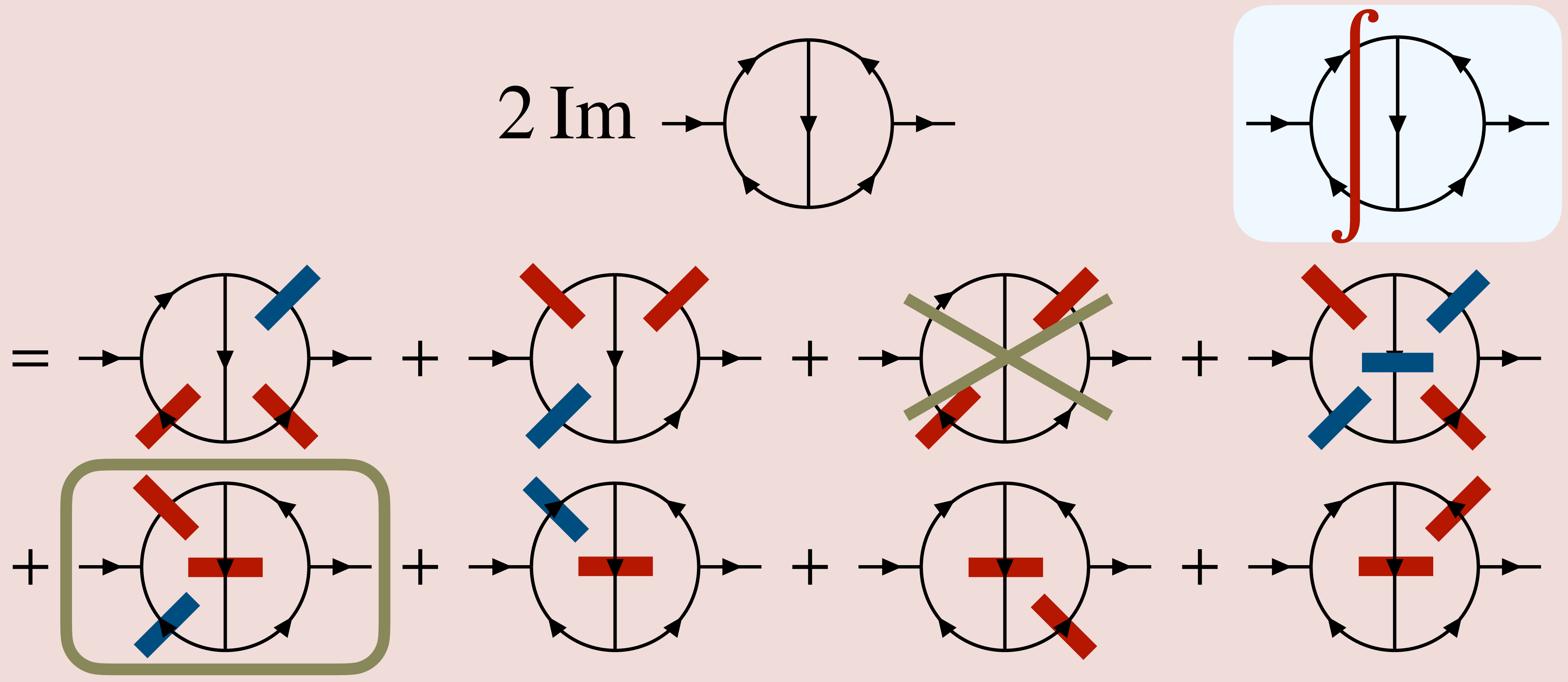

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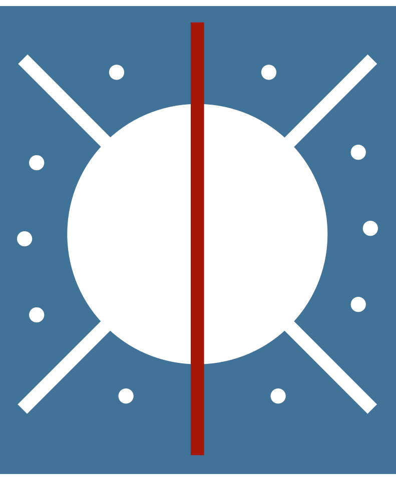

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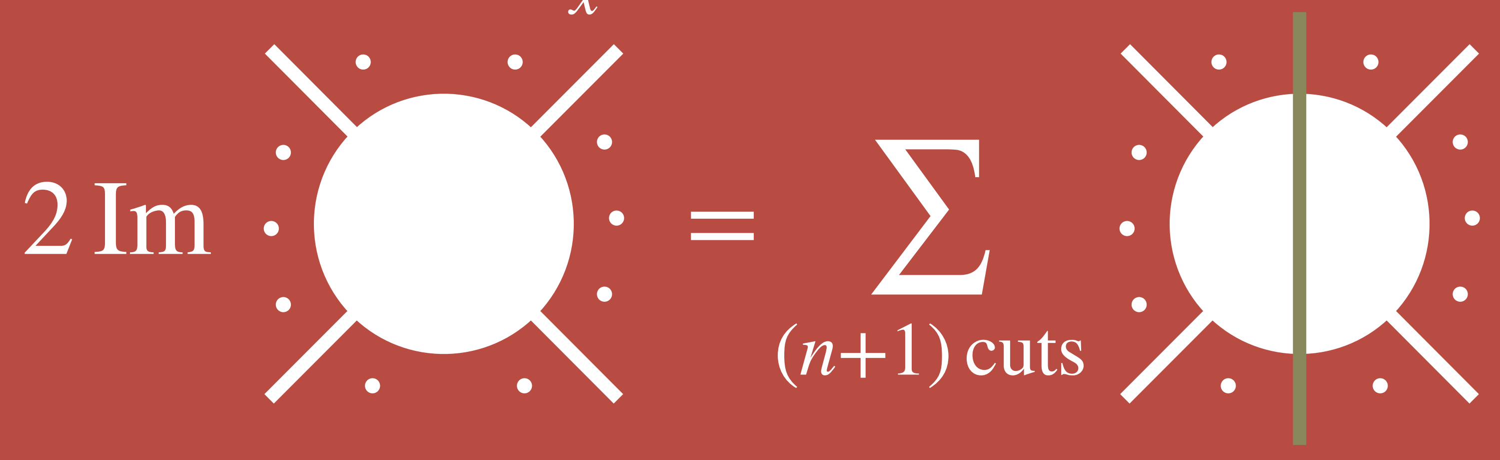

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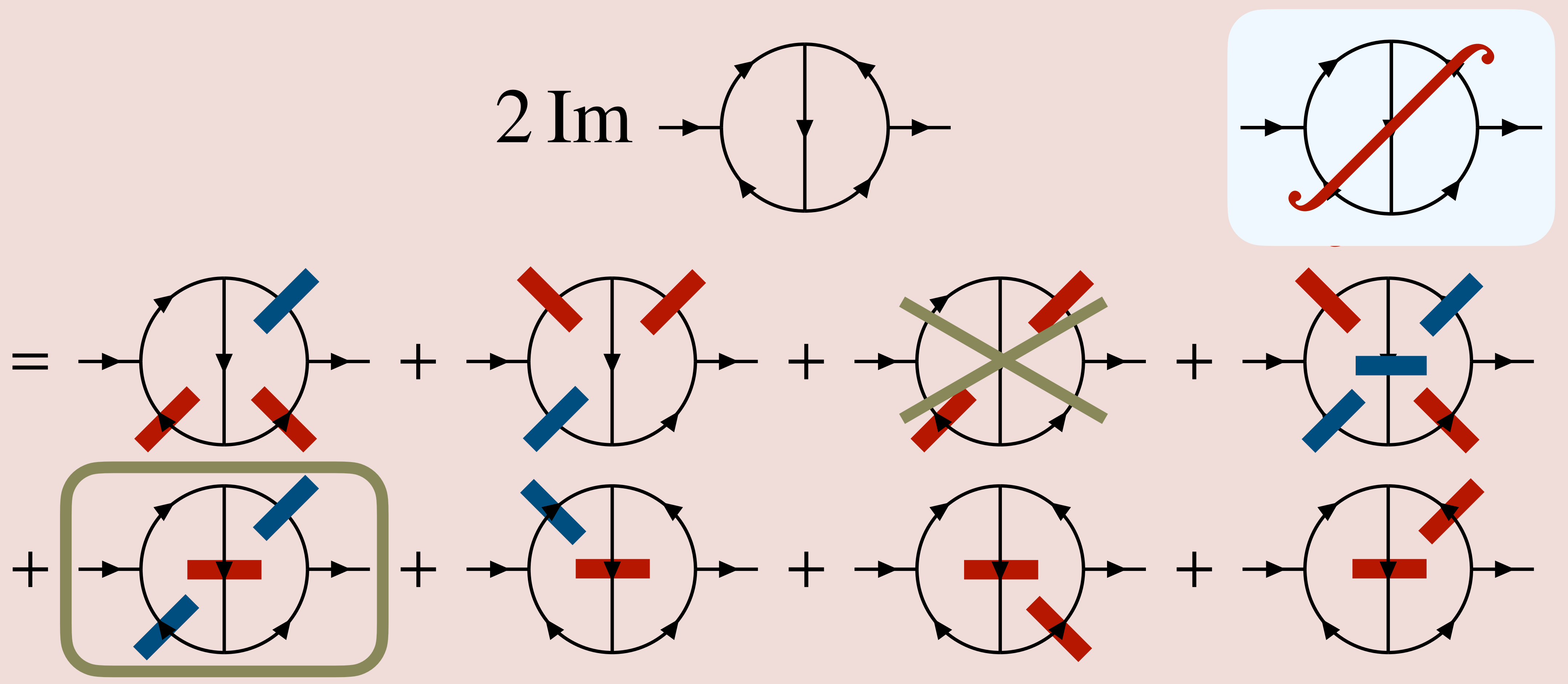

