Beyond eikonal: matrix elements of SM and New Physics in lepton pair production

Status and perspectives

Z. Was*

***Institute of Nuclear Physics, Polish Academy of Sciences, Cracow, Poland**

- In Cracow since more than 40 years now, Monte Carlo programs, phenomenology tools for accelerator experiments are developed.
- In presentation focus was always on practicalities. Principles foundations were left behind. For example our LEP time programs, formal documentation was published many years after use of programs was stopped. **Actually,** final precision data papers were published later.
- It is common opinion that Monte Carlo is bound to rely on approximation and remain inferior to analytic calculations.
- Foundations are rarely underlined and are rarely appreciated even by devoted users.
- Now is a good time to contest $\longrightarrow$ look into principles: new precision challenges of FCC and challenges of manpower/expertise continuity.

My aim is to argue that Monte Carlo techniques are based on strict mathematical rules.

At least that it can be done like that. My talk is of 30 mins, **NOT** of 30 hours.

Introduction, its keyword: eikonal

- **I plan to address several people lifetime effort.**
- **How this can be of help for future efforts?**
- **Selected aspects, formal proofs left to long papers.**
- **Approach backbone eikonal restricted QED. It is (i) solvable, (ii) used in many Monte Carlo designs (iii) SM amplitude level perturbation results can (and must) be represented as corrections to their eikonal parts.**
- **<u>NO eikonal approximation</u> in use.**
- **My talk is addressed to people who may continue efforts toward precision horizons as required e.g. by FCC.**
- **Invitation for further studies, reading....**

Introduction, projects.

• Our programs, like KKMC for $e^+e^- \to l\bar{l}n\gamma$, **Tauola** for τ lepton decays, **photos** for bremsstrahlung in decays of any particle or resonance, **TauSpinner** for weights modifying pp collision samples, imprinting genuine weak corrections, some spin effects or New Physics, became essential for LEP and/or LHC phenomenology.

• Purpose of Monte Carlos: generate series of events including detector response models first. Then compare with results of measurements. Any agreement extend theory applicability, any discrepancy point to New Physics or to experimental or theoretical ambiguities.

• At present these tools are used e.g. in evaluation of W mass measurement ambiguities, where tension between Tevatron and LEP/LHC measurements take place. In FCC feasibility studies, in phenomenology work of Belle II. For $g-2$, ... Previously in precision tests of SM at LEP, Higgs discovery at LHC... Higgs CP sensitive observables using Machine Learning versus optimal variables. Ambiguities for physics, ambiguities for Open AI google software.

• Applications outside accelerator physics, e.g. in cosmic rays experiments.

• **Tower of theories:** eikonal QED,$\to$ QED,$\to$ (contact interaction) $\to$ EW $\to$ SM

• Preceding level has to provide defined parts of the next level amplitudes.

Tension: in lagrangian all fields masless at first.

Phase space first

- Tower of steps for phase space parametrization:
- Riemann cube of manifolds coordinates (random numbers).
- n-body phase space manifolds (phase space slots) of 2,3,4,... states.

Phase-space Jacobians.

- Relating manifolds of distinct multiplicity. CW-complexes; triangulation along lower dimensionality (induced by infrared singularities) phase-space manifolds.
- Match phase space and matrix elements soft/collinear singularities.
- Multi channel singularity presamples.
- Variable number of particles.
- Tangent space formulation and definition of projections.

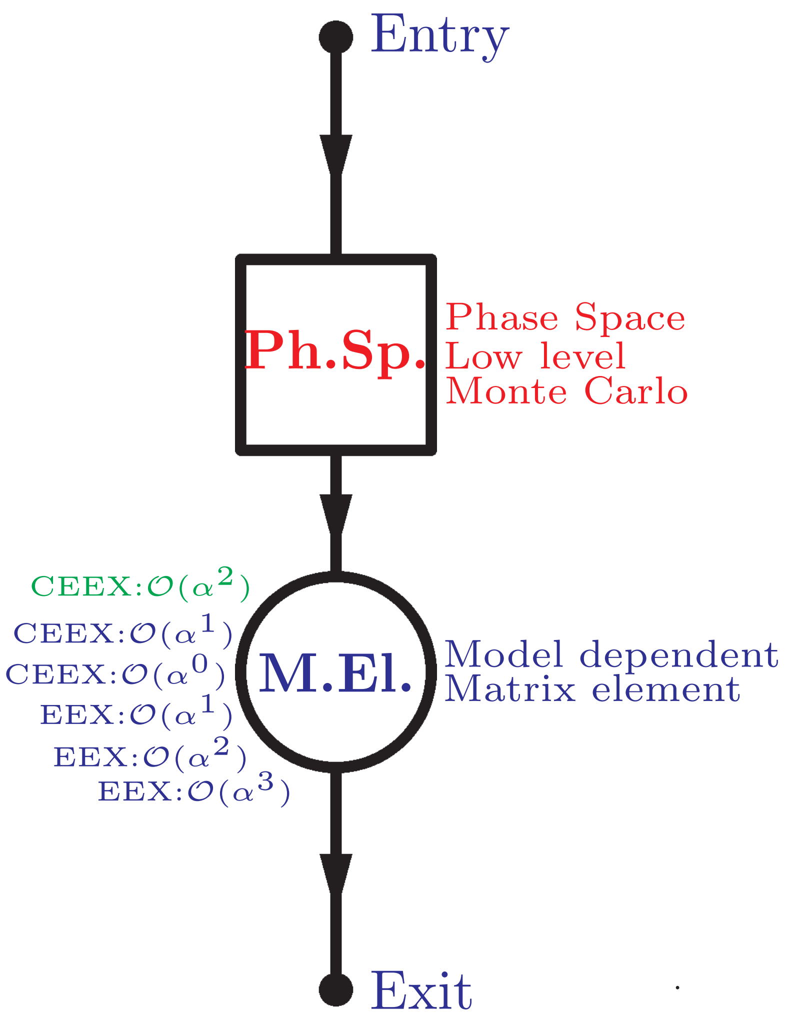
⋆ Beware: match ME singularities with phase space Jacobians minima.

⋆ Start: one dimensional crude distribution with peaks for resonances.

⋆ Fully differential crude distribution.

Phase space first

KKMC follow textbook principle "matrix element × full and exact phase space"



- Phase-space Monte Carlo simulator is a module producing "raw events" (including importance sampling for possible intermediate resonances/singularities).
- Library of Matrix Elements; input for "ME weight"; independent module.
- For Matrix Elements approximations OK. Never for phase space.
- Represent **approximations exactly:** only then ME weights are mathematically sound.
- Lots of technicalities collected in Phys. Rev. D41 (1990) 1425.
- Solutions useful for New Physics too!
- All our programs follow this paradigm.

(i) Phase-space

$$Lips_{n+1} \to Lips_n$$

Orthodox Lorentz-invariant phase space (*Lips*)!

$$dLips_{n+1}(P) =$$
$$\frac{d^3k_1}{2k_1^0(2\pi)^3} \cdots \frac{d^3k_n}{2k_n^0(2\pi)^3} \frac{d^3q}{2q^0(2\pi)^3} (2\pi)^4 \delta^4 \Big(P - \sum_1^n k_i - q\Big)$$
$$= \quad d^4p\delta^4(P-p-q) \frac{d^3q}{2q^0(2\pi)^3} \frac{d^3k_1}{2k_1^0(2\pi)^3} \cdots \frac{d^3k_n}{2k_n^0(2\pi)^3} (2\pi)^4 \delta^4 \Big(p - \sum_1^n k_i\Big)$$
$$= \quad d^4p\delta^4(P-p-q) \frac{d^3q}{2q^0(2\pi)^3} dLips_n(p \to k_1 ... k_n).$$

Introduce factor equal 1: d^4p of four-vector p, times $\delta^4\big(p - \sum_1^n k_i\big)$, and another factor equal 1, integration variable dM_1 times $\delta\big(p^2 - M_1^2\big)$.

(i) Phase-space

Phase Space Formula of Photos

$$dLips_{n+1}(P \to k_1...k_n, k_{n+1}) = dLips_n^{+1\ tangent} \times W_n^{n+1},$$

$$dLips_n^{+1\ tangent} = dk_\gamma d\cos\theta d\phi \times dLips_n(P \to \bar{k}_1...\bar{k}_n),$$
$$\{k_1, \ldots, k_{n+1}\} = \mathbf{T}\big(k_\gamma, \theta, \phi, \{\bar{k}_1, \ldots, \bar{k}_n\}\big). \quad (1)$$

1. If $dLips_n(P)$ was exact, then this formula is exact parametrization of $dLips_{n+1}(P)$
2. Practical implementation: Take the configurations from n-body phase space.
3. Turn it back into some coordinate variables.
4. construct new kinematical configuration from all variables.
5. **Forget about temporary** $k_\gamma \theta \phi$. **Only weight** W_n^{n+1} **and four vectors count.**
6. Several, parallel, $\mathbf{T}$ possible and necessary if more sources – collinear singularities.
7. $\mathbf{T}$ details depend on matrix element: must tangent at singularities, see next slide.

