Spectral densities from Lattice Euclidean correlators

Mattia Bruno

based on MB, Giusti, Saccardi 2407.04141

CERN, Theory department, Switzerland

March 18th 2025

Motivations

Standard Model of particle physics extremely successful

we know is incomplete: neutrino masses, dark matter, etc..

Search for new physics at energy and intensity frontiers

muon anomaly: hadronic vacuum polarization and light-by-light

e.g. $\gamma \to \pi^+\pi^-$, $\pi^0 \to \gamma\gamma$

flavor physics, e.g. CP violation in strange and charm decays

e.g. study ε'/ε from $K \to \pi\pi$

Study of hadronic amplitudes very important

Reliable non-perturbative predictions from QCD (and SM)

$\to$ Lattice QCD

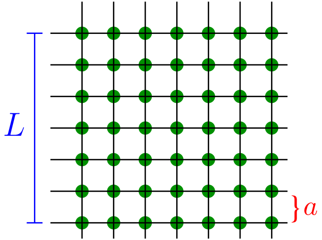
Lattice field theories

lattice spacing $a \to$ regulate UV divergences

finite size $L \to$ infrared regulator

Continuum theory $a \to 0$, $L \to \infty$

Euclidean metric $\to$ Boltzman interpretation of path integral



$$\langle O \rangle = \mathcal{Z}^{-1} \int [DU] e^{-S[U]} O(U) \approx \frac{1}{N} \sum_{i=1}^{N} O[U_i]$$

Very high dimensional integral $\to$ Monte-Carlo methods

Markov Chain of gauge field configs $U_0 \to U_1 \to \cdots \to U_N$

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Some examples where hadronic spectral densities are relevant

- Hadronic τ decays
- Hadronic Vacuum Polarization contribution to $(g-2)_\mu$
- Smeared spectral densities

Hadronic currents, e.g. vector part of $V - A$ $\mathcal{J}_\mu^- = \overline{u}\gamma_\mu d$

Hadronic phase-space factor, i labels hadrons

$$d\Phi_f(p) \equiv (2\pi)^4 \delta^4(p - \sum_i p_i)\, S_f \prod_i \frac{d^3 p_i}{(2\pi)^3 2\omega_i}$$

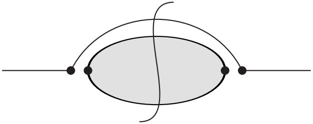
Charged spectral densities

$$\begin{aligned}
\rho^{\mathsf{w}}_{\mu\nu}(p) &= \frac{1}{2\pi} \int d^4x\, e^{ipx} \langle 0|\mathcal{J}_\mu^+(x)\, \mathcal{J}_\nu^-(0)|0\rangle \\
&= \frac{1}{2\pi} \sum_f \int d\Phi_f(p) \langle 0|\mathcal{J}_\mu^+(0)|p_1 \cdots, \mathrm{out}\rangle\langle p_1 \cdots, \mathrm{out}|\mathcal{J}_\nu^-(0)|0\rangle \\
&= (p^2 g_{\mu\nu} - p_\mu p_\nu)\, \rho^{\mathsf{w}}(s) \qquad [s = p_\mu p^\mu]
\end{aligned}$$

Hadronic τ decays

Fermi theory

$$\mathcal{M}_f(P,q,p_1\cdots p_{n_f}) = \frac{G_F V_{ud}}{\sqrt{2}}\,\bar{u}_\nu(-q)\gamma^L_\mu u_\tau(P)\ \langle \text{out}, p_1\cdots p_{n_f}|\mathcal{J}^-_\mu(0)|0\rangle$$



$$d\Gamma = \frac{1}{4m} d\Phi_q \sum_f d\Phi_f \sum_{\text{spin}} |\mathcal{M}_f|^2$$

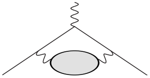
$$= \frac{1}{4m} d\Phi_q \frac{G_F^2 |V_{ud}|^2}{2} \mathcal{L}_{\mu\nu}(P,q)\ \rho^{\text{w}}_{\mu\nu}(p)$$