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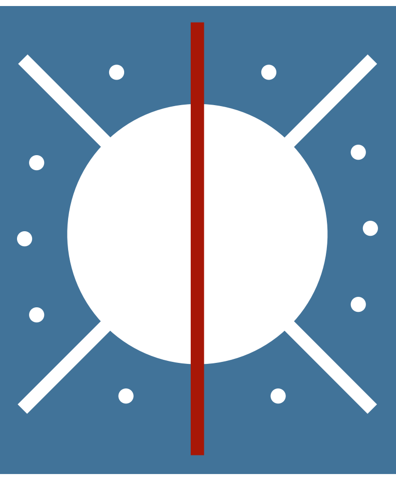

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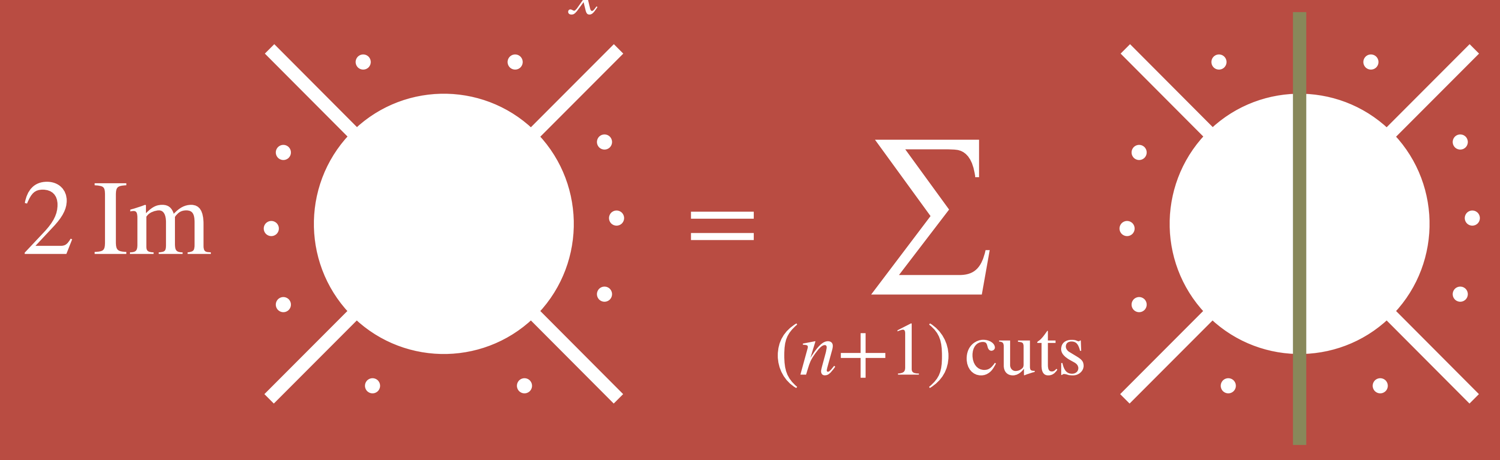

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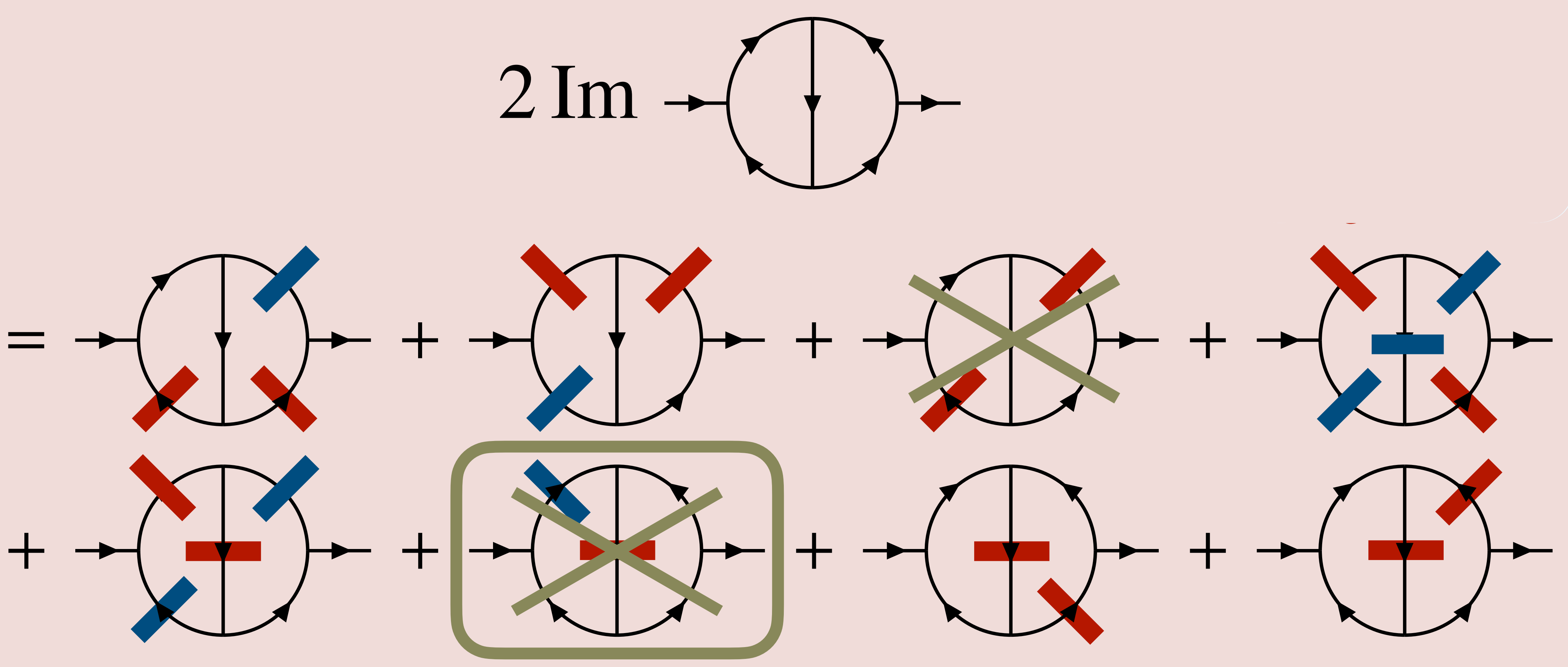

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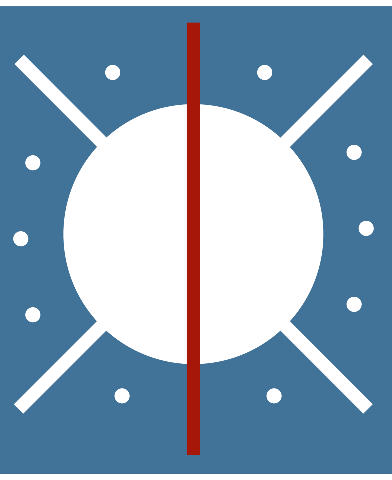

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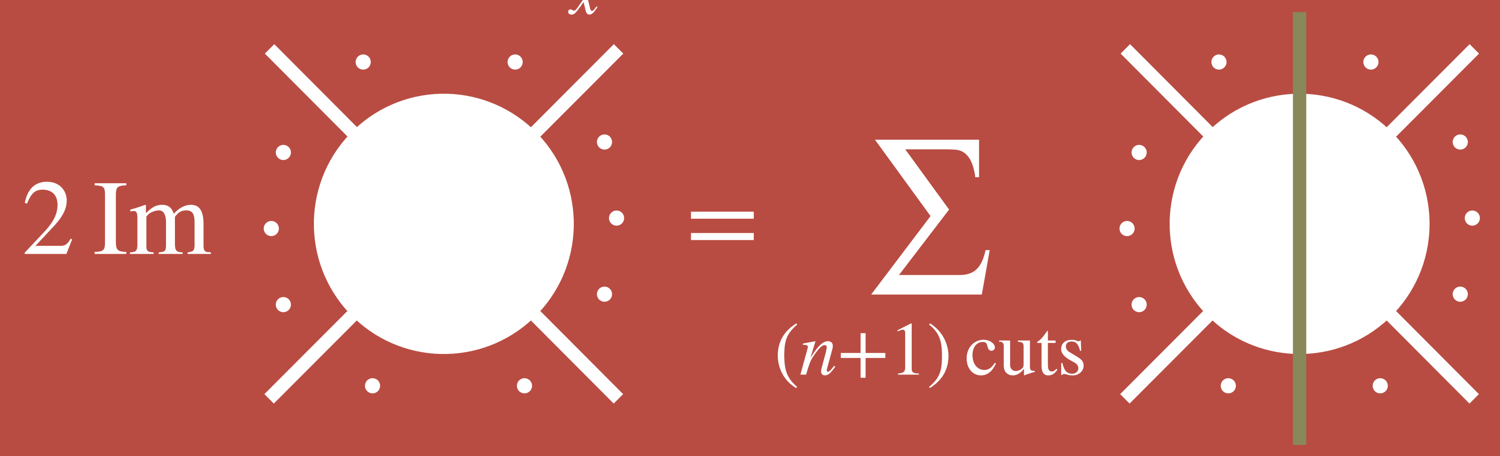