⋆ For W_n^{n+1} and for `KKMC` see the next slide.

Phase Space: (main formula)

If we choose

$$G_n \ : \ M^2_{2\dots n}, \theta_1, \phi_1, M^2_{3\dots n}, \theta_2, \phi_2, \dots, \theta_{n-1}, \phi_{n-1} \ \rightarrow \ \bar{k}_1 \dots \bar{k}_n \tag{2}$$

and

$$G_{n+1} \ : \ k_\gamma, \theta, \phi, M^2_{2\dots n}, \theta_1, \phi_1, M^2_{3\dots n}, \theta_2, \phi_2, \dots, \theta_{n-1}, \phi_{n-1} \ \rightarrow \ k_1 \dots k_n, k_{n+1} \tag{3}$$

then

$$\mathbf{T} = G_{n+1}(k_\gamma, \theta, \phi, G_n^{-1}(\bar{k}_1, \dots, \bar{k}_n)). \tag{4}$$

The ratio of the Jacobians form the phase space weight W_n^{n+1} for the transformation. Such solution is universal and valid for any choice of G's. However, G_{n+1} and G_n has to match matrix element, otherwise algorithm will be inefficient (factor 10^{10} ...).

In case of PHOTOS G_n's

$$W_n^{n+1} = k_\gamma \frac{1}{2(2\pi)^3} \times \frac{\lambda^{1/2}(1, m_1^2/M^2_{1\dots n}, M^2_{2\dots n}/M^2_{1\dots n})}{\lambda^{1/2}(1, m_1^2/M^2, M^2_{2\dots n}/M^2)}, \tag{5}$$

$\star$ In case of KKMC, i.e. masless photons Jacobian $W_n^{n+1} = k_\gamma \frac{1}{2(2\pi)^3}$ is a photon energy.

KKMC: re-scaling photon momentum by η, for Jacobian bring factor η^2.

Phase Space: (multiply iterated)

By iteration, we can generalize formula (1) and add l particles:

$$dLips_{n+l}(P \to k_1...k_n, k_{n+1}...k_{n+l}) = \frac{1}{l!} \prod_{i=1}^{l} \Big[dk_{\gamma_i} d\cos\theta_{\gamma_i} d\phi_{\gamma_i} W^{n+i}_{n+i-1} \Big]$$

$$\times dLips_n(P \to \bar{k}_1...\bar{k}_n), \qquad (6)$$

$$\{k_1, \ldots, k_{n+l}\} = \mathbf{T}\big(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}\big(\ldots, \mathbf{T}\big(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \ldots, \bar{k}_n\}\big)\ldots\big).$$

Note that variables $k_{\gamma_m}, \theta_{\gamma_m}, \phi_{\gamma_m}$ are used at a time of the $m-$th step of iteration only, and are not needed elsewhere in construction of the physical phase space; the same is true for invariants and angles $M^2_{2\ldots n}, \theta_1, \phi_1, \ldots, \theta_{n-1}, \phi_{n-1} \to \bar{k}_1 \ldots \bar{k}_n$ of eqs. (2,3), which are also redefined at each step of the iteration. Also intermediate steps require explicit construction of temporary $\bar{k}'_1 \ldots \bar{k}'_n \ldots \bar{k}'_{n+m}$,

We obtain: exact distribution of **weighted** events over $n + l$ body phase space.

Bosons statistical factor $\frac{1}{l!}$. Photons $W^{n+i}_{n+i-1} = k_i \frac{1}{2(2\pi)^3}$. Conformal symmetry.

Phase Space Formula: multichannels.

Often MC algorithm has to be split into branches. In the most general case, when n different parametrisations of the phase space with different orderings of particles are in use, the cross section can be written as follows:

$$d\Gamma_X = \sum_{\lambda=1}^{n} \int_0^1 \prod_{i=1}^{m} dx_i \ P_\lambda \left[\sum_{\delta=1}^{n} P_\delta J_\delta^{-1}(q_1(\lambda, x_i), ...q_k(\lambda, x_i)) \right]^{-1} \times |M|^2.$$

In the above formula the four-momenta $q_i(\lambda, x_i)$ are calculated from the random numbers x_i according to the parametrization of the phase space of type λ. The Jacobians J_δ have to be calculated for all parametrisations of the phase space at the point q_i; P_λ denotes the probability of choosing the parametrization of type λ in the generation, λ thus takes[a] a role of an additional discrete variable in the generation. Numerical values of probabilities P_λ do not affect the final distributions, but only the efficiency of the generation.

[a] But not δ.

(i) Phase-space

Phase Space case of complex singularity structure

- Several G_{n+1} can be used simultaneously (branching of the generation algorithm).
- Each G_{n+1} can be used to presample distinct singularities chain.
- The price: W_n^{n+1} become more complicated but remain exact.
- HOWEVER: We have observed that while matching Jacobians for the two branches related to collinear singularity of photons along direction of l^+ and l^+ (in Z decay) approximation must be used if more than one photon is present in final state. Otherwise inconsistencies.
- Non Markovian algorithm, whereas matrix element for multi-photon state may be obtained by iteration: `KKMC EEX` variants and `Photos`. Note `KKMC CEEX` is more refined.
- AVOID INCONSISTENCY: in expanding manifold curvature: must be the same for phase space and Matrix Element. Phase space is manifold, Matrix element squared – bi-linear form on it. Truncation of perturbative expansion or iterative solutions mean truncation in powers of Ricci tensor, this has to be consistent.

Message for `Photos` like algorithm, but not for `KKMC`: neither `CEEX` nor `EEX`.

(i) Phase-space

Phase Space: (multiply iterated)

We have generalized formula phase space formula to case of l particles added:

$$dLips_{n+l}(P \to k_1...k_n, k_{n+1}...k_{n+l}) = \frac{1}{l!} \prod_{i=1}^{l} \left[dk_{\gamma_i} d\cos\theta_{\gamma_i} d\phi_{\gamma_i} W^{n+i}_{n+i-1} \right]$$

$$\times dLips_n(P \to \bar{k}_1...\bar{k}_n), \qquad (7)$$

$$\{k_1, \ldots, k_{n+l}\} = \mathbf{T}\big(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}\big(\ldots, \mathbf{T}\big(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \ldots, \bar{k}_n\}\big) \ldots\big).$$

Now we have to start talking about matrix elements: Our relation between n and n+l body phase space is motivated by cancellation of infrared singularities. It provides kind of triangulation. Measure defining distance between points from manifolds of distinct no. of particles. Such phase space points are close if they differ by presence of soft photons only.

Experimental user attention necessary. Can 1 GeV photon be ignored or only 0.1 MeV one.

We will move now from **exact distribution** of **weighted** events over $n + l$ body phase space to case where l is generated too. All remain exact!