Charged spectral densities isospin limit $= \rho^{\text{w},0}$ $\left[d\Phi_q = \frac{d^3 q}{(2\pi)^3 2\omega_q} \right]$

$$\frac{d\Gamma(s)}{ds} = G_F^2 |V_{ud}|^2 \frac{m^3}{16\pi} \kappa(s)\, \theta(m_\tau^2 - s)\, \rho^{\text{w},0}(s)$$

HVP FOR $(g-2)_\mu$

Dispersive

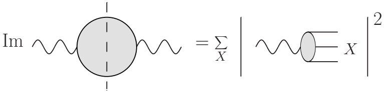


Hadronic Vacuum Polarization (HVP) contribution to a_μ

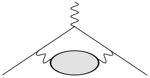
Dispersive

$$a_\mu = \frac{\alpha}{\pi} \int \frac{ds}{s} K(s, m_\mu) \frac{\mathrm{Im}\Pi(s)}{\pi}$$

[Brodsky, de Rafael '68]



$$\frac{4\pi^2\alpha}{s} \frac{\mathrm{Im}\Pi(s)}{\pi} = \sigma_{e^+e^- \to \gamma^\star \to \mathrm{had}}$$



Hadronic Vacuum Polarization (HVP) contribution to a_μ

Lattice

Time-momentum representation [Bernecker, Meyer, '11]

$$G^\gamma(t) = \frac{1}{3}\sum_k \int d\vec{x}\ \langle j_k^\gamma(x) j_k^\gamma(0)\rangle \quad \rightarrow \quad a_\mu = 4\alpha^2 \int_t w_t G^\gamma(t)$$

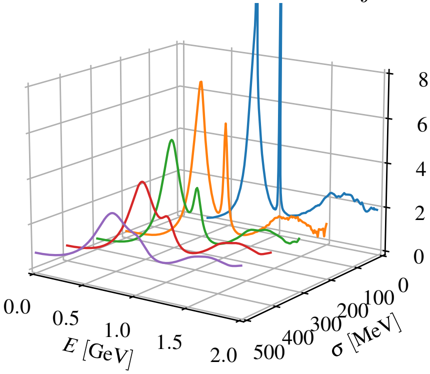
Smeared densities

Lorentzian kernel

ρ_σ analityc continuation in complex plane [Poggio, Quinn, Weinberg '76]

$$\rho_\sigma(\omega) = \int dx \rho(x)\delta_\sigma(x,\omega) = \frac{\sigma}{\pi}\int dx \frac{\rho(x)}{(x-\omega)^2+\sigma^2}$$

$$= \frac{1}{2\pi i}\left[\int_{\mathbb{R}+i\sigma} - \int_{\mathbb{R}-i\sigma}\right] dz \frac{\rho(z)}{z-\omega}$$



R-ratio data [F. Jegelehner]

The problem

"It is of considerable interest to identify the physical quantities, if any, which can be extracted directly from euclidean correlation functions, avoiding analytic continuation" [Maiani, Testa '90] `export NPOINT_CORRELATOR=2` (*)

$\widetilde{J}(t)$ current operator projected to zero total momentum

`export TEMPERATURE=0` (**)

strict zero temperature limit, and $t > 0$

`export STABLE_POLES=0` (*)

spectral density starts w/ branch cut (2 or 3 particles)

(*) Note: extensions to n-point correlators, non-zero momenta straightforward
(**) Note: extension to thermal theory far from trivial

$$C(t) = \langle \widetilde{J}(t)\, \widetilde{J}(0) \rangle = \langle 0 | \widetilde{J}(0)\, e^{-tH}\, \widetilde{J}(0) | 0 \rangle = \int d\omega\, e^{-t\omega}\, \rho(\omega)$$

Inverse Laplace Transform

The problem to solve

Lattice correlator
$\langle \widetilde{J}(t)\, \widetilde{J}(0)\rangle = \int d\omega\, e^{-\omega t}\, \rho(\omega)$

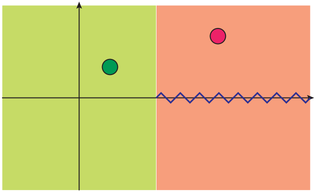
Inverse Laplace
$[e^{-\omega t}] \to [\kappa(\omega)]$

Physical observable
$\rho_\kappa = \int d\omega\, \kappa(\omega)\, \rho(\omega)$

Inverse Laplace Transform

The problem to solve

Can we find $f(t)$ such that $\rho_\kappa = \int dt\, f(t)\, C(t)$?