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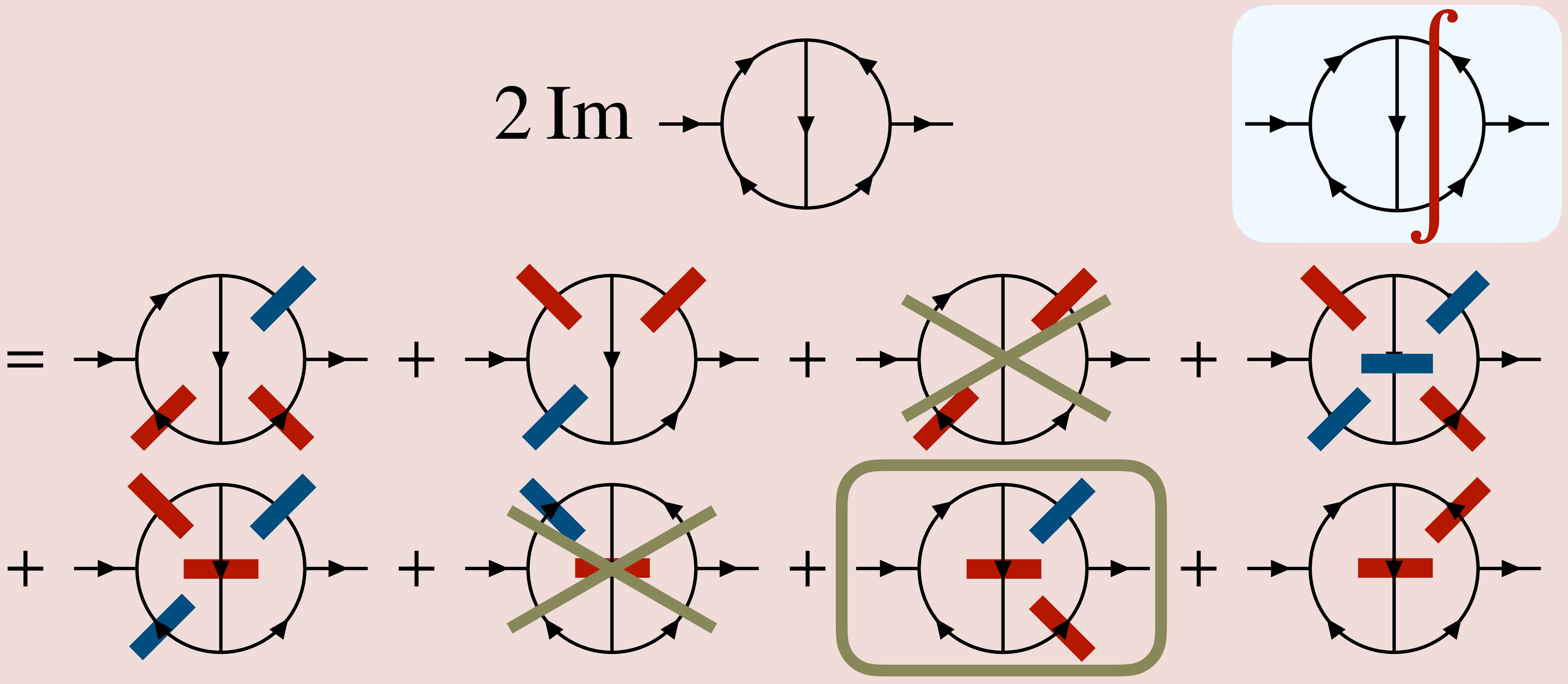

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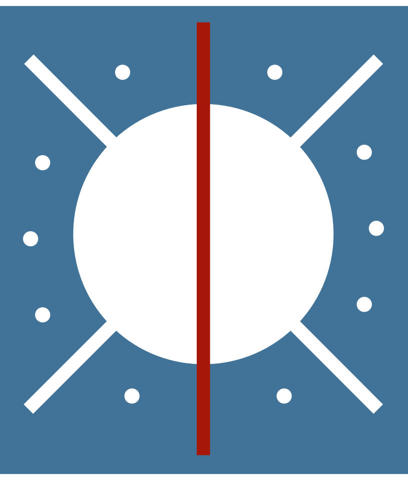

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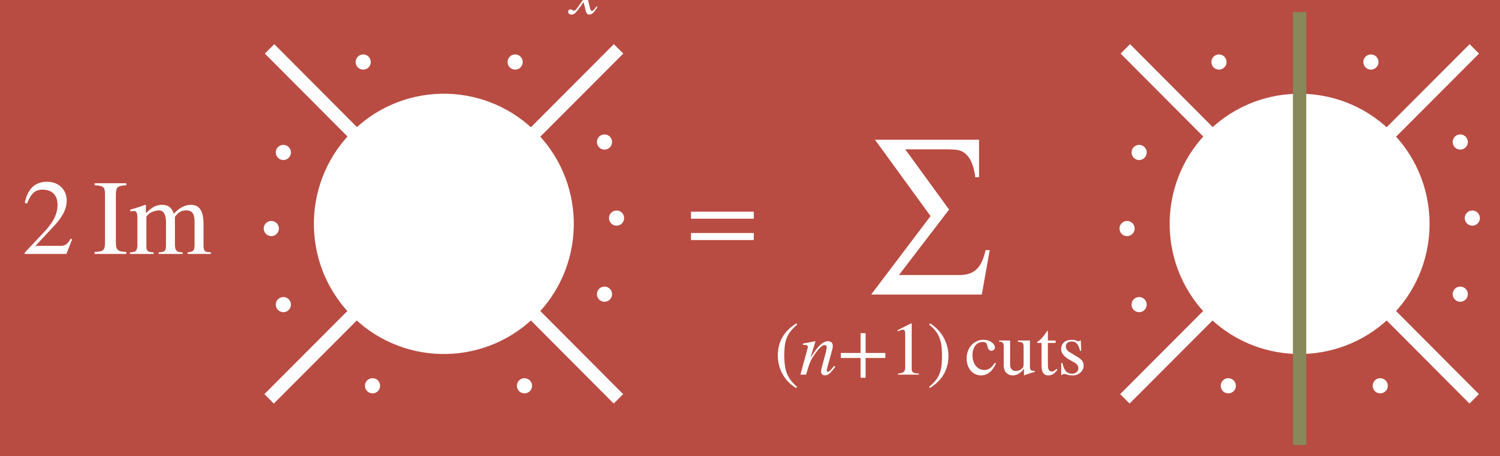

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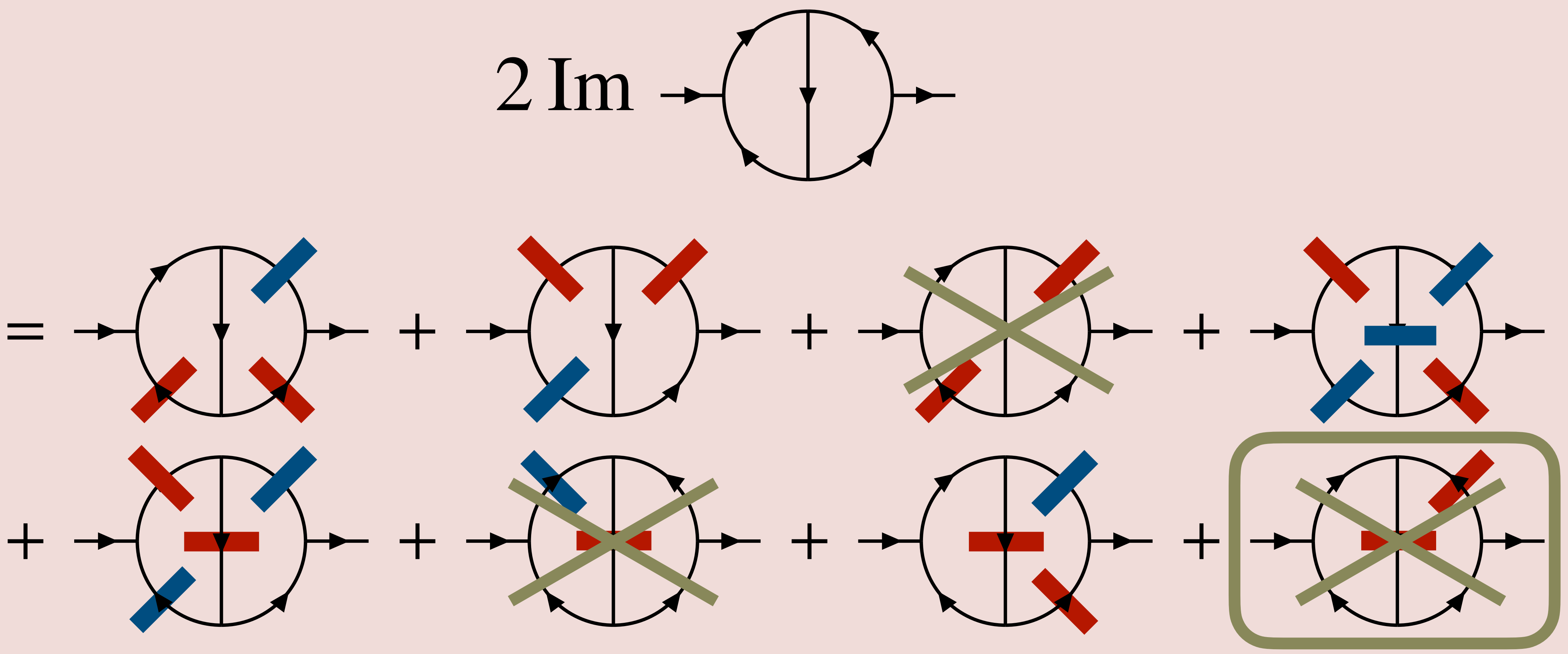

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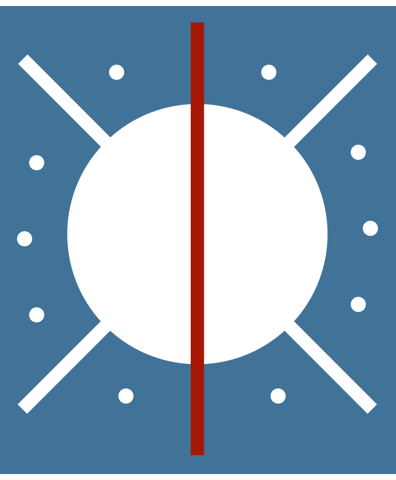