Crude Ddistribution and crude matrix element

If we add arbitrary factors $f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i})$ and sum over l we obtain:

$$\sum_{l=0} \exp(-F)\frac{1}{l!}\prod_{i=1}^{l} f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dLips_{n+l}(P \to k_1...k_n, k_{n+1}...k_{n+l}) =$$

$$\sum_{l=0} \exp(-F)\frac{1}{l!}\prod_{i=1}^{l}\Big[f(k_{\gamma_i}, \theta_{\gamma_i}, \phi_{\gamma_i}) dk_{\gamma_i} d\cos\theta_{\gamma_i} d\phi_{\gamma_i} W^{n+i}_{n+i-1}\Big] \times$$

$$dLips_n(P \to \bar{k}_1...\bar{k}_n), \tag{8}$$

$$\{k_1, \dots, k_{n+l}\} = \mathbf{T}\big(k_{\gamma_l}, \theta_{\gamma_l}, \phi_{\gamma_l}, \mathbf{T}\big(\dots, \mathbf{T}\big(k_{\gamma_1}, \theta_{\gamma_1}, \phi_{\gamma_1}, \{\bar{k}_1, \dots, \bar{k}_n\}\big)\dots\big),$$

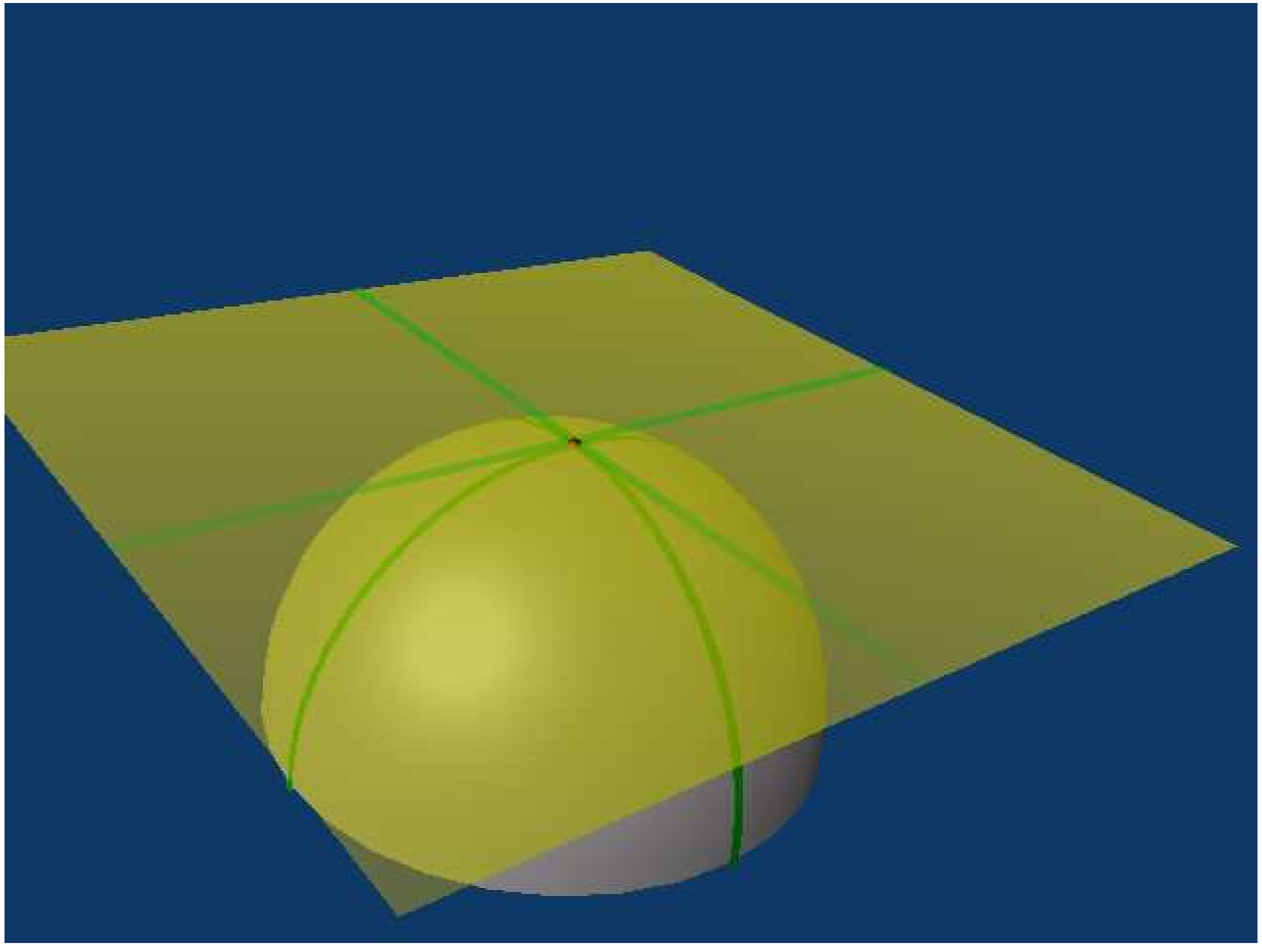
$$F = \int_{k_{min}}^{k_{max}} dk_\gamma d\cos\theta_\gamma d\phi_\gamma f(k_\gamma, \theta_\gamma, \phi_\gamma). \leftarrow \text{KLN good start for Photos}$$

• The olive parts of rhs. alone, give crude distribution over tangent space (orthogonal set of variables k_i, θ_i, ϕ_i). We restrict k_{min} generation (typically 10^{-6} but not by k_{max}.

Heuristic CW complexes

We define our crude distribution over yellow space (surface=1) (represented by sum of: red point, green lines and flat yellow square). Later we do projections into physics space, using **T** and matrix elements.

NOTE: in KKMC YFS exclusive exponentiation – conformal symmetry is used instead.



I was talking mostly about solution of Photos Monte Carlo.

Advantage 1: additional particles can be massive suitable for final state radiation.

Advantage 2: leading contributions of higher orders nicely resummed.

Disadvantage 1: not convenient for processes of intermediate narrow resonances, like in case of $e^+e^- \to l\bar{l}n\gamma$ around intermediate resonances. When initial state bremsstrahlung need to be used.

Disadvantage 2: At present work on interferences was not pursued, this is for case when multiple charged particles are present. Starting from 3 charged final states this is the case even for complete one loop effects implementation.

But this may be starting point to evaluate path for third order matrix element implementation into programs like KKMC.

How it is in KKMC?

• Factor F is not obtained from KLN theorem, but calculated from one loop virtual corrections (Eikonal level).

• For phase space constraints rescaling is used. No rejection of photon candidates needed. Except very soft ones, passing under lower generation phase space boundary. Manageable because factorization works there well.

• Algorithm is useful for initial and final state radiation. Invariant mass of intermediate state Z/γ^* can be generated possibly with beam energy spread.

To continue, properties of matrix elements are necessary.

There are several steps. Both for virtual and real emission amplitudes. Usually off shelf amplitudes and cross sections can not be used.

Let me scratch the topic of YFS exponentiation and its relation to spin amplitudes calculated using Kleiss-Stirling spinor techniques.

Formal proofs and work on virtual corrections is essential, but does not affect as much as real emission amplitudes the way how the programs are being developed.

Matrix Element (starting point):

- Directly starting from Feynman rules one can calculate spin amplitude for any QED/QCD process.
- The case of $Z \to l^+ l^- \gamma$, for Kleiss-Stirling spin amplitudes.
- Single photon amplitude(momentum k_1 polarization e_1 fermion spinors $u(p)$ and $v(q)$ dropped):

$$I = \not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\right] - \left[\frac{1}{2}\frac{\not{e}_1 \not{k}_1}{p\cdot k_1}\right]\not{J} + \not{J}\left[\frac{1}{2}\frac{\not{e}_1 \not{k}_1}{q\cdot k_1}\right]$$

three gauge invariant parts: appear in other processes too.

Pre-property for factorizations of any sorts, deciphered from Lorentz-group layers.

- The fully differential distribution from MUSTRAAL (used also in KORALZ for single photon mode also in TauSpinner) reads:

$$X_f = \frac{Q'^2 \alpha(1-\Delta)}{4\pi^2 s} s^2 \quad \left\{ \frac{1}{(k'_+ k'_-)} \left[\frac{\mathrm{d}\sigma_B}{\mathrm{d}\Omega}(s,t,u') + \frac{\mathrm{d}\sigma_B}{\mathrm{d}\Omega}(s,t',u) \right] \right\}$$

- Here:

$$\begin{aligned} s &= 2p_+ \cdot p_-, \quad s' = 2q_+ \cdot q_-, \\ t &= 2p_+ \cdot q_+, \quad t' = 2p_+ \cdot q_-, \\ u &= 2p_+ \cdot q_-, \quad u' = 2_- \cdot q_+, \\ k'_\pm &= q_\pm \cdot k, \quad x_k = 2E_\gamma/\sqrt{s} \end{aligned}$$

- The Δ term is responsible for final state mass dependent terms, p_+, p_-, q_+, q_-, k denote four-momenta of incoming positron, electron beams, outgoing muons and bremsstrahlung photon.