Example:

ρ has branch cut starting at multi-particle thresholds $E_{\rm thr}$

Kernel κ simple poles s_0 in complex plane

if $\mathrm{Re} s_0 \leq E_{\rm thr}$ $\rightarrow \forall t\ \exists M > 0 \mid f(t)C(t) < e^{-Mt}$
e.g. HVP contribution to $(g-2)_\mu$ [Blum '02][Bernecker-Meyer '11]

if $\mathrm{Re} s_0 > E_{\rm thr}$ $\rightarrow$ inverse problem, no "direct" analytic continuation

A least-square problem

aka Backus-Gilbert-like methods

Ansatz $\kappa(\omega) \approx \sum_t g_t e^{-\omega t}$ (finite number of terms)

1. minimize L^2 norm $\int d\omega \left[\sum_t g_t e^{-\omega t} - \kappa(\omega) \right]^2$
2. define $A(t,t') = \int d\omega e^{-\omega(t+t')} = \dfrac{1}{t+t'}$, $\quad f(t) = \int d\omega \kappa(\omega) e^{-\omega t}$
3. solution is $g_t = \sum_{t'} [A^{-1}]_{t,t'} f(t')$

[Backus, Gilbert '68][Hansen, Lupo, Tantalo '19]

A is finite matrix, but ill-conditioned $\rightarrow$ regularization

define $W[\lambda] = A(1-\lambda) + \lambda B$ and evaluate $g_t = \sum_{t'} [W^{-1}]_{t,t'} f(t')$

B chosen to be covariance matrix

window in λ where syst. $\simeq$ stat. errors

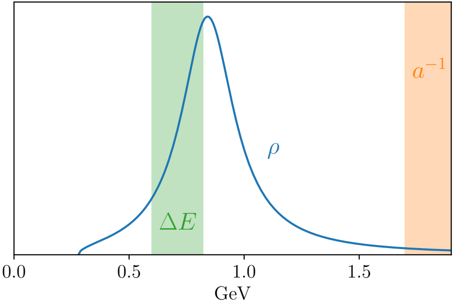
Alternatives Chebyshev, Bayesian methods.. [Bailas et al '20][Del Debbio et al '24]

[Barone et al '24]

Challenges on the Lattice

a superficial overview

1. up, down physical masses ✓ ← algorithmic + technological advances
   strange quark ✓, sea charm effects if small typically controlled



2. lattice cutoff typically $\in [1.7, 4]$ GeV
3. energy resolution $\frac{2\pi}{L} \approx 200$ MeV
4. stat errs grow exponentially at long distances

What is better (on paper) for Lattice QCD?
smeared $\rho = \int d\omega\, \rho(\omega)\, \kappa(\omega)$ w/ broad κ
→ finite-volume errs expected to be exponentially suppressed

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Phenomenological studies

[ETMC '22 '23 '24]

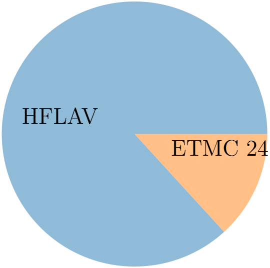
Some recent studies

smeared R-ratio

inclusive τ decays V_{ud}

inclusive τ decays V_{us}

$|V_{us}|$ error



(MY) OBJECTIVES

Similarly to HVP contribution to $(g-2)_\mu$

1. define the problem in the continuum infinite-volume theory
2. define the problem on an infinite-volume lattice
    2.1 study discretization errors w/ Symanzik's EFT
3. define the problem on a continuous torus
    3.1 study finite-volume errors a là Lüscher

[McWhirter, Pike '78][Epstein, Schotland '08]

$$\int_0^\infty dx\, e^{-xy}\, x^a \quad \to z = xy \to \quad \int_0^\infty \frac{dz}{y}\, e^{-z}\, z^{a+1-1}\, y^{-a}$$

$$\int_0^\infty dx\, e^{-xy}\, x^a = \Gamma(a+1) y^{-1-a} \quad \to \quad \mathrm{Re}a = -1/2$$