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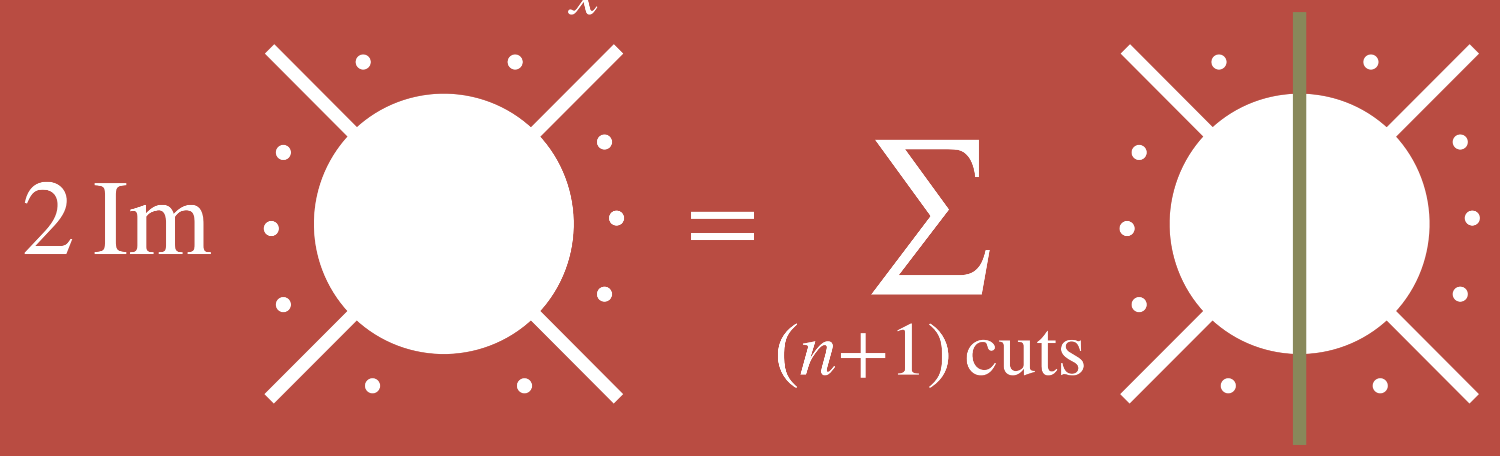

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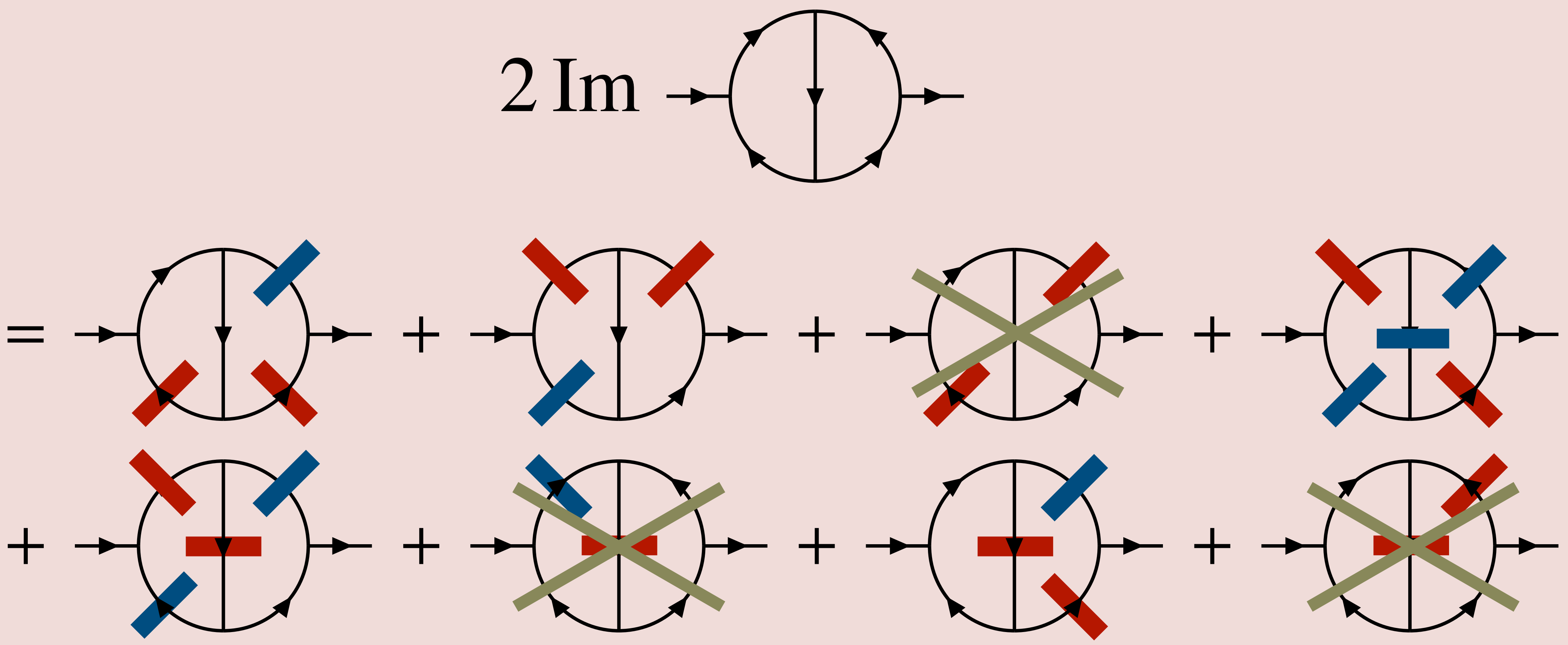

Thresholds
one-loop & two-loop Nf amplitude

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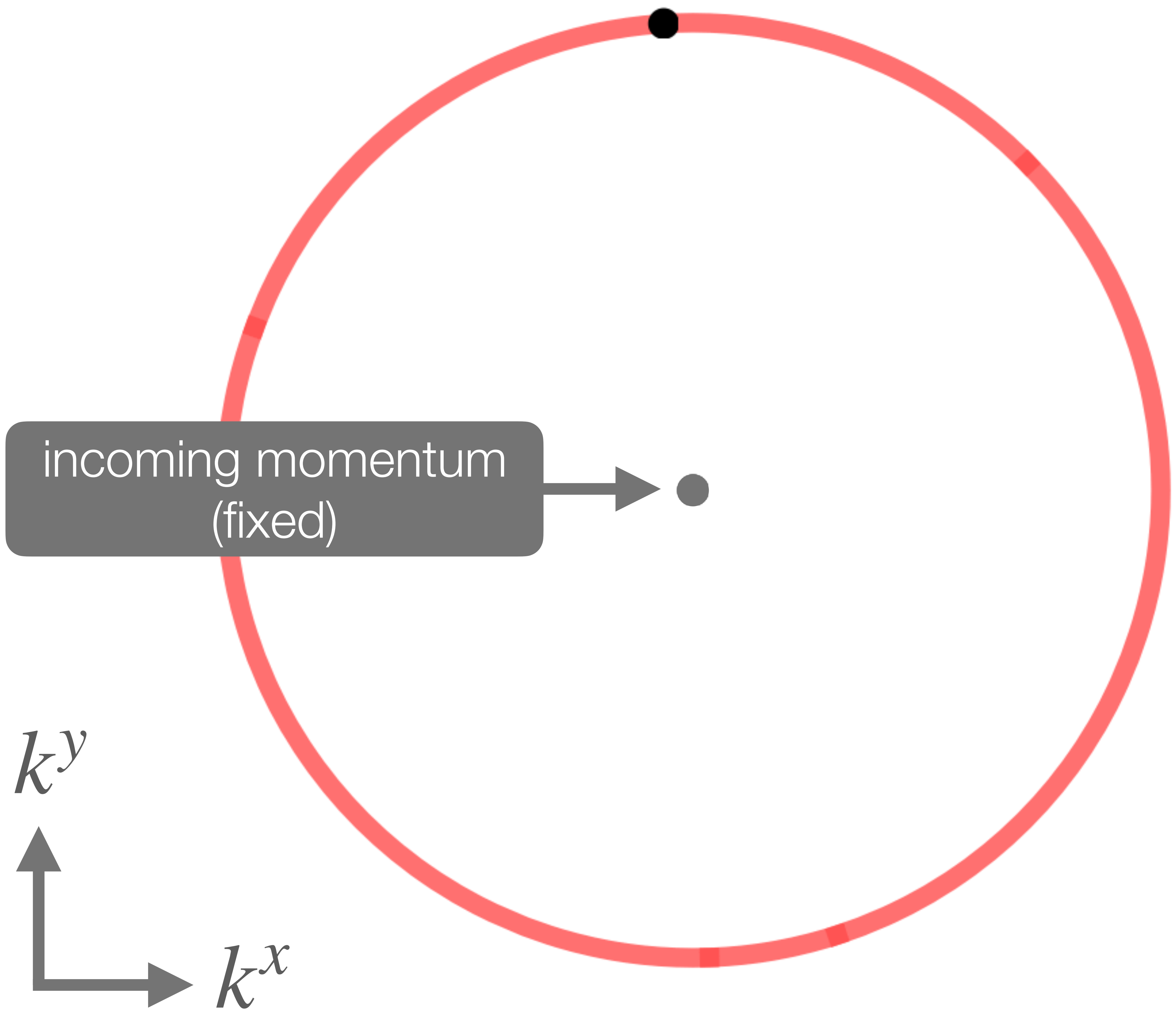
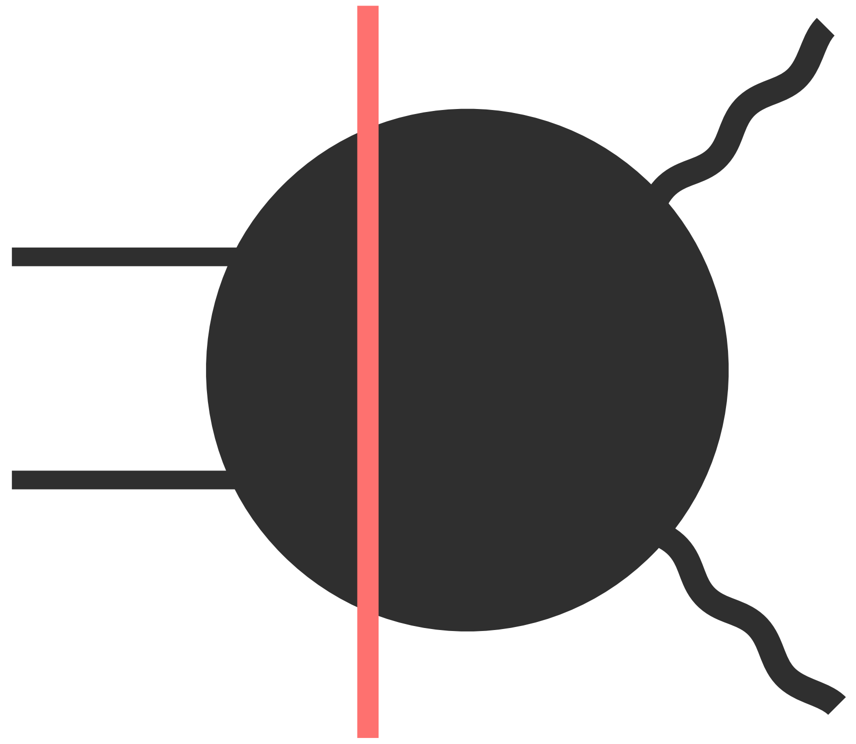