- after trivial manipulation it can be written as:

$$X_f = \frac{Q'^2\alpha(1-\Delta)}{4\pi^2 s} s^2 \quad \Big\{ \frac{1}{(k'_+ + k'_-)} \frac{1}{k'_-} \Big[\frac{d\sigma_B}{d\Omega}(s,t,u') + \frac{d\sigma_B}{d\Omega}(s,t',u) \Big]$$

$$+ \frac{1}{(k'_+ + k'_-)} \frac{1}{k'_+} \Big[\frac{d\sigma_B}{d\Omega}(s,t,u') + \frac{d\sigma_B}{d\Omega}(s,t',u) \Big] \Big\}$$

- In PHOTOS the following expression is used in universal application (AP adj.):

$$X_f^{PHOTOS} = \frac{Q'^2\alpha(1-\Delta)}{4\pi^2 s} s^2 \Big\{$$

$$\frac{1}{k'_+ + k'_-} \frac{1}{k'_-} \Big[(1+(1-x_k)^2) \frac{d\sigma_B}{d\Omega}\Big(s, \frac{s(1-\cos\Theta_+)}{2}, \frac{s(1+\cos\Theta_+)}{2}\Big) \Big] \frac{(1+\beta\cos\Theta_\gamma)}{2}$$

$$+ \frac{1}{k'_+ + k'_-} \frac{1}{k'_+} \Big[(1+(1-x_k)^2) \frac{d\sigma_B}{d\Omega}\Big(s, \frac{s(1-\cos\Theta_-)}{2}, \frac{s(1+\cos\Theta_-)}{2}\Big) \Big] \frac{(1-\beta\cos\Theta_\gamma)}{2} \Big\}$$

$$\text{where}: \Theta_+ = \angle(p_+, q_+),\ \Theta_- = \angle(p_-, q_-)$$

$\Theta_\gamma = \angle(\gamma, \mu^-)$ are defined in (μ^+, μ^-)-pair rest frame

The matrix element weight

- weight for exact matrix element is easy to implement $WT = X_f / X_f^{PHOTOS}$
- also factor $\Gamma^{total}/\Gamma^{Born} = 1 + \frac{3}{4}\frac{\alpha}{\pi}$ defines first order weight, it depends on virtual corrections if non leading mass terms are kept.
- $WT = \frac{X_f}{X_f^{PHOTOS}} \frac{\Gamma^{Born}}{\Gamma^{total}}$

The differences of X_f and X_f^{PHOTOS} are important

- Without process dependent weight PHOTOS is universal and can be combined with any generator rather easily, thus 300+ citations. Last year mainly for B decays and measurements of quark mixing angles.
- Photos weight is then process independent.

$\mathcal{M}$atrix $\mathcal{E}$lement (anything in common?):

- We have seen nice properties of matrix element squared which were factorizing into Born-like distribution and photon factor.
- It was shown many years ago by Ronald Kleiss that such property of distributions does not hold beyond first order!
- Dead end? Not really, just complex weights[a]
- single photon amplitude again:

$$I = \not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\right] - \left[\frac{1}{2}\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J} + \not{J}\left[\frac{1}{2}\frac{\not{e}_1\not{k}_1}{q\cdot k_1}\right]$$

three gauge invariant parts, first is eikonal, other for collinear configuration along q and p

We look for these parts in higher order amplitudes

[a] Also: samples at different level of sophistication can be correlated up to NLO level. That is enough for most of experimental techniques, precision of correlated programs can be higher.

Matrix Element (double emission):

- The structure of exact spin amplitude for single emission looks promising.
- How does it translate to distributions?
- Does it extend to other processes, interactions? Scalar QED QCD as well?
- Does it extent to higher orders?
- Can one decipher anything without enforcing some phase space conditions?
- To identify the building blocks we have used gauge invariance, and we have used also segments localized at lower order.
- For tree diagrams gauge invariance mean in practice that replacement $k \to e$ set expression to zero
- Virtual corrections add complication because of regularization schemes, we will skip that now.

Exact Matrix Element: $e^+e^- \to \nu_\mu \bar{\nu}_\mu \gamma\gamma$ explicitly;

- Expressions are valid for any current J,
- For complete amplitude add fermionic fields, eg. $\bar{u}(p)$ and $v(q)$; 1-st/2-nd photon momenta/polarizations are: k_1/k_2 e_1/e_2.

$$I_1^{\{1,2\}} = \frac{1}{2} \not{J} \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1} \right) \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2} \right) \qquad eikonal$$

$$I_{2l}^{\{1,2\}} = -\frac{1}{4} \left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1} \right) \frac{\not{e}_2 \not{k}_2}{p\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2} \right) \frac{\not{e}_1 \not{k}_1}{p\cdot k_1} \right] \not{J} \qquad \beta_1$$

$$I_{2r}^{\{1,2\}} = \frac{1}{4} \not{J} \left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1} \right) \frac{\not{k}_2 \not{e}_2}{q\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2} \right) \frac{\not{k}_1 \not{e}_1}{q\cdot k_1} \right] \qquad \beta_1$$

$$I_3^{\{1,2\}} = -\frac{1}{8} \left(\frac{\not{e}_1 \not{k}_1}{p\cdot k_1} \not{J} \frac{\not{k}_2 \not{e}_2}{q\cdot k_2} + \frac{\not{e}_2 \not{k}_2}{p\cdot k_2} \not{J} \frac{\not{k}_1 \not{e}_1}{q\cdot k_1} \right) \qquad start for \beta_2 ...$$

$$I_{4p}^{\{1,2\}} = \frac{1}{8}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2}{p\cdot k_1} + \frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1}{p\cdot k_2}\right)\not{J}$$

$$I_{4q}^{\{1,2\}} = \frac{1}{8}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{k}_2\not{e}_2\not{k}_1\not{e}_1}{q\cdot k_1} + \frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2}{q\cdot k_2}\right)$$

$$I_{5pA}^{\{1,2\}} = \frac{1}{2}\not{J}\frac{k_1\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5pB}^{\{1,2\}} = -\frac{1}{2}\not{J}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)$$

$$I_{5qA}^{\{1,2\}} = \frac{1}{2}\not{J}\frac{k_1\cdot k_2}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5qB}^{\{1,2\}} = -\frac{1}{2}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)$$

$$I_{6B}^{\{1,2\}} = -\frac{1}{4}\frac{k_1\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left[+\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}$$

$$I_{7B}^{\{1,2\}} = -\frac{1}{4} \not{J} \frac{k_1 \cdot k_2}{q \cdot k_1 + q \cdot k_2 - k_1 \cdot k_2} \left[+ \left(\frac{q \cdot e_1}{q \cdot k_1} - \frac{k_2 \cdot e_1}{k_1 \cdot k_2} \right) \frac{\not{k}_2 \not{e}_2}{q \cdot k_2} + \left(\frac{q \cdot e_2}{q \cdot k_2} - \frac{k_1 \cdot e_2}{k_1 \cdot k_2} \right) \frac{\not{k}_1 \not{e}_1}{q \cdot k_1} \right]$$

- for the **exponentiation** we have used **separation** into 3 parts only. It is **crystal clear**, also in case of contributions with t-channel W, was very useful for KKMC,

- for PHOTOS kernel, parts $I_3^{\{1,2\}}$, $I_{4p}^{\{1,2\}}$, $I_{4q}^{\{1,2\}}$ were studied separately as well.

- In fact older works on spin amplitudes were used E. Richter-Was Z.Phys.C64:227-240,1994, Z.Phys.C61:323-340,1994.

- Clearly visible but not used for PHOTOS further separation of β_2 terms ...

- Presented above properties of spin amplitudes were used for PHOTOS design to make a choice of phase space parametrization and iteration of consecutive emission kernels that respect numerically as much as possible results of second order amplitudes. Also one want to remain consistent with NLO and exponentiation to all orders.

Matrix Element: $q\bar{q} \rightarrow Jgg$ *- part proportional to* $T^A T^B$ *fermion spinors dropped*

$$I_{lr}^{(1,2)} = \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1 \not{k}_1}{2p\cdot k_1}\right) \not{J} \left(\frac{\not{k}_2 \not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$I_{ll}^{(1,2)} = \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1 \not{k}_1}{2p\cdot k_1}\right) \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2 \not{k}_2}{2p\cdot k_2}\right) \not{J}$$

$$I_{rr}^{(1,2)} = \not{J} \frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2} \left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1 \not{e}_1}{2q\cdot k_1}\right) \left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2 \not{e}_2}{2q\cdot k_2}\right)$$

$$I_{e}^{(1,2)} = \not{J} \left(1 - \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} - \frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\right) \left(\frac{k_1\cdot e_2}{k_1\cdot k_2} \frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right)$$

Remainder:

$$I_{p}^{(1,2)} = -\frac{1}{4} \frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} \left(\frac{\not{e}_1 \not{k}_1 \not{e}_2 \not{k}_2 - \not{e}_2 \not{k}_2 \not{e}_1 \not{k}_1}{k_1\cdot k_2}\right) \not{J}$$

$$I_{q}^{(1,2)} = -\frac{1}{4} \not{J} \frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2} \left(\frac{\not{k}_1 \not{e}_1 \not{k}_2 \not{e}_2 - \not{k}_2 \not{e}_2 \not{k}_1 \not{e}_1}{k_1\cdot k_2}\right)$$