Mellin transform is sort of "eigenfunction" of Laplace trafo

$$u_s(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2+is} = \frac{e^{is\log x}}{\sqrt{2\pi x}}$$

$$\int_0^\infty dx\, e^{-xy}\, u_s(x) = \lambda_s\, u_s^\star(y) \quad \lambda_s = \Gamma\left(\frac{1}{2} + is\right)$$

Note: $t \to Mt$ and $\omega \to \omega/M$

Laplace transform

Some properties

[McWhirter, Pike '78][Epstein, Schotland '08]

$$\int_0^\infty dx\, e^{-xy}\, x^a \quad \to z = xy \to \quad \int_0^\infty \frac{dz}{y}\, e^{-z}\, z^{a+1-1}\, y^{-a}$$

$$\int_0^\infty dx\, e^{-xy}\, x^a = \Gamma(a+1) y^{-1-a} \quad \to \quad \mathrm{Re} a = -1/2$$

Mellin transform is sort of "eigenfunction" of Laplace trafo

$$u_s(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2+is} = \frac{e^{is \log x}}{\sqrt{2\pi x}}$$

$$\int_0^\infty dx\, e^{-xy}\, u_s(x) = \lambda_s\, u_s^\star(y) \quad \lambda_s = \Gamma\left(\frac{1}{2} + is\right)$$

Note: $t \to Mt$ and $\omega \to \omega/M$

[McWhirter, Pike '78][Epstein, Schotland '08]

$$\int_0^\infty dx\, e^{-xy}\, x^a \quad \rightarrow z = xy \rightarrow \quad \int_0^\infty \frac{dz}{y}\, e^{-z}\, z^{a+1-1}\, y^{-a}$$

$$\int_0^\infty dx\, e^{-xy}\, x^a = \Gamma(a+1) y^{-1-a} \quad \rightarrow \quad \mathrm{Re}a = -1/2$$

Mellin transform is sort of "eigenfunction" of Laplace trafo

$$u_s(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2+is} = \frac{e^{is\log x}}{\sqrt{2\pi x}}$$

$$\int_0^\infty dx\, e^{-xy}\, u_s(x) = \lambda_s\, u_s^\star(y) \quad \lambda_s = \Gamma\left(\frac{1}{2} + is\right)$$

Note: $t \rightarrow Mt$ and $\omega \rightarrow \omega/M$

Carleman-Hankel operator

Intermezzo

[Carleman 1923][Groetsch '84][Power '80]

$$\int_t e^{-\omega' t}\, e^{-\omega t} = \frac{1}{\omega' + \omega} = \mathcal{H}(\omega', \omega)$$

1. diagonalize $\mathcal{H}$ w/ "Mellin" basis $\mathcal{H}(\omega, \omega') = \int_s u_s^*(\omega)\, |\lambda_s|^2\, u_s(\omega')$

$$\int_\omega \mathcal{H}(\omega', \omega) u_s(\omega) = \int_\omega \int_t e^{-\omega' t}\, e^{-\omega t}\, u_s(\omega) = \int_t e^{-\omega' t}\, \lambda_s u_s^*(t)$$
$$= |\lambda_s|^2 u_s(\omega')$$

2. $\mathcal{H}$ ill-conditioned $|\lambda_s|^2 = \dfrac{\pi}{\cosh(\pi s)} \overset{s \to \pm\infty}{\to} 0$

3. inverse requires regularization, e.g. a là Tikhonov

$$\mathcal{H}_\alpha^{-1}(\omega, \omega') = \int_s u_s^*(\omega) \frac{1}{|\lambda_s|^2 + \alpha} u_s(\omega')$$

Inverse Laplace transform

A solution

[McWhirter, Pike '78][Epstein, Schotland '08][MB, Giusti, Saccardi '24]