corresponding Cutkosky cuts



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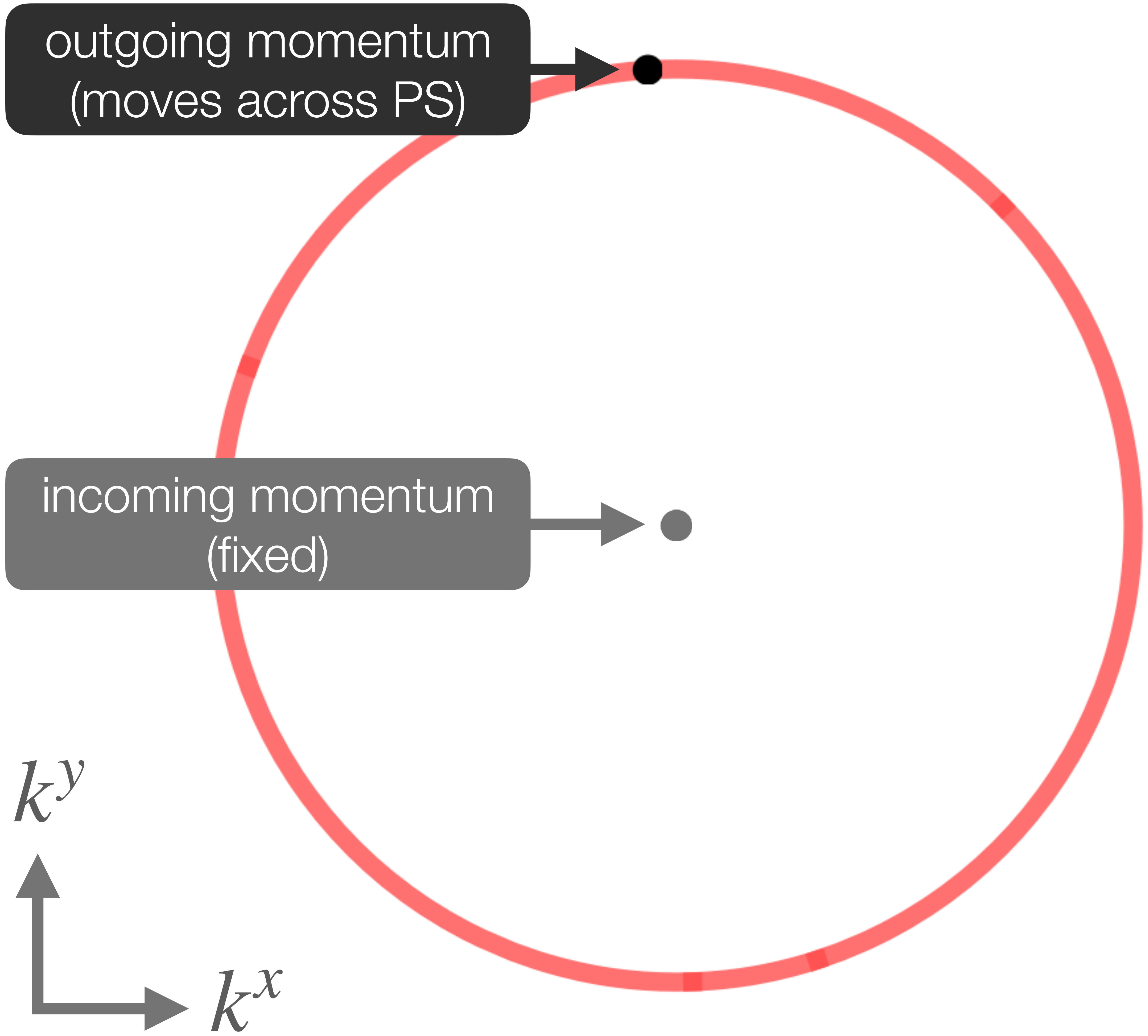

outgoing momentum
(moves across PS)

incoming momentum
(fixed)

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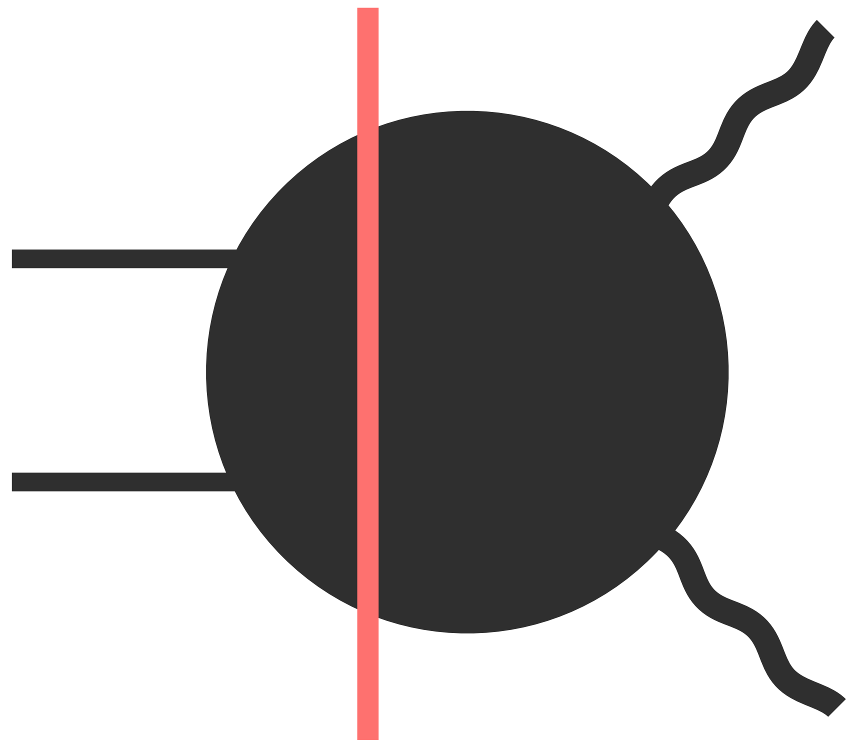