Matrix Element: $q\bar{q} \to Jgg$ - part proportional to $T^B T^A$ fermion spinors dropped

$$I_{lr}^{(2,1)} = \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J}\left(\frac{\not{k}_1\not{e}_1}{2q\cdot k_1} + \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)$$

$$I_{ll}^{(2,1)} = \frac{p\cdot k_1}{p\cdot k_2 + p\cdot k_1 - k_2\cdot k_1}\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}$$

$$I_{rr}^{(2,1)} = \not{J}\,\frac{q\cdot k_2}{q\cdot k_2 + q\cdot k_1 - k_2\cdot k_1}\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right)\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)$$

$$I_{e}^{(2,1)} = \not{J}\left(1 - \frac{p\cdot k_1}{p\cdot k_2 + p\cdot k_1 - k_2\cdot k_1} - \frac{q\cdot k_2}{q\cdot k_2 + q\cdot k_1 - k_2\cdot k_1}\right)\left(\frac{k_2\cdot e_1}{k_2\cdot k_1}\,\frac{k_1\cdot e_2}{k_2\cdot k_1} - \frac{e_2\cdot e_1}{k_2\cdot k_1}\right)$$

$$I_{p}^{(2,1)} = -\frac{1}{4}\,\frac{1}{p\cdot k_2 + p\cdot k_1 - k_2\cdot k_1}\left(\frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1 - \not{e}_1\not{k}_1\not{e}_2\not{k}_2}{k_2\cdot k_1}\right)\not{J}$$

$$I_{q}^{(2,1)} = -\frac{1}{4}\,\not{J}\,\frac{1}{q\cdot k_2 + q\cdot k_1 - k_2\cdot k_1}\left(\frac{\not{k}_2\not{e}_2\not{k}_1\not{e}_1 - \not{k}_1\not{e}_1\not{k}_2\not{e}_2}{k_2\cdot k_1}\right)$$

For QCD we have separation too; 12 gauge invariant parts

- Terms like

$$\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1 \not{k}_1}{2p\cdot k_1}\right) \qquad A$$

once integrated over part of phase space give Atarelli-Parisi kernel

- Terms

$$\frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_2\cdot k_1} \qquad B$$

if combined with phase space Jacobians can be used to redefine fermionic fields from $v(q)$ to $v(q-k_2)$ for example. Term of such type appeared already in scalar QED (normalization of hadronic current).

1. we are using **STANDARD and FORMAL** parametrizations of Lorentz group. One can express it with the help of consecutive boosts and rotations.
2. Convenient for Monte Carlo event construction!
3. For the definition of coordinate system in the P-rest frame the $\hat{x}$ and $\hat{y}$ axes of the laboratory frame boosted to the rest frame of P can be used. The orthogonal right-handed system can be constructed with their help in a standard way.
4. We choose polar angles θ_1 and ϕ_1 defining the orientation of the four momentum $\bar{k}_2$ in the rest frame of P. In that frame $\bar{k}_1$ and $\bar{k}_2$ are back to back[a], see fig. (1).
5. The previous two points would complete the definition of the two-body phase space, if both $\bar{k}_1$ and $\bar{k}_2$ had no measurable spin degrees of freedom visualizing themselves e.g. through correlations of the secondary decay products' momenta. Otherwise we need to know an additional angle ϕ_X to complete the set of Euler angles defining the relative orientation of the axes of the P rest-frame system with the coordinate system used in the rest-frame of $\bar{k}_2$ (and possibly also of $\bar{k}_1$), see fig. (2).

[a] In the case of phase space construction for multi-body decays $\bar{k}_2$ should read as a state representing the sum of all decay products of P but $\bar{k}_1$.

6. If both rest-frames of $\bar{k}_1$ and $\bar{k}_2$ are of interest, their coordinate systems are oriented with respect to P with the help of θ_1, ϕ_1, ϕ_X. We assume that the coordinate systems of $\bar{k}_1$ and $\bar{k}_2$ are connected by a boost along the $\bar{k}_2$ direction, and in fact share axes: $z' \uparrow\downarrow z''$, $x' \uparrow\uparrow x''$, $y' \uparrow\downarrow y''$.

7. For the three-body phase space: We take the photon energy k_γ in P rest frame. We calculate: photon, k_1 and k_2 energies, all in $k_1 + k_2$ frame.

8. We use the angles θ, ϕ, in the rest-frame of the $k_1 + k_2$ pair: angle θ is an angle between the photon and k_1 direction (i.e. $-z''$). Angle ϕ defines the photon azimuthal angle around z'', with respect to x'' axis (of the k_2 rest-frame), see fig. (3).

9. If all k_1, k_2 and $k_1 + k_2$ rest-frames exist, then the x-axes for the three frames are chosen to coincide. It is OK, all frames connected by boosts along z'' see fig. (3).

10. To define orientation of k_2 in P rest-frame coordinate system, and to complete construction of the whole event, we will re-use Euler angles of $\bar{k}_2$: ϕ_X, θ_1 and ϕ_1 (see figs. 4 and 5), defined again of course in the rest frame of P.

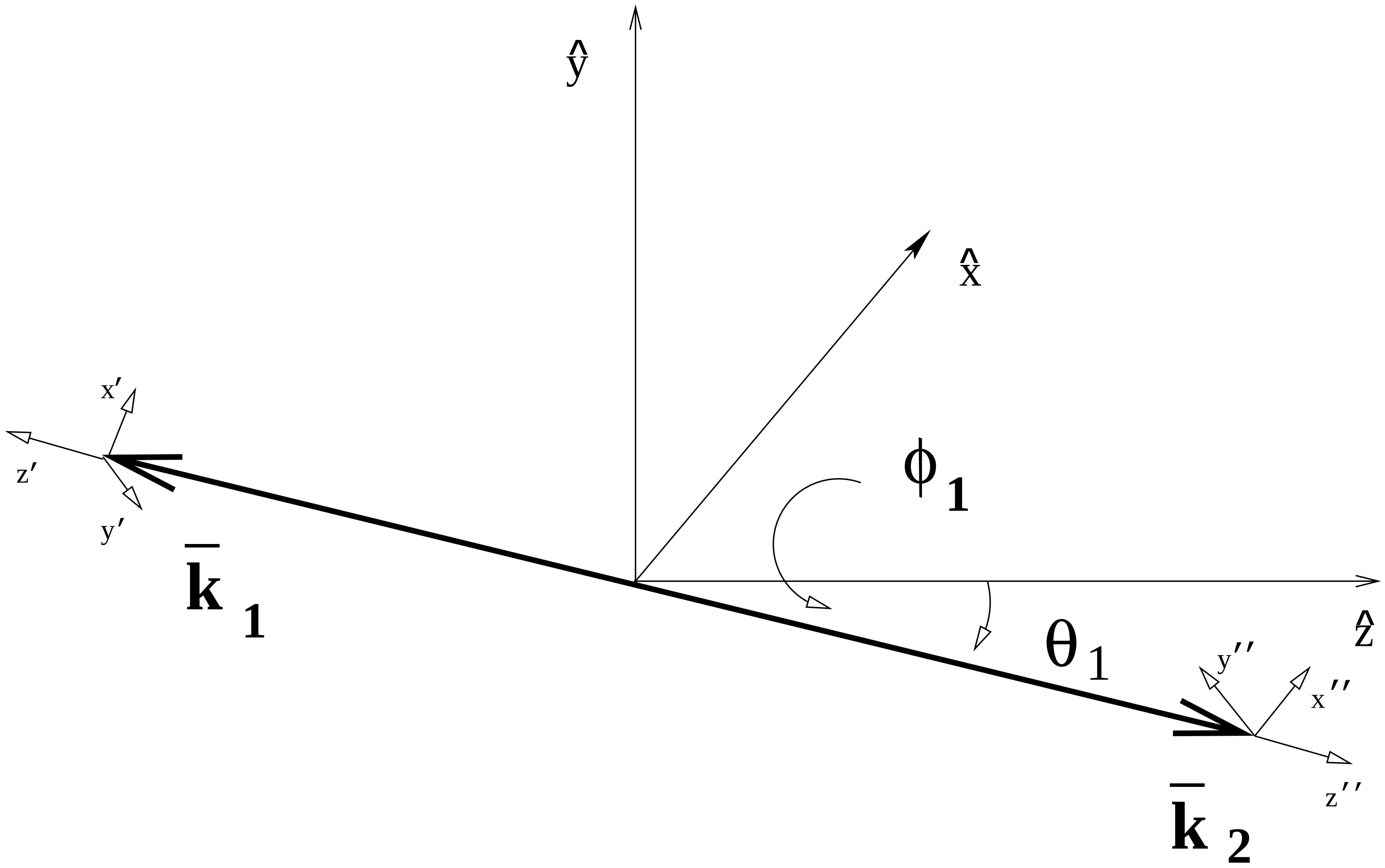


Figure 1: The angles θ_1, ϕ_1 defined in the rest-frame of P and used in parametrization of two-body phase-space.