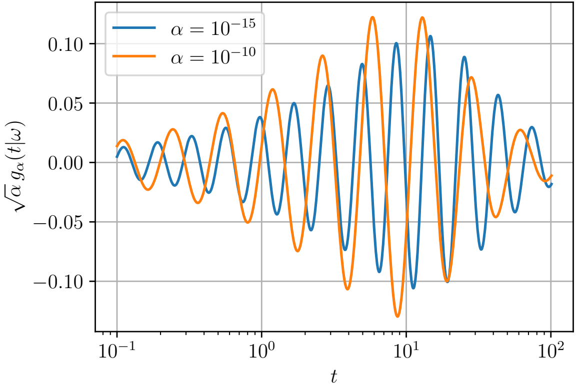
Fredholm integral equation $\int_t e^{-\omega' t}\, C(t) = \int_\omega \mathcal{H}(\omega', \omega)\, \rho(\omega)$

$$
\begin{aligned}
\rho_\alpha(\omega) &\equiv \int_{\omega'} \mathcal{H}_\alpha^{-1}(\omega, \omega') \int_t e^{-\omega' t} C(t) \\
&= \int_s u_s^*(\omega) \frac{\lambda_s}{|\lambda_s|^2 + \alpha} \int_t u_s^*(t)\, C(t) \\
&= \int_t \left[\int_s u_s^*(\omega) \frac{\lambda_s}{|\lambda_s|^2 + \alpha}\, u_s^*(t) \right] C(t) = \int_t g_\alpha(t|\omega)\, C(t) \\
&= \int_{\omega'} \left[\int_s u_s^*(\omega) \frac{|\lambda_s|^2}{|\lambda_s|^2 + \alpha} u_s(\omega) \right] \rho(\omega') = \int_{\omega'} \delta_\alpha(\omega, \omega')\, \rho(\omega')
\end{aligned}
$$

Example

In lattice units

Let's consider $a = 0.12\ \text{fm} = [1.7\ \text{GeV}]^{-1}$ $\quad a\omega = 0.5 \simeq 830\ \text{MeV}$



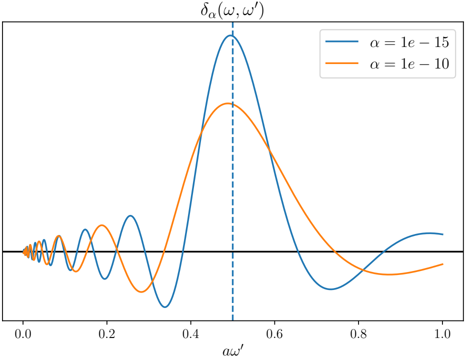
Notice logarithmic scale on x-axis and oscillations $t \ll 0.12$ fm

A minimal kernel

$\rho_\kappa \equiv \int_\omega \rho(\omega)\,\kappa(\omega)$ w/ smearing kernel $\kappa(\omega)$

$$\rho_{\kappa,\alpha} = \int_{\omega,\omega'} \kappa(\omega')\,\delta_\alpha(\omega,\omega')\,\rho(\omega) = \int_t g_\alpha(t|\kappa)\,C(t)$$

$\delta_\alpha(\omega,\omega')$ "minimal smearing" to solve Inverse Laplace Transform (ILT)



$\lim_{\alpha\to 0}\delta_\alpha(\omega,\omega') = \delta(\omega-\omega')$

δ_α acts on $L^2(0,\infty,d\omega)$

if $\rho \in L^2(0,\infty,d\omega)$ then
$\lim_{\alpha\to 0}\rho_\alpha(\omega) = \rho(\omega)$ exists

Least-square problem

A different perspective

Since $\kappa_\alpha(\omega) = \int_{\omega'} \kappa(\omega')\, \delta_\alpha(\omega, \omega') = \int_t g_\alpha(t|\kappa) e^{-\omega t}$

coefficients $g_\alpha(t|\kappa)$ minimize the functional

$$\int_\omega \left[\kappa(\omega) - \kappa_\alpha(\omega)\right]^2 + \alpha \int_t g_\alpha(t|\kappa)^2$$

looks like continuous version of Backus-Gilbert/HLT

Expanding ρ or κ on exponential basis are equivalent points of view

In fact $\rho_\alpha(\omega) = \int_t r_\alpha(t) e^{-\omega t}$ (with $r_\alpha(t)$ trivially related to g_α)

coefficients $r_\alpha(t)$ minimize the functional

$$\int_\omega \left[\rho(\omega) - \rho_\alpha(\omega)\right]^2 + \alpha \int_t r_\alpha(t)^2$$