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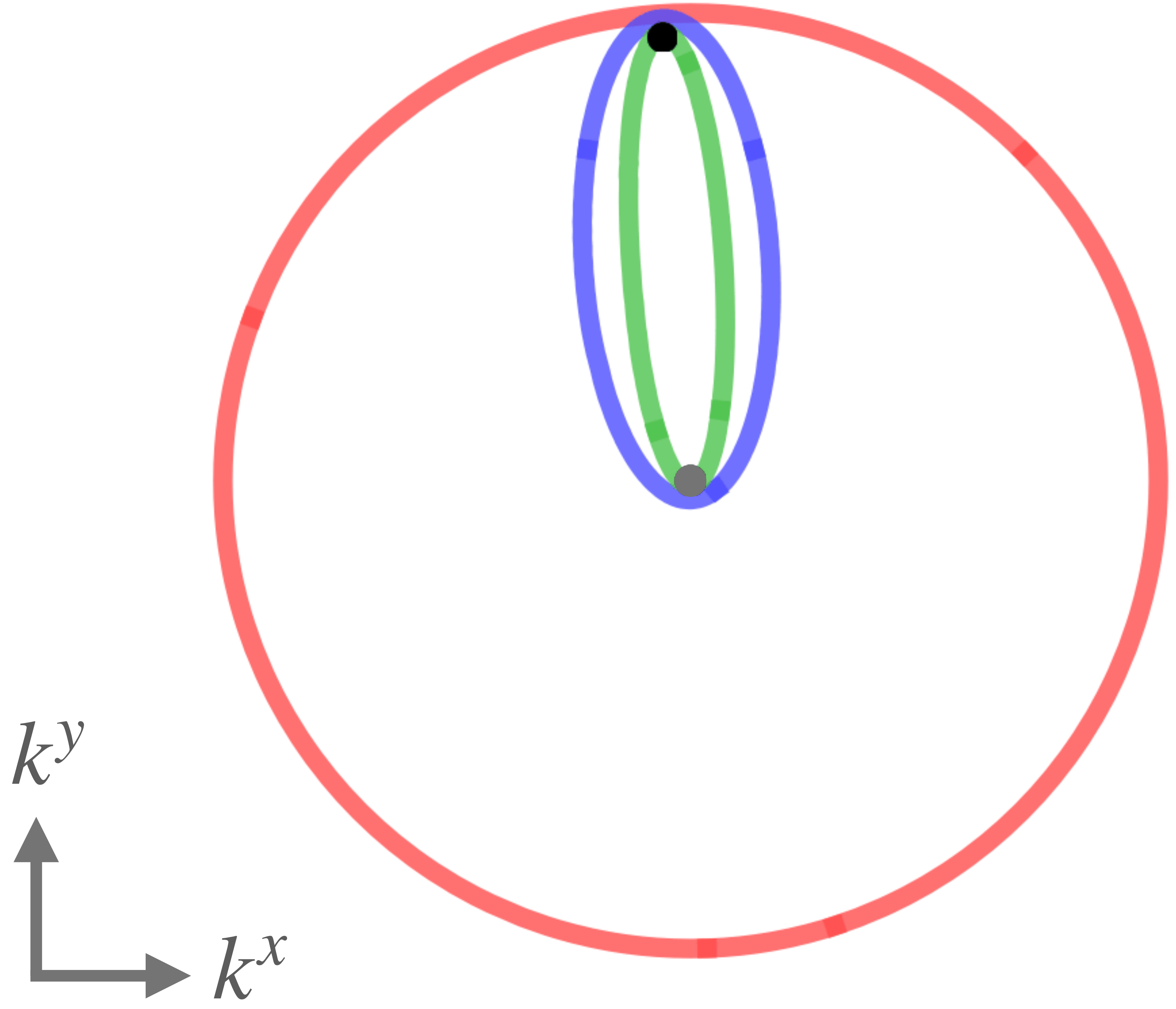

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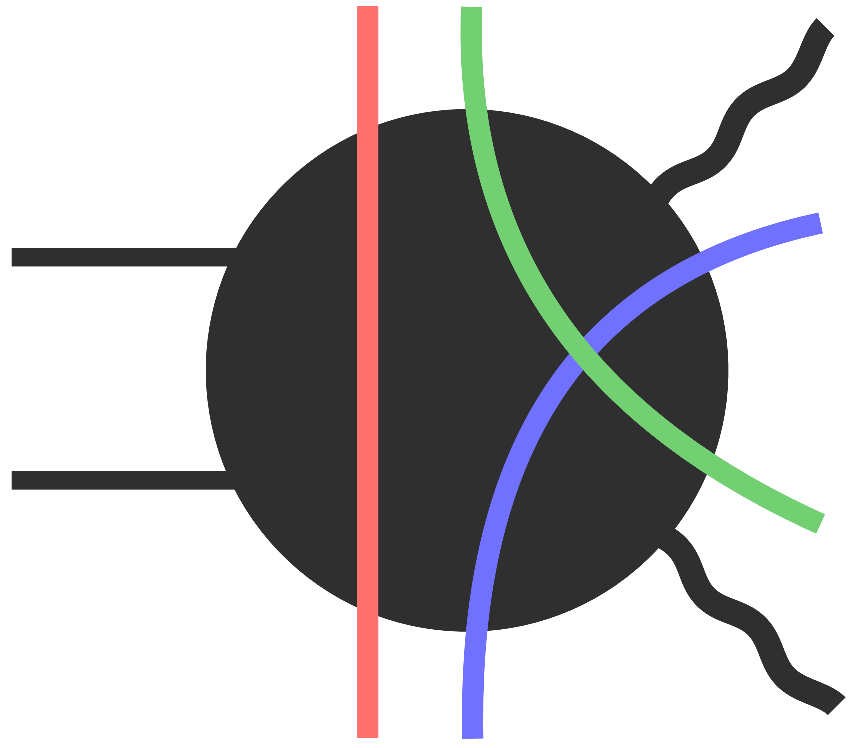

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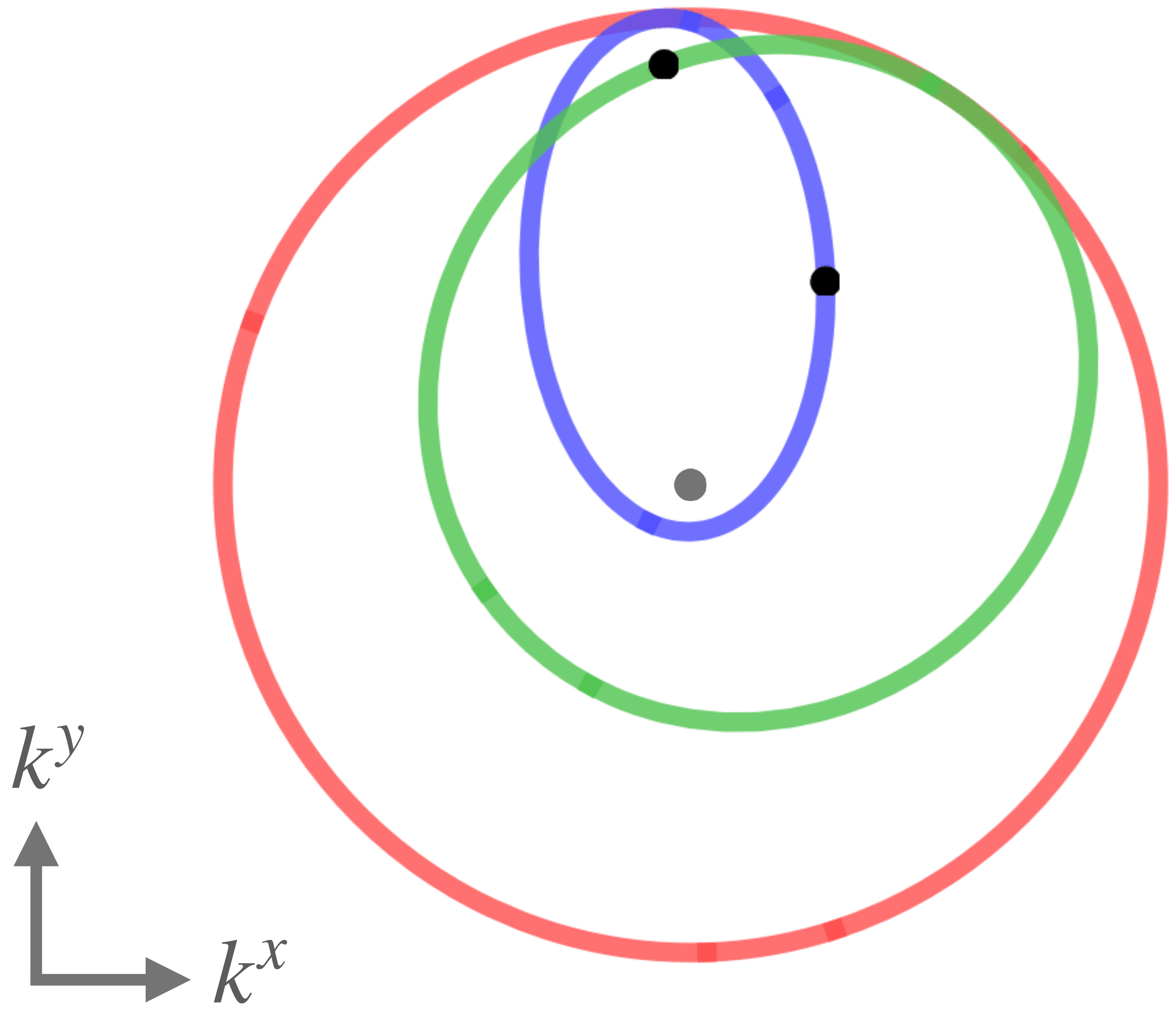

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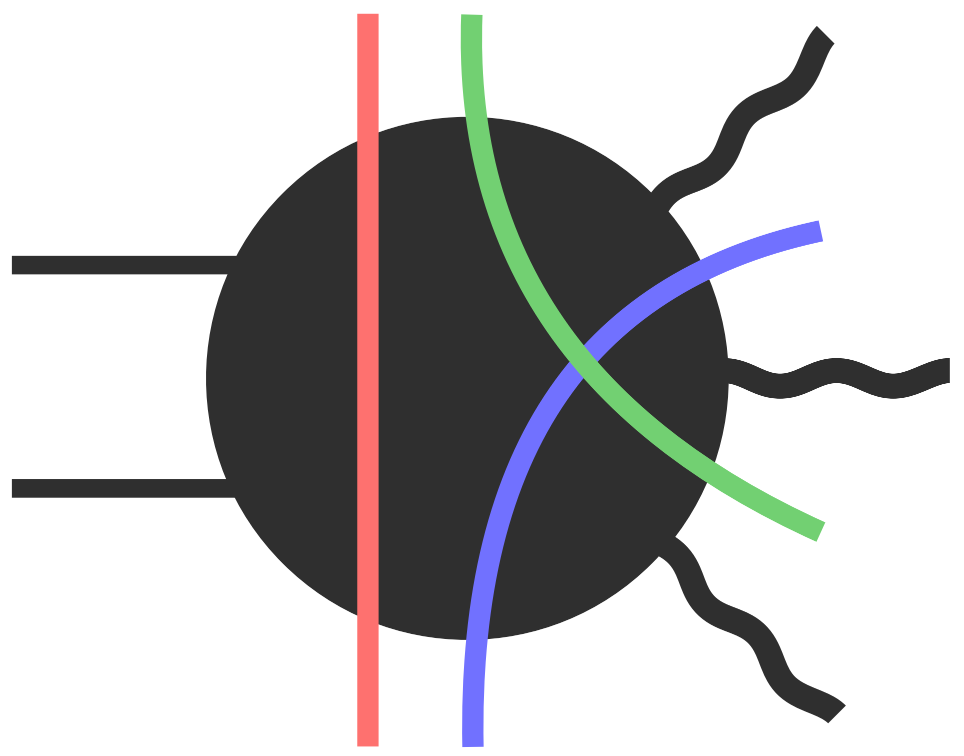