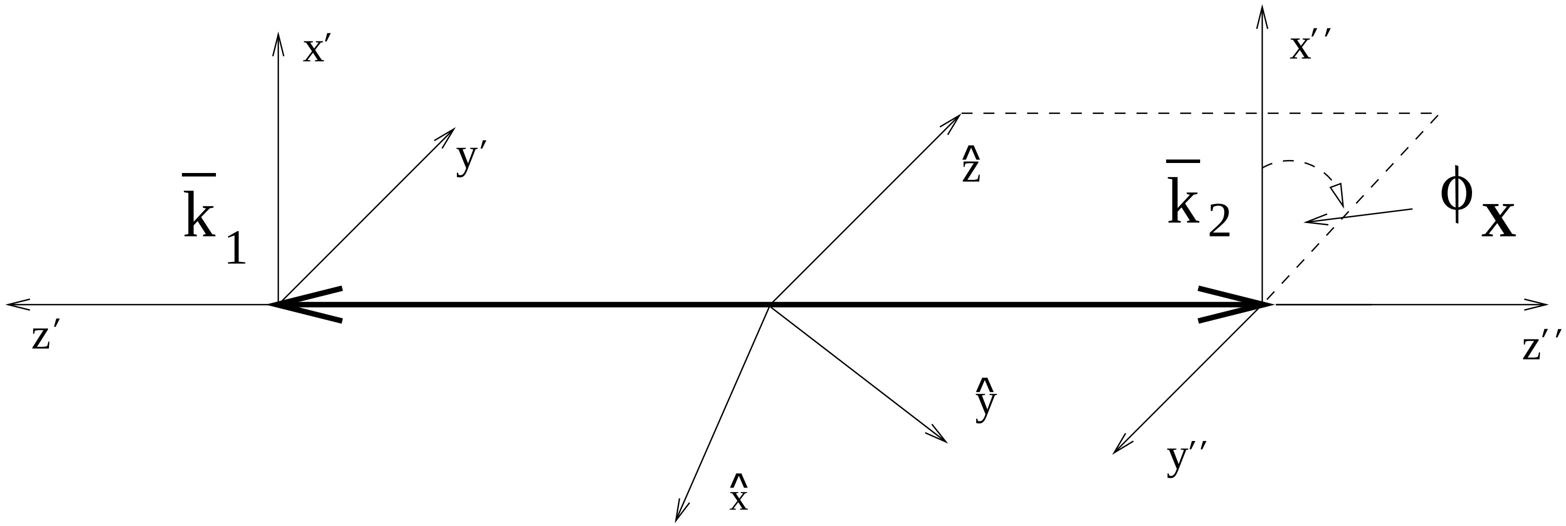


Figure 2: Angle ϕ_X is also defined in the rest-frame of P as an angle between (oriented) planes spanned on: (i) $\bar{k}_1$ and $\hat{z}$-axis of the P rest-frame system, and (ii) $\bar{k}_1$ and x''-axis of the $\bar{k}_2$ rest frame. It completes definition of the phase-space variables if internal orientation of $\bar{k}_1$ system is of interest. In fact, Euler angle ϕ_X is inherited from unspecified in details, parametrization of phase space used to describe possible future decay of $\bar{k}_2$ (or $\bar{k}_1$).

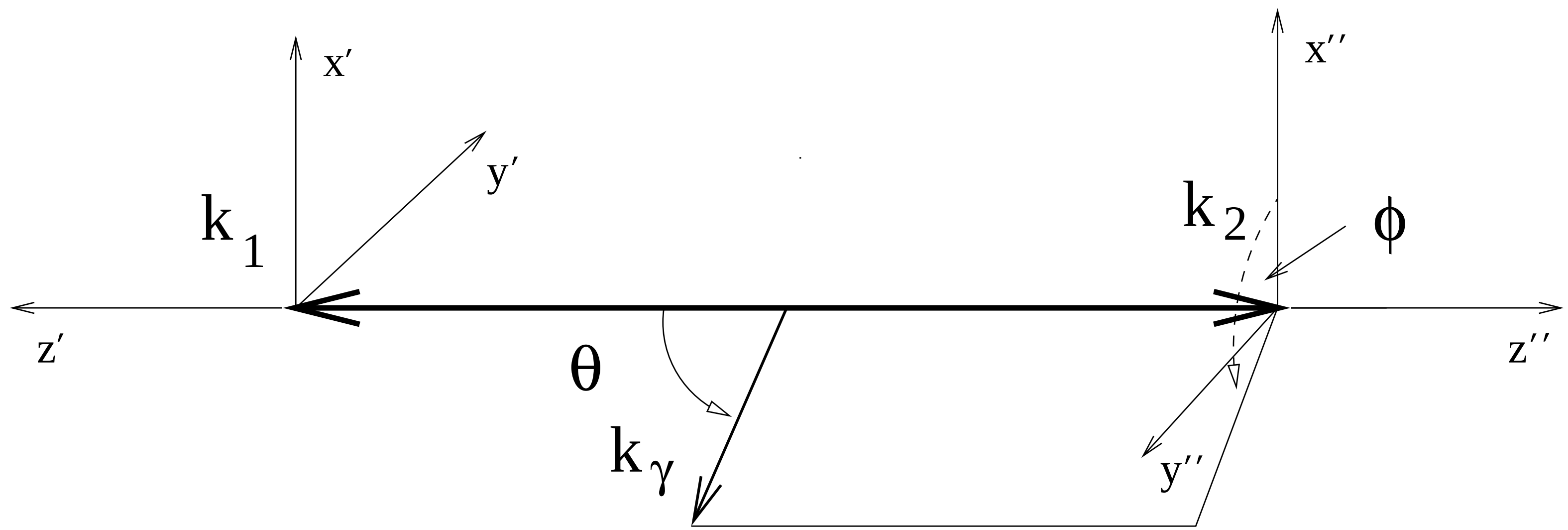


Figure 3: The angles θ, ϕ are used to construct the four-momentum of k_γ in the rest-frame of $k_1 + k_2$ pair (itself not yet oriented with respect to P rest-frame). To calculate energies of k_1, k_2 and photon, it is enough to know m_1, m_2, M and photon energy k_γ of the P rest-frame.