[MB, Giusti, Saccardi '24]

Hyphothesis: $\rho(\omega) \in L^2(0, \infty, d\omega)$

mass gap $\rightarrow$ support over $[\omega_0 > 0, \infty)$

short-distance divergences $\rightarrow \rho \simeq \omega^k \rightarrow$ cannot use $g_\alpha(t|\omega)$

Extension to QFT: change coefficients

$$g_\alpha(t|\omega, p) = \int_s u_s^*(\omega) \frac{\lambda_s}{\lambda_s \lambda_{s,p} + \alpha} u_s^*(t)\, t^p$$

p such that $t^{p-1/2} C(t)$ finite as $t \rightarrow 0$

define ρ_α from $g_\alpha(t|\omega, p)$, take $\alpha \rightarrow 0$

Example: R-ratio and vector correlator

$$C(t) = \int_\omega e^{-\omega t} \omega^2 \rho_s(\omega),\ C(t) \overset{t \simeq 0}{\propto} 1/t^3 \text{ and } \rho_s \overset{\omega \gg 0}{\simeq} \text{const}$$

$$p = \tfrac{5}{2} \quad \rightarrow \quad \rho_s/\sqrt{\omega} \text{ or } p = 3 \quad \rightarrow \quad \rho_s/\omega$$

Lattice, how?

A strategy for a lattice calculation

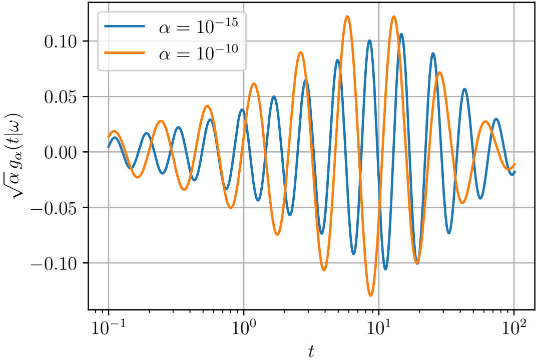
(correlators strictly at zero temperature: keep $e^{-\omega t}$ discard $e^{-\omega(T-t)}$)

1. correlators in the continuum but sampled at discrete $t = a, 2a, 3a \ldots$
   $\overline{C}(t,a) = C(t) + O(a^2)$
2. formulate the problem for an infinite lattice $\sum_{t=a}^{\infty} \ldots C(t)$
3. truncate solution $\sum_{t=a}^{T} \ldots C(t)$
4. address truncation of temporal cut in continuum
   exponential thanks to mass gap

No free lunch

As expected ...

Let's consider $a = 0.12\ \mathrm{fm} = [1.7\ \mathrm{GeV}]^{-1}$ $\quad a\omega = 0.5 \simeq 830\ \mathrm{MeV}$



Oscillations at very short Euclidean times

naive approach
$\int_t g_\alpha \cdots \to \sum_t g_\alpha \ldots$
does not work

need new $\bar{g}_{a,\alpha}$
$\lim_{a\to 0} \bar{g}_{a,\alpha} = g_\alpha$??

Lattice Carleman operator

$$a\sum_{t=a} e^{-\omega' t}\, e^{-\omega t} = \overline{\mathcal{H}}_a = \frac{ae^{-(\omega+\omega')a}}{1-e^{-a(\omega+\omega')}} \quad \overset{a\to 0}{\simeq} \quad \frac{1}{\omega+\omega'} = \mathcal{H}(\omega,\omega')$$

Diagonalization $\overline{\mathcal{H}}_a$ is possible!!

$$\int_{\omega'} \overline{\mathcal{H}}_a(\omega,\omega') v_s(\omega',a) = |\lambda_s|^2 v_s(\omega,a) \quad s\in\mathbb{R}^+$$

$$v_s(\omega,a) \equiv \sqrt{2\pi a}\frac{u_s(1-e^{-a\omega})}{|N_s|} e^{-a\omega}|\lambda_s|^2\, {}_2F_1\left(\begin{matrix}\frac{1}{2}+is, \frac{3}{2}+is\\ 2\end{matrix}\middle| e^{-a\omega}\right)$$