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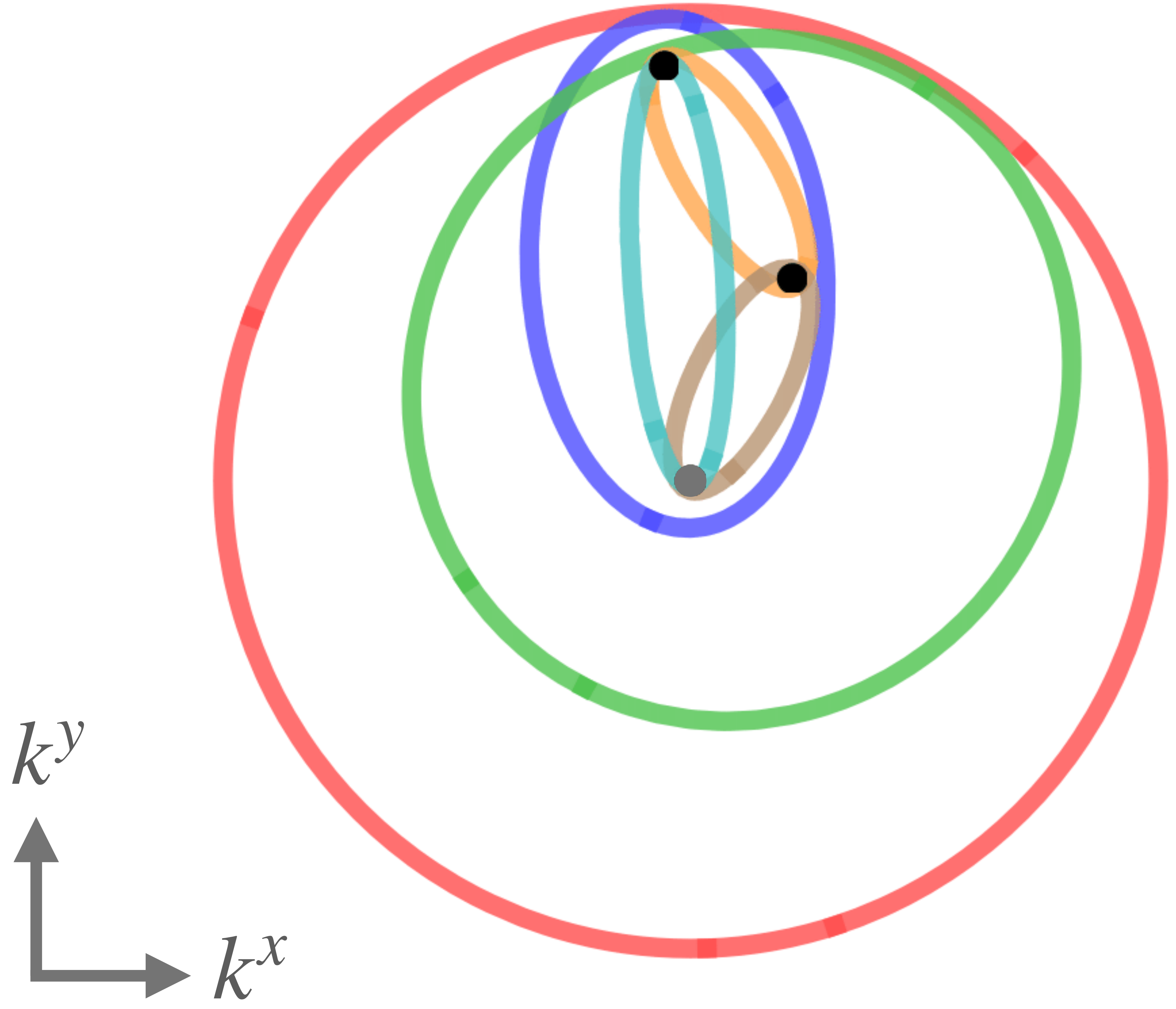

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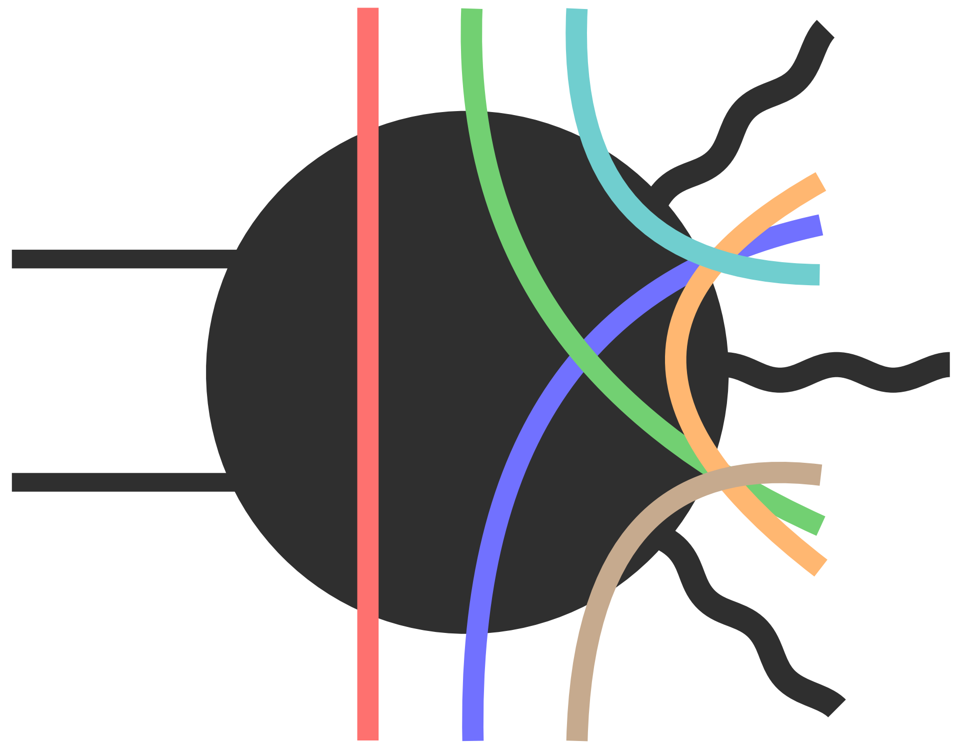

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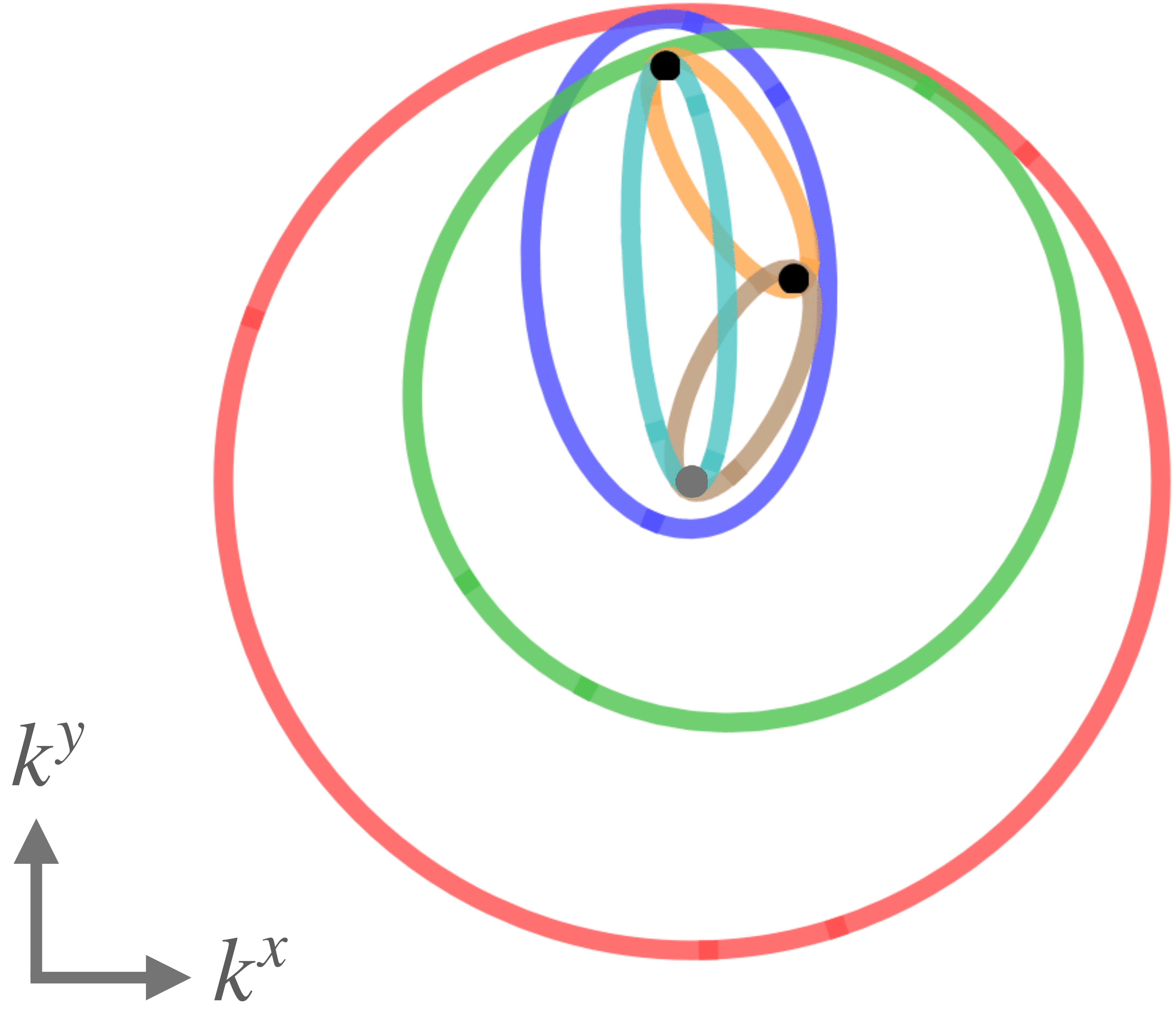