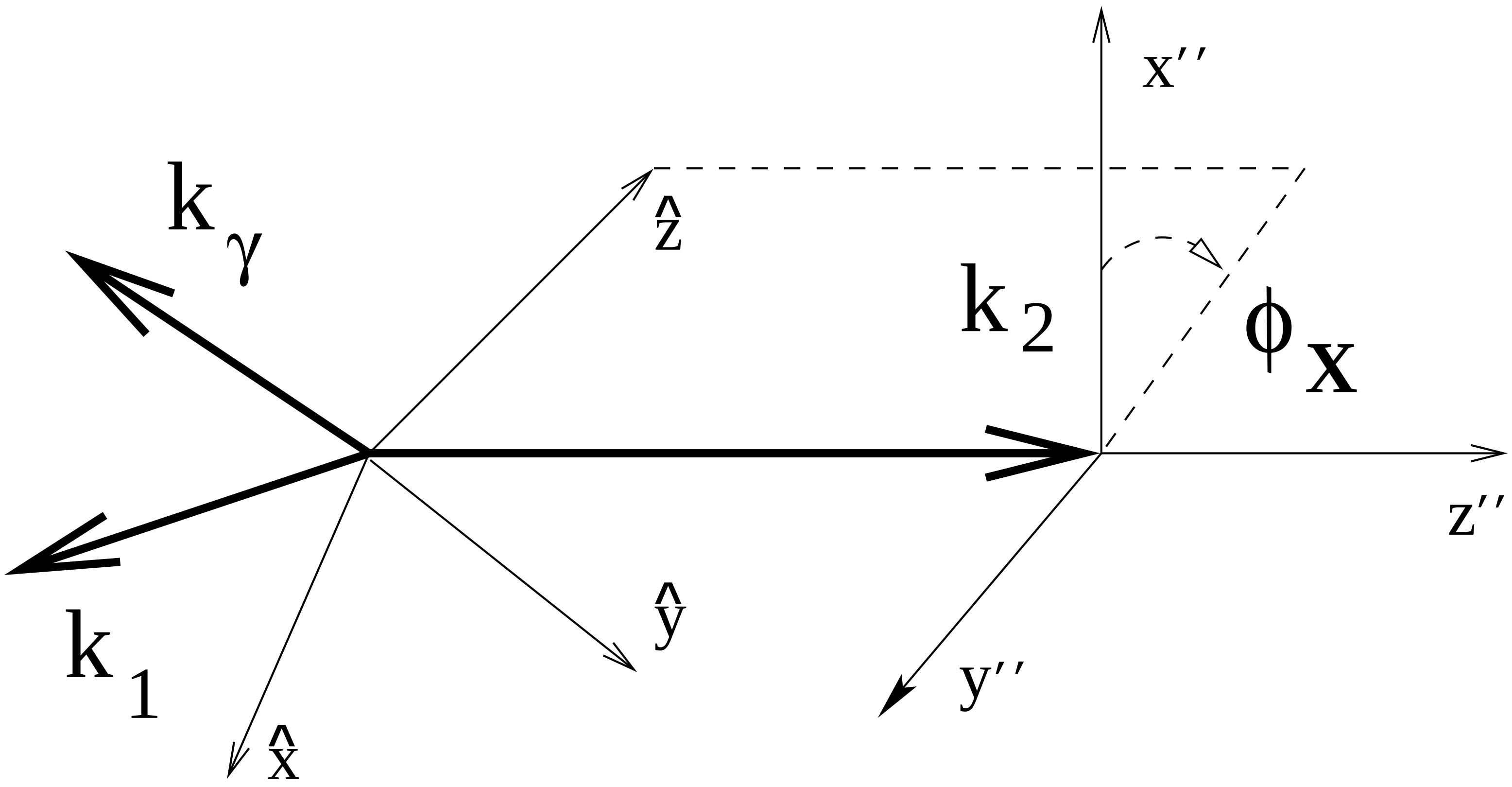


Figure 4: Use of angle ϕ_x in defining orientation of k_1, k_2 and photon in the rest-frame of P. At this step only the plane spanned on P frame axis $\hat{z}$ and k_2 is oriented with respect to $k_2 \times x''$ plane.

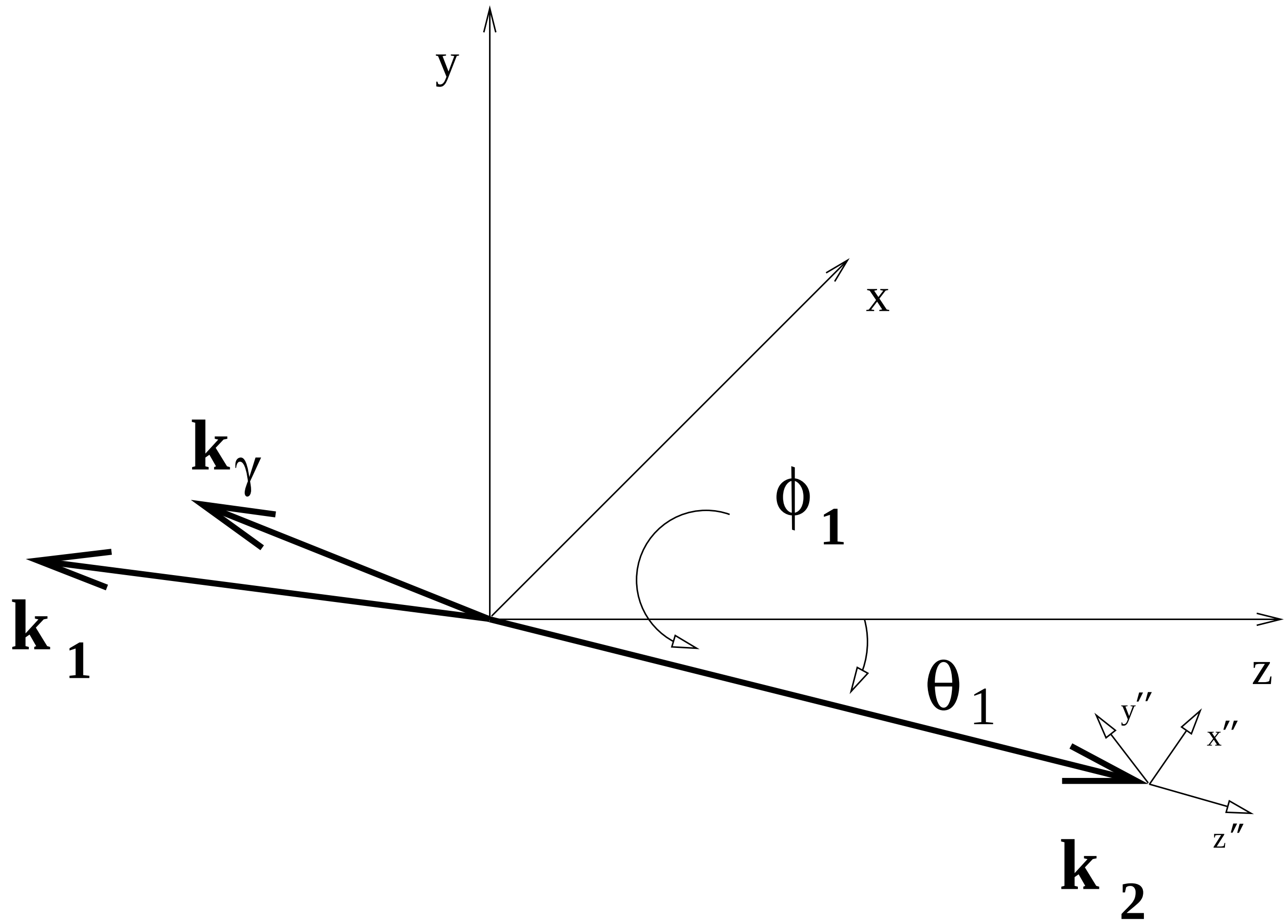
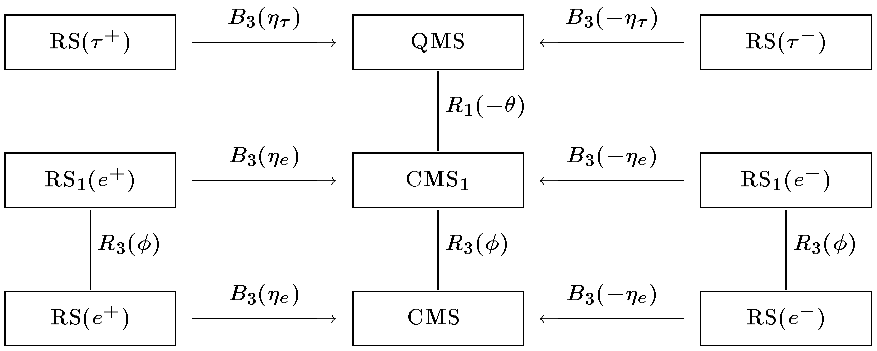


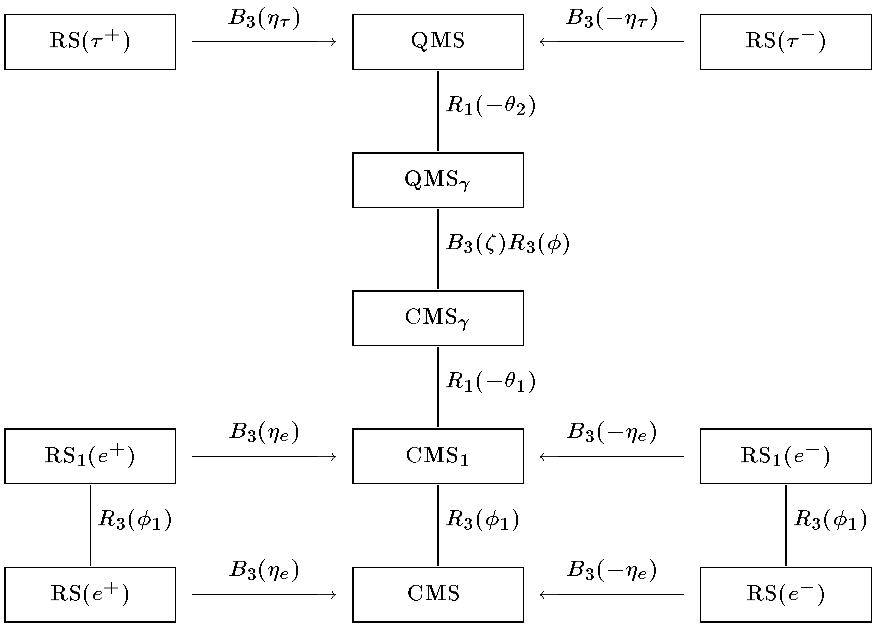
Figure 5: Final step in event construction. Angles θ_1, ϕ_1 are used. The final orientation of k_2 coincide with this of $\bar{k}_2$.