$\overline{\mathcal{H}}_a$ same spectrum as $\mathcal{H}$ and

$$\lim_{a\to 0} v_s(\omega,a) \propto \mathrm{Re}u_s(a\omega) + O(a^2\omega^2)$$

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Lattice Carleman operator

$$a\sum_{t=a} e^{-\omega' t}\, e^{-\omega t} = \overline{\mathcal{H}}_a = \frac{a e^{-(\omega+\omega')a}}{1-e^{-a(\omega+\omega')}} \quad \overset{a\to 0}{\simeq} \quad \frac{1}{\omega+\omega'} = \mathcal{H}(\omega,\omega')$$

Diagonalization $\overline{\mathcal{H}}_a$ is possible!!

$$\int_{\omega'} \overline{\mathcal{H}}_a(\omega,\omega') v_s(\omega',a) = |\lambda_s|^2 v_s(\omega,a) \quad s\in\mathbb{R}^+$$

$$v_s(\omega,a) \equiv \sqrt{2\pi a}\frac{u_s(1-e^{-a\omega})}{|N_s|} e^{-a\omega}|\lambda_s|^2\, {}_2F_1\left(\begin{array}{c} \frac{1}{2}+is, \frac{3}{2}+is \\ 2 \end{array}\middle| e^{-a\omega}\right)$$

$\overline{\mathcal{H}}_a$ same spectrum as $\mathcal{H}$ and

$$\lim_{a\to 0} v_s(\omega,a) \propto \mathrm{Re}\, u_s(a\omega) + O(a^2\omega^2)$$

Continuum: energy and time domains both continuous in $[0, \infty)$

$$\rightarrow \quad u_s(\omega), u_s(t)$$

$$g_\alpha(t|\omega) = \int_{s\in\mathbb{R}} u_s^*(\omega) \frac{\lambda_s}{|\lambda_s|^2 + \alpha} u_s^*(t)$$

$$\delta_\alpha(\omega, \omega') = \int_{s\in\mathbb{R}} u_s^*(\omega) \frac{|\lambda_s|^2}{|\lambda_s|^2 + \alpha} u_s(\omega')$$

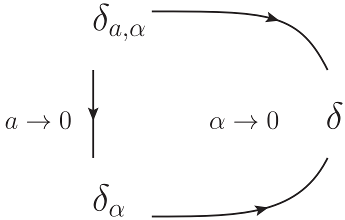
Regular Lattice: asymmetry between continuous ω and discrete times

$$\rightarrow \quad v_s(\omega, a)\, , \overline{v}_s(t, a)$$

$$\overline{g}_{a,\alpha}(t|\omega) = \int_{s\in\mathbb{R}+} v_s(\omega, a) \frac{|\lambda_s|}{|\lambda_s|^2 + \alpha} \overline{v}_s(t - a, a)$$

$$\overline{\delta}_{a,\alpha}(\omega, \omega') = \int_{s\in\mathbb{R}+} v_s(\omega, a) \frac{|\lambda_s|^2}{|\lambda_s|^2 + \alpha} v_s(\omega', a)$$