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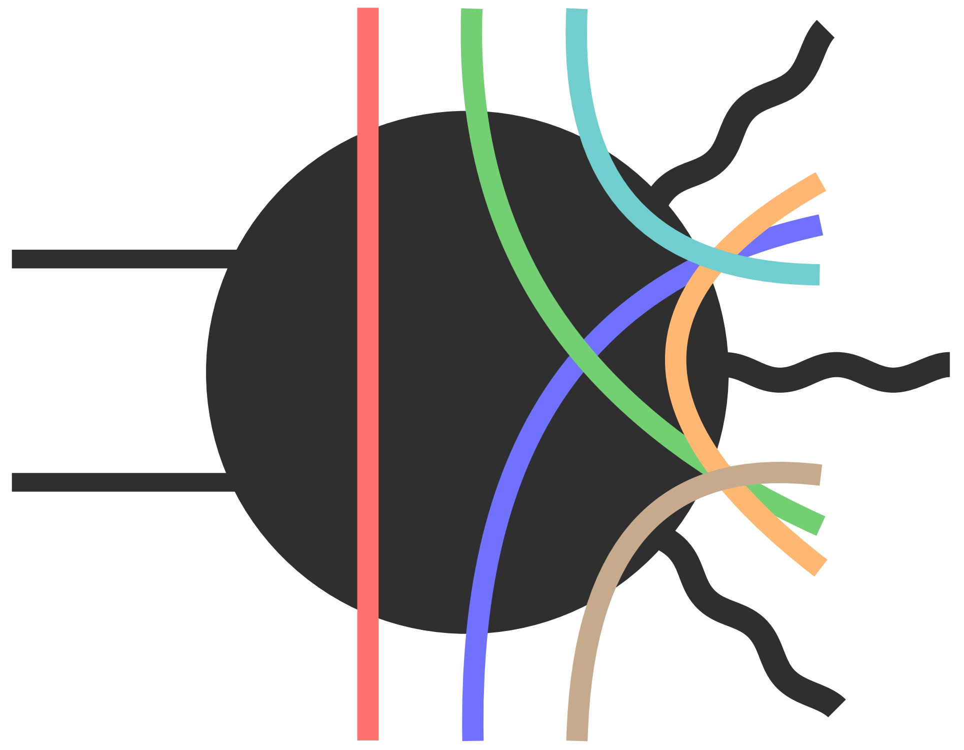
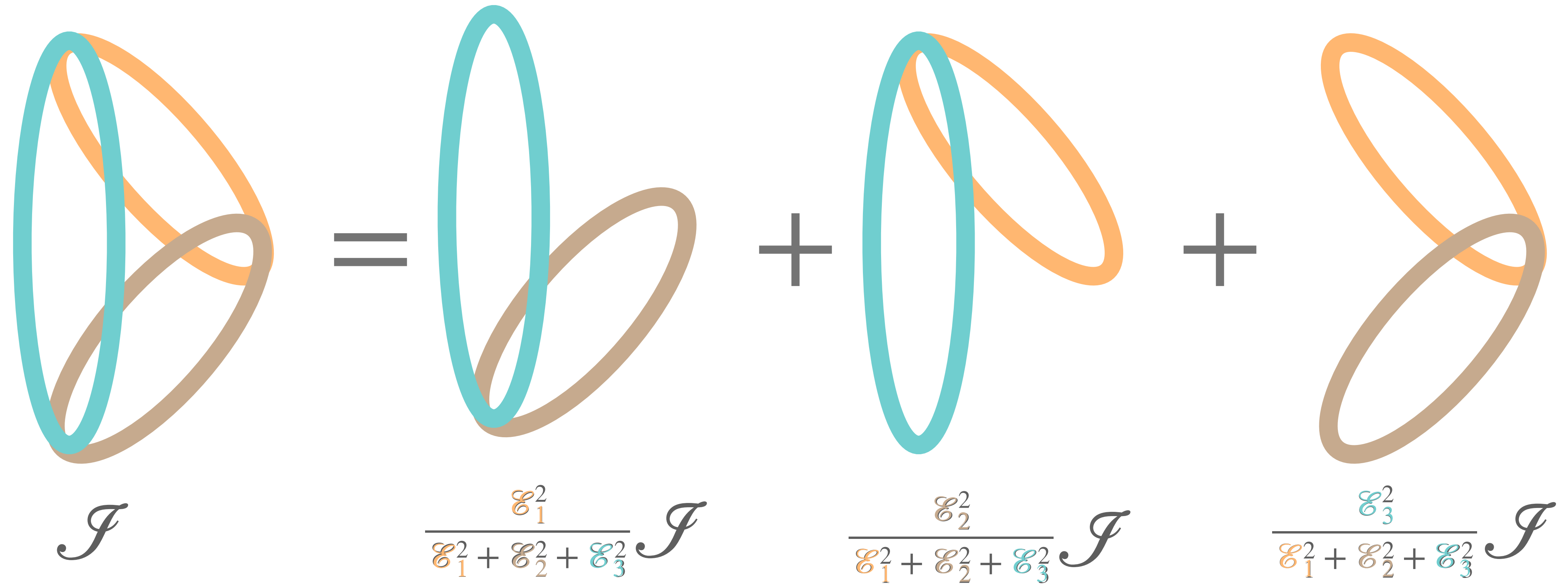


overlapping thresholds
multi-channelling

$$1 = \frac{\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2}{\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2}$$



same pipeline & same computer with 24 cores

DK, Matilde Vicini [2407.18051]
numerical integration over loop & phase space
summed over helicities and convoluted with PDFs

NLO and NNLO-Nf virtual cross sections

	Order	Result [pb]	Δ[%]	total time#
$pp \to \gamma\gamma$	NLO	5.2851 ± 0.0164 e-01	0.3	10 min
	NNLO-Nf	-6.1475 ± 0.0349 e-02	0.6	1 h 30 min
$pp \to \gamma^*\gamma^*$	NLO	4.3172 ± 0.0089 e-01	0.2	2 min
	NNLO-Nf	-3.6943 ± 0.0322 e-02	0.9	40 min
$p_d p_d \to ZZ$	NLO	7.0067 ± 0.0159 e-01	0.2	4 min
	NNLO-Nf	-5.9363 ± 0.0520 e-02	0.9	1 h 30 min
$pp \to \gamma\gamma\gamma$	NLO	1.4874 ± 0.0140 e-04	0.9	2 h 30 min
	NNLO-Nf	-2.5460 ± 0.0237 e-05	0.9	1 day
$pp \to \gamma^*\gamma^*\gamma^*$	NLO	1.4692 ± 0.0144 e-04	1.0	2h 45 min
	NNLO-Nf	-1.4301 ± 0.0137 e-05	1.0	4 days
$p_d p_d \to Z\gamma_1^*\gamma_2^*$	NLO	2.4600 ± 0.0210 e-04	0.9	1 day 12 h
	NNLO-Nf	-2.5301 ± 0.0229 e-05	0.9	1 month

#potential for optimization!

NLO in BLHA
NNLO-Nf in $\overline{\text{MS}}$

NLO cross checked interferences with OpenLoops and cross sections with MadGraph

in agreement with **FivePoint Amplitudes-cpp** Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov [2305.17056]

$\times 3!$ **new!**

masses? no prob!*

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*additional thresholds have to be considered

Summary & Outlook

- ☑ Nf-contribution to NNLO virtual cross section for 3 massive vector boson production
- ☑ First NNLO calculation for the LHC using numerical integration over loop & phase space

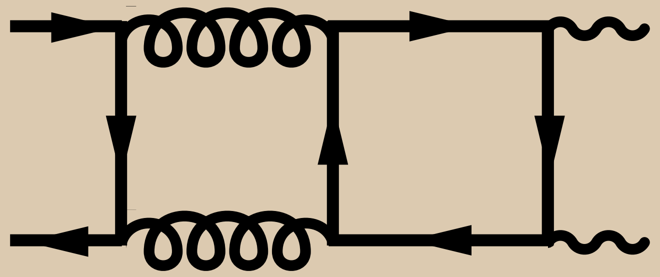
Local IR factorisation & UV renormalisation

Analytic loop energy integration LTD, CFF, TOPT, …

Threshold subtraction

flexible and robust framework suited for automation

- ☐ apply these techniques to the full NNLO virtual contribution
  - ☐ next: other fermion loop contributions
- ☐ combine with real radiation
- ☐ processes with colorful final state
- ☐ …



\+ …

local IR & UV CTs
Anastasiou, Haindl, Karlen, Sterman, Venkata, Yang, Zeng [2403.13712, 2008.12293]

Threshold CTs
DK, Vicini [2407.21511]