Tree of frames used for spin; must be tuned between production and decay

Figure 2



(2a)



(2b)

Let us start with the lowest order coupling constants (without EW corrections) of the Z boson to fermions, $\sin\theta_W^2 = s_W^2 = 1 - m_W^2/m_Z^2$ (on-shell scheme) and T_3^f denotes third component of the isospin.

The vector v_e, v_f and axial a_e, a_f couplings for leptons and quarks are defined with the formulas below:

$$
\begin{aligned}
v_e &= (2 \cdot T_3^e - 4 \cdot q_e \cdot s_W^2)/\Delta \\
v_f &= (2 \cdot T_3^f - 4 \cdot q_f \cdot s_W^2)/\Delta \\
a_e &= (2 \cdot T_3^e)/\Delta \\
a_f &= (2 \cdot T_3^f)/\Delta
\end{aligned}
\tag{9}
$$

where

$$
\Delta = \sqrt{16 \cdot s_W^2 \cdot (1 - s_W^2)} \tag{10}
$$

Factoring genuine weak and strong effects

With this notation, matrix element for the $q\bar{q} \to Z/\gamma^* \to l^+l^-$, ME_{Born}, can be written as:

$$
\begin{aligned}
ME_{Born} &= [\bar{u}\gamma^\mu v g_{\mu\nu} \bar{v}\gamma^\nu u] \cdot (q_e \cdot q_f) \cdot \frac{\chi_\gamma(s)}{s} \\
&+ [\bar{u}\gamma^\mu v g_{\mu\nu} \bar{\nu}\gamma^\nu u \cdot (v_e \cdot v_f) + \bar{u}\gamma^\mu v g_{\mu\nu} \bar{\nu}\gamma^\nu \gamma^5 u \cdot (v_e \cdot a_f) \qquad (11) \\
&+ \bar{u}\gamma^\mu \gamma^5 v g_{\mu\nu} \bar{\nu}\gamma^\nu u \cdot (a_e \cdot v_f) + \bar{u}\gamma^\mu \gamma^5 v g_{\mu\nu} \bar{\nu}\gamma^\nu \gamma^5 u \cdot (a_e \cdot a_f)] \cdot \frac{\chi_Z(s)}{s}
\end{aligned}
$$

Z-boson and photon propagators read respectively as

$$\chi_\gamma(s) = 1 \qquad (12)$$

$$\chi_Z(s) = \frac{G_\mu \dot{M}_z^2}{\sqrt{2} \cdot 8\pi \cdot \alpha_{QED}(0)} \cdot \Delta^2 \cdot \frac{s}{s - M_Z^2 + i \cdot \Gamma_Z \cdot M_Z} \qquad (13)$$

At the peak of resonance $|\chi_Z(s)| \times (v_e \cdot v_f) > (q_e \cdot q_f)$ and as a consequence, angular distribution asymmetries of leptons are proportional to $v_e = (2 \cdot T_3^e - 4 \cdot q_e \cdot s_W^2)$. This gives good sensitivity for s_W^2 measurement.
Above/below resonance – sensitivity to lepton/quark charge or $\alpha_{QED}(s \simeq m_Z^2)$.

$$\mathcal{M}^{IBA} = \frac{e^2 Q_f Q_i}{s} V_{fi}(s,t)\,\gamma_\mu \otimes \gamma^\mu + \left(\frac{g_Z}{2}\right)^2 \frac{Z_{fi}(s,t)}{d(s)}\,\gamma_\mu[v_i(s,t) - a_i\gamma_5] \otimes \gamma^\mu[v_f(s,t) - a_i\gamma_5], \quad (14)$$

$$v_i(s,t) = T_{3i} - 2Q_i s_W^2 K_i(s,t), \qquad v_f(s,t) = T_{3f} - 2Q_f s_W^2 K_f(s,t), \quad (15)$$

$$V_{fi}(s,t) = \Gamma_{vp}(s) + \left(\frac{g_Z}{e}\right)^2 s_W^4 Z_{fi}(s,t) \frac{s}{d(s)} [K_{fi}(s,t) - K_f(s,t) K_i(s,t)], \quad (16)$$

$$\mathcal{M}^{DM} = \frac{e^2 Q_f Q_i}{s} V_{fi}(s,t)\,\gamma_\mu \otimes [A\gamma^\mu + \frac{(p_+ - p_-)^\mu}{2m}(A - iB\gamma_5)] + \left(\frac{g_Z}{2}\right)^2 \frac{Z_{fi}(s,t)}{d(s)}\,\gamma_\mu[v_i(s,t) - a_i\gamma_5] \otimes [X\gamma^\mu + \frac{(p_+ - p_-)^\mu}{2m}(X - iY\gamma_5)], \quad (17)$$

Complete amplitude $\mathcal{M} = \mathcal{M}^{IBA} + \mathcal{M}^{DM}$ (fermions spinors dropped),

Improved Born Approximation (IBA), Dipole Moment (DM).

OK, this looks simple at Born level.

But to obtain such organization,major LEP time effort was necessary.

It does not need to look nicely and intuitive (form-factors in place of couplings?)

Proofs were needed that it represent field theory results with all analytic properties as well as anti-analytic ones (dispersion relations Kutkosky rules) intact.

It was shown to be the case at one loop level. First offending terms at $\mathcal{O}(\alpha^2)$ of no logarithmic enhancements.

I can not review this domain, even give good references to the effort. Personally I profited from discussion with Robin Stuart and long work with W. Hollik group and later D. Bardin group.

- I have presented essential elements of theoretical background for precision Monte Carlos. The focus: eikonal QED .

I have not presented actual effort on writing, managing, user servicing of the programs. Nor the programs or their calculations.

- Massive effort on tests, evaluation what must be included, and what may be left for future more demanding precision was dropped.

- **This work was never single person project** : I should mention first of all Stanislaw Jadach, Bennie Ward but not only. Dimitry Bardin, Bob van Eijk, Y. Shimizu, Johann Kuhn and their research groups provided important elements.

- Some people impacted indirectly the projects.

Sometimes I have realized importance only much later, nonetheless it is worth mentioning now. For example, Dr. Zbigniew Klimek pointed to me some mathematical aspects of Einstein equations solutions: limits of perturbative expansions, due to topological changes. For many years I thought that of no importance, unless accidentally analogy (forgotten inspiration?) was pointed to me.

Excellent training on Lorentz group and representation available in early 80's in Cracow.

Main challenges for future:

- **improve precision to FCC standards, by about a factor of ten,**
- **attract new people and assure that they will stay in the domain.**
- **Preserve expertise and develop new skills.**
- **Assure coherent development:**

— exponentiation require additional effort on fixed oder calculations

— detector granularity (background subtractions) require fine details of phase space treatment: limitations for cone leptons etc.

— ...

Thank you for listening.

Some references which came to my mind, when I was preparing slides:

- Z. Was, “Radiative corrections,” CERN-TH-7154-94.
- S. Jadach, B. F. L. Ward and Z. Was, “Coherent exclusive exponentiation for precision Monte Carlo calculations,” Phys. Rev. D **63**, 113009 (2001)
- S. Jadach, B. F. L. Ward and Z. Was, “The Precision Monte Carlo event generator K K for two fermion final states in e+ e- collisions,” Comput. Phys. Commun. **130**, 260-325 (2000)
- S. Banerjee, A. Y. Korchin and Z. Was, “Spin correlations in τ-lepton pair production due to anomalous magnetic and electric dipole moments,” Phys. Rev. D **106**, no.11, 113010 (2022)
- E. Richter-Was and Z. Was, “Adequacy of Effective Born for electroweak effects and TauSpinner algorithms for high energy physics simulated samples,” Eur. Phys. J. Plus **137**, no.1, 95 (2022)
- T. Przedzinski, [arXiv:2203.11650 [cs.SE]].
- Z. Was, *”On development startegies - case of Precision Standar Model Monte Carlo programs”* at work.