Quick recap
Continuum vs Lattice

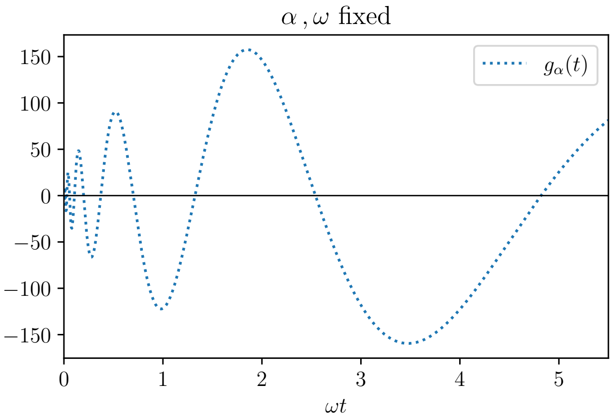


ILT on infinite regular lattice (explicitly) solved

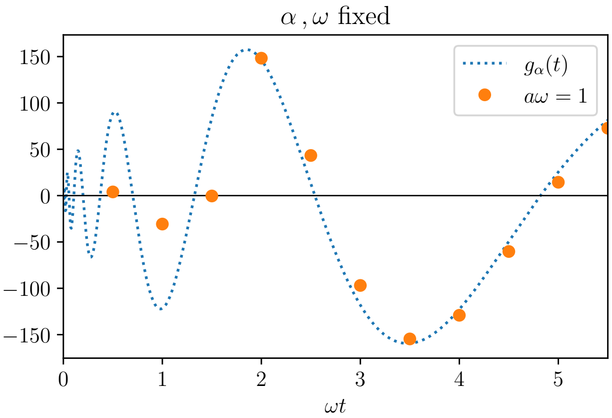
$v_s(\omega, a)$ complete set of orthogonal functions

no discretization errors from ILT if $\alpha \sim 0$, only from correlators

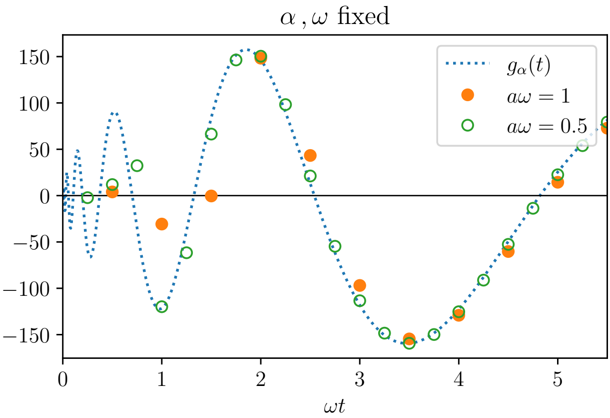
Continuum limit



Continuum limit



Continuum limit



Continuum limit

Thanks to explicit form $\bar{g}_{a,\alpha}(t|\omega) \to g_\alpha(t|\omega) + O((a\omega)^2)$

Coefficients $\bar{g}_{a,\alpha}(t|\omega)$ vs Backus-Gilbert-like methods (HLT)

- depend on Tikhonov not covariance matrix as regulator
- correspond to infinite Hilbert matrix
- we now know their continuum limit at fixed α
  - can study e.g. finite-volume errs in continuum

Conclusions

Explicit formulation of ILT continuum and lattice for $\rho \in L^2$

- $\rho \notin L^2$ still missing for lattice [in progress]
- effects of truncation of integral/sums is exponential
- bound on $|\rho - \rho_\alpha|$

What's next?

- numerical tests and applications
- finite-volume effects starting from correlator [Lüscher][Hansen-Patella]
- statistical errors and covariances
- discretization errors, e.g. [Sommer Lat'22]

Thanks for your attention

Some remarks on errors

Assuming the case $\rho(\omega) \in L^2(0, \infty, d\omega)$

$$\sigma^2_{\rho_\kappa} = \int_{t,t'} g_\alpha(t|\kappa)\,\mathrm{cov}(t,t')\,g_\alpha(t'|\kappa)$$

if $\mathrm{cov} = \mathrm{const}$ the error diverges in the $\alpha \to 0$

if $\mathrm{cov} = e^{-m(t+t')}$ the limit $\alpha \to 0$ is finite

---

$$\rho_\alpha(\omega) - \rho_{\alpha,t_{\max}}(\omega) \leq \frac{|C(t_{\max})|}{2\pi\sqrt{\omega M}\sqrt{M t_{\max}}} \int_s \frac{|\lambda_s|}{|\lambda_s|^2 + \alpha}$$

$$|\rho_\kappa - \rho_{\kappa,\alpha}|^2 \leq \alpha^2 \left[\int_\omega \rho(\omega)^2\right] \int_{\omega,\omega'} \kappa(\omega)\kappa(\omega') \int_s \frac{u_s^*(\omega)u_s(\omega')}{(|\lambda_s|^2 + \alpha)^2}$$